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Deformation and recursion for the $N = 2$ $\alpha = 1$ supersymmetric KdV hierarchy

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Deformation and Recursion for the $N = 2$ $\alpha = 1$ Supersymmetric KdV hierarchy

Alexander S. Sorin and Paul H. M. Kersten

Abstract. A detailed description is given for the construction of the deformation of the $N = 2$ supersymmetric $\alpha = 1$ KdV-equation, leading to the recursion operator for symmetries and the zero-th Hamiltonian structure; the solution to a longstanding problem.

1. Introduction

The $N = 2$ supersymmetric $\alpha = 1$ KdV-equation was originally introduced in [**1**] as a Hamiltonian equation with the $N = 2$ superconformal algebra as a second Hamiltonian structure, and its integrability was conjectured there due to the existence of a few additional nontrivial bosonic Hamiltonians. Then its Lax–pair representation has indeed been constructed in [**2**], and it allowed an algoritmic reconstruction of the whole tower of highest *commutative* bosonic flows and their Hamiltonians belonging to the $N = 2$ supersymmetric $\alpha = 1$ KdV-hierarchy.

Actually, besides the $N = 2$ $\alpha = 1$ KdV-equation there are another two inequivalent $N = 2$ supersymmetric Hamiltonian equations with the $N = 2$ superconformal algebra as a second Hamiltonian structure (the $N = 2$ $\alpha = -2$ and $\alpha = 4$ KdV-equations [**3, 1**]), but the $N = 2$ $\alpha = 1$ KdV-equation is rather exceptional [**4**]. Despite knowledge of its Lax–pair description, there remains a lot of longstanding, unsolved problems which resolution would be quite important for a deeper understanding and more detailed description of the $N = 2$ $\alpha = 1$ KdV hierarchy. Thus, since the time when the $N = 2$ $\alpha = 1$ KdV-equation was proposed, much efforts were made to construct a tower of its *noncommutative* bosonic and fermionic, local and nonlocal symmetries and Hamiltonians, bi-Hamiltonian structure as well as recursion operator (see, e.g. discussions in [**5, 6**] and references therein). Though these rather complicated problems, solved for the case of the $N = 2$ $\alpha = -2$ and $\alpha = 4$ KdV-hierarchies, still wait their complete resolution for the $N = 2$ $\alpha = 1$ KdV- hierarchy, a considerable progress towards their solution arose quite recently. Thus, the *puzzle* [**5, 6**], related to the *"nonexistence"* of higher fermionic flows of the $N = 2$ $\alpha = 1$ KdV-hierarchy, was partly resolved in [**7, 8**] by explicit constructing a few bosonic and fermionic *nonlocal* and *nonpolynomial* flows and Hamiltonains, then their $N = 2$ superfield structure and origin

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were uncovered in [**9**]. A new property, crucial for the existence of these flows and Hamiltonians, making them distinguished compared to all flows and Hamiltonians of other supersymmetric hierarchies constructed before, is their *nonpolynomiality.*
A new approach to a recursion operator treating it as a form–valued vector field which satisfies a generalized symmetry equation related to a given equation was developed in [**10, 11**]. Using this approach the recursion operator of the bosonic limit of the N=2 $\alpha = 1$ KdV-hierarchy was derived in [**12**], and its structure, underlining relevance of these Hamiltonians in the bosonic limit, gives a hint towards its supersymmetric generalization.
The organisation of this paper is as follows.
First the general notions from the mathematical theory of symmetries, nonlocalities, deformations and form-valued vector fields are exposed to some detail in Section 2 and its subsections for the classical KdV-equation. For full details the reader is referred to e.g. [**10, 7**].
In Section 3 we expose all results obtained for the $N = 2$ $\alpha = 1$ supersymmetric KdV-equation in great detail.
In Section 4 we present conclusions, while in Section 5, arranged as an Appendix, results of the Poisson bracket structure are given.

2. Nonlocal Setting for Differential Equations, the KdV Equation

2.1. Nonlocalities. As standard example, to illustrate the notions we are going to discuss for the $N = 2$ $\alpha = 1$ supersymmetric KdV-equation in Section 3, we take the KdV-equation

$$u_t = uu_x + u_{xxx}. \tag{1}$$

For a short theoretical introduction we refer to [**12**], while for more detailed expositions we refer to [**10, 7, 11**].

We consider $Y \subset J^\infty(x, t; u)$ the infinite prolongation of (1), c.f.[**13, 14**], where coordinates in the infinite jet bundle $J^\infty(x, t; u)$ are given by $(x, t, u, u_x, u_t, \cdots)$ and Y is formally described as the submanifold of $J^\infty(x, t; u)$ defined by

$$\begin{aligned} u_t &= uu_x + u_{xxx}, \\ u_{xt} &= uu_{xx} + u_x^2 + u_{xxxx}, \\ &\vdots \quad . \end{aligned} \tag{2}$$

As internal coordinates in Y one chooses $(x, t, u, u_x, u_{xx}, \cdots)$ while $u_t, u_{xt}, \cdots$ are obtained from (2).

The Cartan distribution on Y is given by the total partial derivative vector fields

$$\begin{aligned} \widetilde{D}_x &= \partial_x + \sum_{n\geq 0} u_{n+1}\partial_{u_n}, \\ \widetilde{D}_t &= \partial_t + \sum_{n\geq 0} u_{nt}\partial_{u_n} \end{aligned} \tag{3}$$

where $u = u_0,\ u_1 = u_x,\ u_2 = u_{xx}, \cdots \ ;\ u_{1t} = u_{xt};\ u_{2t} = u_{xxt} \cdots$.

