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Memorandum No. 1592

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November 2001

ISSN 0169-2690

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October 25, 2001

Abstract. We consider sequences of orthogonal polynomials and pursue the question of how (partial) knowledge of the orthogonalizing measure for the *associated polynomials* can lead to information about the orthogonalizing measure for the original polynomials. In particular, we relate the supports of the two measures, and their moments. As an application we analyse the relation between two decay rates connected with a birth-death process.

Keywords and phrases: orthogonal polynomials, associated polynomials, numerator polynomials, birth-death process, decay rate, rate of convergence, first-entrance time

2000 Mathematics Subject Classification: Primary 42C05, Secondary 60J80

1 Introduction

Our point of departure will be the familiar three-terms recurrence relation for orthogonal polynomials. That is, we consider a sequence of monic polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfying

$$\begin{aligned} &P_n(x) = (x - c_n)P_{n-1}(x) - d_n P_{n-2}(x), \quad n > 1, \\ &P_0(x) = 1, \quad P_1(x) = x - c_1, \end{aligned} \tag{1.1}$$

where c_n is real and $d_n > 0$. Then, by Favard's theorem, there exists a positive Borel measure ψ on the real axis (of total mass 1, say) with respect to which the polynomials $\{P_n(x)\}$ are orthogonal, that is,

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)\psi(dx) = k_n\delta_{nm}, \quad n, m \geq 0,$$

with $k_n > 0$. In what follows we shall assume that the Hamburger moment problem (Hmp) associated with the polynomials $\{P_n(x)\}$ is determined, so that ψ is the *unique* orthogonalizing measure for the polynomials $\{P_n(x)\}$.

Given the sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=2}^{\infty}$, one defines the corresponding sequence of *associated polynomials* $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ by a recurrence of the type (1.1) in which c_n and d_n are replaced by c_{n+1} and d_{n+1}, respectively. That is, the associated polynomials satisfy the recurrence relation

$$\begin{aligned} &\tilde{P}_n(x) = (x - c_{n+1})\tilde{P}_{n-1}(x) - d_{n+1}\tilde{P}_{n-2}(x), \quad n > 1, \\ &\tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = x - c_2. \end{aligned} \tag{1.2}$$

Associated polynomials are sometimes called *numerator polynomials* (in Chihara [5] for example) because they are the numerators of the convergents of certain continued fractions (the denominators of which are the polynomials $P_n(x)$).

Clearly, also the associated polynomials $\{\tilde{P}_n(x)\}$ are orthogonal with respect to a Borel measure (of total mass 1) on the real axis. We will denote this measure by $\tilde{\psi}$ and refer to it as the *associated measure*. Our assumption that the Hmp for $\{P_n(x)\}$ is determined implies that $\tilde{\psi}$ is unique, since Shohat and Sherman [20] have shown that the Hmp's corresponding to $\{P_n(x)\}$ and $\{\tilde{P}_n(x)\}$ are determined simultaneously.

Associated polynomials appear already in Stieltjes' seminal work [22] and have been studied by many authors since then (see, for example, Sherman [19], Chihara [5], Belmehdi [1], Van Assche [23], Peherstorfer [16], Ronveaux and Van Assche [17], and the references cited there). The theme of this paper is related to that of [17], namely, we shall be interested in the problem of obtaining information about the measure ψ from (partial) knowledge of the measure $\tilde{\psi}$. Our analysis is motivated by applications in the setting of birth-death processes.

The paper is organized as follows. After collecting some known, but relevant properties in Section 2, we will discuss various aspects of the relation between ψ and $\tilde{\psi}$ in Section 3. Specifically, we will relate the supports, and the moments of the two measures. In Section 4 we will introduce birth-death processes and show how the results of Section 3 can be used to analyse the relation between two decay rates connected with a birth-death process. An example concludes the paper in Section 5.

