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The Ramsey Numbers of Large Star-like Trees versus Large Odd Wheels

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Abstract. For two given graphs G and H, the *Ramsey number* $R(G,H)$ is the smallest positive integer N such that for every graph F of order N the following holds: either F contains G as a subgraph or the complement of F contains H as a subgraph. In this paper, we shall study the Ramsey number $R(T_n, W_m)$ for a star-like tree T_n with n vertices and a wheel W_m with $m+1$ vertices and m odd. We show that the Ramsey number $R(S_n, W_m) = 3n-2$ for $n \geq 2m-4, m \geq 5$ and m odd, where S_n denotes the star on n vertices. We conjecture that the Ramsey number is the same for general trees on n vertices, and support this conjecture by proving it for a number of star-like trees.

Keywords: Ramsey number, star, wheel, tree.

AMS Subject Classifications: 05C55, 05D10.

1 Introduction

Throughout the paper, all graphs are finite and simple. Let G be such a graph. We write $V(G)$ or V for the vertex set of G and $E(G)$ or E for the edge set of G. The graph $\overline{G}$ is the *complement* of the graph G, i.e., the graph obtained from the complete graph $K_{|V(G)|}$ on $|V(G)|$ vertices by deleting the edges of G.

The graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E \cap (V' \times V')$. For any nonempty subset $S \subset V$, the *induced subgraph* by S is the maximal subgraph of G with vertex set S; it is denoted by $G[S]$.

If $e = \{u, v\} \in E$ (in short, $e = uv$), then u is called *adjacent to* v, and u and v are called *neighbors*. For each $x \in V$ and $B \subset V$, define $N_B(x) = \{y \in B : xy \in E\}$. The *degree* $\delta(x)$ of a vertex x is $|N_V(x)|$.

A *cycle* C_n of length $n \geq 3$ is a connected graph on n vertices in which every vertex has degree two. A *star* S_n is a connected graph with one vertex of degree $n-1$, called the *center*, and $n-1$ vertices of degree one. A *wheel* W_n is a graph on $n+1$ vertices obtained from a C_n by adding one vertex x, called the *hub* of

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the wheel, and making x adjacent to all vertices of the C_n, called the *rim* of the wheel.

Given two graphs G and H, the *Ramsey number* $R(G,H)$ is defined as the smallest natural number N such that every graph F on N vertices satisfies the following condition: F contains G as a subgraph or $\overline{F}$ contains H as a subgraph.

Chvátal and Harary [4] studied Ramsey numbers for graphs and established the lower bound: $R(G,H) \geq (\chi(G)-1)(|V(H)|-1)+1$, where $\chi(G)$ is the chromatic number of G. In their paper they also showed that $R(T_n, K_m) = (n-1)(m-1)+1$, where T_n is an arbitrary tree on n vertices and K_m is a complete graph on m vertices.

Several results have been obtained for wheels. For instance, Hendry [8] showed $R(W_3, W_4) = 17$ and $R(W_4, W_4) = 15$ [7]. Faudree and McKay [5] established $R(W_3, W_5) = 19$, $R(W_4, W_5) = 17$ and $R(W_5, W_5) = 17$.

For a combination of cycles and wheels, Burr and Erdős [2] showed that $R(C_3, W_m) = 2m+1$ for each $m \geq 5$. Then Radziszowski and Xia [11] gave a simple and unified method to establish the Ramsey number $R(G, C_3)$, where G is either a path, a cycle or a wheel.

Recently, in [14], it was shown that the Ramsey number $R(S_n, W_4) = 2n-1$ if $n \geq 3$ and n is odd, $R(S_n, W_4) = 2n+1$ if $n \geq 4$ and n is even, and $R(S_n, W_5) = 3n-2$ for each $n \geq 3$.

2 Main Results

In the sequel we will study the Ramsey number $R(T_n, W_m)$, where T_n is a tree on n vertices, and m is odd. We first determine $R(S_n, W_m)$ in the next section, and discuss other trees later.

2.1 Large Stars versus Large Odd Wheels

The aim of this section is to determine the Ramsey number for a combination of a star S_n and a wheel W_m. We show that $R(S_n, W_m) = 3n-2$ for $n \geq 2m-4$, $m \geq 5$ and m odd.

