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Normal, Parametric and Nonparametric Control Charts: a Data Driven Choice

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Abstract Standard control charts are often seriously in error when the distributional form of the observations differs from normality. Recently, control charts have been developed for larger parametric families. A third possibility is to apply a suitable (modified version of a) nonparametric control chart. This paper deals with the question when to switch from the control chart based on normality to a parametric control chart, or even to a nonparametric one. This model selection problem is solved by using the estimated model error as yardstick. It is shown that the new combined control chart asymptotically behaves as each of the specific control charts in their own domain. Simulations exhibit that the combined control chart performs very well under a great variety of distributions and hence it is recommended as an omnibus control chart, nicely adapted to the distribution at hand. The combined control chart is illustrated by an application on real data. The new modified nonparametric control chart is an attractive alternative and can be recommended as well.

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1 Introduction

Classical control charts are based on the assumption that the observations are normally distributed. This assumption is in practice not always fulfilled and the control chart often is seriously in error when the distributional form of the observations differs from normality, see e.g. Chan et al. (1988), Pappanastos and Adams (1996), Albers et al. (2002a, b). An obvious solution to the problem is to assume a larger parametric model, containing normality as a submodel, and to produce a control chart in this new setting. This step has been made in Albers et al. (2002a, b), where the merits of such an approach are comprehensively discussed.

It is clear that as long as we are (very) close to normality, the control chart based on normality is preferable. The first reason is that in that case the performance of this control chart is better, since it is matched to normality of the observations. The second reason is that the normal control chart is easier and more familiar and hence people prefer this chart as long as possible.

However, for distributions farther away from normality, but still close to the larger parametric family, the best choice is the parametric control chart. In that case we should no longer stick to the normal control chart, but move towards the parametric chart.

Of course, there also are distributions so far outside the larger parametric family that the parametric control chart is not satisfactory either and a nonparametric approach should be applied. One may ask why not always apply a nonparametric control chart or a parametric control chart in a very large parametric family. The point is that there are two types of error: the model error (due to the distance between the true distribution and the most suitable distribution in the supposed model) and the stochastic error (caused by estimating parameters or, in the nonparametric case, an extreme quantile). The larger the parametric model, the smaller the model error (with a vanishing model error in the nonparametric control chart), but the larger the stochastic error. For instance, estimating the 0.999-quantile with 100 observations makes no sense in a nonparametric setting.

The theme of this paper is how to choose between the three control charts: the normal control chart, the parametric control chart as developed for the normal power family, cf. Albers et al. (2002a) and the nonparametric control chart, cf. Albers and Kallenberg (2002). The idea is to let the data tell what control chart to use.

A first idea might be to execute a (standard) goodness of fit test to investigate normality. If normality is not rejected, use the normal control chart. If we do reject, apply a goodness of fit test for the normal power family. Again, when not rejecting, apply the parametric control chart and otherwise use the nonparametric control chart (if this makes sense).

Although this way of thinking looks attractive, it has a serious drawback. Standard goodness of fit tests are looking at the majority of the data, and as such concentrate on the middle of the distribution, while here we are not interested in this middle part, but in the (extreme) tail. Therefore, standard goodness of fit tests are not appropriate for the situation at hand. For the same reason, less formal methods like "a good look at the data" or "an inspection of a histogram" are completely insufficient to judge the possible normality in the far tail.

The choice between the three control charts can be seen as a model selection problem. This area in statistics is nowadays in the centre of interest and therefore it looks promising to apply such methods not merely for the three charts mentioned here, but for a whole range of (nested) models. Unfortunately, two problems arise. First of all, it is far from easy to develop control charts in each of these models, see the discussion on several types of models and the corresponding problems in deriving suitable control charts in Albers et al. (2002a). Secondly, again the common selection rules are intended for the bulk of the data and not for the extreme tail.

The motivation to switch from the normal control chart to the parametric control chart or even to the nonparametric control chart is provided by the model error. Let p be the false alarm rate, that is the probability of concluding that the process is out-of-control when in fact it is in control. In this paper we restrict attention to control charts which provide an out-of-control signal when the monitoring random variable (r.v.) X is larger than a certain control limit. For the two-sided case similar results hold. Then the model error is the discrepancy between p and the probability (under the true distribution of X) that X is larger than the "ideal" control limit, being the $(1-p)$-quantile of the "most suitable" distribution in the supposed model. Indeed, it is seen at this point that not the middle of the distribution is coming in, but, due to the (very) small value of p, its far tail. For a more technical description of the model error we refer to Section 3.

As this model error is the discriminating quantity in deciding which control chart to use and since moreover the data should tell us what the appropriate model is, it is natural to base the decision between the three control charts on the estimated model error. To avoid too many technical complications and to keep the control chart as simple as possible we use the (standardized) largest observation to choose between the control charts, thus really looking at the tail of the distribution.

In Section 2 the three control charts are presented. Here also a modified version of the nonparametric control chart is presented. Section 3 is devoted to the choice of the model. The three control charts and the decision rule telling us which one to choose among them, in fact together form a new control chart. An application of this combined control chart on real data is performed in Section 4. It is shown in detail how to do the calculations. In order to study the theoretical behavior of the combined control chart we need so called large deviation results for our estimators of the parameters in the normal power family. In Section 5 and the appendix some large deviation results are presented, which may be of independent interest. The in-control and out-of-control behavior from a theoretical point of view of the combined chart is worked out in Section 6. In contrast to for instance the normal control chart, the combined control chart is valid for all distributions and its out-of-control behavior is asymptotically as good as if we should know to which of the three classes of distributions the true distribution belongs. In addition to the theoretical results we present some simulation results on the new control chart in Section 7. It turns out that the combined control chart behaves very well under a great variety of distributions and therefore it is recommended as an omnibus control chart, nicely adapted to the distribution at hand.

The new modified version of the nonparametric control chart shows good behavior in the in-control situation (as might be expected from a chart close to a real nonparametric chart), but has also a nice performance in the out-of-control in contrast to the standard nonparametric control chart, although under normality some loss has to be accepted w.r.t. the combined control chart. Therefore, this modified nonparametric chart is an attractive alternative omnibus control chart.

2 Three types of control charts

Let $X_1, \ldots, X_n$ be i.i.d. r.v.'s with distribution function F and let X_{n+1} be the monitoring characteristic, having distribution function G. In the in-control situation we have $G = F$. The r.v.'s $X_1, \ldots, X_n$ are the data from Phase I on which the estimators of the unknown parameters or the unknown quantile are based. The monitoring r.v. may in fact be based on m observations, but here we consider the situation $m = 1$ of individual measurements.

Due to estimation the probability of a false alarm is no longer a number, but a r.v. Let P_n be the observed alarm rate, i.e. the probability of an alarm, given $X_1, \ldots, X_n$. Under $G = F$

we want P_n to be close to the prescribed false alarm rate p. To measure the closeness of P_n to p we consider control charts aiming for $Eg(P_n) \approx g(p)$, for some suitable function g. (If not stated otherwise we consider in this section the in-control situation and hence P_n refers to the observed false alarm rate, that is $P_n = \overline{F}(UL)$, where UL denotes the upper control limit.) The most prominent one is simply $g(p) = p$, but also $g(p) = 1/p$ is of interest, being the average of the runlength RL. A third choice which is sometimes used (see e.g. Does and Schriever (1992) or Roes (1995), pp. 102, 103) follows from $g(p) = 1 - (1-p)^k = P(RL \le k)$, where typically $k = [\xi/p]$ (with $[x]$ being the entier of x) for some small ξ like 0.1 or 0.2.

It is well known that simply plugging in estimators in control charts for the unknown parameters is only accurate for very large sample sizes, see e.g. Ghosh, Reynolds and Hui (1981), Quesenberry (1993), Chen (1997), Woodall and Montgomery (1999, p. 379), Chakraborti (2000), Neduraman and Pignatiello (2001), Albers and Kallenberg (2000) and Albers et al. (2002a). In the latter two papers correction terms are proposed that get the behavior of control charts under control again, in the sense that $Eg(P_n) \approx g(p)$, where the correction terms depend on the function g under consideration. Instead of using expectations, alternatively an approach based on exceedance probabilities can be performed, see Albers and Kallenberg (2001) and Albers et al. (2002b). But in the present paper the main interest is on how to choose the right (class of) distribution(s) in order to apply the appropriate control chart, and therefore we do not treat all the possible criteria and their corresponding (corrected) control charts.

Let $\widehat{\mu} = \overline{X} = n^{-1} \sum X_i$ and $\widehat{\sigma} = S = \sqrt{S^2}$ with $S^2 = (n-1)^{-1} \sum \left(X_i - \overline{X}\right)^2$. Based on the data we want to decide which of the following three control charts to choose.

2.1 Normal control chart

Let u_p be the upper p-quantile of the standard normal distribution, that is $u_p = \overline{\Phi}^{-1}(p) = \Phi^{-1}(1-p)$, where Φ denotes the standard normal distribution function and $\overline{\Phi} = 1 - \Phi$. Furthermore, we write φ for the standard normal density. The normal control limit devised for $Eg(P_n) \approx g(p)$ is obtained from the plug-in control limit $\widehat{\mu} + u_p\widehat{\sigma}$ by adding a term c_N to correct for the bias. That is, the normal control chart is given by

$$X_{n+1} > \widehat{\mu} + (u_p + c_N)\widehat{\sigma} \text{ with } c_N \text{ as in Table 1.} \tag{1}$$

Table 1 *Correction terms according to the function g.*

$g(p)$	c_N
p	$\frac{u_p}{4n} + \frac{u_p(u_p^2+2)}{4n}$
$\frac{1}{p}$	$\frac{u_p}{4n} + \frac{u_p^2+2}{4n}\left\{u_p - \frac{2\varphi(u_p)}{p}\right\}$
$1-(1-p)^k$	$\frac{u_p}{4n} + \frac{u_p^2+2}{4n}\left\{u_p - \frac{(k-1)\varphi(u_p)}{1-p}\right\}$

For the derivation of this control chart and its properties we refer to Albers and Kallenberg (2000).

2.2 Parametric control chart

The parametric control chart is based on the normal power family. The t-quantile in the normal distribution is given by $\mu + \sigma\Phi^{-1}(t)$. To embed the normal distributions in a larger family with heavier or thinner tails we consider essentially powers of the standard normal quantiles as the new quantiles. More precisely, replace $\Phi^{-1}(t)$ by

$$K_\gamma^{-1}(t) = c(\gamma)\left|\Phi^{-1}(t)\right|^{1+\gamma} sign(\Phi^{-1}(t)), \tag{2}$$

where $\gamma > -1$ and where $c(\gamma)$ is a normalizing constant given (to make the variance equal to one) by $c(\gamma) = \left\{E\left|Z\right|^{2(1+\gamma)}\right\}^{-1/2} = \pi^{1/4}2^{-(1+\gamma)/2}\Gamma(\gamma+\frac{3}{2})^{-1/2}$ with Z a r.v. having a standard normal distribution. This larger parametric family, called the normal power family, can also be defined by

$$X = \mu + \sigma Z_\gamma \text{ with } Z_\gamma = c(\gamma)\left|Z\right|^{1+\gamma} sign(Z). \tag{3}$$

The corresponding distribution function is denoted by K_γ. It is immediately seen that $\gamma = 0$ leads to the family of normal distributions.

Just adding an extra parameter actually makes it far more complicated to derive control charts. Fortunately, the following recommended control chart in this family can be applied quite straightforwardly. Let $X_{(1)} \leq \ldots \leq X_{(n)}$ be the order statistics of $X_1, \ldots, X_n$. The parametric control chart, which we use in this paper, is given by

$$X_{n+1} > \widehat{\mu} + \widehat{\sigma}\left\{\overline{K}_{\widehat{\gamma}}^{-1}(p) - C1(\widehat{\gamma})\,C2(\widehat{\gamma}) - \frac{C3(\widehat{\gamma})}{n} + \lambda\frac{C4(\widehat{\gamma})}{n}\right\}, \tag{4}$$

where $\overline{K}_\gamma^{-1}(p) = K_\gamma^{-1}(1-p) = \pi^{1/4}2^{-(1+\gamma)/2}\Gamma(\gamma+\frac{3}{2})^{-1/2}u_p^{1+\gamma}$ refers to the normal power family, see (2), where $\lambda = 1, -1, 1-\xi$ according to $g(p) = p$, $g(p) = \frac{1}{p}$, $g(p) = 1-(1-p)^k$, $k = [\xi/p]$, respectively, and where

$$\begin{aligned}
\widehat{\gamma} &= \frac{1}{\log\left(\frac{\Phi^{-1}(0.95)}{\Phi^{-1}(0.75)}\right)}\log\left(\frac{X_{([0.95n+1])} - \overline{X}}{X_{([0.75n+1])} - \overline{X}}\right) - 1 \qquad \left(\frac{1}{\log\left(\frac{\Phi^{-1}(0.95)}{\Phi^{-1}(0.75)}\right)} \approx 1.1218\right),\\
C1(\gamma) &= -1.23 - 0.63\gamma + 0.73\gamma^2 + 0.74u_p - 0.08\gamma u_p - 0.14\gamma^2 u_p,\\
C2(\gamma) &= \left(\frac{\Phi^{-1}\left(\frac{[0.95n+1]}{n+1}\right)}{\Phi^{-1}\left(\frac{[0.75n+1]}{n+1}\right)}\right)^{1+\gamma} - 2.4387^{1+\gamma},\\
C3(\gamma) &= -10.86 - 27.77\gamma - 22.36\gamma^2 + 4.72u_p + 9.98\gamma u_p + 7.29\gamma^2 u_p,\\
C4(\gamma) &= -87.23 - 147.89\gamma - 104.29\gamma^2 + 40.25u_p + 63.69\gamma u_p + 44.47\gamma^2 u_p.
\end{aligned} \tag{5}$$

For the derivation of this control chart and its properties we refer to Albers et al. (2002a).

2.3 Nonparametric control charts

2.3.1 Standard Nonparametric control chart

We start with a formal definition of the inverse of a distribution function: $F^{-1}(t) = \inf\{x : F(x) \geq t\}$. For $\overline{F} = 1 - F$ we define $\overline{F}^{-1}(p) = F^{-1}(1-p) = \inf\{x : F(x) \geq 1-p\} = \inf\{x : \overline{F}(x) \leq p\}$. Applying this to the empirical distribution function F_n we get

$$\overline{F_n}^{-1}(p) = X_{(n-\widetilde{r})} \text{ for } \frac{\widetilde{r}}{n} \leq p < \frac{\widetilde{r}+1}{n}, \widetilde{r} = 0, \ldots, n-1.$$

(Hence $\widetilde{r} = [np]$.) Estimating the unknown quantile $F^{-1}(p)$ in a nonparametric way by its empirical counterpart gives the uncorrected plug-in control chart

$$X_{n+1} > \overline{F_n}^{-1}(p) = X_{(n-[np])}.$$

This implies that we get unconditional false alarm probabilities EP_n equal to $j/(n+1)$ with $j = 1+[np] \geq 1$. An obvious way to correct for the estimation error is to replace p by $p(1 \pm \zeta)$ for some suitably chosen ζ. However, due to (a) the fact that we have always at least a false alarm probability $1/(n+1)$ and (b) the discrete character of the possible false alarm probabilities, we will not get satisfactory results unless n is very large or p not too small.

A possible solution is to make the false alarm probabilities continuous by adding a randomization procedure as follows. Let $U_{(1)} \leq \ldots \leq U_{(n)}$ be the order statistics of the random sample $U_1, \ldots, U_n$ from a uniform distribution on (0,1) and define $U_{(0)} = 0$ and $U_{(n+1)} = 1$. Let g be a monotone function. In particular, we are again interested in $g(p) = p$, $g(p) = \frac{1}{p}$, $g(p) = 1 - (1-p)^k$. For a monotone increasing g define the integer r with $0 \leq r = r(p) \leq n$ by

$$Eg(U_{(r)}) \leq g(p) < Eg(U_{(r+1)}). \tag{6}$$

Let V be a r.v. independent of $X_1, \ldots, X_{n+1}$ taking as values 0 or 1. Replace the control chart by

$$X_{n+1} > VX_{(n-r)} + (1-V)X_{(n-r+1)} \text{ with } P(V=1) = \frac{g(p) - Eg(U_{(r)})}{Eg(U_{(r+1)}) - Eg(U_{(r)})}, \tag{7}$$

where in case $r = 0$ we define $X_{(n+1)} = \infty$. Since $\overline{F}(X_{(n-r)})$ and $\overline{F}(X_{(n-r+1)})$ are distributed as $U_{(r+1)}$ and $U_{(r)}$, respectively, obviously we obtain

$$Eg(P_n) = g(p). \tag{8}$$

(Note that P_n is now defined as the probability of a false alarm, given $X_1, \ldots, X_n$ and V, that is $P_n = V\overline{F}(X_{(n-r)}) + (1-V)\overline{F}(X_{(n-r+1)})$).