2 Preliminaries

It is well known that $P_n(x)$ has n real and simple zeros $x_{n1} < x_{n2} < \ldots < x_{nn}$, and that the zeros of $P_n(x)$ and $P_{n+1}(x)$ mutually separate each other, that is,

$$x_{n+1,i} < x_{ni} < x_{n+1,i+1}, \quad i = 1, 2, \ldots, n, \; n \geq 1. \tag{2.1}$$

Evidently, the real and simple zeros $\tilde{x}_{n1} < \tilde{x}_{n2} < \ldots < \tilde{x}_{nn}$ of the associated polynomials $\{\tilde{P}_n(x)\}$ satisfy a separation property analogous to (2.1). Moreover, the separation result [5, Theorem I.7.2] tells us

$$x_{ni} < \tilde{x}_{ni} < x_{n+1,i+1}, \quad i = 1, 2, \ldots, n, \; n \geq 1. \tag{2.2}$$

It follows (see [5, Theorem III.4.2]) that the limits

$$\xi_i \equiv \lim_{n\to\infty} x_{ni} \;\text{ and }\; \tilde{\xi}_i \equiv \lim_{n\to\infty} \tilde{x}_{ni}, \quad i \geq 1,$$

and the limits

$$\sigma \equiv \lim_{i\to\infty} \xi_i \;\text{ and }\; \tilde{\sigma} \equiv \lim_{i\to\infty} \tilde{\xi}_i,$$

exist and satisfy

$$-\infty \leq \xi_i \leq \tilde{\xi}_i \leq \xi_{i+1} \leq \tilde{\sigma} = \sigma \leq \infty, \quad i \geq 1. \tag{2.3}$$

We also recall (see [5, Theorem II.4.6]) that for $i \geq 0$

$$\xi_i = \xi_{i+1} \;\Rightarrow\; \xi_i = \sigma \quad \text{and} \quad \tilde{\xi}_i = \tilde{\xi}_{i+1} \;\Rightarrow\; \tilde{\xi}_i = \sigma, \tag{2.4}$$

where we use the convention $\xi_0 = \tilde{\xi}_0 \equiv -\infty$.

In what follows we shall assume throughout that $\xi_1 > -\infty$ (so that, by (2.3), also $\tilde{\xi}_1 > -\infty$). Then the quantities ξ_i are closely related to $\text{supp}(\psi)$, the support of the orthogonalizing measure ψ. Indeed, letting $\Xi \equiv \{\xi_1, \xi_2, \ldots\}$, we have

$$\sigma = \infty \;\Rightarrow\; \text{supp}(\psi) = \Xi, \tag{2.5}$$

while

$$\sigma < \infty \;\Rightarrow\; \text{supp}(\psi) \cap (-\infty, \sigma] = \bar{\Xi}, \tag{2.6}$$

a bar denoting closure (see [5, Theorem II.4.5]). Moreover, σ is the smallest limit point of $\text{supp}(\psi)$. Clearly, results analogous to (2.5) and (2.6) are valid for the associated polynomials.

The measures ψ and $\tilde{\psi}$ can be studied conveniently through their Stieltjes transforms

$$F(z) \equiv \int_{-\infty}^{\infty} \frac{\psi(dx)}{z - x}, \quad z \in \mathbb{C} \backslash \text{supp}(\psi),$$

and

$$\tilde{F}(z) \equiv \int_{-\infty}^{\infty} \frac{\tilde{\psi}(dx)}{z - x}, \quad z \in \mathbb{C} \backslash \text{supp}(\tilde{\psi}),$$

respectively, which are analytic in their domains of definition. Indeed, a classical result in the theory of continued fractions (Shohat and Sherman [20], Sherman [19], see also Berg [2]) tells us that the two transforms are related as

$$F(z) = \frac{1}{z - c_1 - d_2 \tilde{F}(z)}. \tag{2.7}$$

So if $\tilde{\psi}$ (and hence $\tilde{F}(z)$) is completely known, we can use (2.7) to find $F(z)$, and then apply the Stieltjes inversion formula (see Widder [24, Corollary VIII.7a])

$$\psi([0,x)) + \frac{1}{2}\psi(\{x\}) = -\frac{1}{\pi}\lim_{y\to 0+}\int_{-\epsilon}^{x} \Im\{F(\xi+iy)\}d\xi, \quad x \geq 0, \tag{2.8}$$

where $\epsilon > 0$, to obtain ψ.

