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τ-Functions for a two-point version of the KP-hierarchy

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τ-FUNCTIONS FOR A TWO-POINT VERSION OF THE KP-HIERARCHY

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Abstract. In this paper a multipoint version of the linearization of the KP-hierarchy is described. Solutions of this system in the form of wave functions are constructed starting from a suitable Grassmann manifold and a group of commuting flows corresponding to the configuration of n points in $\mathbb{C}$. The failure of equivariance at lifting these flows to the determinant bundle over this Grassmann manifold is measured by the determinants of certain Fredholm operators, the so-called τ-functions. In the case of two points an explicit relation between these τ-functions and the wave functions is derived.

1. Introduction

Special functions play a crucial role in all parts of mathematics. Inside the area of integrable systems an intriguing class of such functions is formed by the so-called τ-functions. They are a generalization of theta functions and first occurred in the work of the Kyoto school, see e.g. [DJKM], on the KP-hierarchy and related systems. An analytic setting for the KP-hierarchy and a description of these τ- functions in terms of line bundles over an infinite dimensional Grassmann manifold was given in [SW]. A Lie algebraic interpretation of these functions was given by V.Kac, see e.g. [Kac].

In this paper a multipoint version of the KP-hierarchy is introduced. To give an idea what is meant by that, one recalls several geometric ingredients from the approach in [SW] to construct solutions of the KP-hierarchy. One starts with a splitting $H = H^+ \oplus H^-$ of the space of $\mathcal{L}^2$-boundary values on the unit circle in $\mathbb{P}^1(\mathbb{C})$. Here H^+ is the subspace of boundary values that extend holomorphically to $z = 0$ and H^- consists of those that are holomorphic at infinity and disappear there. To this decomposition was associated a Grassmann manifold of subspaces "similar" to H^+. On this manifold one considers the flows from the group of non-zero holomorphic functions on the unit disc around $z = 0$. To each subspace in the Grassmanian one associates a so-called wavefunction that is a product of an exponential factor, corresponding with the group of flows already mentioned, and a factor that is meromorphic around infinity. This wavefunction satisfies a system of an infinite number of linear evolution equations w.r.t. the parameters of the flows, which is called the *linearization of the KP-hierarchy*. The compatibility conditions for this linear system yield namely the equations of the KP-hierarchy. The role of the points zero and infinity can be interchanged in this setting and this dual geometric picture is the starting point for the generalization to several points.

First one presents for n points $\{a_1, \dots, a_n\}$ in $\mathbb{C}$ and the point infinity, an analytic set-up in the style of [SW] that leads, starting from a Grassmann manifold, to solutions of a similar set of equations. These geometric ingredients are treated in the second section.

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It starts out with a splitting of the space of $\mathcal{L}^2$-boundary values on circles around the $\{a_i\}$ similar to the one mentioned above that determines the Grassmann manifold we will work with. Further it contains a description of the analogue in the present setting of the group of commuting flows for the KP-hierarchy. This group of continuous flows can be complemented with a discrete group Δ that commutes with them.

After the introduction of the geometric set-up in the second section, one continues in the third with the construction of the various wavefunctions and a discussion of the differential equations these functions satisfy.

In the last section τ-functions are introduced that measure the failure of equivariance at lifting these flows to the determinant bundle over the Grassmann manifold. The paper concludes with a description, for the case of two points, of a relation between τ-functions and the wavefunctions.

2. The geometric setting

At the KP-hierarchy, the basic decomposition one starts with, is obtained by choosing besides the point infinity x_∞ another point x_0 on $\mathbb{P}^1(\mathbb{C})$ and a meromorphic functions z on $\mathbb{P}^1(\mathbb{C})$ with divisor $x_0 - x_\infty$. On the unit circle $\{x \mid |z(x)| = 1\}$ one considers then the L^2-boundary values and one splits this space of bounday values in those that extend holomorphically to x_0 and those that are holomorphic around x_∞ and disappear in this point. Subspaces close to this last subspace yield via the construction of wavefunctions solutions of the KP-hierarchy. The first step here will be to decribe the analogue of this setting for a finite number of points.

Consider thereto, besides the point infinity, n different points $a_1, \dots, a_n$ in $\mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$ and let the $\{D_i \mid 1 \le i \le n\}$ be disjoint open discs in $\mathbb{C}$ with center a_i and boundary C_i. As usual, $\underline{\mathcal{O}}$ denotes the sheaf of holomorphic functions. For each closed subset G of $\mathbb{C}$, one introduces, what is called the ring of holomorphic functions on G, as

$$\mathcal{O}(G) = \varinjlim_{\substack{U \text{ open} \\ U \supset G}} \mathcal{O}(U).$$

On each $\mathcal{O}(C_j)$ one takes the innerproduct

$$(f, g)_j = \frac{1}{2\pi i} \oint_{C_j} f(z)\overline{g(z)} \frac{dz}{z - a_j}. \tag{1}$$

Hence, if r_i is the radius of C_i, then we have for all $k \in \mathbb{Z}$

$$((z - a_j)^k, (z - a_j)^k)_j = r_j^{2k}.$$

Let H_j be the completion of $\mathcal{O}(C_j)$ w.r.t. this inner product. Like the elements of $\mathcal{O}(C_j)$, each $f \in H_j$ can be represented by a Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - a_j)^n, \text{ with } \sum_{n \in \mathbb{Z}} |a_n|^2 r_j^{2n} < \infty. \tag{2}$$

In H_j, one considers the closed subspace H_j^+ of H_j of those boundary values that have an analytic continuation to D_j, i.e.

$$H_j^+ = \left\{ f \in H_j \,\middle|\, f(z) = \sum_{n \ge 0} a_n (z - a_j)^n \right\}.$$

Each $f \in H_j$ decomposes as $f = f^+ + f^-$, with $f^+ \in H_j^+$ and $f^- \in H_j^- = (H_j^+)^\perp$. Let H be the direct sum of the Hilbert spaces H_j. If Ω is the complement of $\bigcup_{j=1}^n D_j$ in $\mathbb{P}^1(\mathbb{C})$, then each $f \in \mathcal{O}(\Omega)$ determines an element of H by the diagonal embedding $f \mapsto (f, \dots, f)$. Let H^* be the closure inside H of

$$H_0 = \{(f, \dots, f) |\, f \in \mathcal{O}(\Omega), f(\infty) = 0\}.$$

Subspaces of H "close" to H^* will provide solutions of nonlinear equations that form an analogue of the dual of the KP-hierarchy. Examples of elements in H_0 are the

$$e_{j+(k-1)n} = r_j^k((z - a_j)^{-k}, \cdots, (z - a_j)^{-k}), \text{ with } k > 0.$$

As one will see in this subsection, these elements form a topological basis of H^*. To each element in H^* is associated a holomorphic function on the interior of Ω as one can read off from the following lemma.

