Department of Applied Mathematics
Faculty of EEMCS

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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Basic notions in mathematics: On the "graph" in particular and on ontology in general

C. Hoede

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C. Hoede

Faculty of Mathematical Sciences
University of Twente
P.O. Box 217
7500 AE Enschede, The Netherlands

Abstract

This paper contains material presented on 31 January 2002 to staff and students of the Department of Mathematics of the University of Utrecht. The talk was one in a series of lectures titled Basic Notions in Mathematics.

The paper follows closely the talk, that consisted of six parts; basic definitions, a short survey of important results, some remarks in connection with the Maximum Clique Problem, some relations with other fields in mathematics, knowledge representation and some thoughts on the usefulness of mathematics in science.

Key words: Mathematics, notion, graph, ontology.

AMS Subject Classifications: 00A05, 05C99.

1 The notion of *graph*

The most basic notion in mathematics is probably that of *set*. It suffices to describe graphs, directed graphs and hypergraphs.
A *graph* $G = (V, E)$ consists of a set V of *vertices* and a set E of *edges*. Terminology in graph theory is not unique. Vertices are often called nodes or points, edges are often called lines. V can be seen as a basic set, we will consider finite sets only, and E as something extra, namely as set of unordered pairs chosen from V, so $E \subset V\&V$.
A *directed graph* $\vec{G} = (V, A)$ has a basic set V of vertices too, while now the extra thing is A, a set of ordered pairs chosen from V, so $A \subset V\&V$. Its elements are called *arcs*. A *hypergraph* $H = (V, E)$ again has a basic set V, while now we consider E to be a subset of the power set of V, so $E \subset \mathcal{P}(V)$. E is just a set of subsets of V, called *hyperedges*. If all hyperedges have cardinality 2 we regain an ordinary graph.

Most definitions in graph theory are almost self-evident. One easily gets familiar with the field and when one guesses what a "path", a "cycle" or a "circuit" is, one is almost always right. For terminology I refer to any of many books

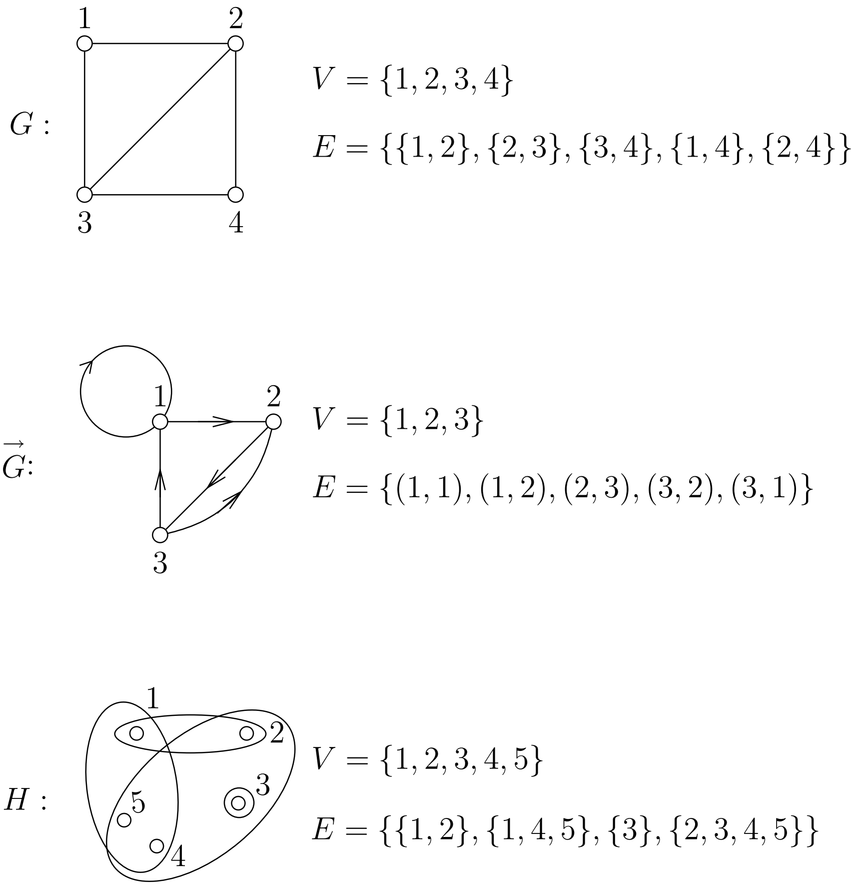


Figure 1: A graph, a directed graph and a hypergraph

on graph theory, e.g., to the book of Bondy and Murty [3]. A graph G can be *isomorphic* to a graph H, and to itself, in which case we speak of auto*morphism.* The automorphisms of G form a group.

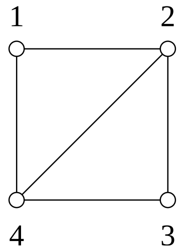
If a set of vertices of G has the property that no pair forms an edge, they are called an *independent* set of vertices, or stable set. A similar notion is that of *independent* set of edges, or matching, that has the property that no two edges have a vertex in common.