The relation (2.7) provides the basis for the analysis in the next section.

3 Relations between ψ and $\tilde{\psi}$

3.1 The support

Maintaining the assumption $\xi_1 > -\infty$, we start off by noting that $F(z)$ can be represented as

$$F(z) = \begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\psi(\{\xi_i\})}{z-\xi_i} & \text{if } \sigma = \infty \\ \displaystyle\sum_{\{i:\ \xi_i<\sigma\}} \frac{\psi(\{\xi_i\})}{z-\xi_i} + \int_{\sigma}^{\infty} \frac{\psi(dx)}{z-x} & \text{if } \sigma < \infty, \end{cases} \tag{3.1}$$

in view of (2.5) and (2.6). This observation enables us to refine the separation result (2.3) in the following theorem, where we use the notation

$$F(y-) \equiv \lim_{\xi\to y-} F(\xi), \quad y \in \mathbb{R},$$

if the limit exists.

Theorem 3.1 *The following statements hold true for $i \geq 1$.*

(i) If $\xi_i < \xi_{i+1} < \xi_{i+2}$ *then* $\xi_i < \tilde{\xi}_i < \xi_{i+1}$.

(ii) If $\xi_i < \xi_{i+1} = \sigma$, *then* $\xi_i < \tilde{\xi}_i < \sigma$ *if* $F(\sigma-) < 0$ *and* $\tilde{\xi}_i = \sigma$ *otherwise.*

Proof. Assuming $\xi_i < \xi_{i+1}$, it is clear that $F(\xi)$ is a strictly decreasing function of ξ in the interval (ξ_i, ξ_{i+1}). Moreover, (3.1) shows that ξ_i is an isolated singularity of $F(z)$, while $F(\xi)$ decreases in the interval (ξ_i, ξ_{i+1}) from $+\infty$ to $F(\xi_{i+1}-)$.

If ξ_{i+1} is an isolated singularity then (3.1) shows that $F(\xi_{i+1}-) = -\infty$. Since, by (2.7), $\tilde{F}(z)$ has singularities at the zeros of $F(z)$, it follows that $\tilde{F}(z)$

has a singularity in the interval (ξ_i, ξ_{i+1}). But, in view of the analogue of (3.1) for $\tilde{F}(z)$, the only candidate for this singularity is $\tilde{\xi}_i$, which proves statement (i).

If $\xi_{i+1} = \sigma$, there will be a zero of $F(\xi)$ in the interval (ξ_i, σ) if $F(\sigma-) < 0$, in which case we must have $\xi_i < \tilde{\xi}_i < \sigma$. If $F(\sigma-) \geq 0$, however, there is no zero of $F(\xi)$, and hence no singularity of $\tilde{F}(z)$, in the interval (ξ_i, σ). Moreover, $\xi_i = \tilde{\xi}_i < \sigma$ is impossible, since, by (2.7), $F(z)$ and $\tilde{F}(z)$ cannot have common poles. It follows that we must have $\xi_i < \tilde{\xi}_i = \sigma$, establishing statement (ii). □

Corollary 3.2 *For all $i \geq 1$ we have $\xi_i \leq \tilde{\xi}_i$ with equality subsisting if and only if $\xi_i = \sigma$.*

Proof. The preceding theorem shows that $\xi_i < \tilde{\xi}_i$ if $\xi_i < \xi_{i+1}$. The result follows in view of (2.3) and (2.4). □

Remark. This corollary may also be obtained from Shohat and Tamarkin [21, Corollary 2.6 and Theorem 2.17], or Chihara [4, Theorem 3], and the results (2.5) and (2.6).

Since our main goal is to obtain information about ψ from knowledge of $\tilde{\psi}$ we also state a converse to the preceding theorem (recall that $\tilde{\xi}_0 \equiv -\infty$).

Theorem 3.3 *The following statements hold true for $i \geq 1$.*

(i) If $\tilde{\xi}_i < \tilde{\xi}_{i+1}$ then $\tilde{\xi}_{i-1} < \xi_i < \tilde{\xi}_i$.

