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A duopoly model with heterogeneous congestion-sensitive customers

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A duopoly model with heterogeneous congestion-sensitive customers

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Abstract

This paper analyzes a model with multiple firms (providers), and two classes of customers. These customers classes are characterized by their attitude towards 'congestion' (caused by other customers using the same resources); a firm is selected on the basis of both the prices charged by the firms, and the 'congestion levels'. The model can be represented by a two-stage game: in the first providers set their prices, whereas in the second the customers choose the provider (or to not use any service at all) for given prices. We explicitly allow the providers to split their resources, in order to serve more than just one market segment. This enables us to further analyze the Paris metro pricing (PMP) proposal for service differentiation in the Internet.

We prove that the stage-2 game (the customers' behavior for given prices, and a given division of the providers' resources) has a unique equilibrium. Insight is gained into the structural properties of the equilibrium. We also show that the objective functions in the stage-1 game are continuous (in the providers' decision variables), thus enabling an efficient search for its equilibrium. We comment on the viability of the PMP proposal.

Key words: Pricing – duopoly – equilibria – game theory – negative externalities – congestion – communication networks – quality-of-service differentiation

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1 Introduction

In various situations the utility experienced by a customer is strongly (negatively) affected by other customers using the same resources. Consider for example a service as Internet access: there customers select a provider not only on the basis of the prices charged, but also on the level of *congestion.* Also in many production systems this type of 'negative externalities' play a crucial role. The fact that customers react to differences in both prices and congestion leads to a complex interaction between suppliers (firms) and customers. If one of the firms increases its price, it becomes less attractive to the customers, but as a consequence also congestion will decrease. Particularly in situations with *heterogeneous customers*, i.e., customers with heterogeneous preferences with respect to price and congestion, this leads to challenging economic questions.

Situations with heterogeneous customers arise naturally in practice. In the example of Internet users, one could think of the difference between business customers and residential users (where business customers are, in general, more 'congestion averse'). Another obvious example relates to first class and second class passengers on a train or plane.

To exploit this intrinsic heterogeneity in preferences, firms could consider the option of 'splitting resources', in order to serve more than just one market segment. In the situation of Internet service, it would mean that the provider splits his network into separated logical subnetworks, each of them charging a different price. The subnetwork with the highest price will attract fewer customers (or, more precisely: a lower congestion level), and is consequently less-congested. As a result, the congestion-sensitive users choose the more expensive network, whereas the congestion-indifferent users opt for the cheaper network. Hence, the 'quality of service' (QoS) the customers experience agrees with the specific preferences of the users.

For Internet pricing, this concept was proposed by Odlyzko [12, 13], who also introduced the term *Paris metro pricing* PMP. Notice that the network's customers are not offered any absolute guarantee of quality of service, but rather a 'soft' relative indication. The major advantage of PMP over traditional QoS-differentiating mechanisms (such as priority queues, see e.g. [8, 10, 11]) is its 'self-organizing' character, in that the QoS differentiation is achieved by providing the network users with economic incentives. We note that often without any QoS-differentiating mechanism essentially just one segment of the market is served – usually the congestion-indifferent users 'push away' the congestion-sensitive users, a phenomenon referred to [5] as *tragedy of the commons.* There is a growing body of literature on QoS differentiation and the related pricing issues – we mention [4, 7, 14, 15], and various chapters of [9]. The recent book by Hassin and Haviv [6] presents a thorough survey on this issue.

System. Motivated by the above, the system considered in the present paper involves competition between two providers. Each provider has the opportunity to split its resource: in the case of the Internet provider, he can split his network into several smaller networks (subnetworks). These networks may differ in capacity and price. Customers choose between networks (and hence implicitly also between firms) based on their congestion levels and prices. If these congestion levels and/or prices are too high, a customer may consider the option of not joining any network at all ('balking'). Evidently, the choice of a customer affects the choice of other customers through the network congestion he incurs. Hence, there is also competition between the customers. This twofold competition (i.e., both among firms and among customers) is modeled as a two-stage noncooperative game. In the first stage, providers set capacities and prices for their networks, whereas in the second stage, customers choose which subnetwork to join, if any (for fixed prices and capacities of the subnetworks). It is noted that in the stage-1 game the firms know the way the customers react to their price and capacity decisions. In our paper we examine equilibrium outcomes in the composite (i.e., two-stage) game.

Literature. The Pmp proposal was further assessed by Gibbens, Mason, and Steinberg [3]. In their paper the focus is on competition between two Pmp-offering providers. In [3] it is shown that – under specific modeling assumptions – *neither provider sub-divides its resources.* In fact both providers focus on just one segment of the market, i.e., they offer a single service class. The other paper that is strongly related to ours, is by Armony and Haviv [1]. They also consider heterogeneous customers in a duopoly environment, but they do not incorporate the possibility of the providers splitting their resources. In [1] congestion is phrased in terms of response times (i.e., queueing delays). Christ and Avi-Itzhak [2] also study a duopoly model, namely a queueing situation in which two servers compete for arriving customers. The servers compete in their service rates and there are no different customer types. Selecting a certain service rate generates a cost. The authors show that if this cost is convex and increasing in the service rate, then there exists a unique Nash equilibrium. We refer to [1] for more references on competition models with congestion-sensitive services.

Modeling assumptions. As mentioned above, Gibbens, Mason, and Steinberg conclude in [3] that both providers offer just a single service class. This result, however, was derived under a number of specific assumptions. In their model providers are identical (their networks have the same capacity). Also, they can only choose between offering either a single or two networks of equal capacity. It is noted that [3] present numerical evidence that suggests that their main conclusion (i.e., Pmp is not sustainable under competition) carries over to the situation in which the providers can divide their capacity in not necessarily equal parts. In their framework there is a continuum of customer types, but the perhaps somewhat unrealistic requirement is imposed that these customers must select a network to join, and cannot opt for leaving.

The modeling assumptions in Armony and Haviv [1] are rather different from those in [3]: just two types of customers are identified, the providers are not necessarily identical, and customers have the option of balking. The procedure followed in [1] to analyze this model, is reminiscent of the two-stage game identified above. For the stage-2 game the authors prove existence and uniqueness of the Nash equilibrium, whereas the stage-1 game is analyzed numerically.

Contribution. Our paper presents a two-stage analysis of the duopoly model with negative externalities (due to congestion). Like in [1], and as opposed to [3], we consider two types of customers, that have the option of balking; we also allow non-symmetric providers who have the opportunity of splitting their network in two (not necessarily equal) parts. The structure followed resembles that of [1] – the major differences are: (i) we allow the providers to split their resources, (ii) congestion is expressed in terms of utilization of the resources, rather than queueing delay. In more detail, our contributions are:

- Existence and uniqueness of the equilibrium of the stage-2 game. This is done for general utility curves (as opposed to [1] where the utility is linear in the mean delay). In addition, we derive a detailed analysis of the equilibrium structure. We present a counterintuitive example that illustrates the phenomenon that the number of customers of a single type may be increasing with the network price.

- For stage 1 it is shown that the profit of each provider is continuous in prices and capacities. This profit function is piece-wise differentiable, and can consequently be computed numerically relatively easily. Unfortunately the profit function need not be quasi-concave, and therefore existence (and uniqueness) of an equilibrium cannot be established by standard machinery.

- Finally, with respect to Pmp under competition, we conclude that the outcome of the model critically depends on the assumptions of the model. In fact, we find, under our specific assumption, that it may pay off for both providers to split their resources, also if the providers can choose between offering either a single network or two networks of equal capacity.

The remainder of our paper is organized as follows. In Section 2 we introduce our model and describe the resulting 2-stage game. The second stage of this game is considered in Section 3, resulting in several structural properties of the customer populations in the various networks. In Section 4 the existence of a unique equilibrium for stage 2 (i.e., customer competition for fixed prices and capacities) is proven. Stage 1 is considered in Section 5. Section 6 illustrates that, under our modeling assumptions, in Pmp both providers may split their resources. Section 7 concludes.

2 Model

This section describes the duopoly model that we study in this paper. The agents are (i) two providers, and (ii) two user groups (with their specific preferences with respect to price and congestion). First, in *stage 1*, the providers set capacities and prices for their networks to maximize their profits. Then in *stage 2*, the customers of both types decide whether or not to enjoy network services, and, if so, from which network. Their goal is to maximize their utility.

Notice that the two stages can be solved sequentially: Given capacities and prices of the networks, the customers decide in stage 2 upon which network services to consume, if any. Knowing these decisions of the customers, in stage 1 the providers set their capacities and prices to maximize their profits. This sequential decision-making is modeled as a two-stage noncooperative game. In the stage-1 game the providers compete for profit, whereas in the stage-2 game the customers compete for network services.

The providers, I and II, are characterized by their respective capacities, C^{I} and C^{II}. Provider I splits his capacity into m^{I} subnetworks, numbered from I,1 to I,m^{I}, with capacities $C^{\mathrm{I},k} \geq 0$ such that $\sum_{k=1}^{m^{\mathrm{I}}} C^{\mathrm{I},k} = C^{\mathrm{I}}$. Each subnetwork charges a price $p^{\mathrm{I},k} \geq 0$ (selected by provider I). Provider II follows an analogous strategy. We assume that there are no costs for providing network services. The goal of each provider is to set his network capacities and prices such that his profit is maximized, given the choices of the other provider.

There are two types of (potential) users of the network resources. The (gross) utility they get from using a network depends on the level of congestion in the network. To be more precise: if they use a network of capacity C with n users, their utility U_i $(i = 1, 2)$ is a continuous function of the 'network utilization parameters' n and C. This is a situation with negative externalities, i.e., for given C the utility is a strictly decreasing function of the number of users n, while for given n the utility is a strictly increasing function of the capacity C. Customers value a network by the *net* utility, that is, gross utility minus price: $U_i - p$. A customer receives a net utility of zero units by rejecting all offered network services. Each customer aims at maximizing his net utility.

The two types of users differ in their attitude towards congestion. The second type of users dislikes congestion more than the first. We model this by assuming that $U_1(n, C) = \alpha \cdot U_2(n, C)$ for some $\alpha > 1$. There are $N_i \leq \infty$ (potential) customers of type i. The networks are ordered by decreasing prices. Let $n_i^{\mathrm{I},k}$ be the number of customers of type i $(i = 1, 2)$ in the kth network of provider I; define $n_i^{\mathrm{II},k}$ analogously. We assume that the customers are infinitely divisible. This implies that $n_i^{\mathrm{I},k}$, $n_i^{\mathrm{II},k}$ need not be integer numbers.