The cardinality of V is called the *order* of G, usually denoted by n and the cardinality of E is called the *size* of G, usually denoted by m.

An edge is said to be *incident* with its two constituent vertices and two vertices are *adjacent* if they form an edge, whereas two edges are *adjacent* if they have a vertex in common. The number of edges incident with a vertex is called the *degree* of that vertex, δ and Δ denoting the minimum respectively maximum degree occurring in a graph.

A *subgraph* $G = (V', E')$ is what the word suggests. Its vertex set V' is a subset of V and its edge set is a subset E' of E. Here caution must be taken, the elements of E' should consist of pairs of V' and not all such edges present in G need be elements of E'. If so, G' is called *induced* subgraph. If $V' = V$, the subgraph is called *spanning.* A *1-factor* is a spanning subgraph, where all

vertices have degree 1, its edges forming a matching. A *2-factor* is a spanning subgraph, where all vertices have degree 2, a graph consisting of cycles.
A graph can be represented by a picture in which the vertices are drawn as points and the edges are drawn as lines between pairs of points in arbitrary way. Quite useful is the representation by the adjacency matrix A, a symmetric $(n \times n)$-matrix with elements 0 or 1,1's indicating the adjacency, or by the incidence matrix M, a $(n \times m)$-matrix in which the rows are labeled by the names of the vertices, usually just numbers 1 to n, and the columns by the edges, so by pairs of numbers from $\{1, 2, \ldots, n\}$.



$$A = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 1 & 1 & 1 & 0 \end{array} \qquad M = \begin{array}{c|ccccc|} & 1 & 1 & 2 & 2 & 3 \\ & 2 & 4 & 3 & 4 & 4 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 0 & 1 & 1 \end{array}$$

Figure 2: Adjacency matrix and incidence matrix

Of the words in italics used sofar, I would like to stress the notions

- *set*
- *morphism*
- *independent*
- *order* and *size*
- *sub-* in "subgraph"

One more notion to mention is that of *operation* on a graph. The important operations are *contraction* and *deletion* of an edge.

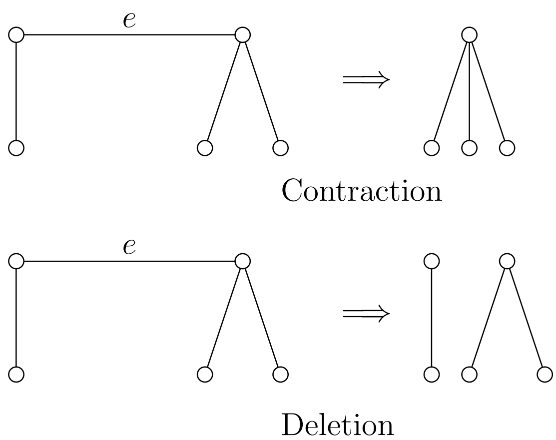


Figure 3: Two operations on graphs

A *minor* of a graph G is a graph obtainable from G by a sequence of deletions and contractions. There are several other operations that have been defined, but these two are particularly mentionable.

2 Classical results

The following is a rather personal choice of results.

a. Menger's theorem

This theorem is chosen first as there are four other, equivalent, theorems. The theorem, as are the others, are stating that some maximum is equal to some minimum. Suppose a graph is *connected*, one of those notions everybody guesses right, then between two vertices x and y there may be different vertex *disjoint* paths. On the other hand x and y may be disconnected from each other, by a "cut" of vertices. We mean by cut that deleting these vertices and their incident edges makes the graph disconnected, unless xy is an edge with x and y in distinct components, what we assume not to be so. There is obviously a minimum number of vertices that form a cut and Menger's theorem is just that the maximum number of vertex disjoint paths between x and y equals the minimum number of vertices in a cut.

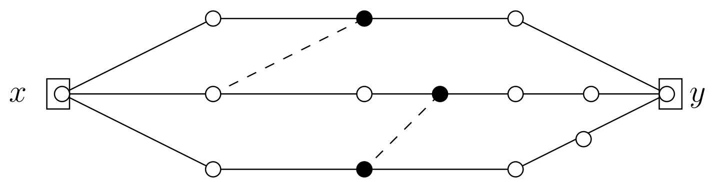


Figure 4: Maximum number of vertex disjoint paths = 3
Minimum number of vertices cutting x from y on deletion = 3.

An equivalent theorem is that of Ford & Fulkerson. The setting is now that of an oriented network, with x and y now called source and sink, of which the arcs are capacitated, e.g. by positive integers. Instead of paths we now have directed paths from x to y carrying a unit "flow" of value 1. The natural question is how large the total flow from x to y can be, taking into account the capacities on the arcs. The maximum flow value equals the minimum value of a "cut" again, but now a cut is a set of arcs and its value is the sum of the capacities of these arcs. Partition V into two sets X and Y, $x \in X$, $y \in Y$ and consider all arcs from a vertex in X to a vertex in Y. These form a cut.