(ii) If $\tilde{\xi}_{i-1} < \tilde{\xi}_i = \sigma$, then $\tilde{\xi}_{i-1} < \xi_i < \sigma$ if $c_1 + d_2\tilde{F}(\sigma-) < \sigma$ and $\xi_i = \sigma$ otherwise.

Proof. From (2.7) we note that $F(z)$ has singularities at the zeros of $z - c_1 - d_2\tilde{F}(z)$, while the latter function is easily seen to be strictly increasing in the interval $(\tilde{\xi}_{i-1}, \tilde{\xi}_i)$. Thus, with $z - c_1 - d_2\tilde{F}(z)$ taking the role of $F(z)$, the proof is similar to the proof of the previous theorem. □

3.2 Moments

We will now turn our attention to the moments

$$m_n \equiv \int_0^\infty x^n \psi(dx) \quad \text{and} \quad \tilde{m}_n \equiv \int_0^\infty x^n \tilde{\psi}(dx), \quad n \geq 0,$$

and their relations. As an aside we note that moments of negative orders (and their relevance for birth-death processes) have been studied in [9]).

We first observe that the system of equations

$$\int_{-\infty}^\infty P_0(x)\psi(dx) = 1, \qquad \int_{-\infty}^\infty P_n(x)\psi(dx) = 0, \quad n \geq 0,$$

can be solved recursively for the moments m_n, $n = 0, 1, \ldots$. In this way we find, for example,

$$\begin{aligned} m_0 &= 1 \\ m_1 &= c_1 \\ m_2 &= c_1^2 + d_2 \\ m_3 &= c_1^3 + (2c_1 + c_2)d_2. \end{aligned}$$

The moments $\tilde{m}_n$ of the associated measure $\tilde{\psi}$ can be found similarly. But we can also express $\tilde{m}_n$ in the moments of ψ, namely,

$$d_2\tilde{m}_n = -\beta_{n+2}, \quad n \geq 0, \tag{3.2}$$

where

$$\beta_n \equiv (-1)^{\frac{1}{2}n(n+1)} \begin{vmatrix} 0 & 0 & \ldots & m_0 & m_1 \\ 0 & 0 & \ldots & m_1 & m_2 \\ \vdots & \vdots & & \vdots & \vdots \\ m_0 & m_1 & \ldots & m_{n-2} & m_{n-1} \\ m_1 & m_2 & \ldots & m_{n-1} & m_n \end{vmatrix}, \quad n \geq 2. \tag{3.3}$$

This result was given (with an error) by Sherman [19], and recently corrected by Berg [2]. Since our main theme is how to obtain information about ψ from $\tilde{\psi}$, we also give the converse result.

Theorem 3.4 *The moments m_n can be expressed in terms of the moments $\tilde{m}_n$ as*

$$m_n = (-1)^{\frac{1}{2}n(n+1)} \begin{vmatrix} 0 & 0 & \dots & \alpha_0 & \alpha_1 \\ 0 & 0 & \dots & \alpha_1 & \alpha_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_0 & \alpha_1 & \dots & \alpha_{n-2} & \alpha_{n-1} \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} & \alpha_n \end{vmatrix}, \quad n \geq 2, \tag{3.4}$$

where

$$\alpha_0 \equiv 1, \ \alpha_1 \equiv -c_1 \quad \textit{and} \quad \alpha_n \equiv -d_2 \tilde{m}_{n-2}, \quad n \geq 2. \tag{3.5}$$

The proof is analogous to the proof of (3.2) (see [2]).

4 Birth-death processes

4.1 Introduction

We consider a birth-death process $\mathcal{X} \equiv \{X(t),\ t \geq 0\}$ taking values in $S \equiv \{0, 1, \dots\}$ with birth rates $\{\lambda_n,\ n \in S\}$ and death rates $\{\mu_n,\ n \in S\}$, all strictly positive except $\mu_0 \geq 0$. When $\mu_0 > 0$ the process may evanesce by escaping from S, via state 0, to an ignored absorbing state -1.