Now, if the utility-constant α, the numbers of customers N_1, N_2, and the capacities C^{I}, C^{II} are known, which capacities and prices will the providers set (and how many users of both types will join the subnetworks)? As argued in the introduction, we have a special focus on the question whether the providers will offer multiple subnetworks, that is, will $m^{\mathrm{I}} \in \{2, 3, \ldots\}$ and $m^{\mathrm{II}} \in \{2, 3, \ldots\}$, or not. In this paper, this question is analyzed by solving the 2-stage problem backwards, i.e., by solving both games sequentially, starting with the stage-2 game.

3 Stage 2: network selection problem

The second stage of the 2-stage problem describes the network selection problem of the customers given the capacities and prices as set by the providers. Suppose there are $m^{\mathrm{I}} + m^{\mathrm{II}} = K$ networks available, which we label, without loss of generality, such that *the sequence of prices is decreasing* – this choice turns out to be convenient when deriving structural properties of the equilibrium. The customers of both types can choose to join one of these K networks, or they can opt for not using any of these. The structural properties derived in this section are used in the next section, where we study existence and uniqueness of the equilibrium.

Each customer joins the network that provides him with the highest nonnegative (net) utility. If all utilities are negative, the customer remains inactive (yielding utility 0). We denote the network population of network j by customers of type i (for given prices and capacities) by

$$n_i^j(\overline{p}, \overline{C}), \quad i = 1, 2, \quad j = 1, \ldots, K,$$

with

$$\overline{p} \equiv (p^1, p^2, \ldots, p^K), \quad \overline{C} \equiv (C^1, C^2, \ldots, C^K),$$

The population profile $n(\overline{p}, \overline{C})$ is an *equilibrium* if it does not pay for any number of customers to join another network. In other words: if customers of type i are present in network k and if they receive a nonnegative utility from doing so, $U_i(n_1^k + n_2^k, C^k) - p^k \geq 0$, then deviating to network ℓ does not increase the utility of the customers:

$$U_i(n_1^k + n_2^k, C^k) - p^k \geq U_i(n_1^\ell + n_2^\ell + \delta_i^k, C^\ell) - p^\ell$$

for all types i and $0 < \delta_i^k \leq n_i^k$. Notice that we use the assumption that customers are infinitely divisible. Because of this assumption we write 'equilibrium' instead of 'Nash equilibrium' since the latter concept refers to individual customers (instead of an arbitrary number of customers) having no incentive to deviate. The following definition is equivalent.

Definition 3.1 *The population profile $n(\overline{p}, \overline{C})$ is an* equilibrium *if*

$$U_i(n_1^k + n_2^k, C^k) - p^k \geq U_i(n_1^\ell + n_2^\ell, C^\ell) - p^\ell$$

for all types i and networks k and ℓ with $n_i^k > 0$ and $U_i(n_1^k + n_2^k, C^k) - p^k \geq 0$.

¿From now on, let $n(\overline{p}, \overline{C})$ denote the equilibrium profile of network populations given the prices and capacities $(\overline{p}, \overline{C})$. The remainder of this section is devoted to the analysis of the specific structure of such an equilibrium profile (in the next section we prove that there exists a unique equilibrium). The first lemma shows that customers use the cheapest networks. Its proof follows directly from the definition of an equilibrium, and is therefore omitted.

Lemma 3.2 *For any value of $j \in \{1, \ldots, K-1\}$, the following situations cannot occur in an equilibrium:*

- $n_1^j > 0$, *and* $n_1^{j+1} = n_2^j = n_2^{j+1} = 0$.
- $n_2^j > 0$, *and* $n_1^j = n_1^{j+1} = n_2^{j+1} = 0$.

The following lemma is based on the strict monotonicity of the utility curves and prices.

Lemma 3.3 *For any value of $j \in \{1, \ldots, K\}$, the following situations cannot occur in an equilibrium:*

- $n_2^j > 0$, *and* $n_1^k > 0$ *for some* $k \in \{j+1, \ldots, K\}$.
- $n_1^j > 0$, *and* $n_2^k > 0$ *for some* $k \in \{1, \ldots, j-1\}$.

Proof. Consider the first claim – the second claim is verified analogously. Suppose the stated is not true. Because the situation is an equilibrium, type 1 customers do not have an incentive to deviate to network j:

$$U_1(n_1^k + n_2^k, C^k) - p^k \geq U_1(n_1^j + n_2^j, C^j) - p^j \ ,$$

due to Definition 3.1. Similarly, type 2 customers do not switch to network k:

$$U_2(n_1^k + n_2^k, C^k) - p^k \leq U_2(n_1^j + n_2^j, C^j) - p^j.$$

This results in

$$\alpha \cdot U_2(n_1^j + n_2^j, C^j) - \alpha \cdot U_2(n_1^k + n_2^k, C^k) \leq p^j - p^k \leq U_2(n_1^j + n_2^j, C^j) - U_2(n_1^k + n_2^k, C^k).$$

However, because of the assumption $p^j - p^k > 0$, this would imply $\alpha \leq 1$. Contradiction. □

From the above lemmas we can draw the following conclusions:

- From Lemma 3.2: If there are empty networks, then these must be the most expensive networks. Let network $j^\star$ be the first non-empty network.
- From Lemma 3.3: Suppose network $j \geq j^\star$ has type 1 customers. Then network 1 up to $j-1$ have no type 2 customers. Suppose network $j \geq j^\star$ has type 2 customers. Then network $j+1$ up to K have no type 1 customers. So there is *at most one network with customers of both types.*

These observations immediately lead to the following corollary describing the structure of the equilibrium.

Corollary 3.4 *Define* $J := \{(j, j^\star) : j \in \{1, \ldots, K\}, j^\star \in \{1, \ldots, j\}\}$. *An equilibrium has of one of the following structures:*

- *it is a* mixed *equilibrium* $M(j, j^\star)$ *if the two types of customers share a network: for some* $(j, j^\star) \in J$,

  (1) $n_1^k = n_2^k = 0$ *for all* $k \in \{1, \ldots, j^\star - 1\}$.

  (2) $n_1^k > 0$ *for all* $k \in \{j^\star, \ldots, j\}$, $n_2^k = 0$ *for all* $k \in \{j^\star, \ldots, j-1\}$, *and*

  (3) $n_1^k = 0$ *for all* $k \in \{j+1, \ldots, K\}$, $n_2^k > 0$ *for all* $k \in \{j, \ldots, K\}$.

- *it is a* separated *equilibrium* $S(j, j^\star)$ *if the two types of customers do not share a network: for some* $(j, j^\star) \in J$, *(1)* $n_1^k = n_2^k = 0$ *for all* $k \in \{1, \ldots, j^\star - 1\}$*.*

(2) $n_1^k > 0$, $n_2^k = 0$ *for all* $k \in \{j^\star, \ldots, j\}$*, and*

(3) $n_1^k = 0$, $n_2^k > 0$ *for all* $k \in \{j+1, \ldots, K\}$*.*

- *it is an* empty *equilibrium if no customers are present in the networks:* $n_1^k = n_2^k = 0$ *for all* k*.*

An equilibrium profile $n(\overline{p}, \overline{C})$ provides information on the utilities of the customers in the various networks, as follows from the next lemmas.

Lemma 3.5 *In an equilibrium,*

- *If* $\sum_{k=1}^K n_i^k < N_i$*, then necessarily* $U_i(n_1^k + n_2^k, C^k) - p^k = 0$ *for all* k *with* $n_i^k > 0$*.*
- *Also* $U_i(n_1^k + n_2^k, C^k) - p^k = U_i(n_1^\ell + n_2^\ell, C^\ell) - p^\ell$ *if both* n_i^k *and* n_i^ℓ *are positive.*

Proof. For the first claim, suppose that the stated it is not true: there exists k with $n_i^k > 0$ such that $U_i(n_1^k + n_2^k, C^k) - p^k > 0$. The utility function being continuous and decreasing in n, there exists $\varepsilon > 0$ such that $U_i(n_1^k + n_2^k + \varepsilon, C^k) - p^k$ is still positive. We conclude that ε customers who were inactive, have an incentive to join network k. Hence $n(\overline{p}, \overline{C})$ cannot be an equilibrium.
Also for the second claim, suppose that the stated is not true. Without loss of generality assume that $U_i(n_1^k + n_2^k, C^k) - p^k > U_i(n_1^\ell + n_2^\ell, C^\ell) - p^\ell$. Using once again that the utility function is decreasing in n, there exists $\varepsilon > 0$ such that $U_i(n_1^k + n_2^k + \varepsilon, C^k) - p^k > U_i(n_1^\ell + n_2^\ell, C^\ell) - p^\ell$. This means that ε customers of network ℓ have an incentive to deviate and join network k. This is in contradiction to $n(\overline{p}, \overline{C})$ being an equilibrium. □

Lemma 3.6 *In an equilibrium,* $\sum_{k=j^\star}^j n_1^k < N_1$ *implies that* $\sum_{k=j}^K n_2^k = 0$*.*

Proof. For a mixed equilibrium this result is seen as follows. Assume that $\sum_{k=j}^K n_2^k > 0$. ¿From $\sum_{k=j^\star}^j n_1^k < N_1$ and Lemma 3.5 it follows that necessarily $U_1(n_1^j + n_2^j, C^j) = p^j > 0$. Hence,

$$\begin{aligned} U_2(n_1^j + n_2^j, C^j) - p^j &= U_2(n_1^j + n_2^j, C^j) - U_1(n_1^j + n_2^j, C^j) \\ &= (1-\alpha) U_2(n_1^j + n_2^j, C^j) = \left(\frac{1}{\alpha} - 1\right) U_1(n_1^j + n_2^j, C^j) < 0. \end{aligned}$$

However, since we are in an equilibrium, also $U_2(n_1^k + n_2^k, C^k) - p^k = U_2(n_1^j + n_2^j, C^j) - p^j \geq 0$. Contradiction.
In case of a separated equilibrium it follows from $\sum_{k=j}^K n_2^k > 0$ that $U_2(n_2^{j+1}, C^{j+1}) - p^{j+1} \geq 0$. Consequently,

$$U_1(n_2^{j+1}, C^{j+1}) - p^{j+1} = \alpha U_2(n_2^{j+1}, C^{j+1}) - p^{j+1} > 0.$$

Therefore there exists a positive number ε of type 1 customers such that $U_1(n_2^{j+1}+\varepsilon, C^{j+1}) - p^{j+1} > 0$. These customers have an incentive to deviate because they can receive a positive utility (instead of zero utility) by joining network $j+1$. We again conclude that $\sum_{k=j}^{K} n_2^k > 0$ cannot hold. □

The above lemma implies that if $\sum_{k=j^\star}^{j} n_1^k < N_1$, then $j = K$. We also conclude that if some customers of type 2 are present in the networks, i.e., $\sum_{k=j}^{K} n_2^k > 0$, then also *all* customers of type 1 must be present, $\sum_{k=j^\star}^{j} n_1^k = N_1$. Hence, the customers of type 1 have some 'dominance' over those of type 2. Another type of dominance is stated in the property below.