Lemma 2.1. *If $f = (f_1, \cdots, f_n) \in H^*$, then there is a holomorphic function $\underline{f}$ on the interior Ω_0 of Ω, such that $\underline{f}$ has Laurent series f_j around a_j.*

Proof. If g belongs to $\mathcal{O}(\Omega)$, then one has for all $\omega \in \Omega_0$ the integral formula

$$g(\omega) = \sum_{j=1}^n \frac{1}{2\pi i} \oint_{C_j} g(z) \frac{dz}{z - \omega}. \tag{3}$$

Let $\{g_j\}$ be a Cauchy sequence in $\mathcal{O}(\Omega)$ w.r.t. the norm $\|\cdot\|$ on H. Then formula (3) implies for each $\omega_0 \in \Omega_0$ that there is a neighborhood M_0 of ω_0 inside such that for all $\omega \in M_0$ and all j and k

$$|g_j(\omega) - g_k(\omega)| \le C\|g_j - g_k\|,$$

for some constant C. This clearly implies uniform convergence on compact subsets of Ω_0 of the $\{g_j\}$. Thus we see that the pointwise limit of the $\{g_j\}$ defines a holomorphic function on Ω_0 that has the required property. This proves the lemma. □

On the holomorphic level we know that $\mathcal{O}(\Omega) \cap H_0 = \{0\}$ and the following proposition shows that this property pertains for the completions of the two spaces. Moreover it also describes a topological basis of H^* and it gives the central decomposition in the paper.

Proposition 2.1. (a) *Let H^+ be the subspace $\oplus_{j=1}^n H_j^+$ of H, then there holds:*

$$H^* \cap H^+ = \{0\}.$$

(b) *The orthogonal projection p_- from H^* to $H^- = \oplus_{j=1}^n H_j^-$ is a homeomorphism between these 2 spaces. The orthogonal projection p_+ from H^* to H^+ is a relatively small operator, for it is Hilbert-Schmidt.*

(c) *The space H decomposes as $H = H^* \oplus H^+$.*

Proof. If $g \in H^* \cap H^+$, then one knows on one hand that g is the boundary value w.r.t. $\|\cdot\|$ of a holomorphic function g_1 on $\bigcup_{j=1}^n D_j$. On the other hand, one sees from the foregoing lemma that g is the boundary value of a holomorphic g_2 on Ω_0. This implies that locally around each point c_0 of $C_j, 1 \le j \le n$, the functions g_1 and g_2 determine the same distributional boundary value. By the edge of the wedge theorem (see [dR], p 75), one may conclude then that locally around each c_0 this distribution is given by a holomorphic function. In other words, g is holomorphic on a neighbourhood of Ω and hence on $\mathbb{P}^1(\mathbb{C})$. Since $g(\infty) = 0$, this shows part (a) of the proposition.

From part (a) one can conclude that the orthogonal projection $p_- : H^* \longrightarrow H^-$ is an injection. It suffices to show still that there is a continuous linear map $\pi : H^- \longrightarrow H^*$ such that $p_- \circ \pi = \mathrm{Id}_{H^-}$. If $f = (f_1, \cdots, f_n)$ is an element of H^-, then one defines for each j, a holomorphic function F_j on the complement of $\overline{D}_j$ by

$$F_j(q) = \frac{1}{2\pi i} \oint_{C_j} f_j(z) \frac{dz}{z-q}.$$

From this formula, one deduces directly that the boundary value $(F_j)_k$ of F_j at C_k belongs to H, that $(F_j)_j = f_j$ and that the map π_j:

$$f_j \longrightarrow ((F_j)_1, \cdots, (F_j)_n)$$

is a continuous map from H_j^- to H. Since for f_j of the form

$$f_j = \sum_{k=1}^{N} a_{kj}(z-a_j)^{-k}$$

the element $\pi_j(f_j)$ clearly belongs to H_0, one gets that $\pi = \oplus_{j=1}^n \pi_j$ is a continuous injection from H^- to H^* and that $p \circ \pi$ equals the identity on all elements of the form

$$\left(\sum_{k=1}^{N} a_k (z-a_1)^{-k}, \cdots, \sum_{k=1}^{N} a_{kn}(z-a_n)^{-k} \right).$$

This proves the first claim in part (b). Note that for all $i \neq j$ and all $k \geq 1$ one has

$$\frac{r_j^k}{(z-a_j)^k} = \frac{r_j^k}{(a_i-a_j)^k} \frac{1}{(1-\frac{z-a_i}{a_j-a_i})^k} = \frac{r_j^k}{(a_i-a_j)^k} \frac{1}{(1-\frac{r_i}{a_j-a_i}\frac{z-a_i}{r_i})^k}$$

$$= \frac{r_j^k}{(a_i-a_j)^k} \sum_{l=0}^{\infty} \frac{(l+k-1)\dots(l+1)}{k!} (\frac{r_i}{a_j-a_i})^l (\frac{z-a_i}{r_i})^l, \text{ if } k \geq 2, \text{ and} \tag{4}$$

$$= \frac{r_j}{a_i-a_j} \sum_{l=0}^{\infty} (\frac{r_i}{a_j-a_i})^l (\frac{z-a_i}{r_i})^l, \text{ if } k = 1.$$

By combining the formula 4 with the property 2 one sees that the matrix coefficients of the orthogonal projection $p_+ : H^* \longrightarrow H^+$ are square integrable. Hence this operator is Hilbert-Schmidt. This proves part (b) of the proposition.

For the third part of the proposition one merely has to decompose an arbitrary $F = (f_1, \cdots, f_n)$ as $F = F^* + F^+$, with $F^* \in H^*$ and $F^+ \in H^+$. From the foregoing discussion it is known that $\sum_{j=1}^n f_j^-$ determines an element of H^* and that $f_j^+ - \sum_{i\neq j} f_i^- \in H_j^+$. Hence there holds

$$F = F^* + F^+, \text{ with } F^* = (\textstyle\sum_{j=1}^n f_j^-, \cdots, \sum_{j=1}^n f_j^-) \text{ and } F^+ = (f_j^+ - \sum_{i\neq j} f_i^-).$$

This proves the various claims of the proposition. □

Remark 2.1. *In* [Di] *it is suggested to take the decomposition* $H = H_1 \oplus H_2$, *where* H_1 *is the closure inside* H *of the space of all holomorphic functions on* Ω *and*

$$H_2 = \{f \mid f \in H^+, \sum_{i=1}^n f(a_i) = 0\}.$$

However this leads to far less nicer formulae at the relation between the τ-functions and the wavefunctions.