Karlin and McGregor [11] have shown that the transition probabilities

$$p_{ij}(t) \equiv \Pr\{X(t) = j \mid X(0) = i\}, \quad t \geq 0,\ i, j \in S,$$

can be represented as

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) \psi(dx), \quad t \geq 0,\ i, j \in S. \tag{4.1}$$

Here $\{\pi_n\}$ are constants given by

$$\pi_0 \equiv 0 \ \text{ and } \ \pi_n \equiv \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}, \quad n > 0,$$

$\{Q_n(x)\}$ is a sequence of polynomials satisfying the recurrence relation

$$\begin{aligned} &\lambda_n Q_{n+1}(x) = (\lambda_n + \mu_n - x) Q_n(x) - \mu_n Q_{n-1}(x), \quad n > 1, \\ &\lambda_0 Q_1(x) = \lambda_0 + \mu_0 - x, \quad Q_0(x) = 1, \end{aligned} \tag{4.2}$$

and ψ – the *spectral measure* of $\mathcal{X}$ – is a measure of total mass 1 on the interval $[0,\infty)$ with respect to which the polynomials $\{Q_n(x)\}$ are orthogonal.

The polynomials $\{Q_n(x)\}$ are related to the polynomials $\{P_n(x)\}$ of the previous sections. For by letting

$$c_{n+1} = \lambda_n + \mu_n \quad \text{and} \quad d_{n+2} = \lambda_n \mu_{n+1}, \qquad n \geq 0, \tag{4.3}$$

we readily see that

$$P_n(x) = (-1)^n \lambda_0 \lambda_1 \dots \lambda_{n-1} Q_n(x), \qquad n > 0.$$

It is known (see Karlin and McGregor [12] and Chihara [6]) that the Hmp associated with the polynomials $\{Q_n(x)\}$ is determined if and only if

$$\sum_{n=0}^{\infty} \pi_{n+1} \left(\sum_{k=0}^{n} (\lambda_k \pi_k)^{-1} \right)^2 = \infty. \tag{4.4}$$

So, assuming (4.4) to prevail, the spectral measure ψ of the birth-death process $\mathcal{X}$ can be identified with the measure ψ of the previous sections, and is uniquely determined by the birth and death rates. It is easy to see that the zeros of $Q_n(x)$ are positive, so that $\xi_1 \geq 0$. This confirms, in view of (2.5) and (2.6), that supp(ψ) is a subset of the interval $[0,\infty)$.

The polynomials $\{\tilde{Q}_n(x)\}$ satisfying the recurrence

$$\begin{aligned} &\lambda_{n+1}\tilde{Q}_{n+1}(x) = (\lambda_{n+1} + \mu_{n+1} - x)\tilde{Q}_n(x) - \mu_{n+1}\tilde{Q}_{n-1}(x), \quad n > 1, \\ &\lambda_1 \tilde{Q}_1(x) = \lambda_1 + \mu_1 - x, \quad \tilde{Q}_0(x) = 1, \end{aligned} \tag{4.5}$$

are related to the associated polynomials $\{\tilde{P}_n(x)\}$ of (1.2) through (4.3) and

$$\tilde{P}_n(x) = (-1)^n \lambda_1 \lambda_2 \dots \lambda_n \tilde{Q}_n(x), \qquad n > 0.$$

As before, $\tilde{\psi}$ will denote the orthogonalizing measure for the associated polynomials, so that $\tilde{\psi}$ is also the spectral measure of the birth-death process $\tilde{\mathcal{X}}$ with birth rates $\{\tilde{\lambda}_n \equiv \lambda_{n+1},\ n \in S\}$ and death rates $\{\tilde{\mu}_n \equiv \mu_{n+1},\ n \in S\}$.