In particular, for $g(p) = p$ we get $r = [p(n+1)]$ and the nonparametric control chart reads as

$$X_{n+1} > VX_{(n-[p(n+1)])} + (1-V)X_{(n-[p(n+1)]+1)} \text{ with } P(V=1) = p(n+1) - [p(n+1)]. \tag{9}$$

Similarly, for a monotone decreasing g define $0 \leq r = r(p) \leq n$ by

$$Eg(U_{(r)}) \geq g(p) > Eg(U_{(r+1)}). \tag{10}$$

The control chart is again given by (7). In particular, for $g(p) = \frac{1}{p}$ we get $r = [np] + 1$ and provided that $r \geq 2$ (that is $np \geq 1$) the nonparametric control chart reads as

$$X_{n+1} > VX_{(n-[np]-1)} + (1-V)X_{(n-[np])} \text{ with } P(V=1) = \frac{([np]+1)(np-[np])}{np}.$$

When $r = 1$ and $g(p) = \frac{1}{p}$ the nonparametric control chart gives an out-of-control signal if $X_{n+1} > X_{(n-1)}$ and hence $P_n = \overline{F}(X_{(n-1)})$, implying $E\frac{1}{P_n} = E\frac{1}{U_{(2)}} = n < \frac{1}{p}$.

For further discussion and results on this and similar nonparametric control charts we refer to Albers and Kallenberg (2002).

2.3.2 Modified nonparametric control chart

The practical implementation of the nonparametric control chart is still questionable for $r = 0$ (for example when $g(p) = p$ then $r = 0$ corresponds to $p(n+1) < 1$), because it implies that with positive probability we will never get an out-of-control signal! Therefore we modify the nonparametric control somewhat in case $r = 0$. We replace the definition $X_{(n+1)} = \infty$ by

$X_{(n+1)} = X_{(n)} + S$. For the in-control situation $X_{(n)}$ is already rather far in the tail of the distribution of X_{n+1} and adding S gives for most distributions a really large value. However, a substantial shift in the out-of-control case can now be detected with higher probability than $P(V=1)$, which equals, for instance, 0.251 when $g(p) = p$, $p = 0.001$ and $n = 250$. Moreover, it is now no longer decided to keep the process running "for ever" when $V = 0$.

The modified nonparametric chart is defined by (7) if $r \geq 1$ and for $r = 0$ by

$$X_{n+1} > X_{(n)} + (1-V)S \text{ with } P(V=1) = \frac{g(p) - g(0)}{Eg(U_{(1)}) - g(0)}. \tag{11}$$

3 Choosing the model

When the observations are close to normality, we want to select the normal control chart. If the departure from normality is too large, we apply the parametric control chart, unless the parametric family also does not fit. In the latter case the (modified) nonparametric control chart comes in. It is argued already in Section 1 (but see also below) that the model error is the guide for choosing between the charts. In principle, the model error can be defined both for the in-control and the out-of-control situation. However, because of the validity of the control chart, our main concern lies in the model error for the in-control case. Therefore, when developing rules for choosing between the three control charts we assume that $X_1, \ldots, X_n, X_{n+1}$ are i.i.d. r.v.'s with common distribution function F.

In deciding, for instance, whether the departure from normality is too large, we have to use a measure for the distance between the realized distribution (the true F) and the supposed model (here normality). This "distance" should be chosen in accordance with the problem at hand, that is the difference between the observed false alarm rate P_n and the prescribed false alarm rate p. The total error consists of the model error and the stochastic error. The stochastic error can be reduced by a correction term (according to the criterion at hand). The model error should be reduced by an appropriate choice of the model and here the notion of "distance" naturally comes in. Furthermore, the data should tell us whether the model error is (too) large. Therefore, the selection between the three possible control charts is based on a kind of *estimation* of the *model error*. To come to an implementation of the preceding ideas, we need a more technical definition of the model error.

In general terms the situation can be described as follows. Denote the supposed model by a parametric family of distribution functions $\{H_\theta : \theta \in \Theta\}$. As before, let F denote the true distribution of X_{n+1} (and $X_1, \ldots, X_n$). If F equals H_θ and θ is known, the control limit simply equals $\overline{H}_\theta^{-1}(p)$. Usually F is unknown and two problems arise: (i) F may be outside the supposed model and (ii) θ is unknown. This leads to two kinds of errors, the model error and the stochastic error. The latter is due to estimation of θ in the control limit by its estimator $\widehat{\theta}$. Therefore, the total error when using the control limit $\overline{H}_{\widehat{\theta}}^{-1}(p)$ can be split up in two parts,

$$P\left(X_{n+1} > \overline{H}_{\widehat{\theta}}^{-1}(p)\right) - p = \left\{\overline{F}\left(\overline{H}_\theta^{-1}(p)\right) - p\right\} + \left\{\overline{F}\left(\overline{H}_{\widehat{\theta}}^{-1}(p)\right) - \overline{F}\left(\overline{H}_\theta^{-1}(p)\right)\right\}. \tag{12}$$

The first term on the right-hand side of (12),

$$\overline{F}\left(\overline{H}_\theta^{-1}(p)\right) - p, \tag{13}$$

is called the model error, the second term is the stochastic error.

Let us start with considering the question whether to use the normal control chart or not. The supposed distribution function H_θ is that of $\mu + \sigma Z$ (with Z having a standard normal distribution). Therefore, the model error equals $\overline{F}(\mu + \sigma u_p) - p$ and we want to check (based on our observations $X_1, \ldots, X_n$) the behavior of $(X - \mu)/\sigma$ in the far tail (p is small!). For instance, when $p = 0.001$ and $n < 1000$, the most obvious quantity to look at is the standardized maximum of our observations: $(X_{(n)} - \widehat{\mu})/\widehat{\sigma}$. In principle we can use for larger p and/or n other order statistics, but to avoid technical complications and to keep the combined control chart simple we restrict ourselves to $(X_{(n)} - \widehat{\mu})/\widehat{\sigma}$.

The next point is the determination of the cut-off points for staying at the normal chart. If $X_1, \ldots, X_n$ are i.i.d. with a standard normal distribution and $\{d_{1N}(n)\}, \{d_{2N}(n)\}$ are sequences of positive numbers satisfying $\lim_{n\to\infty} d_{1N}(n) = \infty, d_{1N}(n) < n$ and $\lim_{n\to\infty} d_{2N}(n) = 0$, then

$$\lim_{n\to\infty} P\left(X_{(n)} < \overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)\right) = 0$$

and

$$\lim_{n\to\infty} P\left(X_{(n)} > \overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right) = 0.$$

Therefore, we will prefer the normal control chart when

$$\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right) \leq \frac{X_{(n)} - \overline{X}}{S} \leq \overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right). \tag{14}$$

Distributions with heavier tails than the normal one give problems with the in-control behavior, leading for common distributions to EP_n being 4 or even 12 times as large as it should be, see Table 1 in Albers et al. (2002a) and hence the control chart is invalid. Distributions with thinner tails are conservative in the in-control case with as consequence a loss in the out-of-control. Because errors in the in-control are more serious than those in the out-of-control case and since a positive model error as large as p or larger can easily occur, whereas the negative model error is at most $-p$, we take the selection rule *unbalanced*. In particular, we will consider $d_{1N}(n) = -0.7 + 0.5 \log n, d_{2N}(n) = 5/\sqrt{n}$, leading to

$$P\left(X_{(n)} < \overline{\Phi}^{-1}\left(\frac{-0.7 + 0.5\log n}{n}\right)\right) = \left(1 - \frac{-0.7 + 0.5\log(n)}{n}\right)^n \approx \exp(0.7 - 0.5\log(n)) \approx \frac{2}{\sqrt{n}}$$

and

$$P\left(X_{(n)} > \overline{\Phi}^{-1}\left(\frac{5}{n\sqrt{n}}\right)\right) = 1 - \left(1 - \frac{5}{n\sqrt{n}}\right)^n \approx 1 - \exp\left(-\frac{5}{\sqrt{n}}\right) \approx \frac{5}{\sqrt{n}},$$

when $X_1, \ldots, X_n$ are i.i.d. having a standard normal distribution.

Remark 3.1 The first step in the selection procedure is to choose the normal or the parametric chart. One might wonder why not to take $\widehat{\gamma}$ as statistic to choose between these two. After all, the normal distribution is a member of the normal power family with $\gamma = 0$ and we simply have to investigate whether $\gamma = 0$ or not. The estimator $\widehat{\gamma}$ seems to be the obvious measure for that. However, presentation in this form is misleading, since we do not need to escape from the normal chart only if we have a member of the normal power family with a substantially different value of γ, but also for distributions outside the normal power family. So, the selection procedure should not be restricted to the framework of the normal power family.

Indeed, it turns out that for several well known distributions such as the Student distribution pretty small values of $\widehat{\gamma}$ are very common, while nevertheless a rather large model error occurs when assuming normality. For example, the standardized Student distribution with 6 degrees of freedom has 0.118 as limiting value of $\widehat{\gamma}$ (when n tends to infinity), while for $p = 0.001$ its model error equals 3.56 and hence $EP_n \approx 4.56p$, see Table 1 in Albers et al. (2002a).

Next we address the question how to choose between the parametric control chart and the nonparametric one. Similarly as for the normal chart we prefer the parametric control chart when

$$\frac{X_{(n)} - \overline{X}}{S} \notin \left[\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right), \overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right], \tag{15}$$
$$\overline{K}_{\widehat{\gamma}}^{-1}\left(\frac{d_{1P}(n)}{n}\right) \leq \frac{X_{(n)} - \overline{X}}{S} \leq \overline{K}_{\widehat{\gamma}}^{-1}\left(\frac{d_{2P}(n)}{n}\right),$$

where $\{d_{1P}(n)\}, \{d_{2P}(n)\}$ are sequences of positive numbers satisfying $\lim_{n\to\infty} d_{1P}(n) = \infty$, $d_{1P}(n) < n$ and $\lim_{n\to\infty} d_{2P}(n) = 0$. In particular, we will consider $d_{1P}(n) = -0.2 + 0.5\log n$, $d_{2P}(n) = 3/\sqrt{n}$, leading to

$$P\left(X_{(n)} < \overline{K}_{\gamma}^{-1}\left(\frac{-0.2 + 0.5\log n}{n}\right)\right) = \left(1 - \frac{-0.2 + 0.5\log(n)}{n}\right)^n$$
$$\approx \exp\left(0.2 - 0.5\log(n)\right) \approx \frac{1.2}{\sqrt{n}}$$

and

$$P\left(X_{(n)} > \overline{K}_{\gamma}^{-1}\left(\frac{3}{n\sqrt{n}}\right)\right) = 1 - \left(1 - \frac{3}{n\sqrt{n}}\right)^n \approx 1 - \exp\left(-\frac{3}{\sqrt{n}}\right) \approx \frac{3}{\sqrt{n}},$$

when $X_1, \ldots, X_n$ are i.i.d. having a normal power family distribution with distribution function K_γ. Note that this specific range has somewhat larger probability of staying at the parametric control chart than the corresponding one for the normal chart, because here the model error is in general lower. The "unbalance-factor" is in both cases approximately the same: $5/2 = 3/1.2$.

When neither (14) nor (15) hold, we choose the modified nonparametric control chart. Moreover, when $p(n+1) \geq 1$ we can make some simplifications by ignoring the correction terms in the normal and parametric control chart (since they are rather small in that case) and replacing in the nonparametric chart the stochastic term V by its deterministic counterpart EV.

In order to avoid too much technicalities we restrict ourselves now to $g(p) = p$ (which is considered in the simulation study). Note however, that with (some) more effort other g's like $g(p) = 1 - (1-p)^k$ or $g(p) = 1/p$ can be analyzed as well.

Consequently, for $g(p) = p$ the combined control chart is defined as follows. Let UL_N, UL_P, UL_{MNP} denote the upper limit of the normal, parametric and modified nonparametric control chart (for $r = 0$), respectively, that is, cf.(1), (4) and (5), and (11),

$$UL_N = \widehat{\mu} + (u_p + c_N)\widehat{\sigma} \text{ with } c_N \text{ as in Table 1,} \tag{16}$$
$$UL_P = \widehat{\mu} + \widehat{\sigma}\left\{\overline{K}_{\widehat{\gamma}}^{-1}(p) - C1(\widehat{\gamma})\,C2(\widehat{\gamma}) - \frac{C3(\widehat{\gamma})}{n} + \lambda\frac{C4(\widehat{\gamma})}{n}\right\},$$
$$UL_{MNP} = X_{(n)} + (1-V)S \text{ with } P(V=1) = p(n+1)$$

and let

$$IN = \left[\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right), \overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right], \tag{17}$$
$$IP = \left[\overline{K}_{\widehat{\gamma}}^{-1}\left(\frac{d_{1P}(n)}{n}\right), \overline{K}_{\widehat{\gamma}}^{-1}\left(\frac{d_{2P}(n)}{n}\right)\right],$$

then the combined control chart for $r = 0$ is given by

$$X_{n+1} > UL_N 1\left(\frac{X_{(n)} - \overline{X}}{S} \in IN\right) + UL_P 1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)} - \overline{X}}{S} \in IP\right) \quad (18)$$
$$+ UL_{MNP} 1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IP\right)$$

with $1(A) = 1$ if A holds and 0 otherwise. Writing $\delta = p(n+1) - r$ the combined control chart for $r \geq 1$ is given by

$$X_{n+1} > \left(\overline{X} + u_p S\right) 1\left(\frac{X_{(n)} - \overline{X}}{S} \in IN\right) \quad (19)$$
$$+ \left(\overline{X} + \overline{K}_{\widehat{\gamma}}^{-1}(p)\, S\right) 1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)} - \overline{X}}{S} \in IP\right)$$
$$+ \left\{\delta X_{(n-r)} + (1-\delta) X_{(n-r+1)}\right\} 1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IP\right).$$

Returning to both the in- an out-of-control situation, that is $X_1, \ldots, X_n$ have distribution function F and X_{n+1} has distribution function G with $G = F$ for the in-control case, we therefore get (with E referring to the expectation under F)

$$EP_n = E\left\{\overline{G}(UL_N^*) 1\left(\frac{X_{(n)} - \overline{X}}{S} \in IN\right)\right\} \quad (20)$$
$$+ E\left\{\overline{G}(UL_P^*) 1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)} - \overline{X}}{S} \in IP\right)\right\}$$
$$+ E\left\{\overline{G}(UL_{NP}^*) 1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IP\right)\right\},$$

where $UL_N^* = UL_N$ if $r = 0$ and $UL_N^* = \overline{X} + u_p S$ if $r \geq 1$, $UL_P^* = UL_P$ if $r = 0$ and $UL_P^* = \overline{X} + \overline{K}_{\widehat{\gamma}}^{-1}(p)\, S$ if $r \geq 1$ and $UL_{NP}^* = UL_{MNP}$ if $r = 0$ and $UL_{NP}^* = \delta X_{(n-r)} + (1-\delta) X_{(n-r+1)}$ if $r \geq 1$. It is easily seen that due to location and scale invariance without loss of generality we may take $\mu = EX_1 = 0$ and $\sigma^2 = var(X_1) = 1$.

We want to show the following results:

1. The combined procedure has good in-control behavior under normality ($F = \Phi$), the normal power family ($F = K_\gamma$ for some γ) and outside the normal power family ($F \neq K_\gamma$ for all γ).

2. During out-of-control there is a gain w.r.t. the nonparametric chart if $F = K_\gamma$ for some γ and hence in particular if $F = \Phi$. There is only a small loss w.r.t. the normal and parametric chart if $F = \Phi$, w.r.t. the parametric chart if $F = K_\gamma$ for some γ, and also only a small loss w.r.t. the nonparametric chart if $F \neq K_\gamma$ for all γ.

¿From the theoretical point of view this will be shown by demonstrating that the combined control chart in each of the three situations (normality, normal power family, outside the normal power family) asymptotically behaves as the specific corresponding control chart. In the simulations this is established by numerical comparison of the specific and the combined control chart for several distributions.