In operations research whole courses are dedicated to this "max flow = min cut" theorem. We will meet the other three theorems later.

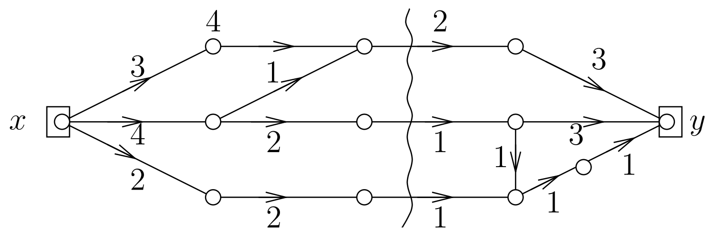


Figure 5: Capacitated oriented network.

b. Kuratowski's theorem

K̲asimir K̲uratowski lend his initials to two small graphs: $K_{3,3}$ and K_5, K_5 is the *complete* graph, or clique, on five vertices (all 10 edges are present) and $K_{3,3}$ is well-known from the puzzle of connecting three houses with three distribution stations, gas, water and electricity, in such a way that no connections cross. This is a (complete) *bipartite* graph, its 6 vertices partitioned into two sets of three (independent) vertices. The solution to the puzzle is that it is impossible.

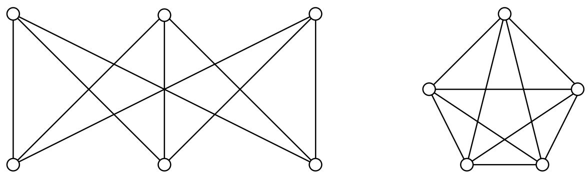


Figure 6: $K_{3,3}$ and K_5 cannot be drawn without crossing edges.

This is a consequence of Kuratowski's characterization of planarity of graphs, their feature to be embeddable is the plane (or on the sphere) without edges crossing. The theorem states that G is planar if and only if G has no subgraph homeomorphic to $K_{3,3}$ of K_5. G is homeomorphic to H if G can be transformed to H by contraction of edges incident with vertices of degree 2. So G is H, apart from vertices that subdivide its edges.

c. Coloring theorems.

Probably the most famous graph is Petersen's graph P.

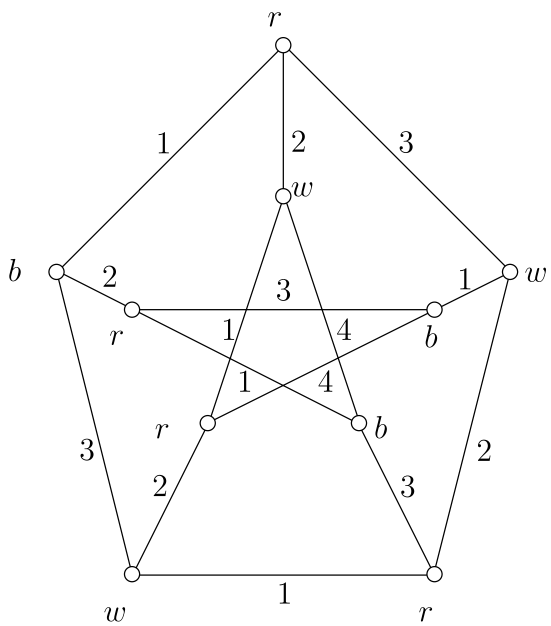


Figure 7: Petersen's graph P, with an edge coloring and a vertex coloring.

The edges of P form a 1-factor and a 2-factor, a matching and two cycles. P is notorious for its ability to provide counterexamples to conjectures. Several results in graph theory have the form "G has this or that property if G does not contain P as a minor". There are books just on the Petersen graph [5]. Particularly interesting are the coloring properties of P.

A (proper) vertex coloring of a graph is a labeling of its vertices with the names of the colors, so that adjacent vertices have different colors. A (proper) edge

coloring of a graph is a labeling of its edges with the names of the colors, so that adjacent edges have different colors, P is 3-vertex-colorable but not 2-vertex-colorable, 4-edge-colorable but not 3-edge-colorable. So 3 is the minimum number of colors needed for a vertex coloring and 4 is the minimum number of colors needed for an edge coloring. We say that the *chromatic number* χ of P is a, and that the *chromatic index* χ' of P is 4.
Of course now we want to know about the possible values for χ and χ'. Vizing has proved that

$$\Delta \leq \chi' \leq \Delta + 1.$$

There are only two values possible for a graph with maximum degree Δ. We therefore speak of class -1 graphs and class -2 graphs. P is a class -2 graph as $\Delta = 3$ and $\chi' = 4$. People have long been searching for cubic class -2 graphs. These are class -2 graphs that are regular, all vertices have the same degree, and this degree is 3. They turned out to be so rare that they were baptized snarks after a poem mentioned in "Alice in Wonderland". P was the first snark found.
The field of coloring graphs developed mainly around the socalled 4-color problem. Given a planar graph, can the faces be colored with 4 colors so that neighbouring faces have different colors? Sofar we considered only vertex colorings and edge colorings. However, representing faces by vertices and joining these vertices whenever the faces are neighbouring, gives the *dual* graph, that is also planar, see Figure 8, and thus the face-coloring problem is turned into a vertex-coloring problem.