4.2 Decay rates

It is well known that the transition probabilities $p_{ij}(t)$ have limits

$$p_j \equiv \lim_{t\to\infty} p_{ij}(t) = \pi_j \psi(\{0\}) = \begin{cases} \dfrac{\pi_j}{\sum_n \pi_n} & \text{if } \mu_0 = 0 \text{ and } \displaystyle\sum_{n=0}^{\infty} \pi_n < \infty \\ 0 & \text{otherwise,} \end{cases}$$

which are independent of the initial state i. If $p_j > 0$, that is, if $\mu_0 = 0$ and $\sum_n \pi_n < \infty$, the process is called *ergodic*. We are interested in the exponential rate of convergence (or *decay rate*) of $p_{ij}(t)$ to its limit p_j, that is, in the quantities

$$\alpha_{ij} \equiv - \lim_{t\to\infty} \frac{1}{t} \log |p_{ij}(t) - p_j|, \quad i, j \in S. \tag{4.6}$$

From Callaert [3] (see also [7]) we know that these limits exist, and that

$$\alpha \equiv \alpha_{00} \leq \alpha_{ij}, \quad i, j \in S, \tag{4.7}$$

with equality whenever $\mu_0 > 0$, and inequality subsisting for at most one value of i or j when $\mu_0 = 0$. The quantity α is therefore indicative of the speed of convergence of the process $\mathcal{X}$. For our purposes it is important to note that α can be expressed in terms of the quantities ξ_i as

$$\alpha = \begin{cases} \xi_2 & \text{if } \xi_2 > \xi_1 = 0 \\ \xi_1 & \text{otherwise} \end{cases} \tag{4.8}$$

(see [7, Theorem 3.1 and Lemma 3.2]). Observe that the process must be ergodic if $\xi_2 > \xi_1 = 0$. On the other hand, if $\mathcal{X}$ is ergodic, we must have either $\xi_2 > \xi_1 = 0$ or $\xi_2 = \xi_1 = 0$, since $\psi(\{0\}) > 0$. Hence, (4.8) may also be formulated as in the next theorem.

Theorem 4.1 *The rate of convergence $\alpha \equiv \alpha_{00}$ of the transition probability $p_{00}(t)$ to its limit p_0 satisfies*

$$\alpha = \begin{cases} \xi_2 & \textit{if } \mathcal{X} \textit{ is ergodic} \\ \xi_1 & \textit{otherwise.} \end{cases}$$

If $\mu_0 > 0$ (so that $\alpha = \xi_1$) one might also be interested in the rates of convergence of the probabilities $p_{i,-1}(t)$, $i \in S$, to their limits. With T_{-1} denoting the (possibly defective) first-entrance time to into state -1, we have

$$p_{i,-1}(\infty) \equiv \lim_{t\to\infty} p_{i,-1}(t) = \Pr\{T_{-1} < \infty \mid X(0) = i\}, \quad i \in S,$$

and

$$p_{i,-1}(\infty) - p_{i,-1}(t) = \Pr\{t < T_{-1} < \infty \mid X(0) = i\}, \quad i \in S,$$

so the rate of convergence of $p_{i,-1}(t)$ is given by

$$\alpha_{i,-1} \equiv -\lim_{t\to\infty} \frac{1}{t} \log \Pr\{t < T_{-1} < \infty \mid X(0) = i\}, \quad i \in S. \tag{4.9}$$

Using [14, Equation (3.7)] it is not difficult to show that this limit exists and

$$\alpha_{i,-1} = \alpha = \xi_1, \quad i \in S. \tag{4.10}$$

This result follows also from Theorem 3.2.2 of Jacka and Roberts [10], which gives, in a more general setting, sufficient conditions for equality of the decay rates $\alpha_{i,-1}$ and α.

Next assuming $\mu_0 = 0$, the question arises what the relation between α and the right-hand side of (4.9) will be if T_{-1} is replaced by T_0, the (possibly defective) first-entrance time into state 0. To answer this question we let

$$F_{i0}(t) \equiv \Pr\{T_0 \le t \mid X(0) = i\} \quad \text{and} \quad F_{i0}(\infty) \equiv \lim_{t\to\infty} F_{i0}(t),$$

so that

$$F_{i0}(\infty) - F_{i0}(t) = \Pr\{t < T_0 < \infty \mid X(0) = i\}, \quad i > 0,$$

and note that by the representation formula (4.1) and a simple probabilistic argument (cf. Karlin and McGregor [12, p. 385])

$$\begin{aligned} \Pr\{t < T_0 < \infty \mid X(0) = i\} &= \mu_1 \int_t^\infty \tilde{p}_{i-1,0}(\tau) d\tau \\ &= \mu_1 \int_t^\infty \int_0^\infty e^{-x\tau} \tilde{Q}_{i-1}(x) \tilde{\psi}(dx) d\tau, \quad i > 0. \end{aligned}$$