For the lower bound, consider the graph $F = 3K_{n-1}$ for $n \geq 2m-4$. Then F has $3n-3$ vertices and it has no star S_n, whereas its complement has no W_m with $m \geq 5$ and m odd. Thus $R(S_n, W_m) \geq 3n-2$. Note that the lower bound is valid for general trees on n vertices.

For the upper bound we will present a proof by induction, starting with the next result for W_5 obtained in [14].

Theorem 1. *For all* $n \geq 3$, $R(S_n, W_5) = 3n-2$.

Theorem 2. *For all* $n \geq 2m-4$, $m \geq 5$ *and* m *odd,* $R(S_n, W_m) = 3n-2$.

Proof. We shall use induction on $m \geq 5$ for all odd m. The start of the induction is implied by Theorem 1: For $m = 5$ we have $R(S_n, W_5) = 3n-2$ if $n \geq 6$. Now assume the theorem holds for $5 < m < k$, k odd, namely, $R(S_n, W_m) = 3n-2$ if $n \geq 2m-4$ and m is odd. We shall show that $R(S_n, W_k) = 3n-2$ if $n \geq 2k-4$. Let F be a graph on $3n-2$ vertices with $n \geq 2k-4$, and suppose F contains no star S_n. We shall show that its complement must contains W_k. To the contrary, assume $\overline{F}$ contains no W_k. By the induction hypothesis, $\overline{F}$ contains a W_{k-2}. Let a_0 denote the hub and $A = \{a_1, a_2, ..., a_{k-2}\}$ the vertex set of the rim of such a

W_{k-2}, in a cyclic ordering. In the remainder of the proof we use $N_S(v)$ to denote the neighbors of v in $S \subset V(F)$ in the graph F. Let $X = V(F)\backslash(A \cup \{a_0\})$ and $Y = X\backslash N_X(a_0)$. See Figure 1, in which edges in F are indicated by lines, and edges in $\overline{F}$ by broken lines; dots between two vertices indicate that there might be more vertices in the same set.

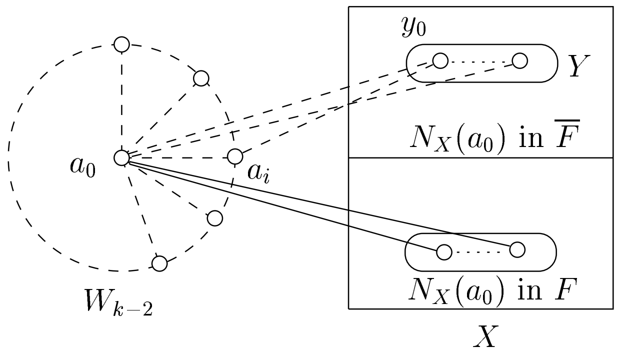

Fig. 1. The set up of the proof of Theorem 2.

Since F contains no S_n, $|Y| \geq |X|-(n-2) = 3n-2-(k-1)-(n-2) = 2n-k+1$. For each $a \in A$ there exists a vertex $y \in Y$ such that $ay \notin E(F)$; otherwise a has at least $2n-k+1 \geq \frac{3}{2}n - 1 \geq n-1$ neighbors, since $k \leq \frac{n+4}{2}$, yielding an S_n. Now, let $y_0 \in Y$ be a nonneighbor in F of $a_i \in A$ for a fixed $i \in \{1, 2, ..., k-2\}$. Define $Y_1 = \{y \in Y : y \text{ is adjacent to } y_0 \text{ in } F\}$ and $Y_2 = \{y \in Y : y \text{ is not adjacent to } y_0 \text{ in } F\}$. Then, $Y_1 \cup Y_2 = Y\backslash\{y_0\}$. Since F contains no S_n, $|Y_1| \leq n-2$ and hence $|Y_2| \geq (2n-k+1)-(n-2)-1 = n-k+2$. Since $\overline{F}$ contains no W_k, we obtain the following fact.

Fact 1. $N_{Y_2}(a_j) = Y_2$ *for* $j = i-1$ *and* $j = i+1$.

Otherwise, replacing for instance $a_i a_{i+1}$ in $\overline{F}$ by $a_i y_0 y^* a_{i+1}$ in $\overline{F}$ for some $y^* \in Y_2\backslash N_{Y_2}(a_{i+1})$, we obtain a W_k in $\overline{F}$.

Since F contains no S_n, we can use Fact 1 to obtain the next fact.

