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MORE ON SPANNING 2-CONNECTED SUBGRAPHS IN TRUNCATED RECTANGULAR GRID GRAPHS

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Abstract

A grid graph is a finite induced subgraph of the infinite 2-dimensional grid defined by $Z \times Z$ and all edges between pairs of vertices from $Z \times Z$ at Euclidean distance precisely 1. An $m \times n$-rectangular grid graph is induced by all vertices with coordinates 1 to m and 1 to n, respectively. A natural drawing of a (rectangular) grid graph G is obtained by drawing its vertices in $\Re^2$ according to their coordinates. We consider a subclass of the rectangular grid graphs obtained by deleting some vertices from the corners. Apart from the outer face, all (inner) faces of these graphs have area one (bounded by a 4-cycle) in a natural drawing of these graphs. We determine which of these graphs contain a Hamilton cycle, i.e. a cycle containing all vertices, and solve the problem of determining a spanning 2-connected subgraph with as few edges as possible for all these graphs.

Keywords: (rectangular) grid graph, Hamilton cycle, spanning 2-connected subgraph

AMS Subject Classifications: 05C40, 05C85

1 Introduction

The *infinite grid graph* G^∞ is defined by the set of vertices $V = \{(x,y) \mid x \in Z,\, y \in Z\}$ and the set of edges $E \subseteq V \times V$ between all pairs of vertices from V at Euclidean distance precisely 1. For any integers $m \geq 1$ and $n \geq 1$, the *rectangular grid graph* $R(m,n)$ is the (finite) subgraph of G^∞ induced by $V(m,n) = \{(x,y) \mid 1 \leq x \leq m, 1 \leq y \leq n, x \in Z, y \in Z\}$ (and just containing all edges from G^∞ between pairs of vertices from $V(m,n)$). This graph $R(m,n)$ is also known as the *product graph* $P_m \times P_n$ of two disjoint paths P_m and P_n. A *grid graph* is a graph that is isomorphic to a subgraph of $R(m,n)$ induced by a subset of $V(m,n)$ for some integers $m \geq 1$ and $n \geq 1$. It is clear that a grid graph $G = (V,E)$ is *a planar graph*, i.e. it can be drawn in the plane $\Re^2$ in such a way that the edges only intersect at the vertices of the graph. In such a drawing, the regions of $\Re^2 \backslash (V \cup E)$ are called the *faces* of G. Exactly one of the faces is unbounded; this is called the *outer face*; the others are its *inner faces*. The *natural drawing* of a grid graph is just described by drawing its vertices in $\Re^2$ according to their coordinates. A *solid grid graph* is a grid graph all of whose inner faces have area one (are bounded by a cycle on four vertices) in a natural drawing. A grid graph that is not solid contains inner faces (in a natural drawing) that have area larger than one; these faces are called *holes*. A subgraph H of a graph $G = (V,E)$ is called a *spanning subgraph* if $V(H) = V$. A connected graph is called 2-*connected* if it remains connected if at most one vertex is removed. A *Hamilton cycle* in a graph $G = (V,E)$ is a cycle containing every vertex of V, i.e. a spanning 2-connected subgraph in which every vertex has degree 2 (the number of edges is $|V|$).

Itai, Papadimitriou and Szwarcfiter [2] proved that deciding whether a given grid graph has a Hamilton cycle is an NP-complete problem. This implies that the problem of finding a spanning 2-connected subgraph with as few edges as possible is also NP-hard for grid graphs. It has been conjectured that the first problem remains NP-complete when it is restricted to solid grid graphs. However, Umans and Lenhart [5] recently proved that this problem is polynomially solvable. For the second problem the complexity is not known when it is restricted to solid grid graphs. It remains an open problem –what the complexity of both problems is –when we restrict ourselves to grid graphs with a fixed number of holes.