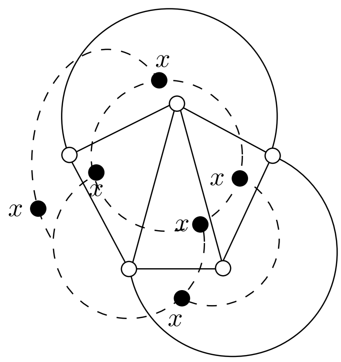


Figure 8: A graph and its dual graph.

That four colors might suffice, is suggested by Kuratowski's theorem and the fact that $\chi(K_5) = 5$ and $\chi(K_{3,3}) = 2$. This problem has been open for about 125 years and was finally solved in 1976 by Appel and Haken, by massive use of the computer. Heesch did important preliminary work.
The counterpart of Vizing's theorem is Brooks' theorem that states that G is Δ-vertex-colorable, unless G is a clique or an odd cycle. We already remarked that $\chi(K_5) = 5$, whereas $\Delta(K_5) = 4$, and one convinces oneself easily that $\chi = 3$ for an odd cycle.

d. Hamilton cycles.

One of the oldest results about cycles and tours is that of Euler (1736) on tours in graphs that contain every edge precisely once. A hamiltonian cycle of a graph G is a cycle that contains every vertex of G.

The five platonic graphs are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. Hamilton invented a puzzle on the dodecahedron, see Figure 9, in which a path of four vertices was to be continued into a Hamilton cycle.

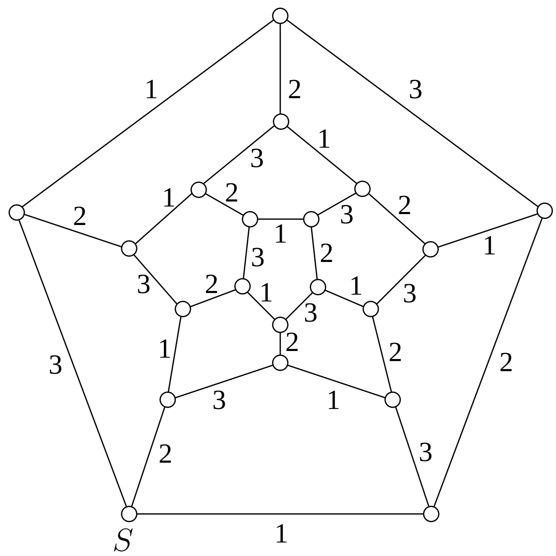


Figure 9: The dodecahedron. From starting point S edges with colors 1 and 2 alternatingly form a Hamilton cycle.

This implies that the dodecahedron D is hamiltonian. Also we have $\chi'(D) = 3$ and $B = 3$, so that D is a class -1 graph.

As D is cubic, 3-regular, and the sum of all degrees of a graph is even, $|E(D)|$ is even and the edges of the Hamilton cycle can be colored alternatingly with two colors. The edges not part of the cycle then can get a third color, hence $\chi'(D) = 3$.

3 Complexity

The notion of complexity in graph theory has two meanings. First there is the notion of complexity of a graph. A *tree* is a connected graph with no cycles. A *spanning* tree of a graph G is a subgraph containing all the vertices of G. The *complexity* of a connected graph G is defined as the number of its spanning trees. The result to mention here is that the number of spanning trees of a labeled complete graph K_n on n vertices is n^{n-2}. With the given definition the complete graph is the most complex graph.

Second the notion of complexity is used for algorithms that solve a graphtheoretical problem. Such a problem may be

Given	:	A graph G on n vertices.
Question	:	Does G contain a Hamilton cycle?

One can just consider all permutations of labels given to the vertices of the graph and check whether there is some ordering of the vertices so that consecutive vertices constitute an edge of G and first and last vertex are adjacent. This algorithm takes an exponential number of steps in terms of the number n of vertices. The algorithm has high complexity. Is there an algorithm that takes a number of steps that is polynomial in n?

This leads us to the important class of $\mathcal{NP}$-complete problems, for which I refer to the book of Garey and Johnson [4]. The Hamilton cycle problem is one of thousands of problems for which no algorithm of polynomial complexity is known. If one of these problems can be solved with a "polynomial" algorithm, then all can. So one can just try to find such an algorithm for one's favorite problem. That approach is very easy. But what if no polynomial algorithm can be found? How to prove that this is the situation? I am not aware of approaches to proving that, in terms of the theory, $\mathcal{P} \neq \mathcal{NP}$.