Fact 2. $|N_{Y_1}(a_j)| \leq k-4$ *for* $j = i-1$ *and* $j = i+1$.

Otherwise, by Fact 1, a_j has at least $n-k+2-(k-3) = n-1$ neighbors in F, yielding an S_n.

Now distinguish the following two cases.

Case 1. a_{i-2} *is not adjacent to* y *for some* $y \in Y_2$.

See Figure 2.

Suppose a_{i-2} is not adjacent to $y_1 \in Y_2$. Since $\overline{F}$ contains no W_k, then $y_1 y \in E(F)$ for each $y \in (Y_1 \cup Y_2)\backslash(N_{Y_1}(a_{i-1}) \cup \{y_1\})$; otherwise, we can either replace

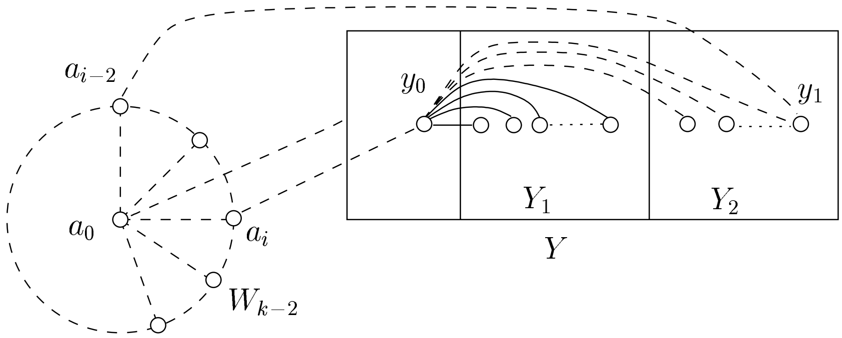

Fig. 2. Case 1 of the proof of Theorem 2.

$a_{i-2}a_{i-1}$ by $a_{i-2}y_1y_2a_{i-1}$ in $\overline{F}$ for some suitable $y_2 \in Y_1$, or replace $a_{i-2}a_{i-1}a_i$ by $a_{i-2}y_1y_2y_0a_i$ in $\overline{F}$ for some suitable $y_2 \in Y_2$, to obtain a W_k in $\overline{F}$. We conclude that $|N_Y(y_1)| \geq |Y_2| - 1 + |Y_1| - |N_{Y_1}(a_{i-1})| = |Y| - 2 - |N_{Y_1}(a_{i-1})| \geq (2n-k-1)-(k-4) = 2n-2k+3 \geq n+1$, yielding an S_n in F, a contradiction.

Case 2. *a_{i-2} is adjacent to all $y \in Y_2$.*

See Figure 3.

Since F contains no S_n, a_{i-2} has at most $(n-2) - |Y_2|$ neighbors in Y_1 in the graph F, hence at least $|Y_1|-(n-2)+|Y_2| = |Y|-n+1 \geq n-k+2$ nonneighbors in Y_1. Using Fact 2, at least $(n-k+2)-(k-4) = n-2k+6 \geq 2$ vertices of Y_1 are nonneighbors in F of both a_{i-1} and a_{i-2}. By symmetry, if we are not in Case 1 for a_{i+2} instead of a_{i-2}, we may assume that at least two vertices of Y_1 are nonneighbors in F of both a_{i+1} and a_{i+2}. It is obvious that we can now find two suitable vertices $y_1, y_2 \in Y_1$, and replace $a_{i-1}a_{i-2}$ in $\overline{F}$ by $a_{i-1}y_1a_{i-2}$ and $a_{i+1}a_{i+2}$ by $a_{i+1}y_2a_{i+2}$, to obtain a W_k in $\overline{F}$, our final contradiction. ■