Since the decomposition $H = H^* \oplus H^+$ is central here, it is convenient to put a different, but equivalent, inner product $< \cdot | \cdot >$ on H such that w.r.t. this new inner product the $\{e_k \mid k \geq 1\}$ are an orthogonal basis of H^* and the space H^* itself is orthogonal to H^+. Namely, we take

$$< F^* + F^+ | G^* + G^+ > := (p_-(F^*), p_-(G^*)) + (F^+, G^+),$$

for all $F^*, G^* \in H^*$ and all $F^+, G^+ \in H^+$. Here $(\cdot, \cdot)$ denotes the original inner product on H. Thus, one completes the orthonormal basis $\{e_\ell | \ell \geq 1\}$ of H^* with the orthonormal basis

$$e_{-(j-1)-kn} = (\cdots, 0, \frac{(z-a_j)^k}{r_j^k}, 0, \cdots), k \geq 0 \text{ and } 1 \leq j \leq n,$$

of H^+ to a Hilbert basis $\{e_\ell | \ell \in \mathbb{Z}\}$ of H.

As in [SW] one associates to the decomposition $H = H^* \oplus H^+$ the Grassmann manifold $Gr(H)$. It consists of all subspaces W in H such that the orthogonal projection $p^+ : W \to H^+$ is a Hilbert-Schmidt operator. Consequently, the orthogonal projection $p^* : W \to H^*$ is then a Fredholm operator, i.e. it has a finite dimensional kernel and cokernel. The connected components of $Gr(H)$ are exactly determined by the index of this operator. They are given by

$$Gr_k(H) = \{W | W \in Gr(H), \text{ index } (p^* | W) = k\}, k \in \mathbb{Z}.$$

In $Gr_0(H)$ we have the open dense subset of all planes W such that $p^* : W \to H^*$ is a bijection. Like in the finite dimensional situation, this is called the big cell of $Gr(H)$ and its elements are called tranverse to H^*. On $Gr(H)$ we have a natural transitive action of the so-called restricted linear group $Gl_{res}(H)$, consisting of

$$Gl_{res}(H) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(H), b \text{ and } c \text{ are Hilbert-Schmidt} \right\},$$

where the elements $g \in Gl(H)$ are decomposed w.r.t. $H = H^* \oplus H^+$. Note that thanks to proposition 2.1 the collection of planes in $Gr(H)$ is the same as the one, one would have got with the initial inner product on H.

Likewise, the Lie algebra, $B_{res}(H)$, of $Gl_{res}(H)$ consists of all bounded operators T on H that decompose w.r.t. $H = H^* \oplus H^+$ as

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ with } b \text{ and } c \text{ Hilbert-Schmidt.}$$

If f is a continuously differentiable function on $\cup_{i=1}^n C_i$, then multiplication with f determines on operator M_f in $B_{res}(H)$. In particular, elements of $\mathcal{O}(\cup_{i=1}^n C_i)$ give rise in this way to operators in $B_{res}(H)$.

Now one is ready to introduce the group of symmetries that form the analogue in our context of the commuting flows of the KP-hierarchy. Its continuous part are the exponentials of holomorphic functions on Ω that are zero at infinity and thus multiplication with them gives you elements in $Gl_{\text{res}}(H)$.

$$\tilde{\Gamma} = \left\{ \exp \sum_{k=1}^{n} \sum_{l=1}^{\infty} t_l^{(k)} (z - a_k)^{-l} \middle| \sum_{k=1}^{n} \sum_{l=1}^{\infty} t_l^{(k)} (z - a_k)^{-l} \in \mathcal{O}(\Omega) \right\},$$

The group $\tilde{\Gamma}$ describes the form of the essential singularities of the wavefunctions around the points $\{a_1, \ldots, a_n\}$ and it clearly stabilizes the space H^*. It is convenient to use a notation related to this group. We will write

$$\xi_k(z,t) = \sum_{l=1}^{\infty} t_l^{(k)}(z-a_k)^{-l} \quad \text{and} \quad \xi(z,t) = \sum_{k=1}^{n} \xi_k(z,t),$$

where t is short for $\left\{t_l^{(k)} | 1 \leq k \leq n, l \geq 1\right\}$.

Besides the continuous group $\tilde{\Gamma}$, there is also a discrete group Δ of transformations of $Gr(H)$ that preserves H^* and that commutes with the action of $\tilde{\Gamma}$. Namely, if $i \neq j$ with $1 \leq i \leq n$ and $1 \leq j \leq n$, let $\delta_{ij} : H \to H$ be defined as

$$\delta_{ij}(f_1(z), \ldots, f_n(z)) = (\frac{r_j(z-a_i)}{r_i(z-a_j)} f_1(z), \ldots, \frac{r_j(z-a_i)}{r_i(z-a_j)} f_n(z)).$$

It is an easy verification to show that δ_{ij} belongs to $Gl_{\text{res}}^{(0)}(H)$ the connected component of the identity in $Gl_{\text{res}}(H)$ and it clearly stabilizes H^*. For $\underline{k} = \{k_{ij} | k_{ij} \in \mathbb{Z}, 1 \leq i < j \leq n\}$, we introduce then the element $\delta(\underline{k})$ by $\prod_{i<j} \delta_{ij}^{k_{ij}}$. The group Δ is now given by

$$\Delta = \{\delta(\underline{k}) | \underline{k} = (k_{ij}) \in \mathbb{Z}^{\frac{1}{2}n(n-1)}\}.$$

As one will see in the next section, the wavefunctions have besides the essential singularity factor e^{ξ}, also a factor that is meromorphic around the points $\{a_1, \ldots, a_n\}$. The group Δ describes the various types of meromorphic behaviour around these points that correspond to elements in $Gl_{\text{res}}^{(0)}(H)$.

The next step will be the introduction of the analogue in this context of the wavefunction of the KP-hierarchy.

3. The wave functions in the multipoint setting

Let W belong to $Gr_0(H)$ and let $p^* : H \to H^*$ as above be the orthogonal projection. In the group Δ one considers

(5) $\quad \Delta_W = \{\delta | \delta \in \Delta, \text{ there is a } \gamma \in \tilde{\Gamma} \text{ such that } p^* : \gamma^{-1}\delta^{-1}W \to H^* \text{ is a bijection}\}$

Note that each δ belongs to Δ_W for each W in the inverse image under δ of the big cell. By using the covering of $Gr_0(H)$ with big cells corresponding to subspaces in $Gr_0(H)$ with a topological basis from the $\{e_k | k \in \mathbb{Z}\}$, one can show that the set Δ_W is non-empty.