I want to use the opportunity to tell about an idea that developed while I tried to find a polynomial algorithm for my favorite, the Maximum Clique Problem.

Given	:	A graph G on n vertices.
Question	:	Is there a complete subgraph K_k in G, where $1 \leq k \leq n$?

Solving this problem for all k gives a maximum value for k. Therefore the name Maximum Clique Problem (MCP).

My experience with "MCP" was that I found a polynomial algorithm A_1, but the next week found a counterexample to A_1. Then I found a polynomial algorithm A_2, that could handle that counterexample for A_1 in polynomial time. The following week I found a counterexample to A_2. You can guess what followed. Again the algorithm was adapted to A_3, that could handle that counterexample again. After this the pattern became clear. *Every algorithm for solving "MCP" seems to have a specific counterexample that needs an exponential number of steps.* But this implies an approach. Given an algorithm that solves some $\mathcal{NP}$-complete problem, like "MCP", say in the form of some Turing machine program. It is imaginable that the program must show some features that ensure that the problem is indeed solved by it. My experience with "MCP" suggests that, whatever the form of the algorithm, the features allow the construction of a specific type of counterexample, in the sense that the program considered must run in exponential time for that specific input.

One of the organizers of this lecture series, Jan Brandts, rightly remarked that this idea had similarity with the diagonalization argument for proving that the real numbers are not countable. I have just used this opportunity to make this remark, maybe somebody can do something with it, but also because the given problems illustrate that $\mathcal{NP}$-completeness can be studied in a very natural way in a graphtheoretical setting.

4 Relations with other fields

Let us now focus of the main theme, the graph as basic notion. Sofar we only said that $G = (V, E)$ is a set V, with something extra, namely a second set E consisting of pairs of elements of V. The notions "set" and "pairs" enable the definition of a graph. A graph is a very simple example of a *structure*, which I would like to define here as *some ground set V with something on top of V*. It is with respect to that what comes on top of the ground set V that different basic notions can be distinguished. We will discuss this more extensively in the next section. In this section some relations with other fields will be discussed. As reference the reader is recommended the wonderful book of Beineke and Wilson [1].

a. Partial orders

The partially ordered set P is another structure, that has considerable relationship with the graph. For our discussion two statements are interesting.

> "A graph is a special kind of partially ordered set"
> "A partially ordered set can be regarded as nothing more than a special kind of directed graph".

Corresponding with these two statements are two transformations. In Figure 10 and 11 we give examples of transforming a graph G into an incidence order $P(G)$, respectively of transforming a partially ordered set P into a bipartite graph $G(P)$.

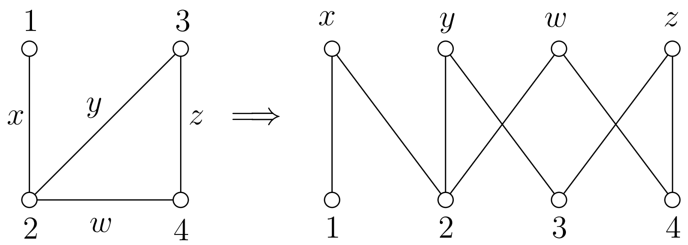


Figure 10: From graph to incidence order.

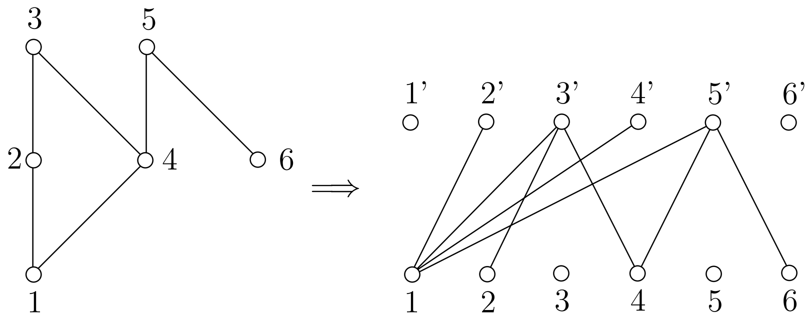


Figure 11: From partially order set to bipartite graph.

The close relationship is, of course, due to the fact that in the notion of directed graph the arcs are *ordered* pairs.

We should now mention the three theorems that are equivalent to the theorems of Menger and Ford and Fulkerson. One of these is Dilworth's theorem: Let $P = (X, <)$ be a finite partially ordered set with height h and width w. Then

(a) the ground-set X can be partioned into h antichains,

(b) the ground-set X can be partioned into w chains,

The *height* of P is the size of a longest chain in P. The *width* of P is the size of a largest antichain.
The other two theorems are:
Hall's theorem, also known as marriage theorem, giving the condition under which in a bipartite graph one class of vertices can be matched in the other: Each subset should have at least as many neighbours in the other class.
König and Egervary's theorem, stating that the maximum number of edges in a matching of a graph is equal to the minimum number of vertices covering all the vertices. This theorem exhibits the typical MAX-MIN form, already met in the theorems of Menger and Ford and Fulkerson.

b. Logic

The relationship with logic is of special importance for the next section on knowledge representation.
Consider the very simple formula $\exists x_1 \exists x_2 E(x_1, x_2)$ in first order propositional logic. We can represent this formula by a graph, see Figure 12.