Lemma 3.7 *In an equilibrium, if N_1 is large enough, then $n_2^k = 0$ for all networks k.*

Proof. Suppose that $n_2^k > 0$ in equilibrium for some network k. Then the customers of type 2 receive utility $U_2(n_1^k + n_2^k, C^k) - p^k \geq 0$. But then $U_1(n_1^k + n_2^k, C^k) - p^k = \alpha U_2(n_1^k + n_2^k, C^k) - p^k > 0$. When increasing N_1, at some point customers of type 1 will become inactive. These customers have an incentive to join k. Therefore this situation cannot be an equilibrium, and consequently $n_2^k = 0$. □

The following algorithm finds the equilibrium profile $n(\overline{p}, \overline{C})$, based on Corollary 3.4, and Lemmas 3.5 and 3.6.

Algorithm 3.8 'Mixed equilibria'. *For $(j, j^\star) \in J$, the equilibrium profile is of type $M(j, j^\star)$ if the following conditions hold. We set $n_i^k = 0$ for $i = 1, 2$ and $k = 1, \ldots, j^\star - 1$. Also $n_2^k = 0$ for $k = j^\star, \ldots, j-1$ and $n_1^k = 0$ for $k = j+1, \ldots, K$.*
In all cases $K - j^\star + 2$ equations in $K - j^\star + 2$ unknowns must be solved.

M_1 : *Solve under $\sum_{k=j^\star}^{j} n_1^k = N_1$ and $\sum_{k=j}^{K} n_2^k = N_2$*

$$U_1(n_1^{j^\star}, C^{j^\star}) - p^{j^\star} = \cdots = U_1(n_1^{j-1}, C^{j-1}) - p^{j-1} U_1(n_1^j + n_2^j, C^j) - p^j \geq 0, \text{ and}$$

$$U_2(n_1^j + n_2^j, C^j) - p^j = U_2(n_2^{j+1}, C^{j+1}) - p^{j+1} = \cdots U_2(n_2^K, C^K) - p^K \geq 0.$$

M_2 : *Solve under $\sum_{k=j^\star}^{j} n_1^k = N_1$*

$$U_1(n_1^{j^\star}, C^{j^\star}) - p^{j^\star} = \cdots = U_1(n_1^{j-1}, C^{j-1}) - p^{j-1} U_1(n_1^j + n_2^j, C^j) - p^j \geq 0, \text{ and}$$

$$U_2(n_1^j + n_2^j, C^j) - p^j = U_2(n_2^{j+1}, C^{j+1}) - p^{j+1} = \cdots U_2(n_2^K, C^K) - p^K = 0.$$

For each of these solutions, check if all $n_i^k \geq 0$, $\sum_{k=1}^{K} n_i^k \leq N_i$, and

(1) $U_1(0, C^{j^\star - 1}) - p^{j^\star - 1} < U_1(n_1^j + n_2^j, C^j) - p^j$,

(2) $U_1(n_2^{j+1}, C^{j+1}) - p^{j+1} < U_1(n_1^j + n_2^j, C^j) - p^j$, *and*

(3) $U_2(n_1^{j-1}, C^{j-1}) - p^{j-1} < U_2(n_1^j + n_2^j, C^j) - p^j$.

'Separated' equilibria. *For* $(j, j^\star) \in J$, *the equilibrium profile is of type* $S(j, j^\star)$ *if the following conditions hold. We set* $n_i^k = 0$ *for* $i = 1, 2$ *and* $k = 1, \ldots, j^\star - 1$, $n_2^k = 0$ *for* $k = j^\star, \ldots, j$ *and* $n_1^k = 0$ *for* $k = j+1, \ldots, K$.
In all cases $K - j^\star + 1$ *equations in* $K - j^\star + 1$ *unknowns must be solved.*

S_1 : *Solve under* $\sum_{k=j^\star}^{j} n_1^k = N_1$ *and* $\sum_{k=j+1}^{K} n_2^k = N_2$

$$U_1(n_1^{j^\star}, C^{j^\star}) - p^{j^\star} = \cdots U_1(n_1^j, C^j) - p^j \geq 0, \text{ and}$$

$$U_2(n_2^{j+1}, C^{j+1}) - p^{j+1} = \cdots U_2(n_2^K, C^K) - p^K \geq 0.$$

S_2 : *Solve under* $\sum_{k=j^\star}^{j} n_1^k = N_1$

$$U_1(n_1^{j^\star}, C^{j^\star}) - p^{j^\star} = \cdots U_1(n_1^j, C^j) - p^j \geq 0, \text{ and}$$

$$U_2(n_2^{j+1}, C^{j+1}) - p^{j+1} = \cdots U_2(n_2^K, C^K) - p^K = 0.$$

$\bar{S}$: *Solve under* $\sum_{k=j^\star}^{K} n_1^k < N_1$ *(*$j = K$ *and* $n_2^k = 0$ *for all* k*)*

$$U_1(n_1^{j^\star}, C^{j^\star}) - p^{j^\star} = \cdots = U_1(n_1^K, C^K) - p^K \geq 0.$$

For each of these solutions, check if all $n_i^k \geq 0$, $\sum_{k=1}^{K} n_i^k \leq N_i$, *and*

(1) $U_1(0, C^{j^\star-1}) - p^{j^\star-1} < U_1(n_1^j, C^j) - p^j$,

(2) $U_1(n_2^{j+1}, C^{j+1}) - p^{j+1} < U_1(n_1^j, C^j) - p^j$, *and*

(3) $U_2(n_1^j, C^j) - p^j < U_2(n_2^{j+1}, C^{j+1}) - p^{j+1}$.

'Empty' equilibrium. *For this equilibrium* $n_i^k = 0$ *for* $i = 1, 2$ *and* $k = 1, \ldots, K$. *Check if* $U_i(0, C^k) - p^k \leq 0$ *for* $i = 1, 2$ *and* $k = 1, \ldots, K$.

A simple counting procedure yields that there are $\frac{1}{2}K(K+1)$ possibilities for a mixed equilibrium M_1, the same number for M_2, S_1 and S_2, K possibilities for an $\bar{S}$ equilibrium, and just 1 for an empty equilibrium. This gives a total number of $(2K+1)(K+1)$ possibilities for the structure of an equilibrium profile.

4 Stage-2 equilibrium

In this section we prove existence and uniqueness of the equilibrium of the stage-2 game, by using the structural results of the previous section.

Theorem 4.1 *There exists a unique equilibrium* $n(\overline{p}, \overline{C})$.

The proof of Theorem 4.1 is divided into two parts: the first part shows the *existence* and the second part shows the *uniqueness* of an equilibrium $n(\overline{p}, \overline{C})$.

Proof of Theorem 4.1: existence. An equivalent definition of an equilibrium in our context can be derived using an iterative procedure. If, at time $t \in \mathbb{N}$, $n_{2,t}^j$ customers of type 2 are present in the networks $j = 1, \ldots, K$, then the type 1 customers respond to this by placing $R_1(n_{2,t}) = n_{1,t+1}$ (a vector of size K). Obviously, $n_{i,t} \in D_i \equiv \{x \in \mathbb{R}^K \mid x \geq 0, \ \sum_{k=1}^K x^k \leq N_i\}$ for $i = 1, 2$. More precisely, the following algorithm determines how many customers of type 1 join networks $1, \ldots, K$.

Algorithm 4.2

1. Determine for all networks j the net utility of network j, $U_1(n_{2,t}^j, C^j) - p^j$, as if there are $n_{2,t}^j$ customers of type 2 and yet no customers of type 1.

2. Sort the networks from highest net utility (network j_1) to lowest net utility (network j_K). If the net utility of network j_1 is nonpositive then all networks have a nonpositive net utility and customers of type 1 do not join these networks: $n_1^k = 0$ for all networks k. Go to step 5. Otherwise, let network j_b be the last network in the ordered list of networks with a positive net utility. Hence, the net utility of network j_k for $k > b$ is nonpositive. Assign $n_1^{j_k} = 0$ for $k = b+1, \ldots, K$ and go to step 3.

3. For m from 1 to $b-1$ apply the following procedure. Assign customers of type 1 to the networks j_1 to j_m such that the net utilities of these networks are equal:

$$U_1(n_1^{j_1} + n_{2,t}^{j_1}, C^{j_1}) - p^{j_1} = \ldots = U_1(n_1^{j_m} + n_{2,t}^{j_m}, C^{j_m}) - p^{j_m}.$$

   Do so until either we run out of customers, i.e., $\sum_{k=1}^m n_1^{j_k} = N_1$, or the net utility level of network j_{m+1} is reached:

$$U_1(n_1^{j_m} + n_{2,t}^{j_m}, C^{j_m}) - p^{j_m} = U_1(n_{2,t}^{j_{m+1}}, C^{j_{m+1}}) - p^{j_{m+1}}.$$

   If we ran out of customers, then set $n_1^{j_k} = 0$ for $k = m+1, \ldots, b$ and go to step 5. Otherwise, increase m by 1. If $m < b$, then repeat step 3, whereas if $m = b$, then go to step 4.