The main point in the theoretical part is to show that the estimators $\overline{X}, S$ and $\widehat{\gamma}$ can essentially be ignored in the selection part of the procedure. This requires a lot of effort, including results on large deviations for these estimators, which are presented in the next section and the appendix. Simplifying things a little bit and taking $d_{1N}(n) = -0.7 + 0.5\log n, d_{2N}(n) = 5/\sqrt{n}, d_{1P}(n) = -0.2 + 0.5\log n, d_{2P}(n) = 3/\sqrt{n}$, for instance, when normality holds the procedure then looks as follows: choose the normal chart except for a probability $\frac{2}{\sqrt{n}}$ (due to possible thinner tail) $+\frac{5}{\sqrt{n}}$ (due to possible heavier tail); when not taking the normal chart we choose with probability $\frac{0.8}{\sqrt{n}} + \frac{2}{\sqrt{n}}$ the parametric chart and otherwise the nonparametric chart.

In general, the behavior of the combined chart is indeed a mixture of the three charts according to the weights of the selection rule with the following refinement: the contribution of the normal chart is similar to the one in the unselected case, due to almost independence of $X_{(n)}$ and $\overline{X}, S$; the contribution of the nonparametric part is lower than in the unselected case, when we enter the nonparametric chart because of large values of $X_{(n)}$, implying that the control limit is also relatively large, and the contribution is larger otherwise; in the parametric part there is some dependence, but not that much. The following simulation results illustrate this phenomenon.

Example 3.1 Assume that $X_1, \ldots, X_{1000}$ are i.i.d. r.v.'s with a normal distribution. The probability of choosing the normal chart then is approximately equal to $1-7/\sqrt{1000} = 0.78$. Our simulation result (based on 100 000 simulations) gives 0.78 as well. The simulated "EP_n" for this part (that is the frequency of the out-of-control signal for these simulations) equals 0.00103, which corresponds with the simulated EP_n using the normal chart for all 100 000 simulations, being also 0.00103.

The probability of choosing the parametric chart, due to large values of $X_{(n)}$ is approximately equal to $2/\sqrt{1000} = 0.06$. The simulated value is 0.07. The contribution of this part equals 0.00076, which is somewhat smaller than the simulated EP_n using the parametric chart for all 100 000 simulations, being 0.00113 . The probability of choosing the parametric chart due to small values of $X_{(n)}$, is approximately equal to $0.8/\sqrt{1000} = 0.03$, which is also the value in the simulation. The simulated "EP_n" for this part equals 0.00165, which indeed is somewhat larger than 0.00113.

The probability of choosing the nonparametric chart due to large values of $X_{(n)}$, is approximately equal to $3/\sqrt{1000} = 0.09$, which is also the simulated value. The simulated "EP_n" for this part is 0.00007, which is much smaller than 0.00101, being the simulated EP_n using the nonparametric chart for all 100 000 simulations. The probability of choosing the nonparametric chart due to small values of $X_{(n)}$, is approximately equal to $1.2/\sqrt{1000} = 0.04$, while the simulation gives 0.03. Its contribution is 0.00401 and this is (as expected) much higher than 0.00101.

The resulting simulated EP_n for the combined chart thus equals $0.78 \times 0.00103 + 0.07 \times 0.00076 + 0.03 \times 0.00165 + 0.09 \times 0.00007 + 0.03 \times 0.00401 = 0.00103$.

4 Application

We apply the new control chart on a real life example concerning the production of electric shavers by Philips. In an electrochemical process razor heads are formed. The measurements concern the thickness of the razor heads. There are two samples each of 835 measurements.

One sample will be used to settle the control chart and then this chart is applied on the second sample. A histogram of the first sample is given in Figure 1.

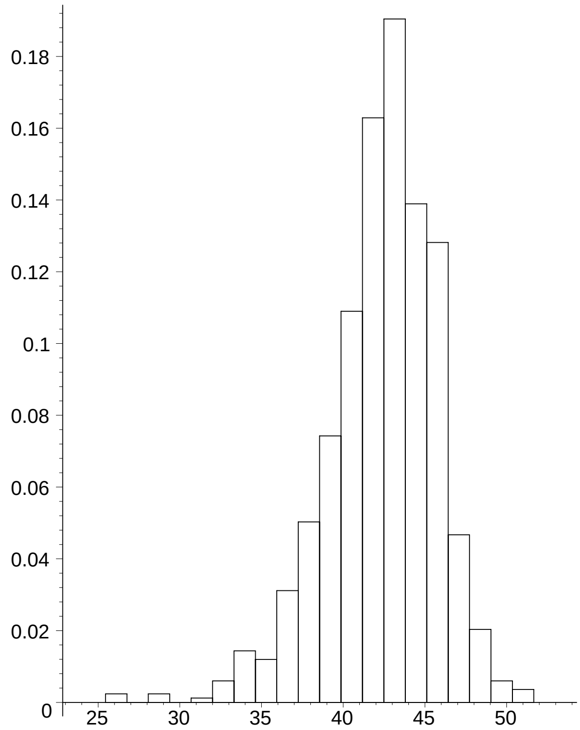

Figure 1. Histogram of the thickness of razor heads for the first sample of 835 measurements.

The control limit is obtained in the following steps

1. We calculate $\frac{X_{(n)}-\overline{X}}{S}$. Since $X_{(n)} = 51.66, \overline{X} = 42.366, S = 3.311$, this leads to

$$\frac{X_{(n)} - \overline{X}}{S} = 2.807.$$

2. We calculate the cut-off points for choosing the normal chart. These values are

$$\overline{\Phi}^{-1}\left(\frac{-0.7 + 0.5 \log n}{n}\right) = 2.728 \text{ and}$$
$$\overline{\Phi}^{-1}\left(\frac{5}{n\sqrt{n}}\right) = 3.531.$$

3. Because $2.807 \in [2.728, 3.531]$ we choose the normal chart and therefore we calculate the corresponding control limit. It is given by $\widehat{\mu} + (u_p + c_N)\widehat{\sigma}$ with $\widehat{\mu} = \overline{X} = 42.366, u_p = 3.090, c_N = \dfrac{u_p}{4n} + \dfrac{u_p(u_p^2 + 2)}{4n} = 0.012$ and $\widehat{\sigma} = S = 3.311$, see (1). As a result the control limit equals 52.635 and an out-of-control signal is given for new observations being large than 52.635.

We apply the control chart on the second sample. Figure 2 shows the result, where the horizontal line gives the control limit 52.635

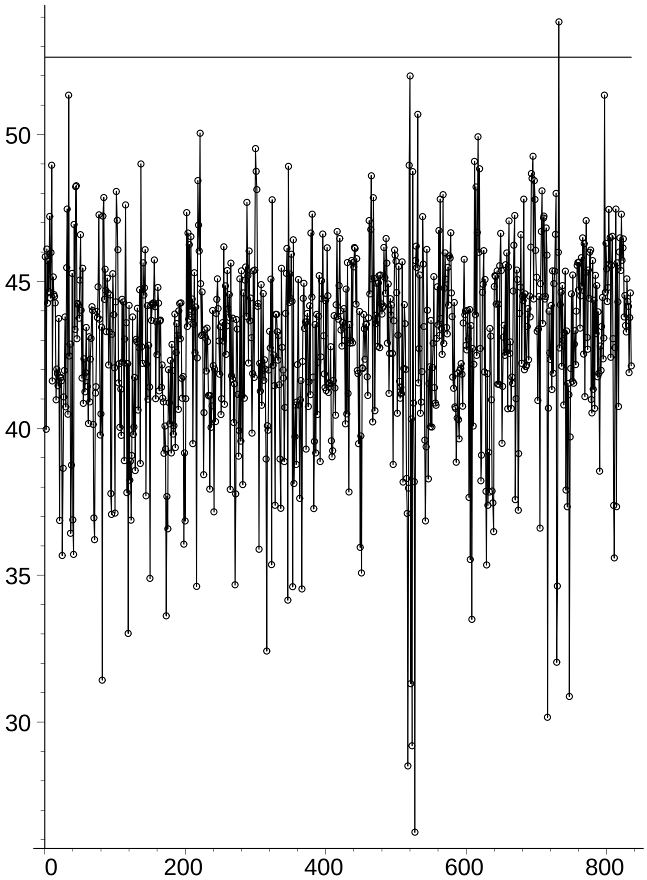

Figure 2. Control chart of the second sample of 835 measurements of the thickness of razor heads.

As is seen there is one observation among the 835 observations which gives an out-of-control signal.

Note that we do not need to calculate the parametric or nonparametric control chart in this case.

If one is interested in controlling the lower values in this example the calculations should be modified in an obvious way. Essentially we consider $-X_1, \dots, -X_n$, calculate for these r.v.'s the upper control limit and apply these control limit on $-X_{n+1}$. An out-of-control signal is given when $-X_{n+1}$ is larger than the obtained upper control limit, or, equivalently, if X_{n+1} is smaller than minus the control limit based on $-X_1, \dots, -X_n$. Here are the details.

1. We calculate $\frac{\overline{X}-X_{(1)}}{S}$. Since $X_{(1)} = 25.45, \overline{X} = 42.366, S = 3.311$, this leads to
$$\frac{\overline{X} - X_{(1)}}{S} = 5.109.$$
2. The cut-off points for choosing the normal chart are again
$$\overline{\Phi}^{-1}\left(\frac{-0.7 + 0.5\log n}{n}\right) = 2.728 \text{ and } \overline{\Phi}^{-1}\left(\frac{5}{n\sqrt{n}}\right) = 3.531.$$
3. Because $5.109 \notin [2.728, 3.531]$ we do not choose the normal chart. Next we calculate the cut-off points for the parametric chart. We are dealing with the lower part and hence
$$\widehat{\gamma} = \frac{1}{\log\left(\frac{\Phi^{-1}(0.05)}{\Phi^{-1}(0.25)}\right)} \log\left(\frac{\overline{X} - X_{(n-[0.95n])}}{\overline{X} - X_{(n-[0.75n])}}\right) - 1.$$

This gives $\widehat{\gamma} = 0.352$. The upper cut-off point for the parametric chart equals $\overline{K}_{\widehat{\gamma}}^{-1}\left(\frac{3}{n\sqrt{n}}\right) =$ 4.957, implying that in this case the (modified) nonparametric chart is chosen. Hence, an out-of-control signal is given if a new observation is smaller than $X_{(1)} - (1-V)S$, which equals 25.45 if $V = 0$ and 22.139 if $V = 1$, where $P(V = 1) = (n+1)p = 0.836$.

The minimum of the second sample is 26.25 and therefore no out-of-control signal is obtained.

5 Large deviations

The large deviation results of this section are used in the proofs of the basic lemmas 3, 5 and 6 in Section 6. We start with a large deviation result on the behavior of the estimators $\overline{X}$ and $\widehat{\gamma}$, see (5), in the normal power family.

Theorem 1 *Let $X_1, \ldots, X_n$ be i.i.d. r.v.'s with a normal power distribution with parameter γ. Then for each $\varepsilon > 0$*

$$\limsup_{n\to\infty} n^{-\min(1,\, 2/(1+\gamma))} \log P\left(\left|\overline{X}\right| > \varepsilon\right) < 0, \tag{21}$$

$$\limsup_{n\to\infty} n^{-\min(1,\, 1/(1+\gamma))} \log P\left(\left|S^2 - 1\right| > \varepsilon\right) < 0 \tag{22}$$

and

$$\limsup_{n\to\infty} n^{-\min(1,\, 2/(1+\gamma))} \log P\left(\left|\widehat{\gamma} - \gamma\right| > \varepsilon\right) < 0. \tag{23}$$

Proof. Denote by $X_{(1)} \leq \ldots \leq X_{(n)}$ the order statistics of $X_1, \ldots, X_n$ and let $U_{(1)} \leq \ldots \leq U_{(n)}$ be the order statistics of the random sample $U_1, \ldots, U_n$ from a uniform distribution on (0,1). Let $0 < s < 1$ be fixed and let $j = j(n)$ satisfy $\lim_{n\to\infty} j/n = s$. Then, for any $\varepsilon > 0$,

$$P(X_{(j)} > K_\gamma^{-1}(s) + \varepsilon) = P(U_{(j)} > K_\gamma(K_\gamma^{-1}(s) + \varepsilon)) = P\left(\sum_{i=1}^{n} 1_{U_i > K_\gamma(K_\gamma^{-1}(s)+\varepsilon)} \geq n - j\right) \tag{24}$$

and by Chernoff's theorem, see e.g. Bahadur (1971), we get

$$\lim_{n\to\infty} n^{-1} \log P\left(\sum_{i=1}^{n} 1_{U_i > K_\gamma(K_\gamma^{-1}(s)+\varepsilon)} \geq n - j\right) < 0.$$

Similarly, we have

$$\lim_{n\to\infty} n^{-1} \log P(X_{(j)} < K_\gamma^{-1}(s) - \varepsilon) < 0$$

and hence

$$\lim_{n\to\infty} n^{-1} \log P\left(\left|X_{(j)} - K_\gamma^{-1}(s)\right| > \varepsilon\right) < 0. \tag{25}$$

If $1 + \gamma \leq 2$, application of Chernoff's theorem easily gives, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{-1} \log P\left(\left|\overline{X}\right| > \varepsilon\right) < 0. \tag{26}$$

For $1 + \gamma > 2$, the moment generating function of X_i does not exist. However, by Nagaev (1969), see also Nagaev (1979, formulas (2.31) and (2.32) on page 764), it follows that

$$\limsup_{n\to\infty} n^{-2/(1+\gamma)} \log P\left(\overline{X} > \varepsilon\right) \leq \lim_{n\to\infty} n^{-2/(1+\gamma)} \log P\left(X_1 > n\varepsilon/2\right) \tag{27}$$
$$= \lim_{n\to\infty} n^{-2/(1+\gamma)} \log P\left(c(\gamma)\left|Z\right|^{1+\gamma} sign(Z) > n\varepsilon/2\right) = -\frac{1}{2}\left(\frac{\varepsilon/2}{c(\gamma)}\right)^{2/(1+\gamma)}.$$

By symmetry, we get

$$P\left(\overline{X} < -\varepsilon\right) = P\left(\overline{X} > \varepsilon\right)$$

and hence we obtain for $\gamma > 1$

$$\limsup_{n\to\infty} n^{-2/(1+\gamma)} \log P\left(\left|\overline{X}\right| > \varepsilon\right) \le -\frac{1}{2}\left(\frac{\varepsilon/2}{c(\gamma)}\right)^{2/(1+\gamma)}. \tag{28}$$

In view of (26) and (28) we arrive at

$$\limsup_{n\to\infty} n^{-\min(1,2/(1+\gamma))} \log P\left(\left|\overline{X}\right| > \varepsilon\right) < 0,$$

thus proving (21).

Since

$$\begin{aligned} P\left(\left|S^2-1\right| > \varepsilon\right) &\le P\left(\left|S^2-1\right| > \varepsilon, \left|\overline{X}\right| < \varepsilon\right) + P\left(\left|\overline{X}\right| > \varepsilon\right) \\ &\le P\left(\left|\sum_{i=1}^{n}\left(X_i^2-1\right)\right| > (n-1)\varepsilon - n\varepsilon^2 - 1\right) + P\left(\left|\overline{X}\right| > \varepsilon\right), \end{aligned}$$

it follows by a similar argument as in the proof of (21) that (22) holds true.

By continuity there exist for each $\varepsilon > 0$ constants $\varepsilon_i = \varepsilon_i(\varepsilon), i = 1, 2, 3$ such that

$$\begin{aligned} &P\left(\left|\widehat{\gamma}-\gamma\right| > \varepsilon\right) \le \\ &P\left(\left|X_{([0.95n+1])} - K_\gamma^{-1}(0.95)\right| > \varepsilon_1\right) + P\left(\left|X_{([0.75n+1])} - K_\gamma^{-1}(0.75)\right| > \varepsilon_2\right) + P\left(\left|\overline{X}\right| > \varepsilon_3\right). \end{aligned}$$

The result now easily follows from (25) and (28). ■

Theorem 1 concerns a fixed deviation. In the next theorem we consider deviations tending to 0, but at a slow rate. Its proof is given in the appendix.