Motivated by the above problems, we study the problem of the existence of a Hamilton cycle and the problem of determining a spanning 2-connected subgraph with as few edges as possible for a number of classes of grid graphs

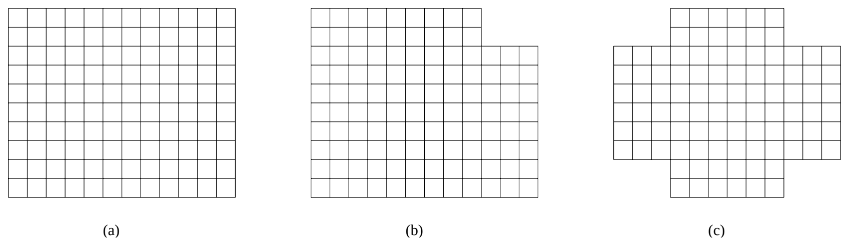

Figure 1: Truncated rectangular grid graphs : (a) $R(11,13)$ (b) $R(11,13)^{-1(2,3)}$ (c) $R(11,13)^{-4(2,3)}$

without holes, obtained from rectangular grid graphs by deleting vertices from the corners in a natural drawing. This is a continuation of the work started in [3]. In a subsequent paper [4] we study the same problem for classes of grid graphs with a few holes. For all graphs of the defined classes we are able to solve the second problem. All solutions are of the same type: first, we use the well-known *Grinberg-condition* to derive a lower bound for the number of edges in a spanning 2-connected subgraph. Secondly, we show by construction that this lower bound is in fact the optimum value.

2 Truncated rectangular grid graphs

We now introduce the classes of grid graphs which we call *truncated rectangular grid graphs*.

For $m \geq 3$, $n \geq 3$, $0 \leq k \leq min\{m-2, n-2\}$ and $0 \leq l \leq min\{m-2, n-2\}$ we define a 1-*corner truncated rectangular grid graph* $R(m,n)^{-1(k,l)}$ as the subgraph obtained from $R(m,n)$ by deleting $k \times l$ vertices from one corner in $V(m,n)$ together with their incident edges in a natural drawing. Notice that $R(m,n)^{-1(0,0)}$ is $R(m,n)$. For illustration, consider $R(11,13)$ and $R(11,13)^{-1(2,3)}$ in Figure 1(a) and 1(b), respectively.

For $m \geq 6$, $n \geq 6$, $1 \leq k \leq min\{\frac{m-4}{2}, \frac{n-4}{2}\}$ and $1 \leq l \leq min\{\frac{m-4}{2}, \frac{n-4}{2}\}$ we define a 4-*corner truncated rectangular grid graph* $R(m,n)^{-4(k,l)}$ as the subgraph obtained from $R(m,n)$ by deleting $k \times l$ vertices from each corner in $V(m,n)$ together with their incident edges in a natural drawing. For illustration, consider $R(11,13)^{-4(2,3)}$ in Figure 1(c).

Our main result characterizes which of the truncated graphs are hamiltonian and shows a spanning 2-connected subgraph with at most three edges more than their number of vertices. We postpone the proofs and constructions (figures) until the next section.

Theorem 1 *Let $R(m,n)^{-1(k,l)}$ and $R(m,n)^{-4(k,l)}$ denote the 1-corner truncated rectangular grid graph and the 4-corner truncated rectangular grid graph as defined above, respectively. Then:*
(i) $R(m,n)^{-1(k,l)}$ contains a spanning 2-connected subgraph with (at most) $|V|+1$ edges and is hamiltonian if and only if both $m \cdot n$ and $k \cdot l$ are even or both $m \cdot n$ and $k \cdot l$ are odd;
(ii)$R(m,n)^{-4(k,l)}$ contains a spanning 2-connected subgraph with (at most) $|V|+3$ edges and is hamiltonian if and only if $m \cdot n$ is even.

3 Proofs and constructions

A useful necessary condition for hamiltonicity is the following result due to Grinberg [1].