4. If $m = b$ then add customers of type 1 to the networks j_1 to j_b such that the net utilities of these networks are equal:

$$U_1(n_1^{j_1} + n_{2,t}^{j_1}, C^{j_1}) - p^{j_1} = \ldots = U_1(n_1^{j_m} + n_{2,t}^{j_m}, C^{j_m}) - p^{j_m}.$$

Do so until either we run out of customers, i.e., $\sum_{k=1}^{m} n_1^{j_k} = N_1$, or the net utility drops to zero:

$$U_1(n_1^{j_m} + n_{2,t}^{j_m}, C^{j_m}) - p^{j_m} = 0.$$

Go to step 5.

5. Re-order the resulting vector $(n_1^{j_1}, \ldots, n_1^{j_K})$, so that we get $(n_1^1, \ldots, n_1^K)$. Finally, set $R_1(n_{2,t}) = n_{1,t+1} = (n_1^1, \ldots, n_1^K)$.

The steps 1 and 2 of the algorithm are based on the fact that a customer does not join a network if this gives him a nonpositive utility. Step 3 corresponds to the second statement in Lemma 3.5: if customers of type i are present in several networks, then they should receive the same net utility from these networks. Further, if we run out of customers in step 3, then the customers who are assigned to the networks j_1 to j_m have no incentive to change networks (since they are assigned to the networks with the highest net utilities, see also Definition 3.1). Finally, step 4 uses the first statement in Lemma 3.5, which says that if some customers of type i decided not to join any network, then the net utility for all those of type i who did join a network should be zero.

The customers of type 2 use an analogous algorithm to respond to $n_{1,t}$ by placing $R_2(n_{1,t}) = n_{2,t}$ customers in the various networks. The iterative procedure has initial value $n_{2,0} = (0, \ldots, 0)$ and is further defined by $n_{1,t+1} = R_1(n_{2,t})$ and $n_{2,t} = R_2(n_{1,t})$. Combining these responses leads to the functions $n_{1,t+1} = R_1(R_2(n_{1,t}))$ and $n_{2,t+1} = R_2(R_1(n_{2,t}))$, or equivalently,

$$n_{i,t+1} = P_i(n_{i,t}),$$

where $P_1 = R_1(R_2)$ and $P_2 = R_2(R_1)$. The iterative procedure has reached an equilibrium if

$$n_{i,t} = P_i(n_{i,t});$$

then no further changes are made. Consequently, equilibria correspond to fixed points of P_i. According to Brouwer's fixed point theorem P_i has a fixed point if $P_i : D_i \to D_i$, P_i is continuous and D_i is a nonempty, compact, and convex set. It is easily checked that D_i is indeed nonempty, compact, and convex. To prove existence, this leaves us with the question whether P_i is continuous.
Continuity of P_i is verified by observing (from steps 1 to 5 in Algorithm 4.2) that a small change in $n_{2,t}$ leads to a small change in $R_1(n_{2,t})$. Hence, R_1 is a continuous function. Similarly, also R_2 is continuous, and hence P_i as well (being composed from R_1 and R_2). □

We now concentrate on the uniqueness claim. In the proof, the following property is useful.

Lemma 4.3 *Suppose there are two equilibria $n'(\overline{p},\overline{C})$ and $n''(\overline{p},\overline{C})$, and let $k' < k'' \leq K$. Suppose that in the first equilibrium the type 2 customers are present in the networks k' till K, i.e., $\sum_{k=k'}^{K}(n_2^k)' = N_2$, whereas in the second equilibrium they are in the networks k'' till K, i.e., $\sum_{k=k''}^{K}(n_2^k)'' = N_2$. Then*

$$U_2((n_1^k)'' + (n_2^k)'', C^k) < U_2((n_1^k)' + (n_2^k)', C^k)$$

for $k = k'', \ldots, K$.

Proof. We establish the claim for $k' = k'' - 1$. Then for $k' < k''$ the stated follows by iteration. Let $1 < k'' \leq K$ and define $\ell := k''$ for ease of notation (i.e., $k' = \ell - 1$). The proof is done in three steps.

- First notice that

$$\sum_{k=\ell}^{K}(n_2^k)' < N_2, \tag{4.1}$$

because we assumed $(n_2^{\ell-1})' > 0$. The population profiles $n'(\overline{p},\overline{C})$ and $n''(\overline{p},\overline{C})$ are equilibrium profiles and therefore, by Lemma 3.5,

$$U_2((n_2^\ell)', C^\ell) - p^\ell = U_2((n_2^k)', C^k) - p^k \text{ and} \tag{4.2}$$

$$U_2((n_1^\ell)'' + (n_2^\ell)'', C^\ell) - p^\ell = U_2((n_2^k)'', C^k) - p^k, \tag{4.3}$$

for $k = \ell + 1, \ldots, K$. This implies in particular

$$U_2((n_1^\ell)'' + (n_2^\ell)'', C^\ell) - U_2((n_2^K)'', C^K) = p^\ell - p^K = U_2((n_2^\ell)', C^\ell) - U_2((n_2^K)', C^K) \tag{4.4}$$

and, for $k = \ell + 1, \ldots, K - 1$,

$$U_2((n_2^k)'', C^k) - U_2((n_2^K)'', C^K) = p^k - p^K = U_2((n_2^k)', C^k) - U_2((n_2^K)', C^K).$$

Rearranging these last equalities gives

$$U_2((n_2^k)', C^k) - U_2((n_2^k)'', C^k) = U_2((n_2^K)', C^K) - U_2((n_2^K)'', C^K) \tag{4.5}$$

for $k = \ell + 1, \ldots, K - 1$.

- Now suppose that

$$U_2((n_2^K)', C^K) \leq U_2((n_2^K)'', C^K). \tag{4.6}$$

Then, by Equation (4.5), $U_2((n_2^k)', C^k) \le U_2((n_2^k)'', C^k)$; this implies $(n_2^k)' \ge (n_2^k)''$, for $k = \ell+1, \dots, K-1$, using that the utility functions are decreasing in n. Using this, and also (4.1), we conclude

$$(n_2^\ell)'' = N_2 - \sum_{k=\ell+1}^{K} (n_2^k)'' \ge N_2 - \sum_{k=\ell+1}^{K} (n_2^k)' > N_2 - (N_2 - (n_2^\ell)') = (n_2^\ell)'.$$

This implies that $U_2((n_1^\ell)'' + (n_2^\ell)'', C^\ell) \le U_2((n_2^\ell)'', C^\ell) < U_2((n_2^\ell)', C^\ell)$. Combining this with (4.6) and (4.3) leads to a contradiction to (4.2):

$$\begin{aligned} U_2((n_2^K)', C^K) - p^K &\le U_2((n_2^K)'', C^K) - p^K = U_2((n_1^\ell)'' + (n_2^\ell)'', C^\ell) - p^\ell \\ &< U_2((n_2^\ell)', C^\ell) - p^\ell. \end{aligned}$$

- The contradiction yields that (4.6) must be invalid, so that $U_2((n_2^K)', C^K) > U_2((n_2^K)'', C^K)$. Then by (4.5), we have that $U_2((n_2^k)', C^k) > U_2((n_2^k)'', C^k)$ for all $k = \ell+1, \dots, K$. Finally, this implies, in conjunction with (4.4), that $U_2((n_2^\ell)', C^\ell) > U_2((n_1^\ell)'' + (n_2^\ell)'', C^\ell)$. This completes the proof. □

Proof of Theorem 4.1: uniqueness. Suppose one of the $(2K+1)(K+1)$ equilibrium structures of Algorithm 3.8 is valid. Then it can be shown that a second equilibrium cannot occur. As the procedure is relatively straightforward, we treat just one example, namely $M_1(1,1)$.
Assume that there is an $M_1(1,1)$ equilibrium denoted by $(\tilde{n}_1, \tilde{n}_2)$. Recall that for this equilibrium

$$\tilde{n}_1^1 = N_1 \text{ and } \sum_{k=1}^{K} \tilde{n}_2^k = N_2;$$

all customers of type 1 are in network 1, while the customers of type 2 are distributed over the K networks. Also,

$$U_1(\tilde{n}_1^1 + \tilde{n}_2^1, C^1) - p^1 > 0,$$

and

$$U_2(\tilde{n}_1^1 + \tilde{n}_2^1, C^1) - p^1 = U_2(\tilde{n}_2^2, C^2) - p^2 = \dots = U_2(\tilde{n}_2^K, C^K) - p^K \ge 0; \tag{4.7}$$

observe that all customers have a nonnegative utility. We show that under these conditions there cannot exist a second equilibrium.

$M_1(j,1)$: First, is it possible to have an $M_1(2,1)$ equilibrium (n_1, n_2) as a second equilibrium? If such an equilibrium would exist, then

$$U_1(n_1^1, C^1) - p^1 = U_1(n_1^2 + n_2^2, C^2) - p^2,$$

i.e., the customers of type 1 receive the same utility from the networks 1 and 2. Rearranging this equality and substituting $U_1 = \alpha U_2$ gives

$$\alpha(U_2(n_1^1, C^1) - U_2(n_1^2 + n_2^2, C^2)) = p^1 - p^2 > 0.$$

From (4.7), we also have $p^1 - p^2 = U_2(\tilde{n}_1^1 + \tilde{n}_2^1, C^1) - U_2(\tilde{n}_2^2, C^2)$, and therefore (use that $\alpha > 1$)

$$0 < U_2(n_1^1, C^1) - U_2(n_1^2 + n_2^2, C^2) < U_2(\tilde{n}_1^1 + \tilde{n}_2^1, C^1) - U_2(\tilde{n}_2^2, C^2). \tag{4.8}$$

From

$$n_1^1 < N_1 = \tilde{n}_1^1 < \tilde{n}_1^1 + \tilde{n}_2^1 \tag{4.9}$$

it follows that $U_2(n_1^1, C^1) > U_2(\tilde{n}_1^1 + \tilde{n}_2^1, C^1)$. But then

$$U_2(n_1^2 + n_2^2, C^2) > U_2(\tilde{n}_2^2, C^2), \tag{4.10}$$

because otherwise (4.8) does not hold. Lemma 4.3 shows that $U_2(n_1^2 + n_2^2, C^2) < U_2(\tilde{n}_2^2, C^2)$. This contradicts (4.10). Conclude that $M_1(1,1)$ and $M_1(2,1)$ cannot occur both.

Along the same lines one can show that there also cannot be an $M_1(j,1)$ equilibrium, $j \in \{3, 4, \ldots\}$.