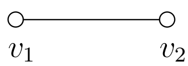


Figure 12: $\exists x_1 \exists x_2 E(x_1, x_2)$.

The predicate $E(x_1, x_2)$ can just be read as "there is an edge between x_1 and x_2".
On the other hand the labeled graph in Figure 13

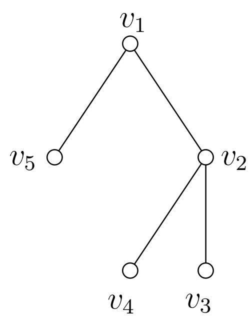


Figure 13: Some labeled graph.

can be expressed by

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 E(x_1, x_2) \wedge E(x_1, x_5) \wedge E(x_2, x_3) \wedge E(x_2, x_4).$$

Charles Sanders Peirce (1893) used socalled *existential* graphs to describe logic. One only needs to express the $\wedge$-operator and the $\neg$-operator, as these two consecutives form a functionally complete set for first order logic. Peirce just put types of "frames" around two propositions p and q, to express $p \wedge q$, respectively around one proposition p, to express $\neg p$.

c. Algebraic graph theory.

There has been considerable effort to describe properties of graphs by the spectrum of eigenvalues of the adjacency matrix A. Several alternatives for A have been proposed, and I would like to mention especially the Seidel matrix $S = J - 2A - I$, where J is the $n \times n$ matrix consisting of 1's only and I is the $n \times n$ unit matrix. Graph theory in Eindhoven has mainly been what is called "crystalline" graph theory. In Twente we focussed more on what is called "amorphous" graph theory.
The reader is referred to the book "Spectra of Graphs" by Cvetkovic, Doob and Sachs for further information.

d. Linear algebra.

A very simple example of a result involving the adjacency matrix A is that $(A^k)_{i,j}$, the i,j-element of the k-th power of A, gives the number of walks of length k from vertex i to vertex j.
A much more important result came from as study of Whitney (1935), who "had the aim of capturing the fundamental properties of *dependence* that are common to graphs and matrices". This led to the notion of *matroid*. Given a graph G one can define a matroid $M(G)$, which is therefore called a *graphic* matroid. But there are matroids for which no graph H can be given such that $M(H)$ is isomorphic to the matroid considered. These matroids are called *non-graphic* matroids. In the definition, there are several other definitions, we meet a returning theme:
A *matroid* M is a pair (E, C) consisting of a finite set E, called the ground set, and a collection C of subsets of E, called circuits, such that:

i) $\emptyset \in C$.

ii) If c_1 and c_2 are members of C, and if $c_1 \subseteq c_2$, then $c_1 = c_2$.

iii) If c_1 and c_2 are distinct numbers of C, and if $e \in c_1 \cap c_2$, then there is a member c_3 of C such that $c_3 \subseteq (c_1 \cup c_2) - \{e\}$.

So we are dealing with a "structure" again. A ground set E, denoted this way as in a graph E is the set of edges, and something on top of that; subsets of E, that are called circuits as in a graph these edge sets form cycles. The interesting thing is the occurrence of non-graphic matroids.
Consider $E = \{1, 2, 3, 4\}$ and $C = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. One can check that the elements of C are indeed circuits of a matroid, called $U_{2,4}$. But it is impossible to construct a graph with four edges such that the circuits occur as cycles (triangles) of that graph.
The name matroid stems from "matrix". Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix},$$

where the column vectors are considered over the set $\mathbb{R}$ of reals. Triples of these four column vectors are dependent. These triples correspond to the circuits of $U_{2,4}$ and form an isomorphic matroid over the ground set of column vectors, called the vector matroid $M[A]$. For graphic matroids we should look upon cycles in the graphs as consisting of a "dependent" set of edges.

Here we have a good place to mention one of the deepest results in graph theory. We already discussed minors of graphs. Also the notion of antichain in a partially ordered set was mentioned. An *antichain* is a set of graphs (or matroids) such that no member of the set is isomorphic to a minor of another member of the set.
The important result meant is the Robertson and Seymour theorem:

There is no infinite antichain of graphs.

e. Groups.

A group is just another structure again. The ground set is formed by the elements and now some rules for mapping two elements on a third element are added. The relationship with graphs is via the notion of automorphism of a graph. These automorphisms form the ground set now, and the automorphisms form a group.
For most graphs the automorphism group is trivial. However, we have

Frucht's theorem: Every group is the automorphism group of some graph.