For each δ in Δ_W one considers the open subset $\Gamma(\delta, W)$ of $\tilde{\Gamma}$ given by

$$\Gamma(\delta, W) = \{\gamma | \gamma \in \tilde{\Gamma}, \gamma^{-1}\delta^{-1}W \text{ is transverse to } H^*\}.$$

Fix some δ in Δ. For each γ in $\Gamma(\delta, W), \gamma^{-1}\delta^{-1}W$ is transverse to H^* and this enables you to define for each $k \geq 1$

$$\hat{\psi}_{W,k}^{(\delta)}(\gamma) = (p^*|_{\gamma^{-1}\delta^{-1}W})^{-1}(e_k).$$

By multiplication with $\gamma\delta$ we translate the element $\hat{\psi}_{W,k}^{(\delta)}(\gamma)$ in $\gamma^{-1}\delta^{-1}W$ back to W and thus we obtain a $\psi_{W,k}^{(\delta)} : \Gamma(\delta, W) \to W$ that can be written as

$$\begin{aligned} \psi_{W,k}^{(\delta)}(\gamma) &= \hat{\psi}_{W,k}^{(\delta)}(\gamma)\delta e^{\xi} \\ &= \{e_k + \textstyle\sum_{s \geq 0} a_{-s,k}(\delta, \gamma) e_{-s}\}\delta e^{\xi}. \end{aligned}$$

The functions $\psi_{W,k}^{(\delta)}$ are called *wavefunctions of type* δ. The collection of functions $\{\psi_{W,k}^{(\delta)}|k \geq 1\}$ determine the space W, since the topological span of the vectors

$$\{\psi_{W,k}^{(\delta)}(\gamma)|k \geq 1, \gamma \in \Gamma(\delta, W)\}$$

equals the space W. One of the first things that one will see in the sequel, is that it already suffices to know the $\{\psi_{W,1}^{(\delta)}, \ldots, \psi_{W,n}^{(\delta)}\}$ to recover W. In that light, one writes for all $\ell \geq 1$ and each $i, 1 \leq i \leq n$, $\hat{\psi}_{W,ij}^{(\delta,\ell)}$ resp. $\psi_{W,ij}^{(\delta,\ell)}$ for the H_j-component of $\hat{\psi}_{W,i+n(\ell-1)}^{(\delta)}$ resp. $\psi_{W,i+n(\ell-1)}^{(\delta)}$.

Often there is no need to stress the W- resp. δ-dependence of the wavefunctions and then one loosens up the notations by leaving them out. E.g. one writes then $\hat{\psi}_{ij}^{(\ell)}$ resp. $\psi_{ij}^{(\ell)}$instead of $\hat{\psi}_{W,ij}^{(\delta,\ell)}$ resp. $\psi_{W,ij}^{(\delta,\ell)}$ and thus one gets the vectors

$$\hat{\psi}_i^{(\ell)} := (\hat{\psi}_{i1}^{(\ell)}, \ldots, \hat{\psi}_{in}^{(\ell)}) \text{ and } \psi_i^{(\ell)} := \hat{\psi}_i^{(\ell)}\delta e^{\xi}.$$

Also the notation for the coefficients of the wavefunctions is compactified by writing $\alpha_{rl}^{(i,j)}(\delta, \gamma)$ for $a_{-(j-1)-rn, i+n(\ell-1)}(\delta, \gamma)$ and then one has

$$\psi_{ij}^{(\ell)}(\gamma) = \{\frac{r_i^\ell}{(z-a_i)^\ell} + \sum_{r\geq 0} \alpha_{r\ell}^{(i,j)}(\delta,\gamma)(\frac{z-a_j}{r_j})^r\}\delta e^{\xi}.$$

The equations that will be derived for the $\{\psi_i^{(\ell)}|\ell \geq 1, 1 \leq i \leq n\}$ in the next subsection amount to nonlinear differential equations for the coeficients $\alpha_{r\ell}^{(i,j)}$, w.r.t. the parameters $\{t_k^{(j)}|k \geq 1, 1 \leq j \leq n\}$. In the sequel one will use the notations $\partial_k^{(i)} = \frac{\partial}{\partial t_k^{(i)}}$ and $\partial_j = \partial_1^{(j)}$ for all $k \geq 1$, $1 \leq i \leq n$ and $1 \leq j \leq n$.

First of all it is shown that instead of considering all the wavefunctions $\{\psi_i^{(\ell)}\ell \geq 1, 1 \leq i \leq n\}$ it suffices to consider a finite number of them

Proposition 3.1. *For each $\ell \geq 1$ and each $i, 1 \leq i \leq n$, there is a unique differential operator $Q_\ell^{(i)}$ of the form*

$$Q_\ell^{(i)} = \sum_{s=0}^{\ell-1} q_{\ell s}^{(i)} \partial_i^s, \quad \textit{with } q_{\ell\ell-1}^{(i)} = r_i^{\ell-1},$$

such that $\psi_i^{(\ell)} = Q_\ell^{(i)}(\psi_i^{(1)})$. The coefficients of $Q_\ell^{(i)}$ are polynomial expressions in the coefficients $\{\alpha_{r1}^{(i,i)}|0 \leq r \leq \ell-2\}$ and their ∂_i-derivatives.

Proof. For each s in $\mathbb{N}$, one has

$$\partial_i^s(\psi_i^{(1)}) = \sum_{r=0}^{s} \left(\begin{array}{c} s \\ r \end{array}\right) \partial_i^{s-r}(\hat{\psi}_i^{(1)}) \frac{1}{(z-a_i)^r}\delta e^{\xi}.$$

This formula implies that, for $j \neq i$, the j-th component of $\partial_i^s(\psi_i^{(1)})$ contains only positive powers of $(z-a_j)$ and that the i-th component of $\partial_i^s(\psi_i^{(1)})$ looks like

$$\left\{\frac{r_i}{(z-a_i)^{s+1}} + \text{ “higher order in } z-a_i\text{”}\right\}\delta e^{\xi}.$$

Therefore, the j-th component of $\psi_i^{(\ell)} - r_i^{\ell-1}\partial_i^{\ell-1}(\psi_i^{(1)})$ for $j \neq i$ belongs to H_j^+ and the i-th component looks like

$$\left\{c_t(z-a_i)^t \ + \ \text{“higher order in } z-a_i\text{”} \right\}\delta e^{\xi}$$

with $t > -\ell$ and c_t a polynomial expression in the $\alpha_{r1}^{(i,i)}, 0 \leq r \leq \ell-2$, and their ∂_i-derivatives. If $t < 0$, then one continues to reduce the order of the singularity around a_i and one considers

$$\psi_i^{(\ell)} - (r_i\partial_i)^{\ell-1}(\psi_i^{(1)}) - r_i^{-1}c_t\partial_i^{-t-1}(\psi_i^{(1)}).$$

Continuing in this fashion, one ends up with a differential operator $Q_\ell^{(i)}$ of the required form such that for all $\gamma \in \Gamma(\delta, W)$

$$\psi_i^{(\ell)}(\gamma) - Q_\ell^{(i)}(\psi_\ell^{(1)})(\gamma) \in H^+\delta\gamma.$$

These vectors, however, also belong to W and, by construction, $W \cap H^+\delta\gamma = \{0\}$. Hence we get

$$\psi_i^{(\ell)} = Q_\ell^{(i)}(\psi_i^{(1)}).$$

The differential operator $Q_\ell^{(i)}$ is unique, because one easily shows, by comparing coefficients of the powerseries involved, that each differential operator $P = \sum \alpha_s\partial_i^s$ such that $P(\psi_i^{(1)}) = 0$ has to be zero. This proves the propositon.