$M_1(j, j^\star)$: Now we check if it is possible to have another $M_1(j, j^\star)$ equilibrium, $j^\star \in \{2, 3, \ldots\}$. Notice that $j^\star > 1$ implies that network 1 is empty. According to Lemma 4.3,

$$U_2(n_1^j + n_2^j, C^j) - p^j < U_2(\tilde{n}_2^j, C^j) - p^j = U_2(\tilde{n}_1^1 + \tilde{n}_2^1, C^1) - p^1 < U_2(\tilde{n}_2^1, C^1) - p^1$$

where the equality follows from (4.7). If there would be $\tilde{n}_2^1$ customers of type 2 in network 1, then they would receive a higher utility than the customers in network j. Hence, there are $\tilde{n}_2^1$ customers of type 2 that have an incentive to deviate to network 1. Conclude that $M_1(j, j^\star)$ cannot be equilibrium, for $j^\star \in \{2, 3, \ldots\}$.

$M_2(j, j^\star)$: Now consider an $M_2(j, 1)$ equilibrium. Here not all the customers of type 2 are present in the K networks, $\sum_{k=j}^K n_2^k < N_2$, and those who did join a network receive a net utility of zero. Notice first that $j = 1$ can be ruled out, because it is impossible that a decrease in the number of customers of type 2, compared to the $M_1(1,1)$ equilibrium, results in a decrease of utility. For $j \in \{2, 3, \ldots\}$, notice that there are $\varepsilon = \min\{N_2 - \sum_{k=j}^K n_2^k, \tilde{n}_1^1 + \tilde{n}_2^1 - n_1^1\}$ customers of type 2 that have an incentive to join network 1 since

$$U_2(n_1^1 + \varepsilon, C^1) - p^1 \geq U_2(\tilde{n}_1^1 + \tilde{n}_2^1, C^1) - p^1 > 0,$$

where the first inequality is due to (4.9). Consequently, $M_2(j, 1)$ cannot be equilibrium.

Along similar lines it can be shown that $M_2(j, j^\star)$ cannot be equilibrium.

$S_1(j, j^\star)$: Now concentrate on an $S_1(j, j^\star)$ equilibrium. If there are customers of type 2 who enjoy network services, $j < K$, then

$$U_2(\tilde{n}_2^1, C^1) - p^1 > U_2(\tilde{n}_1^1 + \tilde{n}_2^1, C^1) - p^1 = U_2(\tilde{n}_2^K, C^K) - p^K > U_2(n_2^K, C^K) - p^K;$$

where the equality is due to (4.7), and the last inequality follows from Lemma 4.3. The above series of inequalities shows that $\tilde{n}_2^1$ customers of type 2 have an incentive to move to network 1. If $j = K$, then the customers of type 1 are present in the networks $j^\star$ until K. The customers of type 2 increase their net utility by joining the empty network 1.

$S_1(j, j^\star)$: There will be no $S_2(j, j^\star)$ equilibrium for the same reasons as for $M_2(j, j^\star)$.

$\bar{S}(K, j^\star)$: Further, there cannot be an $\bar{S}$ equilibrium because in such a situation customers of type 2 have an incentive to join network 1. To conclude, it will be clear that the empty network, in which no customers are present, cannot be an equilibrium if there is already an $M_1(1, 1)$ equilibrium.

Conclude that an $M_1(1, 1)$ equilibrium is unique. In a similar fashion the uniqueness of other equilibrium types can be shown. □

Using the utility function $U_2(n, C) = (C - n)/C$, as elaborated on in Appendix A, it can be shown that the equilibrium number of customers in a network behaves nicely since it is continuous in the network prices and capacities. This simple utility function captures the desired utility degradation due to congestion, cf. the curves used in [3]. At the same time, it enables (relatively) explicit computation of the equilibrium profile, as illustrated below in Example 4.6.

Theorem 4.4 *Under the utility function $U_2(n, C) = (C - n)/C$, the total number of customers in network k, i.e., $n^k(\overline{p}, \overline{C}) := n_1^k(\overline{p}, \overline{C}) + n_2^k(\overline{p}, \overline{C})$, is a continuous function in the prices $\overline{p}$ and the capacities $\overline{C}$ of the networks.*

The proof of this theorem can be found in Appendix B. It indicates that the price range can be cut into segments, where each of these segments has its own specific type of equilibrium. The same holds for the range of capacities. Further, within such a segment, the equilibrium number of customers of a single type depends *linearly* on the network prices.

Lemma 4.5 *Under $U_2(n, C) = (C - n)/C$, the number of customers of type i in network k, $n_i^k(\overline{p}, \overline{C})$, is a piecewise linear but not necessarily continuous function of the network prices.*

Proof. In Corollary 3.4 it is shown that any equilibrium has one of three possible structures, namely it is an equilibrium of type M, type S or an 'empty' equilibrium. In the latter case there is a fixed number of customers present in the networks, namely none at all. From Appendix A one can see that any equilibrium of type M or type S is a piecewise linear function of the network prices. □

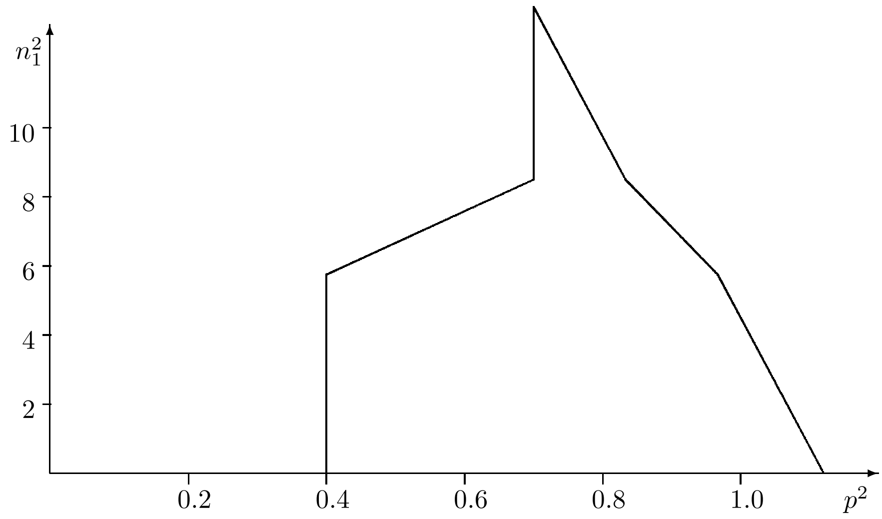

Figure 4.1: This graph shows the number of customers of type 1 in network 2 as a function of the network price. Surprisingly, for $2/5 \le p^2 \le 7/10$ the number of type 1 customers in network 2 increases with the price of this network.

These piecewise linear functions $n_i^k(\overline{p}, \overline{C})$ are not necessarily decreasing or monotone. For certain ranges of prices the number of type i customers in a network may be *increasing* with the network price. The example below demonstrates this phenomenon.

Example 4.6 Two providers offer four networks in total. Provider I owns the networks 1 and 2, and provider II owns 3 and 4. The capacities of the networks are $\overline{C} = (55, 25, 60, 10)$ and the prices equal $\overline{p} = (7/10, p^2, 2/5, 1/5)$. There are $N_1 = 25$ customers of type 1 and $N_2 = 200$ customers of type 2. The gross utility function of type-2 customers is $U_2(n, C) = (C - n)/C$, whereas type-1 customers experience gross utility $U_1(n, C) = \alpha U_2(n, C)$, with $\alpha = 6/5$. Using the results of Appendix A, the number of type 1 customers in network 2 is described by the following piecewise linear function of the network price p^2:

$$n_1^2(p^2) = \begin{cases} 0, & 0 \le p^2 \le \frac{2}{5} \\ 2\frac{1}{12} + 9\frac{1}{6}p^2, & \frac{2}{5} \le p^2 \le \frac{7}{10} \\ 39\frac{3}{4} - 37\frac{1}{2}p^2, & \frac{7}{10} \le p^2 \le \frac{5}{6} \\ 25\frac{11}{16} - 20\frac{5}{8}p^2, & \frac{5}{6} \le p^2 \le \frac{29}{30} \\ 42 - 37\frac{1}{2}p^2, & \frac{29}{30} \le p^2 \le \frac{28}{25} \\ 0, & \frac{28}{25} \le p^2 \le \frac{6}{5} = \alpha. \end{cases}$$

As one can also see in Figure 4.1, the number of type 1 customers in network 2 *jumps* up at $p^2 = 2/5$ and *increases* with the price for $2/5 \le p_2 \le 7/10$.

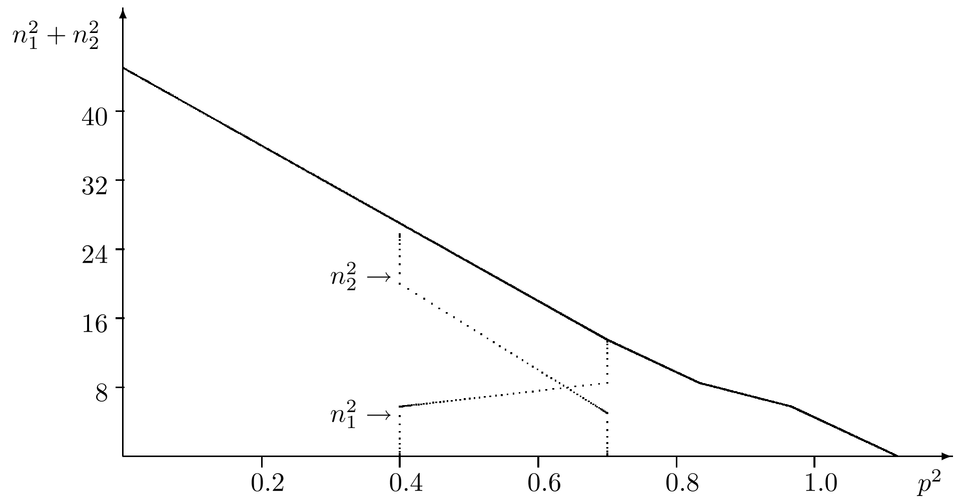

Figure 4.2: This graph shows the number of customers of both types in network 2, as well as their sum. They are given as function of the network price. The solid line represents $n^2 = n_1^2 + n_2^2$, the dotted lines show the n_1^2 and n_2^2, cf. Figure 4.1. The sum is continuous and decreasing in p^2.