Classically the notion of a *generalized* or *higher symmetry* Y of a differential equation $\{F = 0\}$ is defined as a vertical vector field V

$$V = \mathcal{D}_f = f\partial_u + \widetilde{D}_x(f)\partial_{u_1} + \widetilde{D}_x^2(f)\partial_{u_2} + \dots \tag{4}$$

where $f \in C^\infty(Y)$ are such that

$$\ell_F(f) = 0. \tag{5}$$

Here, ℓ_F is the *universal linearisation operator* [**15, 13**], also denoted as Fréchet derivative of F, which reads in the case of the KdV-equation (2)

$$\widetilde{D}_t(f) - u\widetilde{D}_x(f) - u_1 \cdot f - (\widetilde{D}_x)^3(f) = 0. \tag{6}$$

Let now $W \subset \mathbb{R}^m$ with coordinates $(w_1, \cdots w_m)$.
The Cartan distribution on $Y \otimes W$ is given by

$$\begin{aligned} \overline{D}_x &= \widetilde{D}_x + \sum_{j=1}^{m} X^j \partial_{w_j}, \\ \overline{D}_t &= \widetilde{D}_t + \sum_{j=1}^{m} T^j \partial_{w_j} \end{aligned} \tag{7}$$

where $X^j, T^j \in C^\infty(Y \otimes W)$ such that

$$[\overline{D}_x, \overline{D}_t] = 0 \tag{8}$$

which yields the so called *covering condition*

$$\widetilde{D}_x(T) - \widetilde{D}_t(X) + [X, T] = 0$$

whereas in (8) [*,*] is the Lie bracket for vector fields $X = \sum_{j=1}^m X^j \partial_{w_j}$, $T = \sum_{j=1}^m T^j \partial_{w_j}$ defined on W.

A *nonlocal symmetry* is a vertical vector field on $Y \otimes W$, i.e. of the form (4), which satisfies $(f \in C^\infty(Y \otimes W))$

$$\bar{\ell}_F(f) = 0 \tag{9}$$

which for the KdV-equation results in

$$\overline{D}_t(f) - u\overline{D}_x(f) - u_1 \cdot f - (\overline{D}_x)^3(f) = 0. \tag{10}$$

Formally, this is just what is called the *shadow* of the symmetry, i.e., not bothering about the ∂_{w^j} $(j = 1 \dots m)$ components.
In effect the full symmetry should also satisfy the invariance of the equations governing the nonlocal variables w_j $(j = 1, \dots, m)$; i.e.,

$$(w_j)_x = X^j,$$
$$(w_j)_t = T^j.$$

The construction of the associated ∂_{w^j} $(j = 1, \dots, m)$ components is called the *reconstruction problem* [**16**]. For reasons of simplicity, we omit this reconstruction problem, i.e. reconstructing the complete vector field or full symmetry from its shadow.
The classical Lenard recursion operator $\mathcal{R}$ for the KdV-equation,

$$\mathcal{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1} \tag{11}$$

which is just such, that

$$\begin{aligned} f_0 &= u_1, \\ \mathcal{R}f_0 &= f_1 = uu_1 + u_3, \\ \mathcal{R}f_1 &= f_2 = u_5 + \frac{5}{3}u_3u + \frac{10}{3}u_2u_1 + \frac{5}{6}u_1u^2, \end{aligned} \tag{12}$$

i.e., creating the (x,t)–independent hierarchy of higher symmetries, has an action on the vertical symmetry $\mathcal{\partial}_{\overline{f}_{-1}}$ (Gallilei-boost)

$$\begin{aligned} \overline{f}_{-1} &= (1 + tu_1)/3, \\ \mathcal{R}\overline{f}_{-1} &= \overline{f}_0 = 2u + xu_1 + 3t(u_3 + uu_1), \\ \overline{f}_1 = \mathcal{R}\overline{f}_0 &= 3t(f_2) + x(f_0) + 4u_2 + \frac{4}{3}u^2 + \frac{1}{3}u_1 D_x^{-1}(u). \end{aligned} \tag{13}$$

If we introduce the variable p $(= w_1)$ through

$$\begin{aligned} p_x &= u, \\ p_t &= u_2 + \frac{1}{2}u^2, \\ \text{i.e.} \quad D_t(u) &= D_x(u_2 + \frac{1}{2}u^2), \end{aligned} \tag{14}$$

then $\mathcal{\partial}_{\overline{f}_1}$ is the shadow of a nonlocal symmetry in the one-dimensional covering of the KdV-equation by

$$p = w_1,\ X_1 = u,\ T_1 = u_2 + \frac{1}{2}u^2.$$

So, by its action the Lenard recursion operator creates nonlocal symmetries in a natural way.
More applications of nonlocal symmetries can be found in e.g.[**7**].

2.2. A Special Type of Covering: the Cartan-covering. We discuss a special type of the nonlocal setting indicated in the previous section, the so called Cartan-covering. As mentioned before we shall illustrate this by the KdV-equation. Let $Y \subset J^\infty(x,t;u)$ be the infinite prolongation of the KdV-equation (2). Contact one forms on $TJ^\infty(x,t;u)$ are given by

$$\begin{aligned} \alpha_0 &= du - u_1dx - u_tdt, \\ \alpha_1 &= du_1 - u_2dx - u_{1t}dt, \\ \alpha_2 &= du_2 - u_3dx - u_{2t}dt. \end{aligned} \tag{15}$$

From the total partial derivative operators of the previous section we have

$$\widetilde{D}_x(\alpha_0) = \alpha_1,\ \widetilde{D}_x(\alpha_1) = \alpha_2, \dots ,$$

$$\begin{aligned} \widetilde{D}_t(\alpha_0) &= \alpha_0u_x + \alpha_1u + \alpha_3 = \alpha_t, \\ \widetilde{D}_t(\alpha_i) &= (\widetilde{D}_x)^i(\alpha_t). \end{aligned} \tag{16}$$

We now define the Cartan-covering of Y by $Y \otimes \mathbb{R}^\infty$

$$\begin{aligned} D_x^C &= \widetilde{D}_x + \sum_i (\alpha_{i+1})\frac{\partial}{\partial \alpha_i}, \\ D_t^C &= \widetilde{D}_t + \sum_i (\widetilde{D}_x)^i\alpha_t\frac{\partial}{\partial \alpha_i} \end{aligned} \tag{17}$$

where local coordinates are given $(x, t, u, u_1, \dots, \alpha_0, \alpha_1, \dots)$.