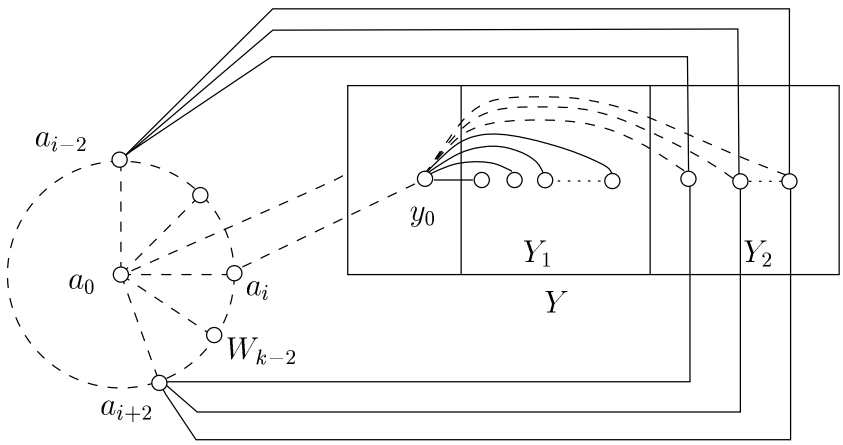

Fig. 3. Case 2 of the proof of Theorem 2.

To conclude this section, we present three conjectures. First of all, we conjecture that for $n \geq m$ we have $R(S_n, W_m) = 3n - 2$ if $m \geq 5$ and m is odd. For even n, we believe the Ramsey number $R(S_n, W_m)$ should be $2n - 1$ if $n \geq 3$ and n is odd, and $2n + 1$ if $n \geq 4$ and n is even. Starting with the results in [14] for W_4 we can use the proof technique from this section to prove an upper bound of $3n-2$ for $n \geq 2m-4$, but to establish a sharper bound one will need a different approach. Finally, we conjecture that the result from this section holds for general trees instead of stars. We support this conjecture by proving it for star-like trees in the next section.

2.2 Star-like Trees versus Odd Wheels

With a *star-like tree* we mean a subdivided star (which is not a path), i.e., a tree with exactly one vertex of degree exceeding two. A star-like tree in which only one of the edges of the star has been subdivided, is sometimes called a *comet* in literature; it is usually denote by $Y_{n,l}$, and consists of a path P_n and l additional vertices of degree one, all adjacent to the same end vertex of the P_n. For this reason, and because of the series of results we will present below, we denote by $Y_{n,l_1,l_2,\ldots,l_k}$ the star-like tree consisting of a P_n, and k additional mutually disjoint paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ all attached by one edge from one of their end vertices to the same end vertex of the P_n. If all l_i are equal to 1, we use the shorter notation $Y_{n,\underline{k}}$ to denote $Y_{n,l_1,l_2,\ldots,l_k}$.

Starting with our result on stars from the previous section, we will show in a number of steps that the same result holds for star-like trees instead of stars. This is done first for $Y_{n,1,1}$, then for $Y_{n,\underline{k}}$, and so on. For convenience, we have split the main result in a number of (weaker) results.

Lemma 1. *$R(Y_{n,1,1}, W_m) = 3(n + 2) - 2$ for $n \geq m \geq 5$ and m odd.*

Proof. We use induction on m. For $m = 5$, we can apply the result in [1] that $R(T_n, W_5) = 3n - 2$ for $n \geq 3$ and $T_n \neq S_n$. Now assume the lemma holds for $5 \leq m < k$, with m and k odd. We will show that $R(Y_{n,1,1}, W_k) = 3(n + 2) - 2$ for $n \geq k$. Consider a graph F on $3(n + 2) - 2$ vertices for $n \geq k$ and suppose F contains no $Y_{n,1,1}$. We will show that $\overline{F}$ contains W_k. To the contrary assume this is not the case. Then it is not difficult to show that F contains a vertex x such that $|N_F(x)| \geq 3$; otherwise the high degrees in $\overline{F}$ easily yield a W_k; we leave the details to the reader. Now consider a $Y_{t,1,1}$ in F which is maximal with respect to t. It is clear that $2 \leq t \leq n - 1$. Denote with y_1, y_2, y_3 the end vertices of $Y_{t,1,1}$, where y_3 is the end vertex of the P_t. By the maximality of t, y_3 is not adjacent in F to a vertex in $X = V(F) \setminus V(Y_{t,1,1})$. We have $|X| = 3(n+2) - 2 - (t+2) \geq 2(n+2) - 1$. By a result in [12], [6], and [10], that $R(C_n, C_k) = 2n - 1$ for $3 \leq k \leq n$, k odd, and $(n, k) \neq (3, 3)$, we obtain that $F[X]$ contains a cycle C_{n+2} with vertex set U, say. Denote $Z = X \setminus U$. Then $|Z| \geq n + 1$. We obtain the following facts.