- The jump can be explained as follows. When p^2 is just below or above $2/5$, the equilibrium $n(\overline{p}, \overline{C})$ is such that the N_1 customers of type 1 are located in the two most expensive networks. Hence, for $p^2 < 2/5 = p^3$ the type 1 customers join the networks 1 and 3, while for $p^2 > 2/5 = p^3$ they join the networks 1 and 2. Hence, at $p^2 = 2/5$ the customers of type 1 move all at once from network 3 to network 2.

- To explain the increase of n_1^2 notice that if $p^2 \in [2/5, 7/10]$, then the equilibria $n(\overline{p}, \overline{C})$ are of the type $M_2(2, 1)$ (see Algorithm 3.8), which means that all N_1 customers of type 1 are present in the networks 1 and 2 and some customers of type 2, less than N_2, are present in the networks 2, 3 and 4. Therefore, the utility of the customers of type 2 who did join a network equals zero (use the first statement in Lemma 3.5). Furthermore, all type 1 customers consume network services and therefore, by the second statement of Lemma 3.5, they experience the same positive utility from the networks 1 and 2.

  If p^2 increases, then the net utility of type-2 customers drops below zero. This implies that they have an incentive to 'balk', yielding a (larger) net utility of zero. Therefore, some type-2 customers leave network 2 until the utility of this network equals zero again. The customers of type 1 in network 2 are faced with a larger network price, that decreases utility, as well as with fewer agents in the network, which increases utility. It turns out that here the positive effect exceeds the negative effect: the utility of type-1 customers in network 2 increases, and it becomes larger than the utility from network 1. This causes some type-1 customers in network 1 to move to network

2 until both networks have the same utility again. This effect causes n_1^2 to increase with p^2, for $p^2 \in [2/5, 7/10]$.

It can be shown that in the same price range the number of type 2 customers in network 2 is decreasing faster than the type 1 customers can join. Hence, the total number of customers in network 2, $n_1^2 + n_2^2$, is decreasing in p^2. In fact, $n_1^2 + n_2^2$ is decreasing for all p^2 as can be seen from Figure 4.2.

5 Stage 1: price and capacity selection problem

In the previous section we have seen how the customers choose their network services when the prices and capacities $(\overline{p}, \overline{C})$ are given. The providers know how the customers react to prices and capacities, i.e., they know the population profiles $n(\overline{p}, \overline{C})$. This knowledge is used by the providers in stage 1, with the objective to maximize their profits. This is done by selecting optimal prices and capacities: provider I maximizes its profit, given the prices and capacities of provider II, and vice versa.
The maximal profit (or revenue, since we abstracted from costs) that provider I can achieve from its m^{I} subnetworks equals

$$\max_{\overline{p}^{\mathrm{I}}, \overline{C}^{\mathrm{I}}} \quad \sum_{k=1}^{m^{\mathrm{I}}} p^{\mathrm{I},k} \left(n_1^{\mathrm{I},k}(\overline{p}^{\mathrm{I}}, \overline{p}^{\mathrm{II}}, \overline{C}^{\mathrm{I}}, \overline{C}^{\mathrm{II}}) + n_2^{\mathrm{I},k}(\overline{p}^{\mathrm{I}}, \overline{p}^{\mathrm{II}}, \overline{C}^{\mathrm{I}}, \overline{C}^{\mathrm{II}}) \right), \tag{5.1}$$

with $\sum_{k=1}^{m^{\mathrm{I}}} C^{\mathrm{I},k} = C^{\mathrm{I}}$, and

$$\overline{p}^{\mathrm{I}} \equiv (p^{\mathrm{I},1}, \ldots, p^{\mathrm{I},m^{\mathrm{I}}}), \quad \overline{C}^{\mathrm{I}} \equiv (C^{\mathrm{I},1}, \ldots, C^{\mathrm{I},m^{\mathrm{I}}}).$$

Similar definitions hold for $\overline{p}^{\mathrm{II}}$ and $\overline{C}^{\mathrm{II}}$. Let $R^{\mathrm{I}}(\overline{p}^{\mathrm{II}}, \overline{C}^{\mathrm{II}})$ be the set of pairs $(\overline{p}^{\mathrm{I}}, \overline{C}^{\mathrm{I}})$ in which the maximum of (5.1) is achieved. These are the best replies of provider I against the choices $(\overline{p}^{\mathrm{II}}, \overline{C}^{\mathrm{II}})$ of his competitor. Similarly, provider II has an optimization problem that leads to his best replies $R^{\mathrm{II}}(\overline{p}^{\mathrm{I}}, \overline{C}^{\mathrm{I}})$. A pair $(\overline{p}, \overline{C})$ is called a price-capacity equilibrium if $(\overline{p}^{\mathrm{I}}, \overline{C}^{\mathrm{I}}) \in R^{\mathrm{I}}(\overline{p}^{\mathrm{II}}, \overline{C}^{\mathrm{II}})$ and $(\overline{p}^{\mathrm{II}}, \overline{C}^{\mathrm{II}}) \in R^{\mathrm{II}}(\overline{p}^{\mathrm{I}}, \overline{C}^{\mathrm{I}})$.
In all the examples that we studied, we found equilibria $(\overline{p}, \overline{C})$. This has led to the following conjecture.

Conjecture 5.1 *There exists a price-capacity equilibrium* $(\overline{p}, \overline{C})$.

We have not been able to prove this conjecture. This is due to the fact that the profit function of a provider is a non-smooth function that is not necessarily quasi-concave and, as a result, the best response function of a provider need not be continuous. Therefore, standard game-theoretic tools for showing the existence of an equilibrium do not apply.
But, as already mentioned before, the price and capacity ranges can be divided into segments such that each segment has its own type of equilbrium. Consequently, the profit function is differentiable within such a segment, and consequently local maxima can be determined easily. These local maxima determine the globally best response of the provider.

6 PMP under competition

As argued in the introduction, an interesting application of our model is Paris metro pricing (PMP). Under their specific assumptions, [3] showed that, in a duopoly, the providers do not have an incentive to split their networks. The example below, where providers have similar restrictions as in [3], shows that identical providers may offer multiple networks under competition.

Example 6.1 Consider two providers, I and II. Both have a total capacity of $C^{\mathrm{I}} = C^{\mathrm{II}} = 55$. There are 25 customers of type 1 and 200 of type 2. The 'utility constant' α equals 6/5. With respect to the optimization over the capacities, we impose the same constraint as in [3]: each provider is only allowed to either offer just a single network, or to divide his capacity equally over two networks.
If both providers divide their capacities equally over two networks, i.e., $\overline{C} = (27.5, 27.5, 27.5, 27.5)$, then provider I chooses prices $p^{\mathrm{I}} = (0.56, 0.46)$, and provider 2 chooses $p^{\mathrm{II}} = (0.57, 0.46)$; all these prices are rounded off to two decimal places and are obtained via 'simulation' of Algorithm 4.2. The profits of these providers are approximately 13.86 for each of them.
Now suppose one provider decides to offer just a single network, while the other supports two networks. Then this single-network provider sets a price of 0.50, whereas the other provider chooses $(0.56, 0.56)$. The single-network provider earns 13.75, against 13.86 for the two-network provider.
Finally, if both providers decide to offer a single network, then the network of provider I gets a price of 0.50, whereas the price of the network of provider II equals 0.56 – notice that we get an asymmetric outcome although the firms are symmetric. The profits are 13.75 for provider I and 13.86 for provider II. The profits are summarized in the table below.

	provider II splits	provider II does not split
provider I splits	13.86 13.86	13.86 13.75
provider I does not split	13.75 13.86	13.75 13.86

The upper number in a cell refers to provider I and the lower one to provider II. One readily sees that provider I prefers to split and offer two networks because then he earns a profit of 13.86 instead of 13.75 no matter the choice of provider II. Provider II also prefers to split.
We conclude that given the restriction on the capacities both providers maximize their profits by dividing their networks into two subnetworks.

We conclude that in our model there exist situations in which both providers maximize their profit by subdividing their resources. In other words: the equilibrium outcome of competition may be such that 'PMP works', i.e., the providers offer differentiated services, in contrast to the conclusion of [3]. Apparantely, conclusions on the viability of PMP critically depend on the precise assumptions of the underlying model.

7 Concluding remarks and discussion

In this paper we studied competition between two providers of network services. The providers set prices and capacities of their subnetworks, with the objective of maximizing their own profits. As a result, the number of customers in their networks depends on these prices and capacities. We proved that a unique equilibrium population profile exists, for given prices and capacities. Using a simple linear utility function, we have derived the continuity of the resulting profit function in the network prices and capacities. It is noted that the network populations need not behave monotone and the profit functions of the providers need not be quasi-concave, and consequently standard arguments do not apply to prove the existence of a price-capacity equilibrium for the providers.

Discussion on the choice of the utility functions. The linear utility functions that we used in the previous sections (and the appendices) are similar to those chosen in in Gibbens *et al.* [3], in that (i) they are a function of the 'load' n/C, and (ii) they are 'uniformly ordered': for fixed prices and capacities, the utility type-1 customers receive from an arbitrary network majorizes the utility of type-2 customers. Armony and Haviv [1] express congestion in terms of the (mean) queueing delay, rather than load, but also assume uniformly ordered utility curves – their (gross) utility for customers of type i reads $R - C_i\,\mathbb{E}D$, where R is a constant (independent of the customers type), $\mathbb{E}D$ is the mean delay, and the C_i denotes the delay cost parameter of class i, for $i = 1, 2$.