It is a straightforward check, and obvious that

$$[D_x^C, D_t^C] = 0, \tag{18}$$

i.e. they form a Cartan distribution on $Y \otimes \mathbb{R}^\infty$.

Note 1:

Since at first α_i $(i = 0, \dots)$ are contact forms, they constitute a Grassmann algebra (graded commutative algebra) $\Lambda(\alpha)$, where

$$\alpha_i \wedge \alpha_j = -\alpha_j \wedge \alpha_i,$$

i.e.,

$$xy = (-1)^{|x||y|} yx$$

where x, y are contact $(*)$-forms of degree $|x|$ and $|y|$, respectively. So in effect we are dealing with a *graded* covering.

Note 2:

Once we have introduced the Cartan-covering by (17) we can forget about the specifics of α_i $(i = 0, \dots)$ and just treat them as (odd) ordinary variables, associated with their differentiation rules.

One can discuss nonlocal symmetries in this type of covering just as in the previous section, the only difference being:

$$f \in C^\infty(Y) \otimes \wedge(\alpha).$$

In the next section we shall combine constructions of the previous subsection and this one, in order to construct the recursion operator for symmetries.

2.3. The Recursion Operator as Symmetry in the Cartan-covering.

We shall discuss the recursion operator for symmetries of the KdV-equation as a geometrical object, i.e., a symmetry in the Cartan-covering.

Our starting point is the four dimensional covering of the KdV-equation in $Y \otimes \mathbb{R}^4$ where

$$\begin{aligned}
\overline{D}_x &= \widetilde{D}_x + u\partial_{w_1} + \frac{1}{2}u^2\partial_{w_2} + (u^3 - 3u_1^2)\partial_{w_3} + w_1\partial_{w_4}, \\
\overline{D}_t &= \widetilde{D}_t + (\frac{1}{2}u^2 + u_2)\partial_{w_1} + (\frac{1}{3}u^3 - \frac{1}{2}u_1^2 + uu_2)\partial_{w_2} \\
&\quad + (\frac{3}{4}u^4 - 6u_1u_3 + 3u^2u_2 - 6uu_1^2 + 3u_2^2)\partial_{w_3} + (u_1 + w_2)\partial_{w_4}.
\end{aligned} \tag{19}$$

$\overline{D}_x, \overline{D}_t$ satisfy the covering condition (8), and note, that due to the fact that the coefficients of ∂_{w_i} $(i = 1, 2, 3)$ in (19) are independent of w_j $(j = 1, 2, 3)$, these coefficients constitute **local** conservation laws for the KdV-equation.

The coefficients w_1 and $u_1 + w_2$ of ∂_{w_4} constitute the first **nonlocal** conservation law of KdV-equation..

We have the following "formal" variables:

$$\begin{aligned} w_1 &= \int u dx, \\ w_2 &= \int \frac{1}{2} u^2 dx, \\ w_3 &= \int (u^3 - 3u_1^2) dx, \\ w_4 &= \int w_1 dx. \end{aligned} \tag{20}$$

where w_4 is of a higher or deeper nonlocality.

We now build the Cartan-covering of the previous section on the covering given by (19) by introduction of the contact forms $\alpha_0, \alpha_1, \alpha_2, \ldots$ (15) and

$$\begin{aligned} \alpha_{-1} &= dw_1 - u dx - (\frac{1}{2}u^2 + u_2) dt, \\ \alpha_{-2} &= dw_2 - \frac{1}{2}u^2 dx - (\frac{1}{3}u^3 - \frac{1}{2}u_1^2 + uu_2) dt \end{aligned} \tag{21}$$

as well as similarly for α_{-3}, α_{-4}. It is straightforward to prove the following relations

$$\begin{aligned} \overline{D}_x(\alpha_{-1}) &= \alpha_0, \quad \overline{D}_t(\alpha_{-1}) = u\alpha_0 + \alpha_0, \\ \overline{D}_x(\alpha_{-2}) &= u\alpha_0, \quad \overline{D}_t(\alpha_{-2}) = u^2\alpha_0 - u_1\alpha_1 + u\alpha_2 + u_2\alpha_0, \\ \overline{D}_x(\alpha_{-3}) &= 3u^2\alpha_0 - 6u_1\alpha_1, \quad \ldots . \end{aligned} \tag{22}$$

We are now constructing symmetries in this Cartan-covering of the KdV-equation which are linear w.r.t. α_i $(i = -4, \ldots, 0, 1, \ldots)$.

The symmetry condition for $f \in C^\infty(Y \otimes \mathbb{R}^4) \otimes \Lambda^1(\alpha)$ is just given by (6)

$$\bar{\ell}_F^C(f) = 0 \tag{23}$$

which for the KdV-equation results in

$$\overline{D}_t^C(f) - u\overline{D}_x^C(f) - u_x f - (\overline{D}_x^C)^3 f = 0.$$

As solutions of these equations we obtained

$$\begin{aligned} f^0 &= \alpha_0, \\ f^1 &= (\frac{2}{3}u)\alpha_0 + \alpha_2 + (\frac{1}{3}u_1)\alpha_{-1}, \\ f^2 &= (\frac{4}{9}u^2 + \frac{4}{3}u_2)\alpha_0 + (2u_1)\alpha_1 + (\frac{4}{3}u)\alpha_2 + \alpha_4 \\ &\quad + \frac{1}{3}(uu_1 + u_3)\alpha_{-1} + \frac{1}{9}(u_1)\alpha_{-2}. \end{aligned} \tag{24}$$