Fact 1. *No vertex of U is adjacent in F to a vertex in $V(Y_{t,1,1}) \cup Z$.*

Otherwise, we clearly get a contradiction with the choice of t.

Fact 2. *$F[U]$ is a complete graph.*

Otherwise, assume there exist nonadjacent vertices $u, v \in U$. Using Fact 1, it is not difficult to construct in $\overline{F}$ a C_k starting at u, alternating between U and Z, and ending, via v, at u. This implies $\overline{F}$ contains a W_k with y_3 as a hub, a contradiction.

By Fact 2, $F[U]$ contains $Y_{n,1,1}$, our final contradiction. ■

Lemma 2. *$R(Y_{n,\underline{k}}, W_m) = 3(n+k) - 2$ for $n \geq 2m-4$, $k \geq 2$, $m \geq 5$, m odd.*

Proof. Let $m \geq 5$ be a fixed odd integer. Let F be a graph on $3(n+k) - 2$ vertices and suppose $\overline{F}$ contains no W_m. We will show that F contains $Y_{n,\underline{k}}$. By the result of the previous section, F contains a star S_{n+k}. If F is disconnected, then one easily shows the existence of $Y_{n,\underline{k}}$ in F, using that $\overline{F}$ is the join of two subgraphs and does not contain W_m. We omit the details. Now assume F is connected and contains no $Y_{n,\underline{k}}$. Consider a $Y_{t,\underline{k}}$ in F chosen in such a way that t is as large as possible. Such a subgraph exists because of the presence of the star S_{n+k} in F. We get that $2 \leq t \leq n-1$, and denote by x_1 and x_t the end vertices of the P_t, and by $y_1, \ldots, y_k$ the other end vertices of $Y_{t,\underline{k}}$, assuming x_1 is the vertex with degree exceeding 2. Now x_t is clearly not adjacent to a vertex in $X = V(F) \setminus V(Y_{t,\underline{k}})$. As in the proof of Lemma 1, we obtain that $F[X]$ contains a C_n. Let $A = V(C_n)$ and $B = X \setminus A$. By similar arguments we find a cycle $C_{\lfloor \frac{n}{2} \rfloor + 1}$ in $F[B]$. We let $D = V(C_{\lfloor \frac{n}{2} \rfloor + 1})$ and $Z = B \setminus D$, and obtain the following facts.

Fact 1. *No vertex of $Y_{t,\underline{k}}$ is adjacent in F to a vertex in A.*

Otherwise, we clearly get a contradiction with the choice of t.

Fact 2. *Each vertex of A is adjacent to at most $k-1$ vertices in B.*

Otherwise, we easily obtain $Y_{n,\underline{k}}$ in F, a contradiction.

Now we distinguish two cases.

Case 1. *No vertex of A is adjacent to a vertex of D.*

By similar arguments as in the proof of Lemma 1, using that $\overline{F}$ contains no W_m, we conclude that both $F[A]$ and $F[D]$ are complete graphs. The connectivity of F now implies there exists a vertex $z \in Z$ that is adjacent to both a vertex of $Y_{t,\underline{k}}$ and a vertex of A. This obviously implies F contains $Y_{n,\underline{k}}$, a contradiction.

Case 2. *Some vertex of A is adjacent to a vertex in D.*

Since F contains no $Y_{n,\underline{k}}$, no vertex of $A \cup D$ is adjacent to a vertex of $Y_{t,\underline{k}}$. Since F is connected, there exists a vertex $z \in Z$ that is adjacent to both a vertex of $Y_{t,\underline{k}}$ and a vertex of $A \cup D$. This again implies F contains $Y_{n,\underline{k}}$, our final contradiction. ■

Below we use $Y_{n,r,\underline{k}}$ to denote $Y_{n,r,1,1,\ldots,1}$, where the number of 1s is k.