'Intersecting' utility curves. Regarding (ii), considering the case of communication networks, it would perhaps be more realistic to assume that the curves *intersect*: for low values of the congestion level, type-1 users have the higher utility (think of users of delay-critical applications, for instance real-time services), whereas for relatively high congestion, type-2 users have the higher utility (think of users of delay-tolerant applications, such as certain data retrieval services). This situation of intersecting utility curves was considered in [8], in the context of a priority queueing model.
With these considerations in mind, one could think of other utility functions. Assume that a type 1 customer dislikes congestion and is willing to pay a high price:

$$U_1(n^j, C^j, p^j) = (C^j - \alpha n^j)/C^j - p^j$$

whereas a type 2 customer is willing to join a 'busy' network for a low price:

$$U_2(n^j, C^j, p^j) = (C^j - n^j)/C^j - \alpha p^j,$$

where $\alpha > 1$. Hence, both types of customers differ in their valuation of network congestion and price. Using these new utility functions our model becomes harder to analyze. This is due to the fact that type-1 customers no longer 'dominate' type-2 customers (cf. Lemma 3.7). Also, the number of potential equilibrium structures increases: the equilibrium profiles are again mixed or separated (or empty, of course), but now there are also equilibria M_3 and S_3 (where $\sum_{k=1}^K n_1^k < N_1$ and $\sum_{k=1}^K n_2^k = N_2$), and M_4 and S_4 (where $\sum_{k=1}^K n_1^k < N_1$ and $\sum_{k=1}^K n_2^k < N_2$).

For these alternative utility curves existence of an equilibrium population profile $n(\overline{p}, \overline{C})$ can still be shown. If it is of type M_4 or of type $S_4(j, j^\star)$ with $p^k = (1+\alpha)^{-1}$ for $k = j$ or $k = j+1$, then there exist multiple equilibria with the property that among these equilibria $n^k = n_1^k + n_2^k$ is constant for all networks k. For all the other types the equilibrium profile is unique.

Finally, with respect to PMP, in first numerical experiments we have found situations where firms are indifferent between dividing their network or not: both options yield the same profit. Notice that this behavior is 'weaker' than the findings of Example 6.1, where both firms prefer to split. Nevertheless, this is again in contrast to the conclusions of [3] where firms 'strictly' prefer not to split their networks.

Appendix A

In this appendix we choose a specific form of the utility curves. We take

$$U_2(n, C) = \frac{C - n}{C}, \text{ and } U_1(n, C) = \alpha \cdot U_2(n, C), \text{ with } \alpha > 1.$$

The hardest case, M_1, we treat in detail; the others are left to the reader.

In case M_1, all customers are present in one of the networks. Network j is shared by both types. It is easy to check that

$$n_1^k = n_1^{j^\star} \frac{C^k}{C^{j^\star}} + C^k \left(\frac{p^{j^\star} - p^k}{\alpha} \right), \quad k = j^\star, \ldots, j-1;$$

$$n_2^k = n_2^K \frac{C^k}{C^K} + C^k \left(p^K - p^k \right), \quad k = j+1, \ldots, K.$$

We get the following four equations with four unknowns:

$$\sum_{k=j^\star}^{j-1} \left(n_1^{j^\star} \frac{C^k}{C^{j^\star}} + C^k \left(\frac{p^{j^\star} - p^k}{\alpha} \right) \right) + n_1^j = N_1;$$

$$\sum_{k=j+1}^{K} \left(n_2^K \frac{C^k}{C^K} + C^k \left(p^K - p^k \right) \right) + n_2^j = N_2;$$

$$n_1^{j^\star} \frac{C^j}{C^{j^\star}} + C^j \left(\frac{p^{j^\star} - p^j}{\alpha} \right) = n_1^j + n_2^j = n_2^K \frac{C^j}{C^K} + C^j \left(p^K - p^j \right).$$

This boils down to

$$\sum_{k=j^\star}^{j} \left(n_1^{j^\star} \frac{C^k}{C^{j^\star}} + C^k \left(\frac{p^{j^\star} - p^k}{\alpha} \right) \right) + \sum_{k=j+1}^{K} \left(n_2^K \frac{C^k}{C^K} + C^k \left(p^K - p^k \right) \right) = N_1 + N_2;$$

$$n_1^{j^\star} = n_2^K \frac{C^{j^\star}}{C^K} + C^{j^\star} \left(p^K - p^j + \frac{p^j}{\alpha} - \frac{p^{j^\star}}{\alpha} \right).$$

This yields:

$M_1(j, j^\star)$:

$$n_1^k = C^k \left(\frac{N_1 + N_2 - \sum_{\ell=j^\star}^{j} C^\ell (p^k - p^\ell)/\alpha - \sum_{\ell=j+1}^{K} C^\ell (p^j - p^\ell + p^k/\alpha - p^j/\alpha)}{\sum_{\ell=j^\star}^{K} C^\ell} \right),$$

$$k = j^\star, \ldots, j-1;$$

$$n_1^j = N_1 - \sum_{k=j^\star}^{j-1} C^k \left(\frac{N_1 + N_2 - \sum_{\ell=j^\star}^{j} C^\ell (p^k - p^\ell)/\alpha - \sum_{\ell=j+1}^{K} C^\ell (p^j - p^\ell + p^k/\alpha - p^j/\alpha)}{\sum_{\ell=j^\star}^{K} C^\ell} \right);$$

$$n_2^j = N_2 - \sum_{k=j+1}^{K} C^k \left(\frac{N_1 + N_2 - \sum_{\ell=j^\star}^{j} C^\ell (p^k - p^j + p^j/\alpha - p^\ell/\alpha) - \sum_{\ell=j+1}^{K} C^\ell (p^k - p^\ell)}{\sum_{\ell=j^\star}^{K} C^\ell} \right);$$

$$n_2^k = C^k \left(\frac{N_1 + N_2 - \sum_{\ell=j^\star}^{j} C^\ell (p^k - p^j + p^j/\alpha - p^\ell/\alpha) - \sum_{\ell=j+1}^{K} C^\ell (p^k - p^\ell)}{\sum_{\ell=j^\star}^{K} C^\ell} \right),$$

$$k = j+1, \ldots, K.$$

$M_2(j, j^\star)$:

$$n_1^k = C^k \left(1 - p^j + \frac{p^j}{\alpha} - \frac{p^k}{\alpha} \right), \;\; k = j^\star, \ldots, j-1;$$

$$n_1^j = N_1 - \sum_{k=j^\star}^{j-1} C^k \left(1 - p^j + \frac{p^j}{\alpha} - \frac{p^k}{\alpha} \right);$$

$$n_2^j = C^j (1 - p^j) - N_1 + \sum_{k=j^\star}^{j-1} C^k \left(1 - p^j + \frac{p^j}{\alpha} - \frac{p^k}{\alpha} \right);$$

$$n_2^k = C^k (1 - p^k), \;\; k = j+1, \ldots, K.$$

$S_1(j, j^\star)$:

$$n_1^k = C^k \left(\frac{N_1 - \sum_{\ell=j^\star}^{j} C^\ell \left(p^k - p^\ell\right)/\alpha}{\sum_{\ell=j^\star}^{j} C^\ell} \right), \;\; k = j^\star, \ldots, j;$$

$$n_2^k = C^k \left(\frac{N_2 - \sum_{\ell=j+1}^{K} C^\ell \left(p^k - p^\ell\right)}{\sum_{\ell=j+1}^{K} C^\ell} \right), \;\; k = j+1, \ldots, K.$$

$S_2(j, j^\star)$:

$$n_1^k = C^k \left(\frac{N_1 - \sum_{\ell=j^\star}^{j} C^\ell \left(p^k - p^\ell\right)/\alpha}{\sum_{\ell=j^\star}^{j} C^\ell} \right), \ \ k = j^\star, \ldots, j;$$

$$n_2^k = C^k(1 - p^k), \ \ k = j+1, \ldots, K.$$

$\bar{S}(K, j^\star)$:

$$n_1^k = C^k(\alpha - p^k)/\alpha, \ \ k = j^\star, \ldots, K;$$

$$n_2^k = 0, \ \ \text{for all } k.$$

Appendix B

In this appendix, we prove Theorem 4.4. In this proof some properties of the utility functions, defined in Appendix A, are needed. Let U_i^k denote the net utility that customers of type i receive from network k, i.e., $U_i^k = U_i(n^k, C^k) - p^k$ where $n^k = n_1^k + n_2^k$. The first lemma shows relations between the net utilities of the customers of types 1 and 2.

Lemma B.1 The following implications hold for the net-utilities of successive networks:

$$U_1^k \leq U_1^{k+1} \Rightarrow U_2^k < U_2^{k+1} \ \text{ and } \ U_2^k \geq U_2^{k+1} \Rightarrow U_1^k > U_1^{k+1}.$$

Further, $U_1^k > U_2^k$ for all k.

The second lemma shows for each type of customers how the net-utilities of the networks are ordered, depending on the type of equilibrium.

Lemma B.2 In case of a mixed equilibrium $M_\ell(j, j^\star)$, $\ell = 1, 2$, the utilities of type 1 customers satisfy

$$U_1^1 < \ldots < U_1^{j^\star-1} < U_1^{j^\star} = U_1^{j^\star+1} = \ldots = U_1^j > U_1^{j+1} > \ldots > U_1^K$$

and for type 2

$$U_2^1 < \ldots < U_2^{j^\star-1} < U_2^{j^\star} < U_2^{j^\star+1} < \ldots < U_2^j = U_2^{j+1} = \ldots = U_2^K.$$

For a separated equilibrium $S_\ell(j, j^\star)$, $\ell = 1, 2$:

$$U_1^1 < \ldots < U_1^{j^\star-1} < U_1^{j^\star} = U_1^{j^\star+1} = \ldots = U_1^j > U_1^{j+1} > \ldots > U_1^K$$

and

$$U_2^1 < \ldots < U_2^{j^\star-1} < U_2^{j^\star} < U_2^{j^\star+1} < \ldots < U_2^{j+1} = U_2^{j+2} = \ldots = U_2^K.$$

(The only difference between M and S is that for a separated equilibrium $U_2^j < U_2^{j+1}$, whereas for a mixed equilibrium $U_2^j = U_2^{j+1}$.)

Proof. The results follow from the second statement in Lemma 3.5 and from the definition of the utility functions in Appendix A. Furthermore, by $U_1^{j^\star-1} < U_1^{j^\star}$ and Lemma B.1 we obtain $U_2^{j^\star-1} < U_2^{j^\star}$. □

This lemma is helpful in tracking the changes in the structure of an equilibrium due to changes in net utility. For example, if in a separated equilibrium $S_\ell(j, j^\star)$, $\ell = 1, 2$, the utility of type 1 customers in the networks $j^\star$ to j decreases sufficiently then it either reaches the value of $U_1^{j^\star-1}$, of U_1^{j+1} or perhaps even 0. In the first case, customers of type 1 will also join network $j^\star - 1$ and so the equilibrium structure is changed to $S_\ell(j, j^\star - 1)$, and so on.
The two lemmas are used in the proof below.