Lemma 2 *Suppose a planar graph G has a Hamilton cycle H. Let G be drawn in the plane, and let r_i denote the number of faces inside H bounded by i edges in this planar embedding. Let $r_i^{'}$ be the number of faces outside H bounded by i edges. Then the numbers r_i and $r_i^{'}$ satisfy the following equation.*

$$\sum_i (i-2)(r_i - r_i^{'}) = 0.$$

We use this lemma to show that $R(m,n)^{-1(k,l)}$ contains no Hamilton cycle if $m \cdot n$ and $k \cdot l$ have a different parity, and that $R(m,n)^{-4(k,l)}$ contains no Hamilton cycle if $m \cdot n$ is odd.

Corollary 3 *$R(m,n)^{-1(k,l)}$ contains no Hamilton cycle if $m \cdot n$ and $k \cdot l$ have a different parity.*

Proof. There is exactly one face with $2(m+n-2)$ edges and there are exactly $(m-1)(n-1)-k \cdot l$ faces with four edges in the planar (natural) drawing of the 1-corner truncated rectangular grid graph $R(m,n)^{-1(k,l)}$. Let this graph be hamiltonian. Then by Lemma 2 we have

$$(2(m+n-2)-2)(-1)+(4-2)(r_4 - r_4^{'}) = 0.$$

Hence

$$r_4 - r_4^{'} = m + n - 3. \qquad (1)$$

It is easy to check that the number of faces with four edges is

$$r_4 + r_4^{'} = (m-1)(n-1) - k \cdot l. \qquad (2)$$

From equation (1) and (2) we obtain

$$2r_4 = m \cdot n - k \cdot l - 2. \qquad (3)$$

So, either $m \cdot n$ and $k \cdot l$ are even or $m \cdot n$ and $k \cdot l$ are odd.

□

Corollary 4 *$R(m,n)^{-4(k,l)}$ contains no Hamilton cycle if $m \cdot n$ is odd.*

Proof. There is exactly one face with $2(m+n-2)$ edges and there are exactly $(m-1)(n-1) - 4k \cdot l$ faces with four edges in the planar (natural) drawing of the 4-corner truncated rectangular grid graph $R(m,n)^{-4(k,l)}$. Let this graph be hamiltonian. Then by Lemma 2 and using a similar method as in the proof of Corollary 3, we obtain

$$2r_4 = m \cdot n - 4k \cdot l - 2. \qquad (4)$$

So, $m \cdot n$ is even.

□

We complete the proof of Theorem 1 by showing, through construction, the existence of a Hamilton cycle or a spanning 2-connected subgraph with at most $|V| + 3$ edges, in all cases where $m = 11$, $n = 12$ or 13, $k = 1, 2$ or 3 and $l = 2$ or 3. Meanwhile, for other values of m, n, k and l, it is not difficult to see from the patterns in the figures that follow how to extend the solutions.

A Hamilton cycle for $R(11,12)^{-1(3,2)}$ is shown in Figure 2(a). The pattern in this figure can be used for finding a Hamilton cycle for the 1-corner truncated rectangular grid graph for either (any numbers m and k, and any even numbers n and l) or (any numbers n and l, and any even numbers m and k). In Figure 2(b) we show a Hamilton cycle for $R(11,12)^{-1(2,3)}$. The pattern in Figure 2(b) can be used for finding a Hamilton cycle for the 1-corner truncated rectangular grid graph for either (any number m, any even numbers n and k, and any odd number l) or (any number n, any even numbers m and l, and any odd number k). Meanwhile, in Figure 2(c) we

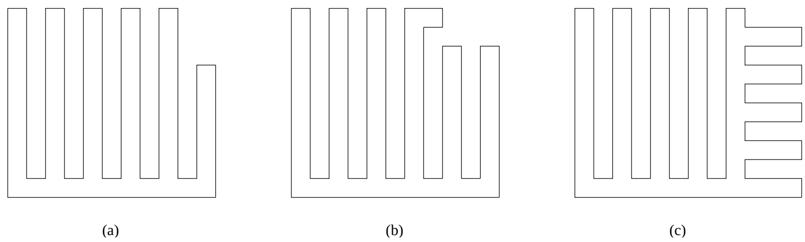

Figure 2: A Hamilton cycle for : (a) $R(11,12)^{-1(3,2)}$ (b) $R(11,12)^{-1(2,3)}$ (c) $R(11,13)^{-1(1,3)}$

Figure 3: A spanning 2-connected subgraph with $|V|+1$ edges for : (a) $R(11,12)^{-1(1,3)}$ (b) $R(11,13)^{-1(3,2)}$

show a Hamilton cycle for $R(11,13)^{-1(1,3)}$. The pattern in Figure 2(c) can be used for finding a Hamilton cycle for the 1-corner truncated rectangular grid graph for any odd numbers m, n, k and l.