Theorem 2 *Let $X_1, \ldots, X_n$ be i.i.d. r.v.'s with a normal power distribution with parameter γ. Let $\{\varepsilon_n\}$ be a sequence of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} n\varepsilon_n^2 = \infty$.*

(i) If $\gamma \le 1$, then

$$\limsup_{n\to\infty} (n\varepsilon_n^2)^{-1} \log P\left(\left|\overline{X}\right| > \varepsilon_n\right) < 0. \tag{29}$$

If $\gamma > 1$, then

$$\limsup_{n\to\infty} \frac{\log P\left(\left|\overline{X}\right| > \varepsilon_n\right)}{\min\{n\varepsilon_n^2, (n\varepsilon_n)^{2/(1+\gamma)}\}} < 0. \tag{30}$$

(ii) If $\gamma \le 0$, then

$$\limsup_{n\to\infty} (n\varepsilon_n^2)^{-1} \log P\left(\left|S^2-1\right| > \varepsilon_n\right) < 0. \tag{31}$$

If $\gamma > 0$, then

$$\limsup_{n\to\infty} \frac{\log P\left(\left|S^2-1\right| > \varepsilon_n\right)}{\min\{n\varepsilon_n^2, (n\varepsilon_n)^{1/(1+\gamma)}\}} < 0. \tag{32}$$

(iii) If $\gamma \le 1$, then

$$\limsup_{n\to\infty} (n\varepsilon_n^2)^{-1} \log P\left(\left|\widehat{\gamma}-\gamma\right| > \varepsilon_n\right) < 0. \tag{33}$$

If $\gamma > 1$, then

$$\limsup_{n\to\infty} \frac{\log P\left(\left|\widehat{\gamma}-\gamma\right| > \varepsilon_n\right)}{\min\{n\varepsilon_n^2, (n\varepsilon_n)^{2/(1+\gamma)}\}} < 0. \tag{34}$$

6 Theoretical behavior of the combined control chart

In this section we study both the in-control and the out-of-control behavior of the combined control chart

$$X_{n+1} > UL_N^* 1\left(\frac{X_{(n)}-\overline{X}}{S} \in IN\right) + UL_P^* 1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)}-\overline{X}}{S} \in IP\right)$$
$$+ UL_{NP}^* 1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IP\right),$$

where $UL_N^* = UL_N$ if $r=0$ and $UL_N^* = \overline{X} + u_p S$ if $r \geq 1$, $UL_P^* = UL_P$ if $r=0$ and $UL_N^* = \overline{X} + \overline{K}_{\widehat{\gamma}}^{-1}(p)\, S$ if $r \geq 1$ and $UL_{NP}^* = UL_{MNP}$ if $r=0$ and $UL_N^* = \delta X_{(n-r)} + (1-\delta) X_{(n-r+1)}$ if $r \geq 1$, see also (16) and (17). We show that the behavior of the combined control chart is asymptotically equivalent to the behavior of the specific control chart for the supposed model. Hence the combined control chart is valid for all distributions and its out-of-control behavior is asymptotically as good as if we should know to which class of distributions the true distribution belongs.

6.1 Normality

Suppose that $X_1, \ldots, X_n$ have distribution function $F = \Phi$ and that X_{n+1} is distributed as $X_1 + \Delta$ for some $\Delta \geq 0$. The in-control situation refers to $\Delta = 0$, while for control charts with only an upper limit the out-of-control case corresponds to $\Delta > 0$. Let $\{d_{1N}(n)\}, \{d_{2N}(n)\}$ be sequences of real numbers satisfying $\lim_{n\to\infty} d_{1N}(n) = \infty, d_{1N}(n) < n$ and $\lim_{n\to\infty} d_{2N}(n) = 0$. In view of (20) we have (with expectations and probabilities referring to $F = \Phi$)

$$\begin{aligned}
EP_n &= E\left\{\overline{\Phi}(UL_N^* - \Delta) 1\left(\frac{X_{(n)}-\overline{X}}{S} \in IN\right)\right\} \\
&+ E\left\{\overline{\Phi}(UL_P^* - \Delta) 1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)}-\overline{X}}{S} \in IP\right)\right\} \\
&+ E\left\{\overline{\Phi}(UL_{NP}^* - \Delta) 1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IP\right)\right\} \\
&= E\left\{\overline{\Phi}(UL_N^* - \Delta)\right\} - E\left\{\overline{\Phi}(UL_N^* - \Delta) 1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IN\right)\right\} \\
&+ E\left\{\overline{\Phi}(UL_P^* - \Delta) 1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)}-\overline{X}}{S} \in IP\right)\right\} \\
&+ E\left\{\overline{\Phi}(UL_{NP}^* - \Delta) 1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IN\right) 1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IP\right)\right\} \\
&= E\left\{\overline{\Phi}(UL_N^* - \Delta)\right\} + E\left\{1\left(\frac{X_{(n)}-\overline{X}}{S} \notin IN\right) h_N\left(X_{(n)}, \overline{X}, S\right)\right\}
\end{aligned}$$

with

$$h_N\left(X_{(n)},\overline{X},S\right)=\left[\overline{\Phi}(UL_P^*-\Delta)-\overline{\Phi}(UL_N^*-\Delta)\right]1\left(\frac{X_{(n)}-\overline{X}}{S}\in IP\right)$$
$$+\left[\overline{\Phi}(UL_{NP}^*-\Delta)-\overline{\Phi}(UL_N^*-\Delta)\right]1\left(\frac{X_{(n)}-\overline{X}}{S}\notin IP\right)$$

and hence, using $\left|h_N\left(X_{(n)},\overline{X},S\right)\right|\leq 1$,

$$\left|EP_n-E\left\{\overline{\Phi}(UL_N^*-\Delta)\right\}\right|\leq P\left(\frac{X_{(n)}-\overline{X}}{S}\notin IN\right). \tag{35}$$

The following lemma gives asymptotic expressions for the probabilities of the right-hand side of (35). Its proof is in the appendix.

Lemma 3 *Let $X_1,\ldots,X_n$ be i.i.d. r.v's with a normal distribution. Let $\{d_{1N}(n)\},\{d_{2N}(n)\}$ be sequences of real numbers satisfying*

$$\lim_{n\to\infty}d_{1N}(n)=\infty,\ \lim_{n\to\infty}d_{1N}(n)\sqrt{\frac{d_{1N}(n)}{n}}\log\left(\frac{d_{1N}(n)}{n}\right)=0,$$
$$\lim_{n\to\infty}d_{2N}(n)=0,\ \lim_{n\to\infty}\sqrt{\frac{|\log d_{2N}(n)|}{n}}\log\left(\frac{d_{2N}(n)}{n}\right)=0.$$

Then

$$P\left(\frac{X_{(n)}-\overline{X}}{S}<\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)\right)=\left(1-\frac{d_{1N}(n)}{n}\right)^n(1+o(1)),$$
$$P\left(\frac{X_{(n)}-\overline{X}}{S}>\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right)=d_{2N}(n)(1+o(1))\ \text{as } n\to\infty.$$

In view of (35) and Lemma 3 we get the following result, which expresses that under normality the combined control chart given by (18) and (19) is asymptotically as good as the normal control chart given in (1). The results of Albers and Kallenberg (2000) on (corrected) normal control charts ensure that $E\overline{\Phi}(UL_N^*)$ is very close to p.

Theorem 4 *Let $X_1,\ldots,X_n$ be i.i.d. r.v's with a normal distribution. Let $\{d_{1N}(n)\},\{d_{2N}(n)\}$ be sequences of real numbers satisfying*

$$\lim_{n\to\infty}d_{1N}(n)=\infty,\ \lim_{n\to\infty}d_{1N}(n)\sqrt{\frac{d_{1N}(n)}{n}}\log\left(\frac{d_{1N}(n)}{n}\right)=0,$$
$$\lim_{n\to\infty}d_{2N}(n)=0,\ \lim_{n\to\infty}\sqrt{\frac{|\log d_{2N}(n)|}{n}}\log\left(\frac{d_{2N}(n)}{n}\right)=0.$$

Then

$$\left|EP_n-E\left\{\overline{\Phi}(UL_N^*-\Delta)\right\}\right|\leq\left\{\left(1-\frac{d_{1N}(n)}{n}\right)^n+d_{2N}(n)\right\}(1+o(1))\ \text{as } n\to\infty.$$

Note that for the choices $d_{1N}(n)=-0.7+0.5\log n, d_{2N}(n)=5/\sqrt{n}$ the conditions of Theorem 4 are satisfied.

6.2 Normal power family

Suppose that $X_1, \ldots, X_n$ have distribution function $F = K_\gamma$ and that X_{n+1} is distributed as $X_1 + \Delta$ for some $\Delta \geq 0$. If $\gamma = 0$, we are in the previous situation. Therefore, assume that $\gamma \neq 0$. Let $\{d_{1P}(n)\}, \{d_{2P}(n)\}$ be sequences of real numbers satisfying $\lim_{n\to\infty} d_{1P}(n) = \infty, d_{1P}(n) < n$ and $\lim_{n\to\infty} d_{2P}(n) = 0$. In view of (20) we have (with expectations and probabilities referring to $F = K_\gamma$)

$$\begin{aligned} EP_n &= E\left\{\overline{K}_\gamma(UL_N^* - \Delta)1\left(\frac{X_{(n)} - \overline{X}}{S} \in IN\right)\right\} \\ &+ E\left\{\overline{K}_\gamma(UL_P^* - \Delta)1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right)1\left(\frac{X_{(n)} - \overline{X}}{S} \in IP\right)\right\} \\ &+ E\left\{\overline{K}_\gamma(UL_{NP}^* - \Delta)1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right)1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IP\right)\right\} \\ &= E\left\{\overline{K}_\gamma(UL_P^* - \Delta)\right\} + Eh_P\left(X_{(n)}, \overline{X}, S\right) \end{aligned}$$

with

$$\begin{aligned} h_P\left(X_{(n)}, \overline{X}, S\right) &= \left[\overline{K}_\gamma(UL_N^* - \Delta) - \overline{K}_\gamma(UL_P^* - \Delta)\right]1\left(\frac{X_{(n)} - \overline{X}}{S} \in IN\right) \\ &+ \left[\overline{K}_\gamma(UL_{NP}^* - \Delta) - \overline{K}_\gamma(UL_P^* - \Delta)\right]1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IP\right)1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right) \end{aligned}$$

and hence

$$\left|EP_n - E\left\{\overline{K}_\gamma(UL_P^* - \Delta)\right\}\right| \leq P\left(\frac{X_{(n)} - \overline{X}}{S} \in IN\right) + P\left(\frac{X_{(n)} - \overline{X}}{S} \notin IP\right). \tag{36}$$

The next two lemmas give the ingredients for the asymptotic expressions of the right-hand side of (36). The proofs of these lemmas can be found in the appendix.

Lemma 5 *Let $X_1, \ldots, X_n$ be i.i.d. r.v's with a distribution from the normal power family. Let $\{d_{1P}(n)\}, \{d_{2P}(n)\}$ be sequences of real numbers satisfying for some $\zeta > 0$*

$$\lim_{n\to\infty} d_{1P}(n) = \infty, \begin{cases} \lim_{n\to\infty} d_{1P}(n)\sqrt{\frac{d_{1P}(n)}{n}}\left|\log\left(\frac{d_{1P}(n)}{n}\right)\right|^{1+\zeta} = 0 \ \textit{if}\ \gamma \leq 0 \\ \lim_{n\to\infty} d_{1P}(n)\max\left(\frac{d_{1P}^{1+\gamma}(n)}{n}, \sqrt{\frac{d_{1P}(n)}{n}}\right)\left|\log\left(\frac{d_{1P}(n)}{n}\right)\right|^{1+\zeta} = 0 \ \textit{if}\ \gamma > 0 \end{cases},$$

$$\lim_{n\to\infty} d_{2P}(n) = 0, \begin{cases} \lim_{n\to\infty} \sqrt{\frac{|\log d_{2P}(n)|}{n}}\left|\log\left(\frac{d_{2P}(n)}{n}\right)\right|^{1+\zeta} = 0 \ \textit{if}\ \gamma \leq 0 \\ \lim_{n\to\infty} \max\left(\frac{|\log d_{2P}(n)|^{1+\gamma}}{n}, \sqrt{\frac{|\log d_{2P}(n)|}{n}}\right)\left|\log\left(\frac{d_{2P}(n)}{n}\right)\right|^{1+\zeta} = 0 \ \textit{if}\ \gamma > 0. \end{cases}.$$

Then

$$\begin{aligned} P\left(\frac{X_{(n)} - \overline{X}}{S} < \overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{1P}(n)}{n}\right)\right) &= \left(1 - \frac{d_{1P}(n)}{n}\right)^n (1 + o(1)), \\ P\left(\frac{X_{(n)} - \overline{X}}{S} > \overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{2P}(n)}{n}\right)\right) &= d_{2P}(n)(1 + o(1)) \ \textit{as}\ n \to \infty. \end{aligned}$$

Lemma 6 *Let $X_1, \ldots, X_n$ be i.i.d. r.v's with distribution function K_γ from the normal power family. Let $\gamma > 0$ and let $\{d_{2N}(n)\}$ be a sequence of real numbers satisfying for some $0 < \zeta < \gamma$*

$$\lim_{n\to\infty} d_{2N}(n) = 0, \lim_{n\to\infty} \frac{|\log d_{2N}(n)|}{(\log n)^{1+\gamma-\zeta}} = 0.$$

Then for some $c > 0$

$$P\left(\frac{X_{(n)} - \overline{X}}{S} \le \overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right) = O\left(\exp\left(-cn^{1/(1+\gamma)}\right)\right) \text{ as } n \to \infty.$$

Let $\gamma < 0$ and let $\{d_{1N}(n)\}$ be a sequence of real numbers satisfying

$$\lim_{n\to\infty} d_{1N}(n) = \infty, \lim_{n\to\infty} d_{1N}(n)\sqrt{\frac{d_{1N}(n)}{n}} \log\left(\frac{d_{1N}(n)}{n}\right) = 0.$$

Then for some $c^* > 0$

$$P\left(\frac{X_{(n)} - \overline{X}}{S} \ge \overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)\right) = O\left(\exp\left\{-c^*(\log n)^{\frac{1}{1+\gamma}}\right\}\right) \text{ as } n \to \infty.$$

Note that

$$\lim_{n\to\infty} d_{2N}(n) = 0, \lim_{n\to\infty} \frac{|\log d_{2N}(n)|}{(\log n)^{1+\gamma-\zeta}} = 0$$

implies

$$\lim_{n\to\infty} d_{2N}(n) = 0, \lim_{n\to\infty} \sqrt{\frac{|\log d_{2N}(n)|}{n}} \log\left(\frac{d_{2N}(n)}{n}\right) = 0.$$

In view of (36), Lemma 5 and Lemma 6 we get the following result, which expresses that within the normal power family the combined control chart given by (18) and (19) is asymptotically as good as the parametric control chart for the normal power family given in (4). Note that

$$\lim_{n\to\infty} \max\left(\frac{|\log d_{2P}(n)|^{1+\gamma}}{n}, \sqrt{\frac{|\log d_{2P}(n)|}{n}}\right) \left|\log\left(\frac{d_{2P}(n)}{n}\right)\right|^{1+\zeta_2} = 0$$

implies

$$\lim_{n\to\infty} \frac{|\log d_{2P}(n)|^{1+\gamma}}{n} = 0$$

and hence

$$\exp\left(-cn^{1/(1+\gamma)}\right) = o\left(d_{2P}(n)\right) \text{ as } n \to \infty.$$

The results of Albers et al. (2002a) on (corrected) parametric control charts ensure that $E\overline{K}_\gamma(UL_P^*)$ is very close to p.