□

As a direct consequence of this result is

Corollary 1. *The wavefunctions $\{\psi_{W,1}^{(\delta)}, \ldots, \psi_{W,n}^{(\delta)}\}$ determine the space W.*

From now on one concentrates on the $\{\psi_i^{(1)} \mid 1 \leq i \leq n\}$ and one will write in the sequel $\hat{\psi}_i$ resp. ψ_i instead of $\hat{\psi}_i^{(1)}$ resp. $\psi_i^{(1)}, 1 \leq i \leq n$. The next step is the determination of the derivatives of the wavefunctions with respect to the parameters $\{t_k^{(j)}\}$ and one starts with the $t_k^{(i)}$-derivatives of ψ_i.

Proposition 3.2. *For all $k \geq 1$ and all $i, 1 \leq i \leq n$, there is a unique differential operator $P_k^{(i)}$ of the form*

$$P_k^{(i)} = \sum_{s=0}^{k} p_{ks}^{(i)}\partial_i^s, \quad \textit{with} \ \ p_{kk}^{(i)} = 1,$$

such that $\partial_k^{(i)}(\psi_i) = P_k^{(i)}(\psi_i)$.

The coefficients of the differential operator $P_k^{(i)}$ are polynomial expressions in the coefficients $\{\alpha_{r1}^{(i,i)} | 0 \leq r \leq k-1\}$ and their ∂_i-derivatives.

Proof. Since

$$\partial_k^{(i)}(\psi_i) = \{\partial_k^{(i)}(\hat{\psi}_i) + \hat{\psi}_i(z-a_i)^{-k}\}\delta e^{\xi},$$

we see that $\{\partial_k^{(i)}(\psi_i) - \partial_j^k(\psi_i)\}\delta^{-1}e^{\xi}$ has no singularities around the points $\{a_j, j \neq i\}$ and has a lower order singularity than $k+1$ around a_i. By successively reducing the order of the singularity around a_i, one obtains again a differential operator $P_k^{(i)}$ of the required form such that

$$\partial_k^{(i)}(\psi_i)(\gamma) - P_k^{(i)}(\psi_i)(\gamma) \in H^+\delta\gamma \cap W = \{0\}.$$

The uniqueness of $P_k^{(i)}$ follows from the same argument as in proposition 3.1. This concludes the proof. $\square$

Next one considers the action of $\partial_k^{(j)}$, with $j \neq i$, and thus obtain the "mixed" equations.

Proposition 3.3. *For all $k \geq 1$ and all $j \neq i$, there exists a unique differential operator $R_k^{(j,i)}$ of the form*

$$R_k^{(j,i)} = \sum_{s=0}^{k-1} r_s^{(j,i)} \partial_j^s$$

satisfying

$$\partial_k^{(j)}(\psi_i) = \frac{1}{(a_i - a_j)^k}\psi_i + R_k^{(j,i)}(\psi_j).$$

The coefficients of the differential operator $R_k^{j,i}$ are polynomial expressions in the functions $\{\alpha_{r1}^{(i,j)} | 0 \leq r \leq k-1\}$, the $\{\alpha_{r1}^{(j,j)} | 0 \leq r \leq k-1\}$ and their ∂_j-derivatives.

Proof. We consider again the singularities of $\{\partial_k^j(\psi_i)\}\delta^{-1}e^{-\xi}$ around the points $\{a_1, \ldots, a_n\}$. Since

$$\partial_k^{(j)}(\psi_i) = \{\partial_k^{(j)}(\hat{\psi}_i) + \hat{\psi}_i(z - a_j)^{-k}\}\delta e^{\xi},$$

there is a k-th order singularity around a_j and a first order around a_i. The first can be eliminated as in the proof of propositions 3.1 and 3.2 with a suitable $(k-1)$-th order differential operator $R_k^{j,i} = \sum_{0 \leq s \leq k-1} r_s^{(j,i)} \partial_j^s$. The second with a proper multiple of ψ_i. Then we have again for all $\gamma \in \Gamma(\delta, W)$

$$\partial_k^{(j)}(\psi_i)(\gamma) - \frac{1}{(a_i - a_j)^k}\psi_i(\gamma) - R_k^{(j,i)}(\psi_j)(\gamma) \in W \cap H^+\delta e^{\xi} = \{0\}.$$

Since the action of the differential operators in ∂_j on ψ_j has no torsion, this equation also implies the uniqueness of $R_k^{(j,i)}$. $\square$

For $k = 1$, the operator $R_1^{(j,i)}$ has a simple form. The equation from proposition 3.3 becomes then

$$\begin{aligned} \partial_j(\psi_i) &= \tfrac{1}{(a_i - a_j)}\psi_i + \left\{ \tfrac{\alpha_{01}^{(i,j)}}{r_j} - \tfrac{r_i r_j^{-1}}{a - i - a_j} \right\} \psi_j \\ &:= \tfrac{1}{(a_i - a_j)}\psi_i + c_W^{(i,j)}\psi_j \end{aligned}$$

Note that for all $i \neq j$, $c_{H^*}^{(i,j)} = \frac{-r_i r_j^{-1}}{a_i - a_j}$ is non-zero. This holds more generally

Lemma 3.1. *For each W in $g_0(H)$ and all $i \neq j$, the function $c_W^{(i,j)}$ is not identical zero.*

From the construction of the wavefunctions, one deduces directly that it suffices to prove this for all W in the big cell corresponding to H^*. Then one uses $c_{H^*}^{(i,j)} \neq 0$, the actual form of the flows in $\tilde{\Gamma}$ and the analyticity of $c_W^{(i,j)}(\gamma)$ in γ to prove the lemma.

Now one restricts the wavefunctions to the dense open part $\Gamma(\delta, W)$ of Γ, given by

$$\tilde{\Gamma}(\delta, W) = \{\gamma | \gamma \in \Gamma(\delta, W), c_W^{(i,j)}(\gamma) \neq 0 \text{ for all } i \neq j\},$$

and one introduces the differential operators T_{ji} by

$$T_{ji} = \frac{1}{c_W^{(i,j)}}(\partial_j - \frac{1}{a_i - a_j}).$$

The interrelations between the wavefunctions $\{\psi_1, \dots, \psi_n\}$ are resumed now in the following proposition

Proposition 3.4. *For $i \neq j$ and $\gamma \in \tilde{\Gamma}(\delta, W)$ one has*

$$T_{ji}(\psi_i)(\gamma) = \psi_j(\gamma).$$

In particular each wavefunction ψ_i is annihilated by each of the second order differential operators $T_{ij}T_{ji} - 1$, with $1 \leq j \leq n$. Thus one has obtained the following refinement of Corollary 3.1

Corollary 2. *Each wavefunction $\psi^{(\delta)}_{W,i}, 1 \leq i \leq n$, determines the space W.*

With the help of the operators $\{T_{ji}\}$ one can put the equations from the propositions 3.2 and 3.3 in the same form: for each $k \geq 1$ and all i and j, there is differential operator $B^{(j,i)}_k$ of order k in ∂_j such that

$$\partial^{(j)}_k(\psi_i) = B^{(j,i)}_k(\psi_i). \tag{6}$$

This is the generalization to the present setting of the linearization of the KP-hierarchy. A natural next step is to consider the compatibility conditions of the equations (6) and this leads to