Lemma 3. *$R(Y_{n,r,\underline{k}}, W_m) = 3(n+r+k) - 2$ for $n \geq 2m-4, n \geq r, m \geq 5$, m odd, and $k + r \geq \lfloor \frac{m}{2} \rfloor + 1$.*

Proof. We use induction on $k+r$. According to Lemma 2, the lemma is true for $k=1$ and $r=1$. Assume the lemma holds for k', r' with $\lfloor \frac{m}{2} \rfloor + 1 \leq k'+r' < k+r$. We shall show that the lemma holds for $k+r$. Let the graph F have $3(n+r+k)-2$ vertices and suppose $\overline{F}$ contains no W_m. We shall show that F must contain $Y_{n,r,\underline{k}}$. If F is disconnected, then it is easy to see that F contains $Y_{n,r,\underline{k}}$, as in the proof of Lemma 2. Now suppose F is connected. By the induction assumption, F contains $Y_{n,r-1,\underline{k}}$, say with x_1 as the vertex with degree exceeding 2; denote by x_n the other end vertex of the path P_n in $Y_{n,r-1,\underline{k}}$ (see Figure 4). Denote by v_{r-1} the end vertex of $Y_{n,r-1,\underline{k}}$ corresponding to the P_{r-1}, and by $y_1, y_2, \ldots, y_l$ the other end vertices of $Y_{n,r-1,\underline{k}}$.

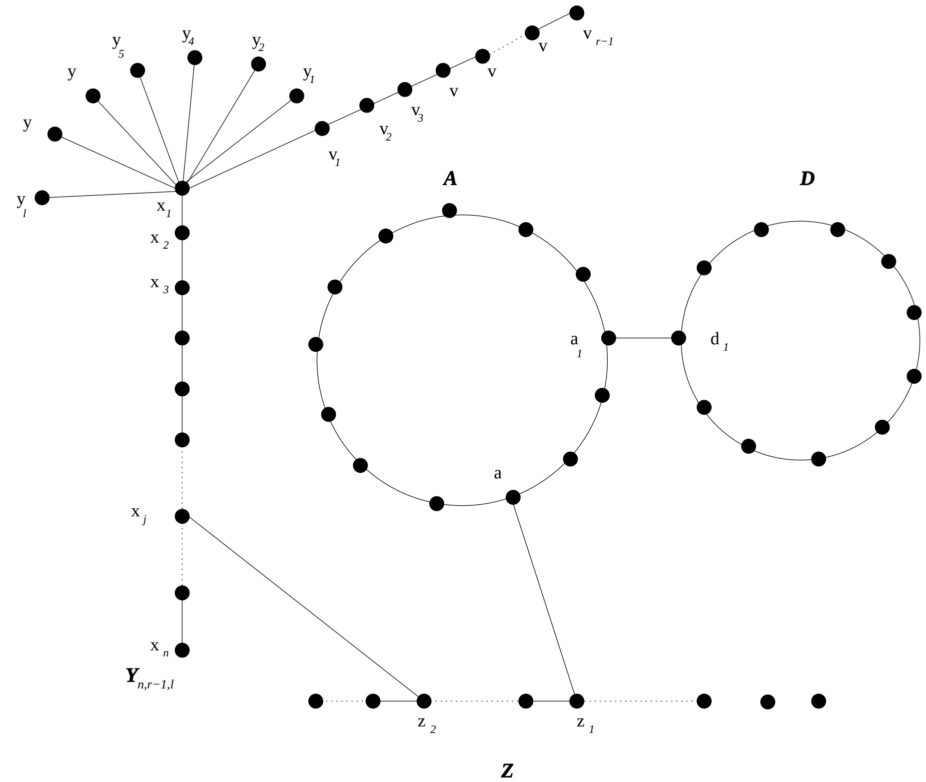

Fig. 4. The proof of Theorem 3

Let $X = V(F) \backslash V(Y_{n,r-1,\underline{k}})$. To the contrary, suppose F contains no $Y_{n,r,\underline{k}}$. Then v_{r-1} is not adjacent to a vertex in X. As in the previous proof, this implies the subgraph $F[X]$ contains two cycles C_n and $C_{\lfloor \frac{n}{2} \rfloor + r+k}$. Let $A = V(C_n), D = V(C_{\lfloor \frac{n}{2} \rfloor + r+k})$ and $Z = X \backslash (A \cup D)$. If C_n is not connected to $C_{\lfloor \frac{n}{2} \rfloor + r+l}$, then, as before, $F[A]$ and $F[D]$ are both complete graphs, and F clearly contains $Y_{n,r,\underline{k}}$. Next suppose C_n is connected to $C_{\lfloor \frac{n}{2} \rfloor + r+k}$, namely, $a_1 d_1 \in E(F)$ for $a_1 \in A, d_1 \in D$. Then we obtain the following facts. We omit the proofs because they are similar to previous proofs.