By Fubini's theorem we may interchange the integrals and obtain

$$\Pr\{t < T_0 < \infty \mid X(0) = i\} = \mu_1 \int_0^\infty \frac{e^{-xt}}{x} \tilde{Q}_{i-1}(x) \tilde{\psi}(dx), \quad i > 0. \tag{4.11}$$

(Alternatively, we could have obtained (4.11) directly by a suitable interpretation of [14, Equation (3.7)].) Denoting the rate of convergence of the first-entrance time distribution function $F_{i0}(t)$ to its limit by γ_i, that is,

$$\gamma_i \equiv -\lim_{t\to\infty} \frac{1}{t} \log \Pr\{t < T_0 < \infty \mid X(0) = i\}, \quad i > 0,$$

and letting $\gamma \equiv \gamma_1$, the following theorem readily emerges.

Theorem 4.2 *For all $i > 0$ the rate of convergence γ_i of the first-entrance time distribution function $F_{i0}(t)$ to its limit $F_{i0}(\infty)$ satisfies $\gamma_i = \gamma = \tilde{\xi}_1$.*

The parity of $\alpha - \gamma$ can now easily be obtained from the Theorems 4.1, 4.2 and 3.1, and Corollary 3.2 as follows.

Corollary 4.3 *(i) If X is ergodic, then $\gamma \leq \alpha$ with equality subsisting if and only if $\sigma = 0$ or*

$$0 < \xi_2 = \sigma \text{ and } \lim_{y \to \sigma -} \int_0^\infty \frac{d\psi(x)}{y - x} \geq 0.$$

(ii) If X is not ergodic, then $\gamma \geq \alpha$ with equality subsisting if and only if $\xi_1 = \sigma$.

It is interesting to relate this corollary to recent work of Martínez and Ycart [15]. Rather than α – the decay rate of $p_{00}(t)$ – these authors study (in a more general setting)

$$\alpha_{tv} \equiv \sup\{x \geq 0 : \sum_{j \in S} |p_{0j}(t) - p_j| = \mathcal{O}(e^{-xt}) \text{ as } t \to \infty\},$$

that is, the decay rate as t goes to infinity of the total variation distance between the distribution at time t and the limit distribution, when the initial state is 0. There are indications (cf. Zeifman [25, Theorem 1] and van Doorn [8, Theorem 3.2 (i)]) that for birth-death processes we will have $\alpha_{tv} = \alpha$, at least under some additional conditions on the birth and death rates. Under these conditions then the above corollary would be a refinement (in the setting at hand) of a result obtained by Martínez and Ycart [15], to the effect that $\gamma \leq \alpha_{tv}$ if $\mathcal{X}$ is ergodic.

As an example Martínez and Ycart consider the process of the number of customers in an $M/M/\infty$ system, which is a birth-death process with birth rates $\lambda_n = \lambda$ and $\mu_n = n\mu$. They show that in this case we indeed have $\alpha_{tv} = \alpha$, and subsequently prove that $\gamma < \alpha$. The latter result follows also directly from the above corollary since the spectral measure the $M/M/\infty$ queue is known to be discrete (see Karlin and McGregor [13, p. 92]), so that $\sigma = \infty$.