One might say, in analogy with the discussion on matroids, that "every group is "graphic"".

f. Knots

A *knot* is defined as an embedding of a circle into R^3. The subject does not immediately make one think of graphs. Where is something like a ground set? The first step in answering this question is to remark that an embedding into R^3 can be represented in the plane by a socalled *link diagram*, see Figure 14.

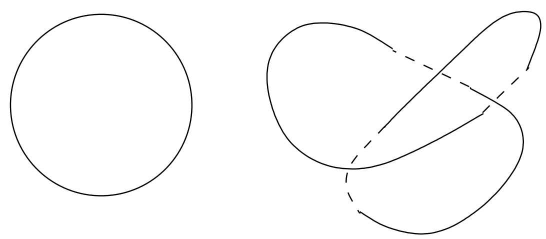


Figure 14: Two non–isotopic link diagrams.

A dotted part of the circle passes underneath another, drawn, part of the circle. Two link diagrams are *isotopic*, when they can be transformed into each other by socalled *Reidemeister moves.* These moves are ways to "unknot" the knot to a simplest form. Each knot in simplest form has a class of isotopic link diagrams associated with it.
Each link diagram determines areas of the plane, the faces, that can be colored black and white, in such a way that no two faces with a common edge have the same color. The ground set chosen now is the set of e.g. the black faces. These

form the vertices of a graph. The edges are added according to whether two black faces share a crossing point of the circle or not. But there are two types of crossings! Well, these are translated into signs "+" or "−" of the edges. In this way we find the theorem. There is a one-one correspondence between link diagrams and signed planar graphs. Of course, the Reidemeister moves can be translated into operations on these signed planar graphs.

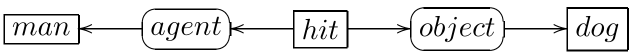
The basic character of the notion of graph within the family of structures should by now be clear. What is put on top of the ground set V, is of extremely simple nature in the case of the graph. Something simpler than "a set of pairs" is not very easily found. "A set of ordered pairs" introduces the notion of ordering as well. So the notion of directed graph can be considered to be less basic than that of graph in this line of reasoning. In the next section the discussion about basic notions will be put in a completely different light.

5 Knowledge representation

Ontology comes from Greek and means something like science of being. Hundreds of ontologies have been developed, sets of basic notions. So what this series of lectures is really about is an ontology for mathematics.

Knowledge representation knows various systems, one of which is well-known. It is the system of *semantic networks*, in which vertices represent concepts and edges represent relationships, that can be of (many) types.

Following Peirce's theory of essential graphs, mentioned in 4.b. Logic, Sowa (1984) introduced *conceptual graphs*, where again the vertices represent concepts, but now the edges represent a restricted number of types. An example of such a representation is:



representing the sentence "man hit dig" and that is to be read as "hit" has agent "man" and object "dog".

Before reporting on our own theory of *knowledge graphs*, developed in Twente and Groningen from 1982 on, we should discuss two of the most famous ontologies.

The ontology of Aristotle:

- quantity
- quality
- relation
- location
- time
- position
- substance
- having
- doing
- being affected.

These are the ten basic notions according to Aristotle. They clearly focus upon physical aspects.

Quite different is the ontology of Kant:

- Quantity : unity, plurality, totality
- Quality : reality, negation, limitation
- Relation : inherence, causality, commonness
- Modality : possibility, existence, necessity.

These four groups of three basic notions clearly focus more on logical aspects, The ontology of knowledge graph theory consists of one basic element, to be represented by a vertex, which is called *something.* Knowledge is represented, as in semantic networks or conceptual graphs by vertices, somethings, that are linked by eight types of binary relationships. These eight types are:

- EQU
- SUB
- ALI
- DIS
- ORD
- CAU
- PAR
- SKO

.

Note that these types are given by triples of letters, not by words. The reason will be discussed in the last section. They are supposed to model:

- equality
- part-of-ness
- alikeness
- disparateness
- ordering
- causality
- attribution
- informational dependency

.

Next to the binary relationship between the vertices, also called tokens, there are four types of "frames", similar to the frames used by Peirce. Any structure consisting of tokens, related by binary relationships, of which possibly some partial structures have been seen as units already, may be seen as a unit, a *frame.* The idea is that impressions lead to a representation in the mind that has the form of a graph, the *mind graph.* Subgraphs of this mind graph can be "framed and named". This is where words come in. The vertices and arcs that are seen as frame can be seen as a one-element n-ary relationship.
However, by the process of framing the vertices and arcs are also put in a relationship with the frame itself. Four types of frames are distinguished:

- FPAR
- POSPAR
- NEGPAR
- NECPAR

.

These four ways of framing parts of the mind graph, allow for the description of logic. If a subgraph of the mind graph represents a proposition p, note that this is not necessarily so, then the four frames around that subgraph represent p, $\neg p$, $\Diamond p$ and $\Box p$ in logic, see van den Berg [2].