Theorem 7 *Let $X_1, \ldots, X_n$ be i.i.d. r.v's with a distribution from the normal power family. Let $\{d_{1N}(n)\}$, $\{d_{2N}(n)\}$ be sequences of real numbers satisfying for some $0 < \zeta_1 < \gamma$*

$$\lim_{n\to\infty} d_{1N}(n) = \infty, \lim_{n\to\infty} d_{1N}(n)\sqrt{\frac{d_{1N}(n)}{n}} \log\left(\frac{d_{1N}(n)}{n}\right) = 0 \textit{ if } \gamma > 0,$$
$$\lim_{n\to\infty} d_{2N}(n) = 0, \lim_{n\to\infty} \frac{|\log d_{2N}(n)|}{(\log n)^{1+\gamma-\zeta_1}} = 0 \textit{ if } \gamma > 0.$$

and let $\{d_{1P}(n)\}, \{d_{2P}(n)\}$ *be sequences of real numbers satisfying for some* $\zeta_2 > 0$

$$\lim_{n\to\infty} d_{1P}(n) = \infty, \begin{cases} \lim_{n\to\infty} d_{1P}(n)\sqrt{\frac{d_{1P}(n)}{n}}\left|\log\left(\frac{d_{1P}(n)}{n}\right)\right|^{1+\zeta_2} = 0 \text{ if } \gamma \le 0 \\ \lim_{n\to\infty} d_{1P}(n)\max\left(\frac{d_{1P}^{1+\gamma}(n)}{n}, \sqrt{\frac{d_{1P}(n)}{n}}\right)\left|\log\left(\frac{d_{1P}(n)}{n}\right)\right|^{1+\zeta_2} = 0 \text{ if } \gamma > 0 \end{cases},$$

$$\lim_{n\to\infty} d_{2P}(n) = 0, \begin{cases} \lim_{n\to\infty} \sqrt{\frac{|\log d_{2P}(n)|}{n}}\left|\log\left(\frac{d_{2P}(n)}{n}\right)\right|^{1+\zeta_2} = 0 \text{ if } \gamma \le 0 \\ \lim_{n\to\infty} \max\left(\frac{|\log d_{2P}(n)|^{1+\gamma}}{n}, \sqrt{\frac{|\log d_{2P}(n)|}{n}}\right)\left|\log\left(\frac{d_{2P}(n)}{n}\right)\right|^{1+\zeta_2} = 0 \text{ if } \gamma > 0. \end{cases}.$$

Then

$$\left|EP_n - E\left\{\overline{K}_\gamma(UL_P^* - \Delta)\right\}\right| \le \left\{\left(1 - \frac{d_{1P}(n)}{n}\right)^n + d_{2P}(n)\right\}(1 + o(1)) \\ + \left[O\left(\exp\left\{-c^*(\log n)^{\frac{1}{1+\gamma}}\right\}\right) \text{ if } \gamma < 0\right] \text{ as } n \to \infty.$$

Note that for the choices $d_{1N}(n) = -0.7 + 0.5\log n, d_{2N}(n) = 5/\sqrt{n}, d_{1P}(n) = -0.2 + 0.5\log n, d_{2P}(n) = 3/\sqrt{n}$ the conditions of Theorem 7 are satisfied.

6.3 Outside the normal power family

Suppose that $X_1, \ldots, X_n$ have distribution function $F \ne K_\gamma$ for all γ and that X_{n+1} is distributed as $X_1 + \Delta$ for some $\Delta \ge 0$. As before, we have $EX_1 = 0$ and $varX_1 = 1$. In view of (20) we have (with expectations and probabilities referring to F)

$$\begin{aligned} EP_n &= E\left\{\overline{F}(UL_N^* - \Delta)1\left(\frac{X_{(n)} - \overline{X}}{S} \in IN\right)\right\} \\ &+ E\left\{\overline{F}(UL_P^* - \Delta)1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right)1\left(\frac{X_{(n)} - \overline{X}}{S} \in IP\right)\right\} \\ &+ E\left\{\overline{F}(UL_{NP}^* - \Delta)1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right)1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IP\right)\right\} \\ &= E\left\{\overline{F}(UL_{NP}^* - \Delta)\right\} + Eh_{NP}\left(X_{(n)}, \overline{X}, S\right) \end{aligned}$$

with

$$h_{NP}\left(X_{(n)}, \overline{X}, S\right) = \left[\overline{F}(UL_N^* - \Delta) - \overline{F}(UL_{NP}^* - \Delta)\right]1\left(\frac{X_{(n)} - \overline{X}}{S} \in IN\right) \\ + \left[\overline{F}(UL_P^* - \Delta) - \overline{F}(UL_{NP}^* - \Delta)\right]1\left(\frac{X_{(n)} - \overline{X}}{S} \notin IN\right)1\left(\frac{X_{(n)} - \overline{X}}{S} \in IP\right)$$

and hence

$$\left|EP_n - E\left\{\overline{F}(UL_{NP}^* - \Delta)\right\}\right| \le P\left(\frac{X_{(n)} - \overline{X}}{S} \in IN\right) + P\left(\frac{X_{(n)} - \overline{X}}{S} \in IP\right). \quad (37)$$

The following result expresses that outside the normal power family the combined control chart given by (18) and (19) is asymptotically as good as the (modified) nonparametric control chart given by (7) if $r \ge 1$ and for $r = 0$ by (11) (with $g(p) = p$). The proof of Theorem 8 is given in the appendix.

Theorem 8 *Let $X_1, \dots, X_n$ be i.i.d. r.v's with distribution function F. Let $\{d_{1N}(n)\}, \{d_{2N}(n)\}, \{d_{1P}(n)\}, \{d_{2P}(n)\}$ be sequences of real numbers satisfying*

$$\begin{aligned} &\lim_{n\to\infty} d_{1N}(n) = \infty, \lim_{n\to\infty} \frac{\log d_{1N}(n)}{\log n} = 0, \\ &\lim_{n\to\infty} d_{2N}(n) = 0, \lim_{n\to\infty} \frac{|\log d_{2N}(n)|}{\log n} = 0, \\ &\lim_{n\to\infty} d_{1P}(n) = \infty, \lim_{n\to\infty} \frac{\log d_{1P}(n)}{\log n} = 0, \\ &\lim_{n\to\infty} d_{2P}(n) = 0, \lim_{n\to\infty} \frac{|\log d_{2P}(n)|}{\log n} = 0 \end{aligned} \tag{38}$$

Let γ be defined as the limit of the estimator $\widehat{\gamma}$ under F, that is by

$$\gamma = \frac{\log\left(\frac{F^{-1}(0.95)}{F^{-1}(0.75)}\right)}{\log\left(\frac{\Phi^{-1}(0.95)}{\Phi^{-1}(0.75)}\right)} - 1.$$

Then, for each $\varepsilon_i, \eta_i, \zeta_i > 0$, $i = 1, \dots 4$, with $\zeta_3, \zeta_4 < 1+\gamma$, we have for sufficiently large n

$$\left|EP_n - E\left\{\overline{F}(UL^*_{NP} - \Delta)\right\}\right| \le \min\{m_1, m_2\} + \min\{m_3, m_4\},$$

where

$$\begin{aligned} m_1 &= F\left(\left(\sqrt{1+\varepsilon_1} + \zeta_1\right)\sqrt{2\log n}\right)^n + P\left(\left|\overline{X}\right| > \eta_1\right) + P\left(\left|S^2 - 1\right| > \varepsilon_1\right), \\ m_2 &= 1 - F\left(\left(\sqrt{1-\varepsilon_2} - \zeta_2\right)\sqrt{2\log n}\right)^n + P\left(\left|\overline{X}\right| > \eta_2\right) + P\left(\left|S^2 - 1\right| > \varepsilon_2\right), \\ m_3 &= F\left(\left(\sqrt{\log n}\right)^{1+\gamma+2\zeta_3}\right)^n + P\left(\left|\overline{X}\right| > \eta_3\right) + P\left(\left|S^2 - 1\right| > \varepsilon_3\right) + P\left(\left|\widehat{\gamma} - \gamma\right| > \zeta_3\right), \\ m_4 &= 1 - F\left(\left(\sqrt{\log n}\right)^{1+\gamma-2\zeta_4}\right)^n + P\left(\left|\overline{X}\right| > \eta_4\right) + P\left(\left|S^2 - 1\right| > \varepsilon_4\right) + P\left(\left|\widehat{\gamma} - \gamma\right| > \zeta_4\right). \end{aligned}$$

Theorem 8 makes only sense if F differs from the normal power family in the sense that for some $\varepsilon > 0$

$$\lim_{n\to\infty}\left[F\left((1+\varepsilon)\sqrt{2\log n}\right)\right]^n = 0$$

(heavier tail than the normal distribution) or

$$\lim_{n\to\infty}\left[F\left((1-\varepsilon)\sqrt{2\log n}\right)\right]^n = 1$$

(thinner tail than the normal distribution) and F is outside the normal power family in the sense that for some $\varepsilon > 0$

$$\lim_{n\to\infty}\left[F\left(\left(\sqrt{\log n}\right)^{1+\gamma+\varepsilon}\right)\right]^n = 0$$

(heavier tail than K_γ) or

$$\lim_{n\to\infty}\left[F\left(\left(\sqrt{\log n}\right)^{1+\gamma-\varepsilon}\right)\right]^n = 1$$

(thinner tail than K_γ).

Note that for the choices $d_{1N}(n) = -0.7 + 0.5\log n, d_{2N}(n) = 5/\sqrt{n}, d_{1P}(n) = -0.2 + 0.5\log n, d_{2P}(n) = 3/\sqrt{n}$ the conditions (38) are satisfied.

7 Simulation

In this section we show by means of simulation the performance for finite sample size of the new control chart, given in (16) – (19) with $d_{1N}(n) = -0.7 + 0.5 \log n$, $d_{2N}(n) = 5/\sqrt{n}$, $d_{1P}(n) = -0.2 + 0.5 \log n$ and $d_{2P}(n) = 3/\sqrt{n}$. We take $g(p) = p$ and hence compare EP_n to p in the in-control situation, while also in the out-of-control case EP_n is our criterion. We choose $p = 0.001$ and for n we take $250, 500, 1000, 1500, 2000$. The number of repetitions in the simulation study equals 100 000.

The theoretical results from Sections 4 and 5 show that the behavior of the new control chart is asymptotically the same as that of the specific control chart adjusted for the distribution at hand. In a simulation study the behavior of the new control chart is investigated to see how well this property continues to hold in the finite sample case. Therefore, we compare the combined (C) control chart with the normal (N) control chart, the parametric (P) chart (both charts with correction for $r = 0$ and without correction for $r \geq 1$) and the nonparametric (NP) control chart, in its modified version for $r = 0$ and for $r \geq 1$ with the stochastic term V replaced by its deterministic counterpart EV, all of this just as in composing the combined control chart, see (16) – (19).

All procedures are location and scale invariant and hence we take for all distributions involved in the simulation the expectation equal to 0 and the variance equal to 1. The distributions can be classified as follows.

1. **The standard normal distribution** (Φ). For this distribution the control chart based on normality will obviously be the favorite one. It is expected that the in-control behavior of all control charts under consideration is sufficiently good. We want to see how much we loose in the out-of-control case when applying the parametric, the nonparametric and, in particular, the combined control chart.

2. **Distributions from the normal power family** (K_γ) with $\gamma = -0.5, 0.5, 1$. It is seen from Table 2 of Albers et al. (2002a) that the normal control chart behaves very badly for these distributions, in the sense that for the in-control situation EP_n differs much from p. In contrast, the parametric, the nonparametric and the combined control chart are expected to behave well when the observations are in-control. With respect to the out-of-control behavior, the interest is in the loss of the nonparametric and especially the combined control chart compared to the parametric control chart.

3. **Distributions outside the normal power family**. We take the Student distribution with 6 degrees of freedom and standardized to unit variance and denote its distribution function by T. Its model error, see (13), when the supposed model is the normal power family, equals 2.08. Furthermore, we consider the random mixture $RM = \frac{1}{2}\Phi + \frac{1}{2}T$ with model error 1.16. The Normal Inverse Gaussian $(2, 1.5, 0, 1)$ distribution, cf. Barndorff-Nielsen (1996), (shortly denoted by $NIG(2, 1.5)$) shows that the member of the normal power family chosen by our estimator does not fit the distribution globally, but it does at the right tail and that is exactly what we want; see Figure 3. Nevertheless, its model error w.r.t. the normal power family still equals 1.93. Note that the model error w.r.t. the normal distribution is 14.65! So, the normal power family gives a great improvement for this distribution.

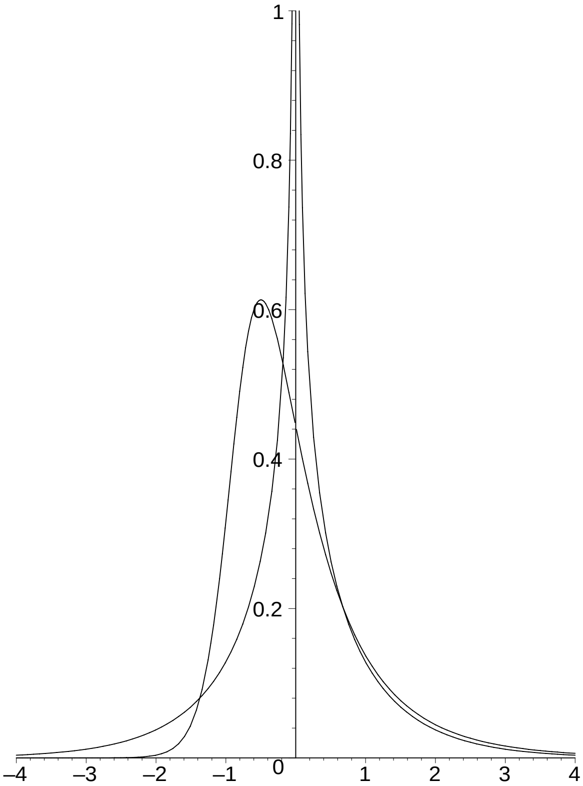

Figure 3. Density of the Normal Inverse Gaussian ($2, 1.5$*) and the corresponding density of the normal power family with* $\gamma = 0.77$*.*

The Normal Inverse Gaussian $(0.5, 0, 0, 1)$ distribution (shortly denoted by $NIG(0.5, 0)$) has model error 2.31 w.r.t. the normal power family. As an example of a (very) negative model error we take the (standardized) Beta $(3, 3.75)$ distribution ($B(3, 3.75)$). Its model error w.r.t. the normal power family equals -0.996.

It is already known from Table 2 in Albers et al. (2002a) that the normal control chart may behave very badly for several of the distributions involved here. Table 7 in Albers et al. (2002a) shows that the parametric control chart gives reasonable total error for these distributions, thus improving the normal control chart tremendously, but still the model errors are not vanishing and considerable deviances remain. It is expected that the nonparametric and the combined control chart behave better when the observations are in-control. Especially for the combined control chart it is of interest to compare the out-of-control results with those of the nonparametric chart for the distributions in this class.

The following table presents the simulation results for the in-control situation.

Table 2 *In-control behavior of the normal* (N)*, parametric* (P) *and nonparametric* (NP) *control charts and the combined control chart* (C)*. Presented is the simulated expected observed false alarm rate, i.e. the simulated* EP_n*, for* $n = 250, 500, 1000, 1500, 2000$*. The unit in the table is* 0.001.

F	250				500			
	N	P	NP	C	N	P	NP	C
Φ	1.00	1.05	1.12	0.97	1.00	1.02	1.03	0.97
$K_{-0.5}$	0.00	0.94	1.00	0.75	0.00	0.97	1.00	0.86
$K_{0.5}$	6.63	1.05	1.67	1.51	6.61	1.02	1.21	1.25
K_1	10.67	1.06	2.20	1.21	10.52	1.02	1.39	1.01
T	4.60	3.02	1.81	2.19	4.59	3.04	1.32	1.79
RM	2.80	2.14	1.58	1.81	2.79	2.15	1.24	1.60
$NIG(2, 1.5)$	16.09	2.98	2.32	1.92	15.88	2.95	1.44	1.71
$NIG(0.5, 0)$	7.91	3.34	2.10	2.28	7.84	3.31	1.38	1.72
$B(3, 3.75)$	0.01	0.16	1.00	0.31	0.01	0.08	1.00	0.46

F	1000				1500				2000			
	N	P	NP	C	N	P	NP	C	N	P	NP	C
Φ	1.03	1.13	1.01	1.03	1.02	1.09	0.90	1.01	1.02	1.07	1.00	1.02
$K_{-0.5}$	0.00	1.21	1.00	1.14	0.00	1.14	0.91	1.08	0.00	1.10	1.00	1.09
$K_{0.5}$	6.69	1.12	0.99	1.17	6.66	1.08	0.89	1.12	6.64	1.06	1.00	1.10
K_1	10.53	1.11	1.00	1.08	10.47	1.08	0.89	1.04	10.45	1.06	1.00	1.05
T	4.64	3.23	1.03	1.48	4.62	3.19	0.88	1.28	4.60	3.16	1.00	1.33
RM	2.83	2.29	1.00	1.40	2.82	2.25	0.87	1.29	2.81	2.23	1.00	1.35
$NIG(2, 1.5)$	15.92	3.17	1.01	1.89	15.82	3.08	0.88	1.88	15.78	3.04	1.00	1.99
$NIG(0.5, 0)$	7.86	3.48	1.00	1.45	7.83	3.43	0.89	1.33	7.81	3.40	1.00	1.39
$B(3, 3.75)$	0.00	0.06	1.00	0.70	0.00	0.04	0.92	0.72	0.00	0.03	1.00	0.80

It is seen in Table 2 that indeed the combined control chart has good in-control behavior under all distributions. The normal control chart cannot be used unless the distribution is very close to normality. The parametric control chart is often a great improvement w.r.t. the normal chart and gives reasonable results, while the nonparametric control chart behaves very well under all distributions.