Proof of Theorem 4.4. Consider prices and capacities $(\overline{p}, \overline{C})$ as set by the providers. According to Theorem 4.1 there is a unique equilibrium $n(\overline{p}, \overline{C})$ in the network selection problem of the customers. As already mentioned in Algorithm 3.8, this equilibrium is one of the following types: $M_1(j, j^\star)$, $M_2(j, j^\star)$, $S_1(j, j^\star)$, $S_2(j, j^\star)$, $\bar{S}(K, j^\star)$ or an empty equilibrium.
First, we consider continuity in the prices. Suppose that the price of a network increases. If this increment is relatively small, then the equilibrium remains the same type. Appendix A shows that for these price changes the numbers n^k are continuous (even linear) in the network prices.
If the price p^k of an empty network k increases, $k \in \{1, \dots, j^\star - 1\}$, then U_i^k will decrease, $i = 1, 2$. This network will remain empty because the net utility is still too low. Hence, n^k remains 0 and the values of the other n^m stay the same.
Next, we consider price increases of nonempty networks. First, suppose that the prices and capacities $(\overline{p}, \overline{C})$ give rise to an equilibrium $n(\overline{p}, \overline{C})$ that is of the mixed type $M_1(j, j^\star)$. If $p^{j^\star}$ increases sufficiently, then (see also Appendix A)

- $n_1^{j^\star}$ decreases towards its lower bound $n_1^{j^\star} = 0$ if $j^\star < j$ (if $j^\star = j$ then $n_1^{j^\star} = N_1$ remains constant),
- n_1^k increases, $k = j^\star + 1, \dots, j$,
- n_2^j decreases towards 0 if $j < K$ (if $j = K$ then $n_2^j = N_2$ remains constant),
- n_2^k increases, $k = j + 1, \dots, K$,

and the various net utilities change as follows:

- $U_1^{j^\star} = \ldots = U_1^j$ decreases towards its lower bound $U_1^{j^\star-1}$ (see Lemma B.2) if $j^\star > 1$ (notice that U_1^{j+1} and 0 cannot be lower bounds, because as long as $U_2^j = U_2^{j+1} \geq 0$ Lemma B.1 implies that $U_1^{j^\star} > U_1^{j+1} > 0$),
- U_2^k decreases, $k = j^\star, \dots, j - 1$ (this has no influence on the equilibrium type since n_2^k was and remains 0),
- U_1^k decreases, $k = j + 1, \dots, K$ (again no influence),

- $U_2^j = \ldots = U_2^K$ decreases towards lower bound 0 (notice that U_2^{j-1} cannot be a lower bound, because as long as $U_1^j = U_1^{j-1}$ Lemma B.1 implies that $U_2^j > U_2^{j-1}$).

The price increase of network $j^\star$ implies that several variables decrease towards their lower bounds. One of these will be the first to actually reach its lower bound if $p^{j^\star}$ increases sufficiently.
Suppose that $n_1^{j^\star}$ is the first to reach its lower bound 0 if $p^{j^\star} = p_a^{j^\star}$. Then the equilibrium of type $M_1(j, j^\star)$ will change to one of type $M_1(j, j^\star + 1)$ as network $j^\star$ will be out of use. Hence, if $p^{j^\star} \uparrow p_a^{j^\star}$ then we have an $M_1(j, j^\star)$ equilibrium in which $n_1^{j^\star} \downarrow 0$. And if $p^{j^\star} \downarrow p_a^{j^\star}$ then we are dealing with an $M_1(j, j^\star + 1)$ equilibrium in which $U_1^{j^\star} \uparrow U_1^{j^\star+1}$.
Therefore, in the limit, if $p^{j^\star} = p_a^{j^\star}$, the equilibrium $n(\overline{p}, \overline{C})$ is one of type $M_1(j, j^\star)$ with $n_1^{j^\star} = 0$, satisfying the following equalities:

$$\begin{cases} U_1^{j^\star} = \ldots = U_1^j, \\ U_2^j = \ldots = U_2^K, \\ N_1 = \sum_{k=j^\star}^{j} n_1^k = \sum_{k=j^\star+1}^{j} n_1^k \ \ \text{(because } n_1^{j^\star} = 0\text{)}, \\ N_2 = \sum_{k=j}^{K} n_2^k, \end{cases} \tag{B.1}$$

and at the same time $n(\overline{p}, \overline{C})$ is an equilibrium of type $M_1(j, j^\star + 1)$ with $U_1^{j^\star} = U_1^{j^\star+1}$:

$$\begin{cases} U_1^{j^\star+1} = \ldots = U_1^j \text{ and } U_1^{j^\star} = U_1^{j^\star+1}, \\ U_2^j = \ldots = U_2^K, \\ N_1 = \sum_{k=j^\star+1}^{j} n_1^k, \\ N_2 = \sum_{k=j}^{K} n_2^k. \end{cases} \tag{B.2}$$

The sets of equations (B.1) and (B.2) are identical. Let $\overline{p}_a$ be the vector of prices $\overline{p}$ in which $p^{j^\star}$ is replaced by $p_a^{j^\star}$. It will now be clear that if $p^{j^\star} \uparrow p_a^{j^\star}$ then $n^k(\overline{p}, \overline{C})$ converges to the value of n^k as determined by (B.1) because the equilibrium type remains $M_1(j, j^\star)$. But this is equal to the value of $n^k(\overline{p}_a, \overline{C})$ as determined by (B.2). We conclude that if $p^{j^\star} \uparrow p_a^{j^\star}$ then $n^k(\overline{p}, \overline{C}) \to n^k(\overline{p}_a, \overline{C})$ for all k.
In the remaining three cases the continuity of n^k can be shown among similar lines. In the left panel of Table B.1 we show all effects that a price increase may have on an equilibrium type. In each of these cases we can show in a similar way as before that the n^k are continuous.
By using similar arguments, one can verify that the numbers of customers in a network are continuous in the capacities. The right panel of Table B.1 gives an overview of possible changes of the equilibrium type under capacity changes. □

References

[1] M. Armony and M. Haviv (2003). Price and delay competition between two service providers. *European Journal of Operational Research*, Vol. 147, pp. 32-50.

Type	$p^m \uparrow$, $m = \dots$	Convergence to
$M_1(j,j^\star)$	$j^\star, j$ $(j^\star < j)$	$M_1(j,j^\star+1)$
	$j^\star,\dots,j$ $(j < K)$	$S_1(j,j^\star)$
	$j+1,\dots,K$ $(j^\star < j)$	$S_1(j-1,j^\star)$
	$j^\star,\dots,K$ $(j^\star > 1)$	$M_1(j,j^\star-1)$
	$j^\star,\dots,K$	$M_2(j,j^\star)$
$M_2(j,j^\star)$	$j^\star, j$ $(j^\star < j)$	$M_2(j,j^\star+1)$
	$j^\star,\dots,j$ $(j^\star < j)$	$S_2(j,j^\star)$
	j $(j^\star > 1)$	$M_2(j,j^\star-1)$
$S_1(j,j^\star)$	$j^\star$ $(j^\star < j)$	$S_1(j,j^\star+1)$
	$j^\star,\dots,j$ $(j^\star > 1)$	$S_1(j,j^\star-1)$
	$j^\star,\dots,j+1$ $(j < K)$	$M_1(j+1,j^\star)$
	$j^\star,\dots,j$ $(j = K)$	$\bar{S}(K,j^\star)$
	$j+1,\dots,K$ $(j < K)$	$M_1(j,j^\star)$
	$j+1,\dots,K$ $(j < K)$	$S_2(j,j^\star)$
$S_2(j,j^\star)$	$j^\star$ $(j^\star < j)$	$S_2(j,j^\star+1)$
	$j^\star,\dots,j$ $(j^\star > 1)$	$S_2(j,j^\star-1)$
	$j^\star,\dots,j+1$ $(j < K)$	$M_2(j+1,j^\star)$
	$j^\star,\dots,j$ $(j = K)$	$\bar{S}(K,j^\star)$
$\bar{S}(K,j^\star)$	$j^\star$ $(j^\star < K)$	$\bar{S}(K,j^\star+1)$
	$j^\star$ $(j^\star = K)$	empty

Type	$C^m \downarrow$, $m = \dots$	Convergence to
$M_1(j,j^\star)$	$j^\star$ $(j^\star < j)$	$M_1(j,j^\star+1)$
	$j^\star,\dots,j$ $(j < K)$	$S_1(j,j^\star)$
	$j,\dots,K$ $(j^\star < j)$	$S_1(j-1,j^\star)$
	$j^\star,\dots,K$ $(j^\star > 1)$	$M_1(j,j^\star-1)$
	$j^\star,\dots,K$	$M_2(j,j^\star)$
$M_2(j,j^\star)$	$j^\star$ $(j^\star < j)$	$M_2(j,j^\star+1)$
	$j^\star,\dots,j$ $(j^\star < j)$	$S_2(j,j^\star)$
$S_1(j,j^\star)$	$j^\star$ $(j^\star < j)$	$S_1(j,j^\star+1)$
	$j^\star,\dots,j$ $(j^\star > 1)$	$S_1(j,j^\star-1)$
	$j^\star,\dots,j$ $(j < K)$	$M_1(j+1,j^\star)$
	$j^\star,\dots,j$ $(j = K)$	$\bar{S}(K,j^\star)$
	$j+1,\dots,K$ $(j < K)$	$M_1(j,j^\star)$
	$j+1,\dots,K$ $(j < K)$	$S_2(j,j^\star)$
$S_2(j,j^\star)$	$j^\star$ $(j^\star < j)$	$S_2(j,j^\star+1)$
	$j^\star,\dots,j$ $(j^\star > 1)$	$S_2(j,j^\star-1)$
	$j^\star,\dots,j$ $(j < K)$	$M_2(j+1,j^\star)$
	$j^\star,\dots,j$ $(j = K)$	$\bar{S}(K,j^\star)$
$\bar{S}(K,j^\star)$	$j^\star$ $(j^\star < K)$	$\bar{S}(K,j^\star+1)$
	$j^\star$ $(j^\star = K)$	empty

Table B.1: Possible equilibrium changes due to increasing prices (left panel) and decreasing capacities (right panel).

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