Definition 1. The equations of the *multipoint KP-hierarchy* are the non linear differential equations for the set of coefficients $\{\alpha^{(j,i)}_{ni} | n \geq 0, 1 \leq j \leq n\}$ of the wavefunction ψ_i that can succinctly be written as

$$\{\partial^{(s)}_m(B^{(j,i)}_k) - \partial^{(j)}_k(B^{(s,i)}_m) + [B^{(j,i)}_k, B^{(s,i)}_m]\}(\psi_i) = 0. \tag{7}$$

The equations (6) are called the *linearization of the multipoint KP-hierarchy.*

Remark 3.1. *In the case of the KP-hierarchy the action on a wavefunction of the differential operators generating the commuting flows was free so that these equations reduce to the so-called zero-curvature or Zacharov-Shabat equations*

$$\partial^{(s)}_m(B^{(j,i)}_k) - \partial^{(j)}_k(B^{(s,i)}_m) + [B^{(j,i)}_k B^{(s,i)}_m] = 0.$$

As one has seen, the action of the mixed derivatives might have torsion in the multipoint setting.

Remark 3.2. *In this paper one restricts oneself to the equations coming from the continuous flows from $\tilde{\Gamma}$. If one would also take the action of Δ into account, then one ends up with a mixture of differential and difference equations in the style of* [HP]. *They will be considered at a later occasion.*

As in the case of the KP-hierarchy, one can show that the wavefunctions can be related to certain Fredholm determinants, the so-called τ-functions. They can also be defined in the present context and there is also a relation with the wavefunctions. This is determined for the case of two points in the next section.

4. The τ-functions

Let W belong to $Gr_0(H)$. Since a Fredholm operator of index zero can be written as the difference of an inverible operator and a finite dimensional one and moreover, the Hilbert-Schmidt operators form a two-sided ideaI in the ring of bounded operators, one can write W as the image of an embedding $w : H^* \to H$ that decomposes w.r.t. $H = H^* \oplus H_+$ as

$$w = \begin{pmatrix} w^* \\ w^+ \end{pmatrix}, \text{ with } w^* - Id \text{ trace class and } w^+ \text{ Hilbert-Schmidt.} \tag{8}$$

We denote the collection of all embeddings of this form by $\mathcal{P}^*$. The component $Gr_0(H)$ can be described then as the quotient $\mathcal{P}^*/\mathcal{T}^*$, where $\mathcal{T}^*$ is the group defined by

$$\mathcal{T}^* = \{t \in Gl(H^*),\ t - Id \text{ is trace class } \}, \tag{9}$$

which by composition on the right on $\mathcal{P}^*$. One can define holomorphic line bundles Det and Det^* over $Gr_0(H)$, by dividing out on the product space $\mathcal{P} \times \mathbb{C}$ respectively the following two actions of $\mathcal{T}^*$,

$$(w, \lambda) \mapsto (w \circ t^{-1}, \lambda det(t)^{-1}), \tag{10}$$

$$(w, \lambda) \mapsto (w \circ t^{-1}, \lambda det(t)). \tag{11}$$

The class of (w, λ) in Det is denoted by $[w, \lambda]_1$ and that in Det^* by $[w, \lambda]_2$. As in the finite dimensional case, the line bundle Det^* has nontrivial sections, e.g. $\sigma : Gr_0(H) \mapsto Det^*$ given by

$$\sigma(\text{ Image of } w) = [w, det(w^*)]_2. \tag{12}$$

The connected component $Gl^{(0)}_{res}(H)$ of $Gl_{res}(H)$ acts transitively on $Gr_0(H)$ and to lift this action to the line bundles Det and Det^* one has to pass to an extension $\mathcal{E}$ of $Gl^{(0)}_{res}(H)$. It is defined by

$$\mathcal{E} = \{(g, q) = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, q), \text{ with } g \in Gl^{(0)}_{res}(H), q \in Gl(H^*) \text{ and } aq^{-1} - Id \text{ trace class } \}.$$

We fix the embedding into $\mathcal{E}$ of the stabilizer $P = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in Gl^{(0)}_{res}(H)\}$ of H^* as follows

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto (\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, a).$$

Note that both the group Γ^* as Δ embed by their action on H into the group P. The group $\mathcal{E}$ acts on Det and Det^* by

$$(g, q).[w, \lambda]_i = [g \circ w \circ q^{-1}, \lambda]_i. \tag{13}$$

This induces an action of $\mathcal{E}$ on the holomorphic sections of Det^*, in particular for σ we get

$$(g, q).\sigma(Im(w)) = [w, det((g^{-1} \circ w \circ q)^*)]_2. \tag{14}$$

Hence, if we define the function τ_w on $\mathcal{E}$ by

$$\tau_w((g, q)) = det((g^{-1} \circ w \circ q)^*) = det((aw^*q + bw^+q)^*), \text{ for } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then this function measures the failure of the $\mathcal{E}$-equivariance of the section σ. Note that choosing another embedding from $\mathcal{P}^*$ with image W only results in multiplying this function with a non zero constant. If one restricts this function to the subgroup P, then we get for a W in the big cell around H^*, i.e. with w^* invertible, that

$$\tau_w((g, a^{-1})) = det(w^*)det(Id + a^{-1}bw^+(w^*)^{-1}). \tag{15}$$

In this case one also writes $W = graph(A)$, with $A = w^+(w^*)^{-1}$.

Like in the case of the KP-hierarchy there are identities linking the wavefunctions of this multipoint version with these τ-functions. Here the case of two points a_1 and a_2 will be treated. Let $x_i = \frac{\zeta_i - a_i}{z - a_i}$, where ζ_i lies inside the circle C_i. Then for all $z \in C_i$ one has $|x_i| < 1$. Define the element $q_i \in \Gamma^*$ by $q_i = (1 - x_i, \ldots, 1 - x_i)$. Note that $q_i = \frac{\zeta_i - a_i}{z - a_i}$ and that q_i stabilizes the plane H^*. We denote the matrix of the action of q_i^{-1} on H by $\begin{pmatrix} \alpha_i & \beta_i \\ 0 & \delta_i \end{pmatrix}$.