In the next two tables the out-of-control situation is presented. We consider shift alternatives, that is $X_1, \ldots, X_n$ have distribution function $F(x)$ and X_{n+1} has distribution function $F(x - \Delta)$. For each F we have selected two values of Δ such that reasonable values of the probability of an out-of-control signal result. When examining Table 3 one should take into account the results in Table 2, because a high value in Table 3 may be caused by an unduly high and thus incorrect value in Table 2. In case of a simulated expected observed false alarm rate under in-control of at least twice as much as p we do not give the (simulated) probability of an out-of-control signal. Such situations are denoted by * in Table 3.

Table 3 *Out-of-control behavior of the corrected normal (N), parametric (P) and nonparametric (NP) control charts and the combined control chart (C). Presented is the simulated expected observed false alarm rate, i.e. the simulated EP_n, for $n = 250, 500, 1000, 1500, 2000$ with $X_1, \ldots, X_n$ having distribution function $F(x)$ and X_{n+1} distribution function $F(x - \Delta)$.*

F	Δ	250				500			
		N	P	NP	C	N	P	NP	C
Φ	2	.133	.120	.090	.115	.135	.128	.096	.122
	3	.450	.410	.311	.396	.457	.436	.327	.417
$K_{-.5}$	1	.000	.191	.071	.145	.000	.208	.120	.179
	2	.157	.496	.330	.437	.164	.499	.370	.462
$K_{.5}$	3	$*$	.080	.115	.107	$*$	.080	.088	.093
	4	$*$	.293	.344	.299	$*$	.305	.294	.294
K_1	3	$*$	.021	$*$	.031	$*$	.020	.030	.021
	4	$*$	.071	$*$	.082	$*$	.060	.110	.063
T	2	$*$	$*$	.039	$*$	$*$	$*$	.027	.040
	3	$*$	$*$	.157	$*$	$*$	$*$	.119	.170
RM	2	$*$	$*$	.058	.076	$*$	$*$	.048	.068
	3	$*$	$*$	.219	.285	$*$	$*$	.190	.262
$NIG(2, 1.5)$	4	$*$	$*$	$*$	.089	$*$	$*$	.063	.076
	5	$*$	$*$	$*$	.237	$*$	$*$	.175	.218
$NIG(.5, 0)$	3	$*$	$*$	$*$	$*$	$*$	$*$	.050	.067
	4	$*$	$*$	$*$	$*$	$*$	$*$	.176	.232
$B(3, 3.75)$	1	.016	.019	.023	.021	.017	.020	.033	.025
	2	.148	.149	.140	.161	.151	.161	.174	.174

F	Δ	1000				1500				2000			
		N	P	NP	C	N	P	NP	C	N	P	NP	C
Φ	2	.139	.140	.121	.132	.138	.140	.120	.134	.138	.139	.129	.137
	3	.464	.464	.412	.443	.464	.464	.418	.451	.464	.464	.437	.459
$K_{-.5}$	1	.000	.227	.208	.218	.000	.227	.211	.222	.000	.227	.218	.225
	2	.172	.501	.499	.500	.172	.501	.500	.500	.172	.501	.500	.501
$K_{.5}$	3	$*$	.088	.077	.090	$*$	.086	.071	.088	$*$	.085	.079	.087
	4	$*$	.338	.287	.321	$*$	.328	.272	.318	$*$	.322	.303	.320
K_1	3	$*$	.021	.020	.021	$*$	.020	.017	.020	$*$	.020	.019	.020
	4	$*$	.062	.065	.065	$*$	.059	.051	.059	$*$	.057	.058	.058
T	2	$*$	$*$	.019	.032	$*$	$*$	.016	.027	$*$	$*$	.018	.027
	3	$*$	$*$	.093	.139	$*$	$*$	.079	.121	$*$	$*$	.092	.127
RM	2	$*$	$*$	.040	.060	$*$	$*$	.034	.054	$*$	$*$	.040	.057
	3	$*$	$*$	.180	.236	$*$	$*$	.166	.222	$*$	$*$	.195	.243
$NIG(2, 1.5)$	4	$*$	$*$	.040	.085	$*$	$*$	.033	.084	$*$	$*$	.037	.087
	5	$*$	$*$	.115	.252	$*$	$*$	.094	.251	$*$	$*$	.107	.263
$NIG(.5, 0)$	3	$*$	$*$	.030	.053	$*$	$*$	.098	.183	$*$	$*$	.027	.046
	4	$*$	$*$	.121	.204	$*$	$*$	.327	.394	$*$	$*$	.112	.188
$B(3, 3.75)$	1	.017	.023	.052	.039	.017	.023	.052	.043	.017	.023	.053	.047
	2	.155	.175	.257	.220	.154	.174	.259	.234	.154	.174	.263	.244

The comparison of the out-of-control behavior of the several charts is obscured by the differences in the in-control behavior. Indeed, when the in-control rate of chart 1 is $1.3p$ and that of chart 2 equals p, we may expect also a higher out-of-control rate for chart 1. Therefore, we have made a more "fair" comparison in the next table. Let E_0P_n be the in-control rate of Table 2. The probability of detecting the shift Δ with this in-control rate when the distribution is completely known equals

$$\widetilde{p}_n = \overline{F}\left(\overline{F}^{-1}\left(E_0P_n\right) - \Delta\right).$$

This $\widetilde{p}_n$ serves as a bench mark of what can be obtained in the out-of-control case with in-control rate E_0P_n. The out-of-control rate is related to the in-control rate by presenting EP_n of Table 3 as a percentage of $\widetilde{p}_n$. Firstly, this indicates how well the chart performs in the out-of-control case on its own and secondly, it makes comparison between several charts more fair. However, still we have to realize that at the first place the in-control behavior should be controlled. Therefore, again a * is denoted when a simulated expected observed false alarm rate under in-control is at least twice as much as p. Also when the simulated in-control rate E_0P_n equals (exactly) 0 a * is denoted.

Table 4 *Out-of-control behavior of the corrected normal (N), parametric (P) and non-parametric (NP) control charts and the combined control chart (C). Presented is $(EP_n)/\widetilde{p}_n$ (as percentage) for $n = 250, 500, 1000, 1500, 2000$ with $X_1, \ldots, X_n$ having distribution function $F(x)$ and X_{n+1} distribution function $F(x-\Delta)$.*

F	Δ	250				500				1000			
		N	P	NP	C	N	P	NP	C	N	P	NP	C
Φ	2	96	85	62	85	98	92	69	90	99	96	87	94
	3	97	87	65	86	99	93	70	91	99	97	89	95
$K_{-0.5}$	1	*	85	31	67	*	92	53	81	*	96	91	93
	2	*	99	66	87	*	100	74	92	*	100	100	100
$K_{0.5}$	3	*	94	92	93	*	97	92	95	*	98	94	97
	4	*	94	65	68	*	100	83	81	*	103	96	93
K_1	3	*	107	*	137	*	103	113	114	*	102	106	104
	4	*	126	*	125	*	111	141	119	*	104	124	112
T	2	*	*	108	*	*	*	113	111	*	*	112	113
	3	*	*	82	*	*	*	92	90	*	*	102	94
RM	2	*	*	82	92	*	*	90	94	*	*	100	97
	3	*	*	68	80	*	*	72	81	*	*	83	80
$NIG(2, 1.5)$	4	*	*	*	115	*	*	117	114	*	*	113	113
	5	*	*	*	104	*	*	112	111	*	*	117	113
$NIG(0.5, 0)$	3	*	*	*	*	*	*	135	135	*	*	124	134
	4	*	*	*	*	*	*	114	105	*	*	132	122
$B(3, 3.75)$	1	66	53	42	51	74	64	59	56	83	77	93	78
	2	81	69	52	69	87	78	65	71	92	87	96	86

F	Δ	1500				2000			
		N	P	NP	C	N	P	NP	C
Φ	2	99	97	91	97	100	98	93	98
	3	100	98	93	97	100	98	94	98
$K_{-0.5}$	1	*	97	94	96	*	98	96	97
	2	*	100	100	100	*	100	100	100
$K_{0.5}$	3	*	99	96	98	*	99	97	99
	4	*	103	100	96	*	102	101	98
K_1	3	*	101	104	102	*	101	103	101
	4	*	103	112	106	*	102	110	104
T	2	*	*	113	116	*	*	109	112
	3	*	*	108	98	*	*	104	98
RM	2	*	*	103	97	*	*	100	97
	3	*	*	88	81	*	*	89	85
$NIG(2, 1.5)$	4	*	*	108	112	*	*	107	109
	5	*	*	113	113	*	*	110	110
$NIG(0.5, 0)$	3	*	*	116	131	*	*	112	124
	4	*	*	127	126	*	*	121	121
$B(3, 3.75)$	1	87	81	96	86	86	84	96	90
	2	93	90	97	91	93	91	98	94

The limiting value $\widetilde{p}_n$ indicates what out-of-control probability can be obtained for the distribution at hand if the distribution is completely known. It is seen in Table 4 that as a rule all charts and hence in particular the combined control chart and the (modified) nonparametric control chart compare very well w.r.t. this bench mark. Moreover, Tables 3 and 4 show that

1. if $F = \Phi$, the combined control chart has a substantial gain w.r.t. the nonparametric control chart and only a small loss w.r.t. the normal and parametric control chart;
2. if $F = K_\gamma$ for some γ, the combined control chart has for not too large n and γ a pretty gain w.r.t. the nonparametric control chart and in general only a small loss w.r.t. the parametric control chart; the normal control chart cannot be applied unless γ is very close to 0; for positive γ's this is due to its bad in-control behavior and for negative γ's it has a very low alarm rate under out-of-control;
3. outside the normal power family, the combined control chart exhibits only a small loss w.r.t. the nonparametric control chart, while the normal control chart cannot be applied in this case due to its bad in-control behavior or its low alarm rate (unless the distribution is very close to normality); the same holds for the parametric control chart, albeit to a much smaller extent.

Conclusion

¿From the theory and the simulations presented in this paper as well as from additional simulations that we have performed, we conclude that the new combined control chart can be recommended as an omnibus control chart with a well-behaved performance under a great variety of distributions, both for the in-control and for the out-of-control situation. The modified nonparametric chart can be recommended as well, although under normality a price has to be paid for its simpler form.

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Appendix Some proofs of Theorems and Lemmas

Proof of Theorem 2. Denote by $X_{(1)} \leq \ldots \leq X_{(n)}$ the order statistics of $X_1, \ldots, X_n$ and let $U_{(1)} \leq \ldots \leq U_{(n)}$ be the order statistics of the random sample $U_1, \ldots, U_n$ from a uniform distribution on (0,1). Let $\{\varepsilon_n\}$ be a sequence of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} n\varepsilon_n^2 = \infty$. Let $0 < s < 1$ be fixed and let $j = j(n)$ satisfy $j/n = s + O(\varepsilon_n)$ as $n \to \infty$. Then, cf.(24),

$$P(X_{(j)} > K_\gamma^{-1}(s) + \varepsilon_n) = P\left(\sum_{i=1}^{n} 1_{U_i > K_\gamma(K_\gamma^{-1}(s)+\varepsilon_n)} \geq n - j\right)$$

and by standard large deviation theory, cf. e.g. Feller(1971) p. 553,

$$\limsup_{n\to\infty} (n\varepsilon_n^2)^{-1} \log P(X_{(j)} > K_\gamma^{-1}(s) + \varepsilon_n) < 0.$$

Similarly, we have

$$\limsup_{n\to\infty} (n\varepsilon_n^2)^{-1} \log P(X_{(j)} < K_\gamma^{-1}(s) - \varepsilon_n) < 0$$

and hence

$$\limsup_{n\to\infty} (n\varepsilon_n^2)^{-1} \log P\left(\left|X_{(j)} - K_\gamma^{-1}(s)\right| > \varepsilon_n\right) < 0. \tag{39}$$

If $1 + \gamma \leq 2$, again by standard large deviation theory, cf. e.g. Feller(1971) p. 553, it follows that

$$\lim_{n\to\infty} (n\varepsilon_n^2)^{-1} \log P\left(\left|\overline{X}\right| > \varepsilon_n\right) < 0.$$

For $1 + \gamma > 2$, the moment generating function of X_i does not exist. However, by Nagaev (1969), see also Nagaev (1979, formulas (2.31) and (2.32) on page 764), it follows that

$$\limsup_{n\to\infty} \frac{\log P\left(\overline{X} > \varepsilon_n\right)}{\min\{n\varepsilon_n^2, (n\varepsilon_n)^{2/(1+\gamma)}\}} \leq -\min\left\{1/20, \frac{1}{2}\left(\frac{1/2}{c(\gamma)}\right)^{2/(1+\gamma)}\right\},$$

where we have used

$$\lim_{n\to\infty} \frac{\log n}{(n\varepsilon_n)^{2/(1+\gamma)}} = \lim_{n\to\infty} \frac{\log n}{n^{1/(1+\gamma)}(n^{1/2}\varepsilon_n)^{2/(1+\gamma)}} = 0.$$

By symmetry we get

$$\limsup_{n\to\infty} \frac{\log P\left(\left|\overline{X}\right| > \varepsilon_n\right)}{\min\{n\varepsilon_n^2, (n\varepsilon_n)^{2/(1+\gamma)}\}} < 0,$$

which completes the proof of (30).

Since

$$\begin{aligned} P\left(\left|S^2 - 1\right| > \varepsilon_n\right) &\leq P\left(\left|S^2 - 1\right| > \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n\right) + P\left(\left|\overline{X}\right| > \varepsilon_n\right) \\ &\leq P\left(\left|\sum_{i=1}^{n} \left(X_i^2 - 1\right)\right| > (n-1)\varepsilon_n - n\varepsilon_n^2 - 1\right) + P\left(\left|\overline{X}\right| > \varepsilon_n\right), \end{aligned}$$

it follows by a similar argument as in the proof of (29) and (30) that (31) and (32) hold true.

There exists a constant $c > 0$ such that

$$P(|\widehat{\gamma} - \gamma| > \varepsilon_n) \leq P\left(\left|X_{([0.95n+1])} - K_\gamma^{-1}(0.95)\right| > c\varepsilon_n\right) + P\left(\left|X_{([0.75n+1])} - K_\gamma^{-1}(0.75)\right| > c\varepsilon_n\right) + P\left(\left|\overline{X}\right| > \varepsilon_n\right).$$

Note that $[n+1-qn]/n - 1 - q = O(1/n) = o(\varepsilon_n)$ for any fixed $0 < q < 1$. The results given in (33) and (34) now easily follow from (39), (29) and (30). ■

Before proving lemma 3 we present the following lemma on the behavior in the tail of the standard normal distribution.