Fact 1. *No $w \in A \cup D$ is adjacent to a vertex in $\{x_1, y_1, \ldots y_l, v_1, \ldots v_{r-1}\}$.*

Fact 2. *There exists vertices* z_1 *and* z_2 *in a path* $P_l \subseteq F[Z]$ *such that* z_1 *is adjacent to a vertex in* A *and* z_2 *to a vertex in* $x_j \in \{x_2, x_3, ..., x_n\}$.

Fact 3. z_1 *is not adjacent to a vertex in* $\{x_1, y_1, ...y_l, v_1, ...v_{r-1}\}$ *and* $|N_D(z_1)| \leq k-1$.

Fact 4. *The complement of the subgraph of* F *induced by* $\{x_1, y_1, ...y_l, v_1, ...v_{r-1}\} \cup D \backslash N_D(z_1)$ *contains* C_m.

Thus, we obtain a W_m with z_1 as a hub, a contradiction. This completes the proof. ■

We are now prepared to present the main result of this section.

Theorem 3. $R(Y_{n,l_1,l_2,...,l_k}, W_m) = 3(\sum_{i=1}^{k} l_i) - 2$ *for* $n \geq 2m-4, n \geq l_i$ *for each* $i = 1, 2, ..., k$, $m \geq 5$ *odd, and* $\lfloor \frac{m}{2} \rfloor + 1 \leq \sum_{i=1}^{k} l_i$.

Proof. The proof of this theorem is similar to that of Lemma 3, using induction on $\sum_{i=1}^{k} l_i$. We omit the details. ■

References

1. E. T. Baskoro, Surahmat, S.M. Nababan, and M. Miller, On Ramsey Graph Numbers for all Trees versus W_4 or W_5, *Preprint* (2001).
2. S. A. Burr and P. Erdős, Generalization of a Ramsey-Theoretic Result of Chvátal, *Journal of Graph Theory* **7** (1983) 39-51.
3. Chen Guantao, A Result on C_4-Star Ramsey Numbers, *Discrete Mathematics* **163** (1997) 243-246.
4. V. Chvátal and F. Harary, Generalized Ramsey Theory for Graphs, III. Small off Diagonal Numbers, *Pas. Journal Math.* **41** (1972) 335-345.
5. R. J. Faudree and B. D. McKay, A Conjecture of Erdős and the Ramsey Number $R(W_6)$, *Journal of Combinatorial Mathematics and Combinatorial Computing* **13** (1993) 23-31.
6. R. J. Faudree and R. H. Schelp, All Ramsey Numbers for Cycles in Graphs, *Discrete Mathematics* **8** (1974) 313-329.
7. H. Harborth and I. Mengersen, All Ramsey Number for Five Vertices and Seven or Eight edges, *Discrete Mathematics* **73** (1988/1989) 91-98.
8. G.R.T. Henry, The Ramsey Numbers $R(K_2+\overline{K}_3, K_4)$ and $R(K_1+C_4, K_4)$, *Utilitas Mathemathica* **41** (1992) 181-203.
9. C. J. Jayawardene and C. C. Rousseau, Ramsey Number $R(C_5, G)$ for all Graphs G of Order Six, *Preprint.*
10. G. Karolyi and V. Rosta, Generalized and Geometric Ramsey Numbers for Cycles, *Preprint* (1998).
11. S. P. Radziszowski and J. Xia, Paths, Cycles and Wheels without Antitriangles, *Australasian Journal of Combinatorics* **9** (1994) 221-232.
12. V. Rosta, On a Ramsey Type Problem of J.A. Bondy and P. Erdős, *Journal of Combinatorial Theory (B)* **15** (1973) 94-120.
13. Y J Sheng, H Y Ru and Z K Min, $R(C_n, K_4) = 3(n-1)+1, (n \geq 4)$, *Preprint.*
14. Surahmat and Edy Tri Baskoro, On The Ramsey Number of a Path or a Star versus W_4 or W_5, *Proceedings of the 12-th Australasian Workshop on Combinatorial Algorithms*, Bandung, Indonesia, July 14-17 (2001) 165-170.

15. H. L. Zhou, The Ramsey Number of an Odd Cycles With respect to a Wheel (in chinese), *Journal of Mathematics, Shuxu Zazhi (Wuhan)* **15** (1995), 119-120.