Since one wants to compute $det(Id + \alpha_i^{-1}\beta_i A)$ for suitable bounded operators $A : H^* \mapsto H^+$, the first step is to compute $\alpha_i^{-1}\beta_i$. It will turn out to be an operator of rank one. For each $l \geq 0$ one has

$$q_1^{-1}(((z - a_1)^l, 0)) = (\frac{(z - a_1)^l}{1 - x_1}, 0) = (\zeta_1 - a_1)^l(\frac{x_1^{-l}}{1 - x_1}, 0) = \tag{16}$$

$$(\zeta_1 - a_1)^l[(\frac{x_1}{1 - x_1}, \frac{x_1}{1 - x_1}) + (\frac{x_1^{-l} - x_1}{1 - x_1}, -\frac{x_1}{1 - x_1})]. \tag{17}$$

where one has used the splitting of $\frac{x_1^{-l}}{1-x_1}$ in its singular and nonsingular part at a_1

$$\left(\frac{x_1^{-l}}{1 - x_1}\right)_1^+ = \frac{x_1^{-l} - x_1}{1 - x_1} \text{ and } \left(\frac{x_1^{-l}}{1 - x_1}\right)_1^- = \frac{x_1}{1 - x_1}. \tag{18}$$

By combining this formula with the fact that the map $\alpha_1 : H^* \mapsto H^*$ is just given by multiplication with $(1 - x_1)^{-1}$, one obtains for the action of the operator $\alpha_1^{-1}\beta_1$ on the element e_{-2l}

$$\begin{aligned} \alpha_1^{-1}\beta_1(e_{-2l}) &= \frac{(\zeta_1 - a_1)^l}{r_1^l}(1 - x_1)(\frac{x_1}{1 - x_1}, \frac{x_1}{1 - x_1}) = \\ &= \frac{(\zeta_1 - a_1)^l}{r_1^l}(x_1, x_1) = \frac{(\zeta_1 - a_1)^{l+1}}{r_1^{l+1}}e_1. \end{aligned}$$

It is moreover clear that for all $l \geq 0$, the element $(1 - x_1)^{-1}e_{-2l-1}$ belongs to H^+, so that we have $\alpha_i^{-1}\beta_i(e_{-2l-1}) = 0$. This shows that $\alpha_1^{-1}\beta_1$ maps H^* onto the line through e_1. From the foregoing calculations follows that , if $(f_1(z), f_2(z)) \in H^+$, with $f_1(z) = \sum_{p \geq 0} f_{1p}(z - a_1)^p$ and $f_2(z) = \sum_{p \geq 0} f_{2p}(z - a_2)^p$, then

$$\alpha_1^{-1}\beta_1(f_1(z), f_2(z)) = \frac{(\zeta_1 - a_1)}{r_1}f_1(\zeta_1)e_1.$$

Now one can compute $det(Id + \alpha_1^{-1}\beta_1 A)$ for any bounded operator $A : H^* \mapsto H^+$. Suppose that $A((z - a_1, 0)) = (f_1^1(z), f_1^2(z))$. Since $\alpha_1^{-1}\beta_1 \circ A : H^* \mapsto H^*$ is an operator of rank one with image in $< e_1 >$, there holds $det(Id+\alpha_1^{-1}\beta_1 A) = 1+tr(\alpha_1^{-1}\beta_1 A)$. Thus we have found that

$$det(Id+\alpha_1^{-1}\beta_1 A) = 1+(\zeta_1-a_1)f_1^1(\zeta_1) = \frac{(\zeta_1-a_1)}{r_1}\hat{\psi}_{11}(\zeta_1). \tag{19}$$

Similar computations hold for q_2, so that one gets for $i=1,2$

$$det(Id+\alpha_i^{-1}\beta_i A) = \frac{(\zeta_i-a_i)}{r_i}\hat{\psi}_{ii}(\zeta_i). \tag{20}$$

Next one combines the actions of q_1 and q_2 and considers the element $q = q_1 q_2$. The action of its inverse on the space H is given by the matrix

$$\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} = \begin{pmatrix}\alpha_1 & \beta_1\\ 0 & \delta_1\end{pmatrix}\begin{pmatrix}\alpha_2 & \beta_2\\ 0 & \delta_2\end{pmatrix}.$$

The aim is again to compute $det(Id+\alpha^{-1}\beta A)$. From this matrix equation one sees that $\alpha^{-1}\beta = \alpha_2^{-1}\beta_2 + \alpha_2^{-1}\alpha_1^{-1}\beta_1\delta_2$. One computes directly that

$$\delta_2(0,(z-a_2)^l) = (\zeta_2-a_2)^l(\frac{x_2}{1-x_2}, \frac{x_2^{-l}-x_2}{1-x_2})$$

and this leads to

$$\alpha_2^{-1}\alpha_1^{-1}\beta_1\delta_2(0,(z-a_2)^l) = (\zeta_2-a_2)^l\alpha_2^{-1}\alpha_1^{-1}\beta_1(\frac{x_2}{1-x_2}, \frac{x_2^{-l}-x_2}{1-x_2}) = \tag{21}$$

$$(\zeta_2-a_2)^l\alpha_2^{-1}\left(\frac{-x_2}{1-x_2}|_{z=\zeta_1}\right)\frac{(\zeta_1-a_1)}{r_1}\alpha_2^{-1}(e_1) = \tag{22}$$

$$(\zeta_2-a_2)^l\frac{a_2-\zeta_2}{\zeta_1-\zeta_2}\frac{(\zeta_1-a_1)}{r_1}(1-x_2)e_1 = \tag{23}$$

$$-(\zeta_2-a_2)^{l+1}\frac{a_2-\zeta_2}{\zeta_1-\zeta_2}\left(\frac{\zeta_2-a_1}{r_1(a_2-a_1)}e_1 - \frac{\zeta_2-a_2}{r_2(a_2-a_1)}e_2\right). \tag{24}$$

By using also the relation $\alpha_2^{-1}\beta_2(0,(z-a_2)^l) = \frac{(\zeta_2-a_2)^{l+1}}{r_2}e_2$, one obtains in total

$$\alpha^{-1}\beta(e_{-2l-1}) = \frac{(\zeta_2-a_2)^{l+1}}{r_2^l}\left(\frac{(\zeta_1-a_2)(\zeta_2-a_1)}{r_2(\zeta_1-\zeta_2)(a_2-a_1))}e_2 - \frac{(\zeta_1-a_1)(\zeta_2-a_1)}{r_1(\zeta_1-\zeta_2)(a_2-a_1)}e_1\right). \tag{25}$$

A similar computation yields $\alpha^{-1}\beta(e_{-2l})$

$$\alpha^{-1}\beta(e_{-2l}) = \frac{(\zeta_1-a_1)^{l+1}}{r_1^l}\left(\frac{(\zeta_1-a_2)(\zeta_2-a_1)}{r_1(\zeta_1-\zeta_2)(a_2-a_1))}e_1 - \frac{(\zeta_1-a_2)(\zeta_2-a_2)}{r_2(\zeta_1-\zeta_2)(a_2-a_1)}e_2\right). \tag{26}$$