As we mentioned above we are working in effect with form-valued vector fields $\partial_{f^0}, \partial_{f^1} \partial_{f^2}$. For these objects one can define Frölicher-Nijenhuis and (by contraction) Richardson-Nijenhuis brackets [**10**],[**7**] Without going into details, for which the reader is referred to [**10**], we can construct the contraction of a (generalized) symmetry and a form-valued symmetry e.g.

$$R = (\frac{2}{3}u\alpha_0 + \alpha_2 + \frac{1}{3}u_1\alpha_{-1})\frac{\partial}{\partial u} + \ldots . \tag{25}$$

The contraction formally defined by

$$V \lrcorner R_u = \sum_\alpha V_\alpha R_u^\alpha$$

where α runs over all local and nonlocal variables, is given by

$$(V \lrcorner R) = (V \lrcorner R_u)\partial_u + \overline{D}_x^C (V \lrcorner R_u)\partial_{u_1} + \dots \ . \tag{26}$$

Start now with

$$V_1 = u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \dots \tag{27}$$

whose prolongation in the setting $Y \otimes \mathbb{R}^4$ is

$$\overline{V}_1 = u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \dots + u \frac{\partial}{\partial w_1} + \frac{1}{2} u^2 \frac{\partial}{\partial w_2} + (u^3 - 3u_1^2) \frac{\partial}{\partial w_3} + w_1 \frac{\partial}{\partial w_4} \tag{28}$$

then

$$\begin{aligned}(\overline{V}_1 \lrcorner R) &= [(\frac{2}{3}u)u_1 + 1 \cdot u_3 + \frac{1}{3}u_1 \cdot u] \frac{\partial}{\partial u} + \dots \\ &= (u_3 + uu_1)\frac{\partial}{\partial u} + \dots = V_3\end{aligned} \tag{29}$$

and similarly

$$(\overline{V}_3 \lrcorner R) = (u_5 + \frac{5}{3}u_3 u + \frac{10}{3}u_2 u_1 + \frac{5}{6}u^2 u_1)\frac{\partial}{\partial u} + \dots = V_5. \tag{30}$$

The result given above means that the well known Lenard recursion operator for symmetries of the KdV-equation is represented as a *symmetry*, ϑ_{f_1} , in the Cartan-covering of this equation and in effect is a geometrical object.

3. The $N=2$ $\alpha=1$ Supersymmetric KdV Equation

In this section we shall discuss all computations leading at the end to the complete integrability of the $N=2$ $\alpha=1$ supersymmetric KdV-equation.
The $N=2$ $\alpha=1$ supersymmetric KdV-equation is described by

$$J_t = \{J_{zz} + 3J[D,\overline{D}]J + J^3\}_z \tag{31}$$

in the $N=2$ superspace with a coordinate $Z=(z,\theta,\overline{\theta})$, $_z$ denotes derivative with respect to z and $D,\overline{D}$ are the fermionic covariant derivatives of the $N=2$ supersymmetry, governed by definitions

$$D=\frac{\partial}{\partial\theta}-\frac{1}{2}\overline{\theta}\frac{\partial}{\partial z},\quad \overline{D}=\frac{\partial}{\partial\overline{\theta}}-\frac{1}{2}\theta\frac{\partial}{\partial z},\quad D^2=\overline{D}^2=0,\quad \{D,\overline{D}\}=-\frac{\partial}{\partial z}\equiv-\partial.$$

Formally, the nonlocal setting for differential equations of the previous section was done for classical equations, but applies too for supersymmetric equations([**7**]).
In order to discuss conservation laws, symmetries and deformations we choose local coordinates in the infinite jet bundle $Y((z,\theta,\overline{\theta});J)$, where we choose as local even coordinates

$$z,t,J,D\overline{D}J,J_z,D\overline{D}J_z,J_{zz},D\overline{D}J_{zz},J_{zzz},\ldots$$

and odd coordinates

$$\theta,\overline{\theta},DJ,\overline{D}J,DJ_z,\overline{D}J_z,DJ_{zz},\overline{D}J_{zz},\ldots\ .$$

In order to have a complete setting we first describe the construction of conservation laws and the associated introducton of nonlocal variables.
In effect we construct an abelian covering of the equation structure.
In the first subsection we shall discuss conservation laws and the associated nonlocalities.
In the next subsection we obtain nonlocal symmetries associated to these nonlocalities, which turn out to be $(\theta,\overline{\theta})$ - dependent and arise in so called quadruplets, similar to the nonlocalities.
In the subsection 3.3 we derive an explicit deformation of the equation structure leading to the recursion operator for symmetries.
Finally in the last subsection we obtain the factorization of the recursion operator as product of the second Hamiltonian operator and the inverse of the zero-th Hamiltonian operator.

3.1. Conservation laws and nonlocal variables. Here we shall construct conservation laws for (31) in order to arrive at an abelian covering.
So we construct $X=X(z,t,J,\ldots)$, $T=T(z,t,J,\ldots)$ such that

$$D_z(T)=D_t(X) \tag{32}$$

and in a similar way we construct nonlocal conservation laws by the requirement

$$\bar{D}_z(\bar{T})=\bar{D}_t(\bar{X}) \tag{33}$$

where $\bar{D}_z,\bar{D}_t$ are defined by

$$\begin{aligned}\bar{D}_z &= \partial_z + J_z\partial_J + D\overline{D}J_z\partial_{D\overline{D}J}+\ldots\\ &\qquad + DJ_z\partial_{DJ}+\overline{D}J_z\partial_{\overline{D}J}+\ldots\\ \bar{D}_t &= \partial_t + J_t\partial_J + D\overline{D}J_t\partial_{D\overline{D}J}+\ldots\\ &\qquad + DJ_t\partial_{DJ}+\overline{D}J_t\partial_{\overline{D}J}+\ldots\ .\end{aligned} \tag{34}$$