Lemma 9 *Let $\{x_n\}, \{\varepsilon_n\}$ be sequences of real numbers satisfying $\lim_{n\to\infty} x_n = \infty$, $\lim_{n\to\infty} x_n^2 \varepsilon_n = 0$, where ε_n may be positive as well as negative. Then*

$$\overline{\Phi}\left(x_n\left(1+\varepsilon_n\right)\right) = \overline{\Phi}\left(x_n\right)\left(1 + O\left(x_n^2 \varepsilon_n\right)\right) \text{ as } n \to \infty.$$

If moreover, $\lim_{n\to\infty} n\overline{\Phi}(x_n)\, x_n^2 \varepsilon_n = 0$, then

$$\left\{1 - \overline{\Phi}\left(x_n\left(1+\varepsilon_n\right)\right)\right\}^n = \left(1 - \overline{\Phi}\left(x_n\right)\right)^n \left(1 + o(1)\right) \text{ as } n \to \infty.$$

Proof. By Taylor expansion we get

$$\overline{\Phi}\left(x_n\left(1+\varepsilon_n\right)\right) = \overline{\Phi}\left(x_n\right) - x_n \varepsilon_n \varphi\left(\xi_n\right)$$

with ξ_n between x_n and $x_n(1+\varepsilon_n)$. For $x \to \infty$ we have

$$\overline{\Phi}(x) = \frac{\varphi(x)}{x}(1 + o(1))$$

and hence, noting that $\lim_{n\to\infty} x_n^2 \varepsilon_n = 0$ implies $\lim_{n\to\infty} \frac{\varphi(\xi_n)}{\varphi(x_n)} = 1$, we obtain

$$\begin{aligned}\frac{\overline{\Phi}\left(x_n\left(1+\varepsilon_n\right)\right)}{\overline{\Phi}\left(x_n\right)} &= 1 - x_n^2 \varepsilon_n \frac{\varphi\left(\xi_n\right)}{\varphi\left(x_n\right)}(1 + o(1)) \\ &= 1 + O\left(x_n^2 \varepsilon_n\right) \text{ as } n \to \infty.\end{aligned}$$

If moreover, $\lim_{n\to\infty} n\overline{\Phi}(x_n)\, x_n^2 \varepsilon_n = 0$, then

$$\begin{aligned}n \log\left(1 - \overline{\Phi}\left(x_n\left(1+\varepsilon_n\right)\right)\right) &= n \log\left(1 - \overline{\Phi}\left(x_n\right) + O\left(\overline{\Phi}\left(x_n\right) x_n^2 \varepsilon_n\right)\right) \\ &= n \log\left(1 - \overline{\Phi}\left(x_n\right)\right) + O\left(n\overline{\Phi}\left(x_n\right) x_n^2 \varepsilon_n\right) \\ &= n \log\left(1 - \overline{\Phi}\left(x_n\right)\right) + o(1),\end{aligned}$$

which completes the proof. ■

Proof of Lemma 3. For $x \to 0$ we have

$$\overline{\Phi}^{-1}(x) = \sqrt{-2\log x}\,(1 + o(1))$$

and hence

$$\lim_{n\to\infty} d_{1N}(n) \sqrt{\frac{d_{1N}(n)}{n}} \log\left(\frac{d_{1N}(n)}{n}\right) = 0$$

implies

$$\lim_{n\to\infty} d_{1N}(n)\sqrt{\frac{d_{1N}(n)}{n}}\left\{\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)\right\}^2 = 0.$$

Therefore, there exists a sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} a_n = \infty,\ \lim_{n\to\infty} a_n d_{1N}(n)\sqrt{\frac{d_{1N}(n)}{n}}\left\{\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)\right\}^2 = 0. \tag{40}$$

Let

$$\varepsilon_n = a_n\sqrt{\frac{d_{1N}(n)}{n}},$$

then we have $\lim_{n\to\infty}\varepsilon_n = 0$ and $\lim_{n\to\infty} n\varepsilon_n^2\left(d_{1N}(n)\right)^{-1} = \infty$. By Theorem 2 we get

$$\lim_{n\to\infty}\frac{P\left(\left|\overline{X}\right| > \varepsilon_n\right)}{\left(1-\frac{d_{1N}(n)}{n}\right)^n} = 0,\ \lim_{n\to\infty}\frac{P\left(\left|S^2-1\right| > \varepsilon_n\right)}{\left(1-\frac{d_{1N}(n)}{n}\right)^n} = 0$$

and hence

$$\begin{aligned}&P\left(\frac{X_{(n)}-\overline{X}}{S} < \overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)\right) \\ &= P\left(\frac{X_{(n)}-\overline{X}}{S} < \overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right), \left|S^2-1\right| < \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n\right) + o\left(\left(1-\frac{d_{1N}(n)}{n}\right)^n\right).\end{aligned} \tag{41}$$

Writing

$$z_{1n} = \overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)(1-\varepsilon_n),\ z_{2n} = \overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)(1+\varepsilon_n)$$

we obtain for sufficiently large n,

$$\begin{aligned}&P\left(X_{(n)} < z_{1n}, \left|S^2-1\right| < \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n\right) \\ &\le P\left(\frac{X_{(n)}-\overline{X}}{S} < \overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right), \left|S^2-1\right| < \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n\right) \le P\left(X_{(n)} < z_{2n}\right).\end{aligned} \tag{42}$$

Using

$$P\left(X_{(n)} < z_{2n}\right) = P\left(\overline{\Phi}\left(X_{(n)}\right) > \overline{\Phi}\left(z_{2n}\right)\right) = \left(1-\overline{\Phi}\left(z_{2n}\right)\right)^n$$

and writing $x_n = \overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)$, the conditions of Lemma 9 are fulfilled, see also (40). Application of Lemma 9 yields

$$P\left(X_{(n)} < z_{2n}\right) = \left(1-\overline{\Phi}\left(z_{2n}\right)\right)^n = \left(1-\frac{d_{1N}(n)}{n}\right)^n(1+o(1)) \text{ as } n\to\infty. \tag{43}$$

Because

$$P\left(X_{(n)} < z_{1n}, \left|S^2-1\right| < \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n\right) = P\left(X_{(n)} < z_{1n}\right) + o\left(\left(1-\frac{d_{1N}(n)}{n}\right)^n\right),$$

another application of Lemma 9 gives

$$P\left(X_{(n)} < z_{1n}, \left|S^2-1\right| < \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n\right) = \left(1-\frac{d_{1N}(n)}{n}\right)^n(1+o(1)) \text{ as } n\to\infty. \tag{44}$$

Combination of (41) – (44) leads to

$$P\left(\frac{X_{(n)}-\overline{X}}{S}<\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)\right)=\left(1-\frac{d_{1N}(n)}{n}\right)^{n}(1+o(1)) \text{ as } n\to\infty.$$

Since

$$\lim_{n\to\infty}\sqrt{\frac{|\log d_{2N}(n)|}{n}}\log\left(\frac{d_{2N}(n)}{n}\right)=0,$$

there exists a sequence $\{b_n\}$ such that

$$\lim_{n\to\infty}b_n=\infty,\ \lim_{n\to\infty}b_n\sqrt{\frac{|\log d_{2N}(n)|}{n}}\left\{\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right\}^{2}=0. \tag{45}$$

Let

$$\eta_n=b_n\sqrt{\frac{|\log d_{2N}(n)|}{n}},$$

then we have $\lim_{n\to\infty}\eta_n=0$ and $\lim_{n\to\infty}n\eta_n^2\left(|\log d_{2N}(n)|\right)^{-1}=\infty$. By Theorem 2 we get

$$\lim_{n\to\infty}\frac{P\left(\left|\overline{X}\right|>\eta_n\right)}{d_{2N}(n)}=0,\ \lim_{n\to\infty}\frac{P\left(\left|S^2-1\right|>\eta_n\right)}{d_{2N}(n)}=0$$

and hence

$$\begin{aligned}&P\left(\frac{X_{(n)}-\overline{X}}{S}>\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right) \\ &=P\left(\frac{X_{(n)}-\overline{X}}{S}>\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right),\left|S^2-1\right|<\eta_n,\left|\overline{X}\right|<\eta_n\right)+o\left(d_{2N}(n)\right).\end{aligned} \tag{46}$$

Writing

$$y_{1n}=\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)(1+\eta_n),\ y_{2n}=\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)(1-\eta_n)$$

we obtain for sufficiently large n,

$$\begin{aligned}&P\left(X_{(n)}>y_{1n},\left|S^2-1\right|<\varepsilon_n,\left|\overline{X}\right|<\varepsilon_n\right) \\ &\le P\left(\frac{X_{(n)}-\overline{X}}{S}>\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right),\left|S^2-1\right|<\eta_n,\left|\overline{X}\right|<\eta_n\right)\le P\left(X_{(n)}>y_{2n}\right).\end{aligned} \tag{47}$$

Using

$$P\left(X_{(n)}>y_{2n}\right)=P\left(\overline{\Phi}\left(X_{(n)}\right)<\overline{\Phi}\left(y_{2n}\right)\right)=1-\left(1-\overline{\Phi}\left(y_{2n}\right)\right)^{n}$$

and writing $x_n=\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)$, the conditions of the first part of Lemma 9 (with in the lemma ε_n replaced by $-\eta_n$) are fulfilled, see also (45). Application of Lemma 9 yields

$$P\left(X_{(n)}>y_{2n}\right)=1-\left(1-\overline{\Phi}\left(y_{2n}\right)\right)^{n}=1-\left(1-\frac{d_{2N}(n)(1+o(1))}{n}\right)^{n}=d_{2N}(n)(1+o(1)) \tag{48}$$

as $n\to\infty$. Because

$$P\left(X_{(n)}>y_{1n},\left|S^2-1\right|<\eta_n,\left|\overline{X}\right|<\eta_n\right)=P\left(X_{(n)}>y_{1n}\right)+o\left(d_{2N}(n)\right),$$

another application of Lemma 9 gives

$$P\left(X_{(n)} > y_{1n}, \left|S^2 - 1\right| < \eta_n, \left|\overline{X}\right| < \eta_n\right) = d_{2N}(n)(1 + o(1)) \text{ as } n \to \infty. \tag{49}$$

Combination of (46) – (49) gives

$$P\left(\frac{X_{(n)} - \overline{X}}{S} > \overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right) = d_{2N}(n)(1 + o(1)) \text{ as } n \to \infty,$$

thus completing the proof of the lemma. ■

The following lemma describes the tail behavior in the normal power family and is in fact a generalization of Lemma 9.

Lemma 10 *Let* $\{x_n\}, \{\varepsilon_n\}$ *be sequences of real numbers satisfying* $\lim_{n\to\infty} x_n = \infty$, $\lim_{n\to\infty} x_n^{\frac{2}{1+\gamma}} \varepsilon_n = 0$*, where* ε_n *may be positive as well as negative. Then*

$$\overline{K}_\gamma\left(x_n\left(1 + \varepsilon_n\right)\right) = \overline{K}_\gamma\left(x_n\right)\left(1 + O\left(x_n^{\frac{2}{1+\gamma}} \varepsilon_n\right)\right) \text{ as } n \to \infty.$$

If moreover, $\lim_{n\to\infty} n\overline{K}_\gamma\left(x_n\right) x_n^{\frac{2}{1+\gamma}} \varepsilon_n = 0$*, then*

$$\left\{1 - \overline{K}_\gamma\left(x_n\left(1 + \varepsilon_n\right)\right)\right\}^n = \left(1 - \overline{K}_\gamma\left(x_n\right)\right)^n (1 + o(1)) \text{ as } n \to \infty.$$

Proof. In view of (the proof of) Lemma 9 we have

$$\overline{K}_\gamma\left(x_n\left(1 + \varepsilon_n\right)\right) = \overline{\Phi}\left(\left(\frac{x_n\left(1 + \varepsilon_n\right)}{c(\gamma)}\right)^{\frac{1}{1+\gamma}}\right) = \overline{\Phi}\left(\left(\frac{x_n}{c(\gamma)}\right)^{\frac{1}{1+\gamma}}\left(1 + \varepsilon_n\right)^{\frac{1}{1+\gamma}}\right)$$

$$= \overline{\Phi}\left(\left(\frac{x_n}{c(\gamma)}\right)^{\frac{1}{1+\gamma}}\right)\left(1 + O\left(x_n^{\frac{2}{1+\gamma}} \varepsilon_n\right)\right) = \overline{K}_\gamma\left(x_n\right)\left(1 + O\left(x_n^{\frac{2}{1+\gamma}} \varepsilon_n\right)\right) \text{ as } n \to \infty,$$

which gives the first part of the lemma. The proof of the second part is similar to the proof of the second part of Lemma 9. ■

Note that for $\gamma = 0$, that is the normal distribution, indeed Lemma 10 gives the result of Lemma 9.

Proof of Lemma 5. We have

$$\begin{cases} \lim_{n\to\infty} d_{1P}(n)\sqrt{\frac{d_{1P}(n)}{n}}\left|\log\left(\frac{d_{1P}(n)}{n}\right)\right|^{1+\zeta} = 0 \text{ if } \gamma \le 0 \\ \lim_{n\to\infty} d_{1P}(n)\max\left(\frac{d_{1P}^{1+\gamma}(n)}{n}, \sqrt{\frac{d_{1P}(n)}{n}}\right)\left|\log\left(\frac{d_{1P}(n)}{n}\right)\right|^{1+\zeta} = 0 \text{ if } \gamma > 0 \end{cases}$$

and therefore there exists a sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} a_n = \infty, \begin{cases} \lim_{n\to\infty} a_n d_{1P}(n)\sqrt{\frac{d_{1P}(n)}{n}}\left|\log\left(\frac{d_{1P}(n)}{n}\right)\right|^{1+\zeta} = 0 \text{ if } \gamma \le 0 \\ \lim_{n\to\infty} a_n d_{1P}(n)\max\left(\frac{d_{1P}^{1+\gamma}(n)}{n}, \sqrt{\frac{d_{1P}(n)}{n}}\right)\left|\log\left(\frac{d_{1P}(n)}{n}\right)\right|^{1+\zeta} = 0 \text{ if } \gamma > 0 \end{cases} \tag{50}$$

Let

$$\varepsilon_n = \begin{cases} a_n \sqrt{\frac{d_{1P}(n)}{n}} \text{ if } \gamma \le 0 \\ a_n \max\left(\frac{d_{1P}^{1+\gamma}(n)}{n}, \sqrt{\frac{d_{1P}(n)}{n}}\right) \text{ if } \gamma > 0 \end{cases}, \tag{51}$$

then we have $\lim_{n\to\infty} \varepsilon_n = 0$ and by Theorem 2

$$\lim_{n\to\infty} \frac{P\left(\left|\overline{X}\right| > \varepsilon_n\right)}{\left(1 - \frac{d_{1P}(n)}{n}\right)^n} = 0, \lim_{n\to\infty} \frac{P\left(\left|S^2 - 1\right| > \varepsilon_n\right)}{\left(1 - \frac{d_{1P}(n)}{n}\right)^n} = 0, \lim_{n\to\infty} \frac{P\left(\left|\widehat{\gamma} - \gamma\right| > \varepsilon_n\right)}{\left(1 - \frac{d_{1P}(n)}{n}\right)^n} = 0$$

and hence

$$\begin{aligned} &P\left(\frac{X_{(n)} - \overline{X}}{S} < \overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{1P}(n)}{n}\right)\right) \\ &= P\left(\frac{X_{(n)} - \overline{X}}{S} < \overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{1P}(n)}{n}\right), \left|S^2 - 1\right| < \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n, \left|\widehat{\gamma} - \gamma\right| < \varepsilon_n\right) \\ &+ o\left(\left(1 - \frac{d_{1P}(n)}{n}\right)^n\right). \end{aligned} \tag{52}$$

Writing

$$\begin{aligned} z_{1n} &= \overline{K}_{\gamma}^{-1}\left(\frac{d_{1P}(n)}{n}\right)\left(1 - \varepsilon_n \overline{\Phi}^{-1}\left(\frac{d_{1P}(n)}{n}\right)^{2\zeta}\right), \\ z_{2n} &= \overline{K}_{\gamma}^{-1}\left(\frac{d_{1P}(n)}{n}\right)\left(1 + \varepsilon_n \overline{\Phi}^{-1}\left(\frac{d_{1P}(n)}{n}\right)^{2\zeta}\right) \end{aligned}$$

we obtain for sufficiently large n,

$$\begin{aligned} &P\left(X_{(n)} < z_{1n}, \left|S^2 - 1\right| < \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n, \left|\widehat{\gamma} - \gamma\right| < \varepsilon_n\right) \\ &\le P\left(\frac{X_{(n)} - \overline{X}}{S} < \overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{1P}(n)}{n}\right), \left|S^2 - 1\right| < \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n, \left|\widehat{\gamma} - \gamma\right| < \varepsilon_n\right) \\ &\le P\left(X_{(n)} < z_{2n}\right). \end{aligned} \tag{53}$$

Using

$$P\left(X_{(n)} < z_{2n}\right) = P\left(\overline{K}_{\gamma}\left(X_{(n)}\right) > \overline{K}_{\gamma}\left(z_{2n}\right)\right) = \left(1 - \overline{K}_{\gamma}\left(z_{2n}\right)\right)^n$$

and writing $x_n = \overline{K}_{\gamma}^{-1}\left(\frac{d_{1P}(n)}{n}\right)$, the conditions of Lemma 10 (with ε_n replaced by $\varepsilon_n \overline{\Phi}^{-1}\left(\frac{d_{1P}(n)}{n}\right)^{2\zeta}$) are fulfilled, see also (50) and (51). Application of Lemma 10 yields

$$P\left(X_{(n)} < z_{2n}\right) = \left(1 - \overline{K}_{\gamma}\left(z_{2n}\right)\right)^n = \left(1 - \frac{d_{1P}(n)}{n}\right)^n (1 + o(1)) \text{ as } n \to \infty. \tag{54}$$