5 Example

We consider a birth-death process $\mathcal{X}$ with unspecified values of λ_0 and μ_0, but constant rates $\lambda_n = \lambda$ and $\mu_n = \mu$ for $n \geq 1$. The coefficients in the recurrence relation (4.5) for the associated polynomials $\{\tilde{Q}_n(x)\}$ are therefore constant, and it follows that these polynomials can be represented as

$$\tilde{Q}_n(x) = \left(\frac{\mu}{\lambda}\right)^{n/2} U_n\left(\frac{\lambda+\mu-x}{2\sqrt{\lambda\mu}}\right), \quad n \geq 0, \tag{5.1}$$

where $\{U_n(x)\}$ are the Chebysev polynomials of the second kind. Moreover, the associated measure $\tilde{\psi}$ satisfies

$$\tilde{\psi}(dx) = \frac{1}{2\pi\lambda\mu}\sqrt{4\lambda\mu - (\lambda+\mu-x)^2}dx$$

in the interval $|\lambda+\mu-x| < 2\sqrt{\lambda\mu}$, and is zero outside this interval. It follows in particular that

$$\tilde{\xi}_1 = \tilde{\sigma} = \lambda + \mu - 2\sqrt{\lambda\mu}. \tag{5.2}$$

Finally, the Stieltjes transform of $\tilde{\psi}$ is given by

$$\tilde{F}(z) = \frac{1}{2\lambda\mu}\left(z - \lambda - \mu + \sqrt{(z-\lambda-\mu)^2 - 4\lambda\mu}\right), \tag{5.3}$$

for values of $z < \lambda + \mu - 2\sqrt{\lambda\mu}$. (See, for example, Karlin and McGregor [13, Equations (5.6) - (5.8)] for the above results.)

We now wish to establish for which values of λ_0 and μ_0 we have $\xi_1 < \sigma\ (= \tilde{\sigma})$, that is, for which values of λ_0 and μ_0 the spectral measure ψ of $\mathcal{X}$ has an isolated point mass to the left of σ, the smallest limit point of the support of ψ. To this end we first note that

$$\tilde{F}(\sigma-) = \tilde{F}(\tilde{\sigma}-) = -\frac{1}{\sqrt{\lambda\mu}}. \tag{5.4}$$

Subsequently applying Theorem 3.3 we obtain after some algebra

$$\xi_1 < \sigma \iff \lambda_0 + \mu_0 - \lambda_0\sqrt{\mu/\lambda} < \left(\sqrt{\lambda} - \sqrt{\mu}\right)^2, \tag{5.5}$$

in view of (5.3) and (5.2). If $\mu_0 = 0$ we can reformulate this result as

$$\xi_1 < \sigma \iff \lambda < \mu \text{ or } \left(\lambda > \mu \text{ and } \lambda_0 < \lambda - \sqrt{\lambda\mu}\right). \tag{5.6}$$

Obviously, the process is ergodic if and only if $\mu_0 = 0$ and $\lambda < \mu$, in which case we have $0 = \xi_1 < \xi_2 = \sigma$, by the preceding result. The Theorems 4.1 and 4.2 tell us that $\gamma = \tilde{\xi}_1$ and $\alpha = \xi_2$ in this case, while Corollary 4.3 states that $\gamma \leq \alpha$ with equality subsisting unless $F(\sigma-) < 0$. But, by (2.7), (5.4) and a little algebra, it is easily seen that under the prevailing conditions $F(\sigma-) > 0$, so we always have $\gamma = \alpha$ if $\mu_0 = 0$ and $\lambda < \mu$.

If $\mu_0 > 0$ or $\lambda \geq \mu$, then, by the Theorems 4.1 and 4.2 and Corollary 4.3 again, we have $\alpha = \xi_1 \leq \gamma = \tilde{\xi}_1$, with inequality subsisting if and only if $\xi_1 < \sigma$. The necessary and sufficient condition for this to occur is given in (5.5).

Remarks. (*i*) The theory available for perturbed Chebysev polynomials may be employed to calculate the measure ψ explicitly (see Sansigre and Valent [18] and the references cited there). Alternatively, the polynomials $\{P_n(x)\}$ may be regarded as the *anti-associated polynomials* of the polynomials $\{\tilde{P_n}\}(x)\}$, a point of view which also enables one to calculate the measure ψ explicitly (see Ronveaux and Van Assche [17, Section 6]).
(*ii*) The result (5.5) may also be derived by using chain-sequence techniques (see, in particular, Chihara [5, Exercise III.5.1 and Theorem IV.2.1]).

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