Moreover $\bar{X}$,$\bar{T}$ are dependent on local variables z, t, $J, \ldots$, as well as the already determined nonlocal variables, denoted here by p_*, which are associated to the conservation laws (X, T) by the formal definition

$$D_z(p_*) = (p_*)_z = X,$$
$$D_t(p_*) = (p_*)_t = T.$$

Proceeding in this way, we obtained a number of conservation laws, which arise as multiplets of four conservation laws each. The corresponding nonlocal variables are

$$\begin{aligned} &P_0, DP_0, \overline{D}P_0, D\overline{D}P_0,\\ &Q_{\frac{1}{2}}, DQ_{\frac{1}{2}}, \overline{D}Q_{\frac{1}{2}}, D\overline{D}Q_{\frac{1}{2}},\\ &\overline{Q}_{\frac{1}{2}}, D\overline{Q}_{\frac{1}{2}}, \overline{DQ}_{\frac{1}{2}}, D\overline{DQ}_{\frac{1}{2}},\\ &P_1, DP_1, \overline{D}P_1, D\overline{D}P_1 \end{aligned} \tag{35}$$

where their defining equations are given by

$$\begin{aligned} (P_0)_z &= J,\\ (Q_{\frac{1}{2}})_z &= e^{(+)}DJ,\\ (\overline{Q}_{\frac{1}{2}})_z &= e^{(-)}\overline{D}J,\\ (P_1)_z &= e^{(+)}(\overline{D}J)(DP_0) + e^{(-)}(\overline{D}J)(Q_{\frac{1}{2}}). \end{aligned} \tag{36}$$

In (36) $e^{(+)}$ and $e^{(-)}$ refer to $e^{(+2P_0)}$ and $e^{(-2P_0)}$ respectively.
It should be noted that the quadru-plet P_0 satisfies differentiation rules as follows:

$$\begin{aligned} D(P_0) &= DP_0,\\ D(DP_0) &= 0,\\ D(\overline{D}P_0) &= D\overline{D}P_0,\\ D(D\overline{D}P_0) &= 0,\\ \\ \overline{D}(P_0) &= \overline{D}P_0,\\ \overline{D}(DP_0) &= -(P_0)_z - D\overline{D}P_0,\\ \overline{D}(\overline{D}P_0) &= 0,\\ \overline{D}(D\overline{D}P_0) &= -\overline{D}(P_0)_z, \end{aligned} \tag{37}$$

and similarly for other quadru-plets $Q_{\frac{1}{2}}, \overline{Q}_{\frac{1}{2}}, P_1$.
It should be noted that for the two other $N=2$ KdV-hierarchies (α=4 and -2), despite their original $N=2$ supersymmetric structure, their conservation laws do not form supersymmetric multiplets.
So in effect we have at this moment sixteen local and nonlocal conservation laws leading to a similar number of new nonlocal variables.
They perfectly match with those ones obtained previously ([**7**])(page 340,(7.78)).
If we arrange them according to their respectively degrees we arrive at:

$$\begin{aligned}
&0 : P_0;\\
&\frac{1}{2} : DP_0, \overline{D}P_0, Q_{\frac{1}{2}}, \overline{Q}_{\frac{1}{2}};\\
&1 : D\overline{D}P_0, DQ_{\frac{1}{2}}, D\overline{Q}_{\frac{1}{2}}, \overline{D}Q_{\frac{1}{2}}, \overline{DQ}_{\frac{1}{2}}, P_1;\\
&\frac{3}{2} : D\overline{D}Q_{\frac{1}{2}}, D\overline{DQ}_{\frac{1}{2}}, DP_1, \overline{D}P_1;\\
&2 : D\overline{D}P_1.
\end{aligned}$$

In Subsection 3.2 we shall discuss local and nonlocal symmetries of the (31) while in Subsection 3.3 we construct the recursion operator or deformation of the equation structure.

3.2. Local and nonlocal symmetries. In this section we shall present results for the construction of local and nonlocal symmetries of (31). In order to construct these symmetries, we consider the system of partial differential equations obtained by the infinite prolongation.
First we present the **local** symmetries as they are required for explicit formulae for the coefficients which arise in the form-valued vector field of the next subsection leading to the recursion operator for symmetries.

$$Y_1 := J_z,$$

$$Y_3 := \{J_{zz} + 3J[D,\overline{D}]J + J^3\}_z,$$

$Y_5 := -10\cdot\overline{D}J\cdot DJ_{zz}\cdot J-20\cdot\overline{D}J\cdot DJ_z\cdot J^2-10\cdot\overline{D}J\cdot DJ_z\cdot J_z-10\cdot DJ\cdot\overline{D}J_{zz}\cdot J+20\cdot DJ\cdot\overline{D}J_z\cdot J^2-10\cdot DJ\cdot\overline{D}J_z\cdot J_z+40\cdot DJ\cdot\overline{D}J\cdot J\cdot J_z+5\cdot J^4\cdot J_z+20\cdot J^3\cdot D\overline{D}J_z+10\cdot J^3\cdot J_{zz}+60\cdot J^2\cdot D\overline{D}J\cdot J_z+30\cdot J^2\cdot J_z^2+10\cdot J^2\cdot J_{zzz}+80\cdot J\cdot D\overline{D}J\cdot D\overline{D}J_z+40\cdot J\cdot D\overline{D}J\cdot J_{zz}+5\cdot J\cdot J_{zzzz}+40\cdot J\cdot J_z\cdot D\overline{D}J_z+50\cdot J\cdot J_z\cdot J_{zz}+10\cdot J\cdot D\overline{D}J_{zzz}+40\cdot D\overline{D}J^2\cdot J_z+40\cdot D\overline{D}J\cdot J_z^2+10\cdot D\overline{D}J\cdot J_{zzz}+J_{zzzzz}+15\cdot J_z^3+20\cdot J_z\cdot D\overline{D}J_{zz}+15\cdot J_z\cdot J_{zzz}+20\cdot D\overline{D}J_z\cdot J_{zz}+10\cdot J_{zz}^2,$