Because

$$P\left(X_{(n)} < z_{1n}, \left|S^2 - 1\right| < \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n, \left|\widehat{\gamma} - \gamma\right| < \varepsilon_n\right) = P\left(X_{(n)} < z_{1n}\right) + o\left(\left(1 - \frac{d_{1P}(n)}{n}\right)^n\right),$$

another application of Lemma 10 gives

$$P\left(X_{(n)} < z_{1n}, \left|S^2 - 1\right| < \varepsilon_n, \left|\overline{X}\right| < \varepsilon_n\right) = \left(1 - \frac{d_{1P}(n)}{n}\right)^n (1 + o(1)) \text{ as } n \to \infty. \tag{55}$$

Combination of (52) – (55) leads to

$$P\left(\frac{X_{(n)}-\overline{X}}{S}<\overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{1P}(n)}{n}\right)\right)=\left(1-\frac{d_{1P}(n)}{n}\right)^{n}(1+o(1)) \text{ as } n\to\infty.$$

Since

$$\begin{cases}\lim\limits_{n\to\infty}\sqrt{\frac{|\log d_{2P}(n)|}{n}}\left|\log\left(\frac{d_{2P}(n)}{n}\right)\right|^{1+\zeta}=0 \text{ if } \gamma\le 0\\ \lim\limits_{n\to\infty}\max\left(\frac{|\log d_{2P}(n)|^{1+\gamma}}{n},\sqrt{\frac{|\log d_{2P}(n)|}{n}}\right)\left|\log\left(\frac{d_{2P}(n)}{n}\right)\right|^{1+\zeta}=0 \text{ if } \gamma>0.\end{cases}$$

there exists a sequence $\{b_n\}$ such that

$$\lim_{n\to\infty}b_n=\infty,\begin{cases}\lim\limits_{n\to\infty}b_n\sqrt{\frac{|\log d_{2P}(n)|}{n}}\left|\log\left(\frac{d_{2P}(n)}{n}\right)\right|^{1+\zeta}=0 \text{ if } \gamma\le 0\\ \lim\limits_{n\to\infty}b_n\max\left(\frac{|\log d_{2P}(n)|^{1+\gamma}}{n},\sqrt{\frac{|\log d_{2P}(n)|}{n}}\right)\left|\log\left(\frac{d_{2P}(n)}{n}\right)\right|^{1+\zeta}=0 \text{ if } \gamma>0.\end{cases} \tag{56}$$

Let

$$\eta_n=\begin{cases}b_n\sqrt{\frac{|\log d_{2P}(n)|}{n}} \text{ if } \gamma\le 0\\ b_n\max\left(\frac{|\log d_{2P}(n)|^{1+\gamma}}{n},\sqrt{\frac{|\log d_{2P}(n)|}{n}}\right) \text{ if } \gamma>0\end{cases}, \tag{57}$$

then we have $\lim\limits_{n\to\infty}\eta_n=0$ and by Theorem 2 we get

$$\lim_{n\to\infty}\frac{P\left(\left|\overline{X}\right|>\eta_n\right)}{d_{2P}(n)}=0,\ \lim_{n\to\infty}\frac{P\left(\left|S^2-1\right|>\eta_n\right)}{d_{2P}(n)}=0,\ \lim_{n\to\infty}\frac{P\left(\left|\widehat{\gamma}-\gamma\right|>\eta_n\right)}{d_{2P}(n)}=0$$

and hence

$$\begin{aligned}&P\left(\frac{X_{(n)}-\overline{X}}{S}>\overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{2P}(n)}{n}\right)\right) \\ &=P\left(\frac{X_{(n)}-\overline{X}}{S}>\overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{2P}(n)}{n}\right),\left|S^2-1\right|<\eta_n,\left|\overline{X}\right|<\eta_n,\left|\widehat{\gamma}-\gamma\right|<\eta_n\right)+o\left(d_{2P}(n)\right).\end{aligned} \tag{58}$$

Writing

$$y_{1n}=\overline{K}_{\gamma}^{-1}\left(\frac{d_{2P}(n)}{n}\right)\left(1-\eta_n\overline{\Phi}^{-1}\left(\frac{d_{2P}(n)}{n}\right)^{2\zeta}\right),$$

$$y_{2n}=\overline{K}_{\gamma}^{-1}\left(\frac{d_{2P}(n)}{n}\right)\left(1+\eta_n\overline{\Phi}^{-1}\left(\frac{d_{2P}(n)}{n}\right)^{2\zeta}\right)$$

we obtain for sufficiently large n,

$$\begin{aligned}&P\left(X_{(n)}>y_{1n},\left|S^2-1\right|<\eta_n,\left|\overline{X}\right|<\eta_n,\left|\widehat{\gamma}-\gamma\right|<\eta_n\right) \\ &\le P\left(\frac{X_{(n)}-\overline{X}}{S}>\overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{2P}(n)}{n}\right),\left|S^2-1\right|<\eta_n,\left|\overline{X}\right|<\eta_n,\left|\widehat{\gamma}-\gamma\right|<\eta_n\right) \\ &\le P\left(X_{(n)}>y_{2n}\right).\end{aligned} \tag{59}$$

Using

$$P\left(X_{(n)}>y_{2n}\right)=P\left(\overline{K}_{\gamma}\left(X_{(n)}\right)<\overline{K}_{\gamma}\left(y_{2n}\right)\right)=1-\left(1-\overline{K}_{\gamma}\left(y_{2n}\right)\right)^{n}$$

and writing $x_n = \overline{K}_\gamma^{-1}\left(\frac{d_{2P}(n)}{n}\right)$, the conditions of the first part of Lemma 10 (with in the lemma ε_n replaced by $-\eta_n \overline{\Phi}^{-1}\left(\frac{d_{2P}(n)}{n}\right)^{2\zeta}$) are fulfilled, see also (56) and (57). Application of Lemma 10 yields

$$P\left(X_{(n)} > y_{2n}\right) = 1 - \left(1 - \overline{K}_\gamma\left(y_{2n}\right)\right)^n = 1 - \left(1 - \frac{d_{2P}(n)(1+o(1))}{n}\right)^n \tag{60}$$
$$= d_{2P}(n)(1+o(1)) \text{ as } n \to \infty.$$

Because

$$P\left(X_{(n)} > y_{1n}, \left|S^2 - 1\right| < \eta_n, \left|\overline{X}\right| < \eta_n, |\widehat{\gamma} - \gamma| < \eta_n\right) = P\left(X_{(n)} > y_{1n}\right) + o\left(d_{2P}(n)\right),$$

another application of Lemma 10 gives

$$P\left(X_{(n)} > y_{1n}, \left|S^2 - 1\right| < \eta_n, \left|\overline{X}\right| < \eta_n, |\widehat{\gamma} - \gamma| < \eta_n\right) = d_{2P}(n)(1+o(1)) \text{ as } n \to \infty. \tag{61}$$

Combination of (58) – (61) gives

$$P\left(\frac{X_{(n)} - \overline{X}}{S} > \overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{2P}(n)}{n}\right)\right) = d_{2P}(n)(1+o(1)) \text{ as } n \to \infty,$$

thus completing the proof of the lemma. ■

Proof of Lemma 6. Consider first $\gamma > 0$. Let $\varepsilon > 0$. It follows from Theorem 1 that for some $c > 0$

$$P\left(\frac{X_{(n)} - \overline{X}}{S} \le \overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right) \tag{62}$$
$$\le P\left(X_{(n)} \le \sqrt{1+\varepsilon}\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right) + \varepsilon\right) + O\left(\exp\left(-cn^{1/(1+\gamma)}\right)\right)$$
$$= \left\{1 - \overline{\Phi}\left(\left(\frac{\sqrt{1+\varepsilon}\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right) + \varepsilon}{c(\gamma)}\right)^{\frac{1}{1+\gamma}}\right)\right\}^n + O\left(\exp\left(-cn^{1/(1+\gamma)}\right)\right).$$

Since

$$\lim_{n\to\infty} d_{2N}(n) = 0, \lim_{n\to\infty} \frac{|\log d_{2N}(n)|}{(\log n)^{1+\gamma-\zeta}} = 0$$

and for $x \to 0$ we have

$$\overline{\Phi}^{-1}(x) = \sqrt{-2\log x}\,(1+o(1)),$$

we obtain for sufficiently large n

$$\frac{\sqrt{1+\varepsilon}\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right) + \varepsilon}{c(\gamma)} \le \sqrt{(\log n)^{1+\gamma-\zeta}}$$

and hence, taking $0 < \eta < 1 - \frac{1}{1+\gamma}$, we get for sufficiently large n

$$\left\{1 - \overline{\Phi}\left(\left(\frac{\sqrt{1+\varepsilon}\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right) + \varepsilon}{c(\gamma)}\right)^{\frac{1}{1+\gamma}}\right)\right\}^n \le \left\{1 - \overline{\Phi}\left((\log n)^{\frac{1+\gamma-\zeta}{2(1+\gamma)}}\right)\right\}^n \tag{63}$$
$$\le \left\{1 - \exp\left\{-(\log n)^{1-\frac{\zeta}{2(1+\gamma)}}\right\}\right\}^n \le \left\{1 - n^{-\eta}\right\}^n \le \exp\left(-n^{1-\eta}\right) = O\left(\exp\left(-cn^{1/(1+\gamma)}\right)\right).$$

Combination of (62) and (63) yields

$$P\left(\frac{X_{(n)}-\overline{X}}{S}\leq\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right)=O\left(\exp(-cn^{1/(1+\gamma)}\right).$$

Next let $\gamma<0$. It follows from Theorem (1) that for some $c>0$

$$\begin{aligned}&P\left(\frac{X_{(n)}-\overline{X}}{S}\geq\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)\right) \qquad (64)\\ &\leq P\left(X_{(n)}\geq\sqrt{1-\varepsilon}\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)-\varepsilon\right)+O\left(\exp\left(-cn^{1/(1+\gamma)}\right)\right)\\ &=1-\left\{1-\overline{\Phi}\left(\left(\frac{\sqrt{1-\varepsilon}\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)-\varepsilon}{c(\gamma)}\right)^{\frac{1}{1+\gamma}}\right)\right\}^{n}+O\left(\exp\left(-cn^{1/(1+\gamma)}\right)\right).\end{aligned}$$

Since

$$\lim_{n\to\infty}d_{1N}(n)=\infty,\ \lim_{n\to\infty}d_{1N}(n)\sqrt{\frac{d_{1N}(n)}{n}}\log\left(\frac{d_{1N}(n)}{n}\right)=0,$$

we obtain for sufficiently large n

$$\frac{\sqrt{1-\varepsilon}\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)-\varepsilon}{c(\gamma)}\geq\sqrt{\widetilde{c}\log n}$$

for some $\widetilde{c}>0$. Hence, taking $0<\eta<1-\frac{1}{1+\gamma}$, we get for sufficiently large $n$we get for sufficiently large n

$$\begin{aligned}&1-\left\{1-\overline{\Phi}\left(\left(\frac{\sqrt{1-\varepsilon}\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)-\varepsilon}{c(\gamma)}\right)^{\frac{1}{1+\gamma}}\right)\right\}^{n}\leq1-\left\{1-\overline{\Phi}\left(\left(\sqrt{\widetilde{c}\log n}\right)^{\frac{1}{1+\gamma}}\right)\right\}^{n} \qquad (65)\\ &\leq1-\left\{1-\exp\left\{-\frac{1}{2}\left(\widetilde{c}\log n\right)^{\frac{1}{1+\gamma}}\right\}\right\}^{n}\leq\exp\left\{-c^{*}\left(\log n\right)^{\frac{1}{1+\gamma}}\right\}\end{aligned}$$

for some $c^{*}>0$. Combination of (64) and (65) yields

$$P\left(\frac{X_{(n)}-\overline{X}}{S}\geq\overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)\right)=O\left(\exp\left\{-c^{*}\left(\log n\right)^{\frac{1}{1+\gamma}}\right\}\right).$$

■

Proof of Theorem 8. We have for each $\varepsilon,\eta>0$

$$\begin{aligned}&P\left(\frac{X_{(n)}-\overline{X}}{S}\leq\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right)\\ &\leq P\left(X_{(n)}\leq\sqrt{1+\varepsilon}\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)+\eta\right)+P\left(\left|\overline{X}\right|>\eta\right)+P\left(\left|S^{2}-1\right|>\varepsilon\right).\end{aligned}$$

Since

$$\lim_{n\to\infty}d_{2N}(n)=0,\ \lim_{n\to\infty}\frac{|\log d_{2N}(n)|}{\log n}=0$$

we get

$$\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right) = \sqrt{-2\log\left(\frac{d_{2N}(n)}{n}\right)}(1+o(1)) = \sqrt{2\log n}(1+o(1))$$

and hence for each $\zeta > 0$ we get for sufficiently large n

$$\sqrt{1+\varepsilon}\overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right) + \eta \leq \left(\sqrt{1+\varepsilon}+\zeta\right)\sqrt{2\log n}.$$

Therefore, we obtain for each $\varepsilon, \eta, \zeta > 0$ and sufficiently large n

$$\begin{aligned} &P\left(\frac{X_{(n)}-\overline{X}}{S} \leq \overline{\Phi}^{-1}\left(\frac{d_{2N}(n)}{n}\right)\right) \\ &\leq F\left(\left(\sqrt{1+\varepsilon}+\zeta\right)\sqrt{2\log n}\right)^n + P\left(\left|\overline{X}\right| > \eta\right) + P\left(\left|S^2-1\right| > \varepsilon\right). \end{aligned}$$

Similarly, we get for each $\varepsilon, \eta, \zeta > 0$ and sufficiently large n

$$\begin{aligned} &P\left(\frac{X_{(n)}-\overline{X}}{S} \geq \overline{\Phi}^{-1}\left(\frac{d_{1N}(n)}{n}\right)\right) \\ &\leq 1-F\left(\left(\sqrt{1-\varepsilon}-\zeta\right)\sqrt{2\log n}\right)^n + P\left(\left|\overline{X}\right| > \eta\right) + P\left(\left|S^2-1\right| > \varepsilon\right). \end{aligned}$$

Using

$$\overline{K}_\gamma^{-1}(t) = c(\gamma)\left\{\overline{\Phi}^{-1}(t)\right\}^{1+\gamma} \text{ for } 0 < t < \frac{1}{2},$$

we get for each $0 < \zeta < 1+\gamma$ (with ξ between γ and $\widetilde{\gamma}$) as $t \to 0$

$$\begin{aligned} &\sup_{|\widetilde{\gamma}-\gamma|\leq\zeta} \overline{K_{\widetilde{\gamma}}}^{-1}(t) \\ &= \sup_{|\widetilde{\gamma}-\gamma|\leq\zeta}\left\{\overline{K}_\gamma^{-1}(t) + (\widetilde{\gamma}-\gamma)\left[c'(\xi)\left\{\overline{\Phi}^{-1}(t)\right\}^{1+\xi} + c(\xi)\left\{\overline{\Phi}^{-1}(t)\right\}^{1+\xi}\log\left(\overline{\Phi}^{-1}(t)\right)\right]\right\} \\ &\leq \left\{\overline{K}_\gamma^{-1}(t)\right\}^{1+\frac{1.5\zeta}{1+\gamma}} \end{aligned}$$

we obtain for sufficiently large n

$$\begin{aligned} &P\left(\frac{X_{(n)}-\overline{X}}{S} \leq \overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{2P}(n)}{n}\right)\right) \\ &\leq F\left(\left(\sqrt{\log n}\right)^{1+\gamma+2\zeta}\right)^n + P\left(\left|\overline{X}\right| > \eta\right) + P\left(\left|S^2-1\right| > \varepsilon\right) + P\left(\left|\widehat{\gamma}-\gamma\right| > \zeta\right). \end{aligned}$$

Similarly, we get for sufficiently large n

$$\begin{aligned} &P\left(\frac{X_{(n)}-\overline{X}}{S} \geq \overline{K_{\widehat{\gamma}}}^{-1}\left(\frac{d_{1P}(n)}{n}\right)\right) \\ &\leq 1-F\left(\left(\sqrt{\log n}\right)^{1+\gamma-2\zeta}\right)^n + P\left(\left|\overline{X}\right| > \eta\right) + P\left(\left|S^2-1\right| > \varepsilon\right) + P\left(\left|\widehat{\gamma}-\gamma\right| > \zeta\right). \end{aligned}$$

■

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