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A note on set games and cost sharing problems

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Abstract

The concept of set games is shown to pay a role in cost sharing problems and to give an approach to those problems.

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1 Introduction

Set games were introduced by the author [Hoede, 1992] and values for set games were studied by Aarts, Funaki and Hoede [Aarts *et al.*, 1997, 2002].
We refer to the Ph.D.-thesis of Aarts [Aarts, 1994] for an introduction to set games or to the two mentioned papers.

The essential difference with the cooperative games usually studied is that the value of a coalition S of the N players is not some real number but is a set of elements taken from a universe of elements U.
As an example, that we will use later, consider three players, $N = \{1, 2, 3, \}$, and three elements, $U = \{a, b, c\}$. The eight coalitions may now have values

$$\begin{aligned} v(\phi) &= \phi \\ v(1) &= \{a\} \\ v(2) &= \{a, b\} \\ v(3) &= \{a, b, c\} \\ v(12) &= \{a, b\} \\ v(13) &= \{a, b, c\} \\ v(23) &= \{a, b, c\} \\ v(123) &= \{a, b, c\}. \end{aligned}$$

This defines a set game.
The allocation $\phi_i(v)$, $i \in \{1, 2, 3\}$, the solution of the set game, is found by some value ϕ for which several alternatives can be found in [Aarts, 1994].
Let us consider the marginalistic value on page 141, that reads

$$\phi_i(v) = \bigcup_{T \ni i} (v(t) \backslash v(T \backslash \{i\})) \qquad \text{,for all } i \in N.$$

For our example this yields

$$\begin{aligned}\phi_1(v) &= (\{a\}\backslash\phi) \cup (\{a,b\}\backslash\{a,b\}) \cup (\{a,b,c\}\backslash\{a,b,c\}) \cup (\{a,b,c\}\backslash\{a,b,c\}) \\ &= \{a\}.\end{aligned}$$

$$\begin{aligned}\phi_2(v) &= (\{a,b\}\backslash\phi) \cup (\{a,b\}\backslash\{a\}) \cup (\{a,b,c\}\backslash\{a,b,c\}) \cup (\{a,b,c\}\backslash\{a,b,c\}) \\ &= \{a,b\}.\end{aligned}$$

$$\begin{aligned}\phi_3(v) &= (\{a,b,c\}\backslash\phi) \cup (\{a,b,c\}\backslash\{a\}) \cup (\{a,b,c\}\backslash\{a,b\}) \cup (\{a,b,c\}\backslash\{a,b\}) \\ &= \{a,b,c\}.\end{aligned}$$

2 Splitting off the set game aspect

This note was inspired by a lecture of Stef Tijs at the workshop on game theory, held in Twente University in June 2002. In his talk "Cost sharing in low budget situations", he gave various examples of cost sharing problems. The important aspect for this note was that *basic units* were considered. Players may be interested in having a joint project that possesses certain basic units. In the case of a graph, a player may be interested in the presence of a certain subset of its edges. Now these basic units may have specific costs. Let U denote these basic units, then $u \in U$ has some specific cost $c(u)$. Building the structure that includes all basic units in U involves some total cost $C = \sum_{u \in U} c(u)$. The problem of cost sharing is to distribute this total cost C over the N players.

The solution to this problem, that is proposed in this note, is the following. The players indicate which basic units they want to be present. This defines the values of the singleton coalitions. The empty player set gets value ϕ. The coalition S gets value

$$v(S) = \bigcup_{i \in S} v(i).$$

Now a set game has been defined.
Suppose now that an allocation method has been chosen for this set game, then we obtain solutions of the form

$$\phi_i(v) = A_i,$$

where $A_i \subseteq U$.
We can interpret these allocations A_i as elements that the players are interested in and *hence should pay for*.
Now we look at the solution in a different way. We choose a specific basic unit u and determine the set $P(u)$ of players that have element u in their allocation. These players should share the cost $c(u)$! Doing this for all basic units gives a total cost for each player and with that a solution to the cost sharing problem!

3 Example

An interesting application is given by the landing fee problem. An example is the following. Let three planes need three different runways. A small plane 1 needs a strip of length a, say with cost $c(a) = 99$. A medium sized plane 2 needs a strip of length $a + b$, where $c(b) = 70$ say, and some jumbo plane 3 needs a strip of length $a + b + c$, where $c(c) = 30$. The total strip of length $a + b + c$ has cost $C = 199$ and the problem is to distribute this cost over the owners of the three planes. The method presented in this note is to split off the set game aspect! We can use the demands of the planes, players, to determine a set game. The basic units are a, b and c.
The set game expressing the demands of strips is precisely as given in the example set game before. The solution turned out to be, for the chosen value,

$$\begin{aligned} \phi_1 &= \{a\} \\ \phi_2 &= \{a, b\} \\ \phi_3 &= \{a, b, c\}. \end{aligned}$$

Note that no costs were involved sofar. Only the set game aspect, the combinatorial aspect, was considered.
Now, determine the player sets interested in each basic unit and obtain

$$\begin{aligned} P(a) &= \{1, 2, 3\} \\ P(b) &= \{2, 3\} \\ P(c) &= \{3\}. \end{aligned}$$

This means that 1,2 and 3 have to share the cost 99 for a, each having to pay 33. 2 and 3 share the cost 70 for b, each having to pay $35 \cdot 3$ is the only one to pay for c, so 3 has to pay 30. The cost of 199 is shared in the following way:

$$\begin{array}{lclclclcl} c_1 & = & 33 & + & & & & = & 33 \\ c_2 & = & 33 & + & 35 & & & = & 68 \\ c_3 & = & 33 & + & 35 & + & 30 & = & 98 \\ \hline & & & & & & \text{Total } C & = & 199. \end{array}$$

This method gives an answer to the problem how the cost C should be distributed over the players.
Any further considerations, as for example the fact that the players do not have sufficient budget, is brought back to the problem of having an ideal cost sharing vector $\underline{c}_{\text{ideal}}$, given by our method, and deciding what to do if the available cost sharing vector $\underline{c}_{\text{budget}}$ is given.

The combinatorial aspect has been taken care of by the set game. Now other considerations can be taken into account. If profits are known for the players a vector $\underline{c}_{\text{profit}}$ way be taken into account too. The problem has been reduced to the interplay of three vectors: $\underline{c}_{\text{ideal}}, \underline{c}_{\text{budget}}, \underline{c}_{\text{profit}}$. The structure of the project, however, has been taken into account by the set game.

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