$Y_7 := (-42\cdot\overline{D}J_z\cdot DJ_{zzz}\cdot J-126\cdot\overline{D}J_z\cdot DJ_{zz}\cdot J^2-28\cdot\overline{D}J_z\cdot DJ_{zz}\cdot J_z-42\cdot DJ_z\cdot\overline{D}J_{zzz}\cdot J+126\cdot DJ_z\cdot\overline{D}J_{zz}\cdot J^2-28\cdot DJ_z\cdot\overline{D}J_{zz}\cdot J_z+364\cdot DJ_z\cdot\overline{D}J_z\cdot J\cdot J_z-28\cdot\overline{D}J\cdot DJ_{zzzz}\cdot J-70\cdot\overline{D}J\cdot DJ_{zzz}\cdot J^2-42\cdot\overline{D}J\cdot DJ_{zzz}\cdot J_z-70\cdot\overline{D}J\cdot DJ_{zz}\cdot J^3-168\cdot\overline{D}J\cdot DJ_{zz}\cdot J\cdot D\overline{D}J-350\cdot\overline{D}J\cdot DJ_{zz}\cdot J\cdot J_z-28\cdot\overline{D}J\cdot DJ_{zz}\cdot J_{zz}-84\cdot\overline{D}J\cdot DJ_z\cdot J^4-336\cdot\overline{D}J\cdot DJ_z\cdot J^2\cdot D\overline{D}J-378\cdot\overline{D}J\cdot DJ_z\cdot J^2\cdot J_z-168\cdot\overline{D}J\cdot DJ_z\cdot J\cdot D\overline{D}J_z-294\cdot\overline{D}J\cdot DJ_z\cdot J\cdot J_{zz}-168\cdot\overline{D}J\cdot DJ_z\cdot D\overline{D}J\cdot J_z-238\cdot\overline{D}J\cdot DJ_z\cdot J_z^2-14\cdot\overline{D}J\cdot DJ_z\cdot J_{zzz}-28\cdot DJ,12)\cdot J+70\cdot DJ\cdot\overline{D}J_{zzz}\cdot J^2-42\cdot DJ\cdot\overline{D}J_{zzz}\cdot J_z-70\cdot DJ\cdot\overline{D}J_{zz}\cdot J^3-168\cdot DJ\cdot\overline{D}J_{zz}\cdot J\cdot D\overline{D}J+182\cdot DJ\cdot\overline{D}J_{zz}\cdot J\cdot J_z-28\cdot DJ\cdot\overline{D}J_{zz}\cdot J_{zz}+84\cdot DJ\cdot\overline{D}J_z\cdot J^4+336\cdot DJ\cdot\overline{D}J_z\cdot J^2\cdot D\overline{D}J-42\cdot DJ\cdot\overline{D}J_z\cdot J^2\cdot J_z-168\cdot DJ\cdot\overline{D}J_z\cdot J\cdot D\overline{D}J_z+126\cdot DJ\cdot\overline{D}J_z\cdot J\cdot J_{zz}-168\cdot DJ\cdot\overline{D}J_z\cdot D\overline{D}J\cdot J_z+70\cdot DJ\cdot\overline{D}J_z\cdot J_z^2-14\cdot DJ\cdot\overline{D}J_z\cdot J_{zzz}+336\cdot DJ\cdot\overline{D}J\cdot J^3\cdot J_z+336\cdot DJ\cdot\overline{D}J\cdot J^2\cdot D\overline{D}J_z+168\cdot DJ\cdot\overline{D}J\cdot J^2\cdot J_{zz}+672\cdot DJ\cdot\overline{D}J\cdot J\cdot D\overline{D}J\cdot J_z+336\cdot DJ\cdot\overline{D}J\cdot J\cdot J_z^2+84\cdot DJ\cdot\overline{D}J\cdot J\cdot J_{zzz}+140\cdot DJ\cdot\overline{D}J\cdot J_z\cdot J_{zz}+7\cdot J^6\cdot J_z+42\cdot J^5\cdot D\overline{D}J_z+21\cdot J^5\cdot J_{zz}+210\cdot J^4\cdot D\overline{D}J\cdot J_z+105\cdot J^4\cdot J_z^2+35\cdot J^4\cdot J_{zzz}+504\cdot J^3\cdot D\overline{D}J\cdot D\overline{D}J_z+252\cdot J^3\cdot D\overline{D}J\cdot J_{zz}+35\cdot J^3\cdot J_{zzzz}+252\cdot J^3\cdot J_z\cdot D\overline{D}J_z+350\cdot J^3\cdot J_z\cdot J_{zz}+70\cdot J^3\cdot D\overline{D}J_{zzz}+756\cdot J^2\cdot D\overline{D}J^2\cdot J_z+756\cdot J^2\cdot D\overline{D}J\cdot J_z^2+224\cdot J^2\cdot D\overline{D}J\cdot J_{zzz}+21\cdot J^2\cdot J_{zzzzz}+315\cdot J^2\cdot J_z^3+406\cdot J^2\cdot J_z\cdot D\overline{D}J_{zz}+315\cdot J^2\cdot J_z\cdot J_{zzz}+420\cdot J^2\cdot D\overline{D}J_z\cdot J_{zz}+210\cdot J^2\cdot J_{zz}^2+840\cdot J\cdot D\overline{D}J^2\cdot D\overline{D}J_z+420\cdot J\cdot D\overline{D}J^2\cdot$