For the discussion of basic notions in mathematics, it was interesting for me to see to which basic notions the ontology of knowledge graph theory would lead. The basic element, token, that is represented by a vertex and called something, the top element of any type hierarchy, has the character of a *prenotion.* Awareness of two or more somethings can be seen as the awareness of a notion like *set.*

The four types of binary relationships EQU, SUB, ALI and DIS were chosen into the ontology as there are four ways two sets can be related. The corresponding mathematical notions are:

EQU : *equality* (notions with names ISO- or EQU-)
SUB : *subset* (notions with names SUB-)
ALI : *similarity* (the notion of non-empty intersection of two sets)
DIS : *disjointness* (the notion of empty intersection of two sets).

The two binary relationships of type ORD and CAU were chosen into the ontology in reflection on the space-time aspects of the world. The ordering relationship seems the dominant relationship, as comes forward in many words like "before", "after", "from", "to", etcetera. According to the philosopher Hume the causal relationship is a composite of the ordering relationship and other types. We have maintained it as it is so important in the application field of expert systems. So

ORD : *ordering*
(CAU) : (*causality*) .

The two last types of binary relationships chosen are due to the fact that both attribution and informational dependency are notions that have to do with the functioning of minds. In the sentence "This is a nice pitbull", it is a problem to bring about a relationship between "nice" and "pitbull". It is clearly an attribution by the owner, not a part of the dog, as expressed by the *sub*-relationship, nor a part of the definition of pitbull, as expressed by the FPAR-relationship. In the theory three merological (part of-) relationships are distinguished and we make clear distinction between:

part (of) : SUB
property (of) : FPAR
attribute (of) : PAR .

The SKO-relationship was introduced by van den Berg and Willems, in connection with the problem to express universal quantification in a graphtheoretical way. The name is derived from the name Skolem, a logician. Mathematical notions based on the two types are

PAR : *number*, etc.
SKO : *mapping* .

Especially in the category of attribution, described by the PAR-relationships, many notions "came to mind" (which is to be taken quite literally).

From the point of view of knowledge graph theory basic notions in logic are

FPAR : *proposition*
NEGPAR : *negation*
POSPAR : *possibility*
NECPAR : *necessity* .

Finally, something should be said about language. The content of a frame may represent a proposition. However, when a specific subgraph of the mind graph is "framed and named", the name of the frame is a notion that is *defined* by the subgraph, also when that subgraph does not express a well-defined formula, like in Section 4.b. Each *word* has a corresponding *word graph*, and each sentence a corresponding *sentence graph*. This gives the possibility to develop a knowledge graph theory of language, that stresses semantics, see Willems [7] and Liu [6].

6 Philosophical aftermath

The choice of the ontology in knowledge graph theory can be made without justification. The relationship types seem reasonable and we can just see how

far we come with the chosen ontology. Can we represent all knowledge by graphs? In semantic network theory the attitude is to just add a new type of relationship if one gets into trouble. But the goal is more ambitious.
Ontologies for Aristotle and Kant definitely were intended to cover "everything". The same holds for the knowledge graph ontology. Therefore the attempt to represent all words, in whatever language, by word graphs. Now words were said to come in when "framing and naming". So before any words are used the mind graph is assumed to exist already. That is why triples of letters were used to describe the types! The relationships are considered to be processed by the brain. The neural networks involved, recognize an ordering relationship for example. What we want to do is make a guess about the different types of neural networks, really.
In the course of millions of years the brains of animals (including homo sapiens) have developed in reflection upon or reaction to external stimuli. The nature of the world must therefore have played a crucial role in the development of neural networks.
A slogan in knowledge graph theory is that "Thinking is linking somethings". The linking takes place by neural networks recognizing types of relationships (on the sub-word level). What now is the guess that we can make about the types that have developed? Neurophysiologists in 2100 might tell us, but at the moment we have to make a guess.
The philosophy is the following:

A. The world has a granular structure (due to its quantummechanical nature). This must have led to recognition of *sets* and relationships between sets.

B. The world has a space-time aspect. This must have led to recognition of *ordering*. Animals that cannot process ordering correctly soon starve.

C. There are minds in the world. This must have led to recognition of *attribution*.

This is the reasoning behind the choice of the eight binary relationships types.

This philosophy has a strange drawback on the discussion of basic notions in mathematics!! What do mathematicians do? They consider structures, like graphs, and are animals equipped with a certain set of neural networks, developed in the course of millions of years in reaction to outer-world stimuli.
The notions they develop are conditioned by these neural networks. So of course they consider sets, orderings, mappings. The "framing and naming" ability introduces both language and logic. Having brains that are completely conditioned by the outer-world stimuli it is extremely difficult not to think in terms of the basic relationships their brains process.

The strange drawback is: It is NO miracle that mathematics is so useful in other sciences! Whatever mathematicians develop as theory is more or less doomed to be applicable!!!

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