A spanning 2-connected subgraph with $|V|+1$ edges for $R(11,12)^{-1(1,3)}$ is shown in Figure 3(a). The pattern in this figure can be used for finding such a spanning subgraph with $|V|+1$ edges for the 1-corner truncated rectangular grid graph for any even number m or n and for any odd numbers k and l. In Figure 3(b) we show a spanning 2-connected subgraph with $|V|+1$ edges for $R(11,13)^{-1(3,2)}$. The pattern in Figure 3(b) can be used for finding a spanning 2-connected subgraph with $|V|+1$ edges for the 1-corner truncated rectangular grid graph for any odd numbers m and n and for any even number k or l.

Hamilton cycles for $R(11,12)^{-4(3,2)}$ and $R(11,12)^{-4(2,3)}$ are shown in Figure 4. The pattern in Figure 4(a) can be used for finding a Hamilton cycle for the 4-corner truncated rectangular grid graph for either (any numbers m and k, and any even numbers n and l) or (any numbers n and l, and any even numbers m and k). Meanwhile, the pattern in Figure 4(b) can

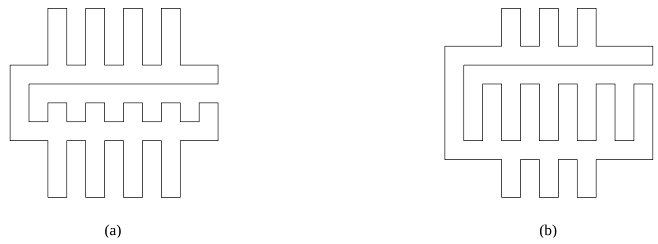

Figure 4: A Hamilton cycle for : (a) $R(11,12)^{-4(3,2)}$ (b) $R(11,12)^{-4(2,3)}$

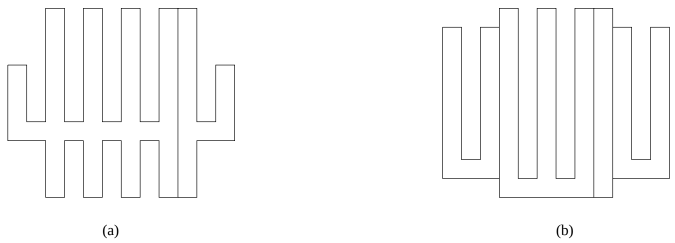

Figure 5: A spanning 2-connected subgraph for : (a) $R(11,13)^{-4(3,2)}$ with $|V|+1$ edges (b) $R(11,13)^{-4(1,3)}$ with $|V|+3$ edges

be used for finding a Hamilton cycle for the 4-corner truncated rectangular grid graph for either (any numbers m and k, any even number n, and any odd number l) or (any numbers n and l, any even number m, and any odd number k).

In Figure 5(a), we show a spanning 2-connected subgraph with $|V|+1$ edges for $R(11,13)^{-4(3,2)}$. The pattern in this figure can be used for finding a spanning 2-connected subgraph with $|V|+1$ edges for the 4-corner truncated rectangular grid graph for any odd numbers m and n, and for any even number k or l. Finally, in Figure 5(b) we show a spanning 2-connected subgraph with $|V|+3$ edges for $R(11,13)^{-4(1,3)}$. The pattern in this last figure can be used for finding a spanning 2-connected subgraph with $|V|+3$ edges for the 4-corner truncated rectangular grid graph for any odd numbers m, n, k and l. We are not absolutely sure that this is the optimum value for the minimum number of edges in a spanning 2-connected subgraph.

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