Again both expressions can be fit into one formula. For $(f_1(z), f_2(z)) \in H^+$, there holds

$$\begin{aligned}(27)\qquad & \alpha^{-1}\beta(f_1(z), f_2(z)) = \\ & f_1(\zeta_1)\left(\frac{(\zeta_1-a_2)(\zeta_2-a_1)(\zeta_1-a_1)}{r_1(\zeta_1-\zeta_2)(a_2-a_1))}e_1 - \frac{(\zeta_1-a_2)(\zeta_1-a_1)(\zeta_2-a_2)}{r_2(\zeta_1-\zeta_2)(a_2-a_1)}e_2\right) + \\ & +f_2(\zeta_2)\left(-\frac{(\zeta_2-a_2)(\zeta_1-a_1)(\zeta_2-a_1)}{r_1(\zeta_1-\zeta_2)(a_2-a_1)}e_1 + \frac{(\zeta_1-a_2)(\zeta_2-a_1)(\zeta_2-a_2)}{r_2(\zeta_1-\zeta_2)(a_2-a_1))}e_2\right) = \\ & \frac{(\zeta_1-a_1)(\zeta_2-a_2)(\zeta_1-a_2)(\zeta_2-a_1)}{(\zeta_1-\zeta_2)(a_2-a_1))}. \\ & .\left(\left(\frac{f_1(\zeta_1)}{r_1(\zeta_2-a_2)} - \frac{f_2(\zeta_2)}{r_1(\zeta_1-a_2)}\right)e_1 - \left(\frac{f_1(\zeta_1)}{r_2(\zeta_2-a_1)} - \frac{f_2(\zeta_2)}{r_2(\zeta_1-a_1)}\right)e_2\right).\end{aligned}$$

In particular this formula implies that $\alpha^{-1}\beta A : H^* \mapsto H^*$ is an operator of rank two with image the span of the vectors e_1 and e_2. Thus at the computation of the determinant one has to take two terms into account, since

$$(28)\qquad det(Id + \alpha^{-1}\beta A) = 1 + tr(\alpha^{-1}\beta A) + tr(\alpha^{-1}\beta A \wedge \alpha^{-1}\beta A).$$

If one has $A(e_i) = (f_{i1}(z), f_{i2}(z))$ for $i = 1, 2$, then formula (27) implies that

$$\begin{aligned}(29)\qquad tr(\alpha^{-1}\beta A) = & \frac{(\zeta_1-a_1)(\zeta_2-a_2)(\zeta_1-a_2)(\zeta_2-a_1)}{(\zeta_1-\zeta_2)(a_2-a_1)}. \\ & \left(\frac{f_{11}(\zeta_1)}{(\zeta_2-a_1)} - \frac{f_{12}(\zeta_2)}{(\zeta_1-a_2)} - \frac{f_{21}(\zeta_1)}{(\zeta_2-a_1)} + \frac{f_{22}(\zeta_2)}{(\zeta_1-a_1)}\right)\end{aligned}$$

and likewise

$$\begin{aligned}(30)\qquad & tr(\alpha^{-1}\beta A \wedge \alpha^{-1}\beta A) = \\ & \frac{(\zeta_1-a_1)(\zeta_2-a_2)(\zeta_1-a_2)(\zeta_2-a_1)}{(\zeta_1-\zeta_2)(a_2-a_1)}.(f_{11}(\zeta_1)f_{22}(\zeta_2) - f_{21}(\zeta_1)f_{12}(\zeta_2)).\end{aligned}$$

By using that $r_i^{-1}\hat{\psi}_{ik}(z) = (z-a_i)^{-1} + f_{ik}(z)$ one can compute $det(Id + \alpha^{-1}\beta A)$ and it is equal to

$$(31)\qquad \frac{(\zeta_1-a_1)(\zeta_2-a_2)(\zeta_1-a_2)(\zeta_2-a_1)}{r_1r_2(\zeta_1-\zeta_)(a_2-a_1))}(\hat{\psi}_{11}(\zeta_1)\hat{\psi}_{22}(\zeta_2) - \hat{\psi}_{21}(\zeta_1)\hat{\psi}_{12}(\zeta_2)).$$

The expression between brackets one recognizes as a determinant and thus one gets the formula

$$(32)\qquad det(Id + \alpha^{-1}\beta A) = \frac{(\zeta_1-a_1)(\zeta_2-a_2)(\zeta_1-a_2)(\zeta_2-a_1)}{r_1r_2(\zeta_1-\zeta_2)(a_2-a_1))}\begin{vmatrix}\hat{\psi}_{11}(\zeta_1) & \hat{\psi}_{12}(\zeta_2)\\ \hat{\psi}_{21}(\zeta_1) & \hat{\psi}_{22}(\zeta_2)\end{vmatrix}$$

This equation contains as a special case the equation (20). By substituting in the equation (32) namely that

$$(33)\qquad \lim_{\zeta_i \mapsto a_i} \frac{(\zeta_i - a_i)}{r_i}\hat{\psi}_{ji}(\zeta_i) = \delta_{ij}.$$

one obtains the equation (20).

Let $w \in \mathcal{P}^*$ have W in $Gr_0(H)$ as its image. Take any $\delta \in \Delta_W$, and let the action of δ^{-1} on $H = H^* \oplus H^+$ decompose as

$$\delta^{-1} = \begin{pmatrix} \delta_{11} & \delta_{12} \\ 0 & \delta_{22} \end{pmatrix}.$$

Consider the embedding $v = \delta^{-1} \circ w \circ \delta_{11}^{-1} \in \mathcal{P}^*$ with image $\delta^{-1}W$. With the help of formula (15), one can identify the left hand side of equation (31) as the quotient of $\tau_v(\gamma.q)$ and $\tau_v(\gamma)$, for each $\gamma \in \Gamma(\delta, W)$. This leads to the final result

Theorem 1. *All the notations being as introduced above, then there holds for all $\gamma \in \Gamma(\delta, W)$ and all ζ_i inside C_i the relation*

$$\frac{\tau_v(\gamma.q)}{\tau_v(\gamma)} = \frac{(\zeta_1 - a_1)(\zeta_2 - a_2)(\zeta_1 - a_2)(\zeta_2 - a_1)}{r_1 r_2(\zeta_1 - \zeta_2)(a_2 - a_1))} \begin{vmatrix} \hat{\psi}^{(\delta)}_{W,11}(\gamma, \zeta_1) & \hat{\psi}^{(\delta)}_{W,12}(\gamma, \zeta_2) \\ \hat{\psi}^{(\delta)}_{W,21}(\gamma, \zeta_1) & \hat{\psi}^{(\delta)}_{W,22}(\gamma, \zeta_2) \end{vmatrix}$$

Remark 4.1. *The left hand side of the relation in theorem 1 can be expressed in the coordinates $\{t_j^{(k)}\}$ on the group $\tilde{\Gamma}$. For, if one writes the element q as*

$$q = exp(log(1 - x_1) + log(1 - x_2)) = exp\left(\sum_{k=1}^{2} \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{\zeta_k - a_k}{z - a_k}\right)^j\right)$$

then the τ-function quotient in theorem 1 becomes

$$\frac{\tau_v((t_j^{(k)} - \frac{1}{j}\left(\frac{\zeta_k - a_k}{z - a_k}\right)^j))}{\tau_v((t_j^{(k)}))},$$

a formula that reminds of those in the setting of the KP-hierarchy and its matrix versions, see [DJKM], [SW] *and* [HP].

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