$J_{zz}+98\cdot J\cdot D\overline{D}J\cdot J_{zzzz}+840\cdot J\cdot D\overline{D}J\cdot J_z\cdot D\overline{D}J_z+1148\cdot J\cdot D\overline{D}J\cdot J_z\cdot J_{zz}+196\cdot J\cdot D\overline{D}J\cdot D\overline{D}J_{zzz}+147\cdot J\cdot J_{zzzz}\cdot J_z+14\cdot J\cdot D\overline{D}J_{zzzzz}+7\cdot J\cdot J_{zzzzzz}+742\cdot J\cdot J_z^2\cdot D\overline{D}J_z+735\cdot J\cdot J_z^2\cdot J_{zz}+98\cdot J\cdot J_z\cdot D\overline{D}J_{zzz}+420\cdot J\cdot D\overline{D}J_z\cdot D\overline{D}J_{zz}+210\cdot J\cdot D\overline{D}J_z\cdot J_{zzz}+210\cdot J\cdot J_{zz}\cdot D\overline{D}J_{zz}+245\cdot J\cdot J_{zz}\cdot J_{zzz}+280\cdot D\overline{D}J^3\cdot J_z+420\cdot D\overline{D}J^2\cdot J_z^2+84\cdot D\overline{D}J^2\cdot J_{zzz}+14\cdot D\overline{D}J\cdot J_{zzzzz}+350\cdot D\overline{D}J\cdot J_z^3+364\cdot D\overline{D}J\cdot J_z\cdot D\overline{D}J_{zz}+266\cdot D\overline{D}J\cdot J_z\cdot J_{zzz}+336\cdot D\overline{D}J\cdot D\overline{D}J_z\cdot J_{zz}+168\cdot D\overline{D}J\cdot J_{zz}^2+42\cdot J_{zzzz}\cdot D\overline{D}J_z+56\cdot J_{zzzz}\cdot J_{zz}+42\cdot D\overline{D}J_{zzzz}\cdot J_z+28\cdot J_{zzzzz}\cdot J_z+J_{zzzzzzz}+105\cdot J_z^4+182\cdot J_z^2\cdot D\overline{D}J_{zz}+210\cdot J_z^2\cdot J_{zzz}+280\cdot J_z\cdot D\overline{D}J_z^2+448\cdot J_z\cdot D\overline{D}J_z\cdot J_{zz}+280\cdot J_z\cdot J_{zz}^2+70\cdot J_{zz}\cdot D\overline{D}J_{zzz}+70\cdot D\overline{D}J_{zz}\cdot J_{zzz}+35\cdot J_{zzz}^2)$.

It should be noted that the symmetries Y_5 and Y_7 are rather massive, containing 28 and 104 terms respectively. We made our choice for a presentation as shown above, in order to have expressions, which are quite huge, as readable as possible.

Now, we present the four multiplets of nonlocal $(\theta,\bar{\theta})$–dependent symmetries as they were constructed in the above described nonlocal setting.

The nonlocal $(\theta,\overline{\theta})$–dependent symmetries at level $0, 1/2, 1, 3/2, 2$ are given as follows:
This represents the first quadru-plet associated to P_0
$Y_{P_0,\theta\overline{\theta}} := \theta\overline{\theta}\cdot J_z+\theta\cdot DJ-\overline{\theta}\cdot\overline{D}J$,

$Y_{P_0,\overline{\theta}} := \overline{\theta}\cdot J_z+DJ$,

$Y_{P_0,\theta} := \theta\cdot J_z+\overline{D}J$,

$Y_{P_0} := J_z$.

The second quadru-plet associated to $Q_{\frac{1}{2}}$ is represented by
$Y_{Q_{\frac{1}{2}},\theta\overline{\theta}} := (2\cdot\theta\overline{\theta}\cdot e^{(+)}\cdot(-2\cdot DP_0\cdot D\overline{D}J-2\cdot DP_0\cdot J_z-DJ_z+2\cdot DJ\cdot DP_0\cdot\overline{D}P_0+DJ\cdot D\overline{D}P_0)+2\cdot\theta\overline{\theta}\cdot(Q_{\frac{1}{2}}\cdot J_z+\overline{D}J\cdot DQ_{\frac{1}{2}}+DJ\cdot\overline{D}Q_{\frac{1}{2}})-2\cdot\theta\cdot e^{(+)}\cdot DJ\cdot DP_0+2\cdot\theta\cdot DJ\cdot Q_{\frac{1}{2}}-2\cdot\overline{\theta}\cdot e^{(+)}\cdot(DJ\cdot\overline{D}P_0+D\overline{D}J+J_z)-2\cdot\overline{\theta}\cdot\overline{D}J\cdot Q_{\frac{1}{2}}-e^{(+)}\cdot DJ)/2$,

$Y_{Q_{\frac{1}{2}},\overline{\theta}} := \overline{\theta}\cdot e^{(+)}\cdot(-2\cdot DP_0\cdot D\overline{D}J-2\cdot DP_0\cdot J_z-DJ_z+2\cdot DJ\cdot DP_0\cdot\overline{D}P_0+DJ\cdot D\overline{D}P_0)+\overline{\theta}\cdot(Q_{\frac{1}{2}}\cdot J_z+\overline{D}J\cdot DQ_{\frac{1}{2}}+DJ\cdot\overline{D}Q_{\frac{1}{2}})-e^{(+)}\cdot DJ\cdot DP_0+DJ\cdot Q_{\frac{1}{2}}$,

$Y_{Q_{\frac{1}{2}},\theta} := \theta\cdot e^{(+)}\cdot(-2\cdot DP_0\cdot D\overline{D}J-2\cdot DP_0\cdot J_z-DJ_z+2\cdot DJ\cdot DP_0\cdot\overline{D}P_0+DJ\cdot D\overline{D}P_0)+\theta\cdot(Q_{\frac{1}{2}}\cdot J_z+\overline{D}J\cdot DQ_{\frac{1}{2}}+DJ\cdot\overline{D}Q_{\frac{1}{2}})+e^{(+)}\cdot(DJ\cdot\overline{D}P_0+D\overline{D}J+J_z)+\overline{D}J\cdot Q_{\frac{1}{2}}$,

$Y_{Q_{\frac{1}{2}}} := e^{(+)}\cdot(-2\cdot DP_0\cdot D\overline{D}J-2\cdot DP_0\cdot J_z-DJ_z+2\cdot DJ\cdot DP_0\cdot\overline{D}P_0+DJ\cdot D\overline{D}P_0)+Q_{\frac{1}{2}}\cdot J_z+\overline{D}J\cdot DQ_{\frac{1}{2}}+DJ\cdot\overline{D}Q_{\frac{1}{2}}$.

The third quadru-plet associated to $\overline{Q}_{\frac{1}{2}}$ is represented by

$Y_{\overline{Q}_{\frac{1}{2}},\theta\overline{\theta}} := (2\cdot\theta\overline{\theta}\cdot e^{(+)}\cdot(\overline{Q}_{\frac{1}{2}}\cdot J_z+\overline{D}J\cdot D\overline{Q}_{\frac{1}{2}}+DJ\cdot\overline{DQ}_{\frac{1}{2}})+2\cdot\theta\overline{\theta}\cdot(2\cdot\overline{D}P_0\cdot D\overline{D}J+\overline{D}J_z+2\cdot\overline{D}J\cdot DP_0\cdot\overline{D}P_0-\overline{D}J\cdot J-\overline{D}J\cdot D\overline{D}P_0)+2\cdot\theta\cdot e^{(+)}\cdot DJ\cdot\overline{Q}_{\frac{1}{2}}+$

$2 \cdot \theta \cdot (\overline{D}J \cdot DP_0 + D\overline{D}J) - 2 \cdot \overline{\theta} \cdot e^{(+)} \cdot \overline{D}J \cdot \overline{Q}_{\frac{1}{2}} + 2 \cdot \overline{\theta} \cdot \overline{D}J \cdot \overline{D}P_0 - \overline{D}J)/(2 \cdot e^{(+)}),$

$Y_{\overline{Q}_{\frac{1}{2}},\overline{\theta}} := (\overline{\theta} \cdot e^{(+)} \cdot (\overline{Q}_{\frac{1}{2}} \cdot J_z + \overline{D}J \cdot D\overline{Q}_{\frac{1}{2}} + DJ \cdot \overline{DQ}_{\frac{1}{2}}) + \overline{\theta} \cdot (2 \cdot \overline{D}P_0 \cdot D\overline{D}J + \overline{D}J_z + 2 \cdot \overline{D}J \cdot DP_0 \cdot \overline{D}P_0 - \overline{D}J \cdot J - \overline{D}J \cdot D\overline{D}P_0) + e^{(+)} \cdot DJ \cdot \overline{Q}_{\frac{1}{2}} + \overline{D}J \cdot DP_0 + D\overline{D}J)/e^{(+)},$

$Y_{\overline{Q}_{\frac{1}{2}},\theta} := (\theta \cdot e^{(+)} \cdot (\overline{Q}_{\frac{1}{2}} \cdot J_z + \overline{D}J \cdot D\overline{Q}_{\frac{1}{2}} + DJ \cdot \overline{DQ}_{\frac{1}{2}}) + \theta \cdot (2 \cdot \overline{D}P_0 \cdot D\overline{D}J + \overline{D}J_z + 2 \cdot \overline{D}J \cdot DP_0 \cdot \overline{D}P_0 - \overline{D}J \cdot J - \overline{D}J \cdot D\overline{D}P_0) + e^{(+)} \cdot \overline{D}J \cdot \overline{Q}_{\frac{1}{2}} - \overline{D}J \cdot \overline{D}P_0)/e^{(+)},$

$Y_{\overline{Q}_{\frac{1}{2}}} := (e^{(+)} \cdot (\overline{Q}_{\frac{1}{2}} \cdot J_z + \overline{D}J \cdot D\overline{Q}_{\frac{1}{2}} + DJ \cdot \overline{DQ}_{\frac{1}{2}}) + 2 \cdot \overline{D}P_0 \cdot D\overline{D}J + \overline{D}J_z + 2 \cdot \overline{D}J \cdot DP_0 \cdot \overline{D}P_0 - \overline{D}J \cdot J - \overline{D}J \cdot D\overline{D}P_0)/e^{(+)}.$

The fourth quadru-plet associated to P_1 is represented by

$Y_{P_1,\theta\overline{\theta}} := (\theta\overline{\theta} \cdot e^{2(+)} \cdot (4 \cdot DP_0 \cdot \overline{Q}_{\frac{1}{2}} \cdot D\overline{D}J + 4 \cdot DP_0 \cdot \overline{Q}_{\frac{1}{2}} \cdot J_z + 2 \cdot DJ_z \cdot \overline{Q}_{\frac{1}{2}} - DJ \cdot D\overline{DQ}_{\frac{1}{2}} - 2 \cdot DJ \cdot \overline{Q}_{\frac{1}{2}} \cdot D\overline{D}P_0 + 2 \cdot DJ \cdot \overline{D}P_0 \cdot D\overline{Q}_{\frac{1}{2}} - 4 \cdot DJ \cdot DP_0 \cdot \overline{D}P_0 \cdot \overline{Q}_{\frac{1}{2}} - 2 \cdot DJ \cdot DP_0 \cdot \overline{DQ}_{\frac{1}{2}} + 2 \cdot D\overline{D}J \cdot D\overline{Q}_{\frac{1}{2}} + 2 \cdot J_z \cdot D\overline{Q}_{\frac{1}{2}}) + \theta\overline{\theta} \cdot e^{(+)} \cdot (-2 \cdot Q_{\frac{1}{2}} \cdot \overline{Q}_{\frac{1}{2}} \cdot J_z - 2 \cdot \overline{D}J \cdot \overline{Q}_{\frac{1}{2}} \cdot DQ_{\frac{1}{2}} + 2 \cdot \overline{D}J \cdot Q_{\frac{1}{2}} \cdot D\overline{Q}_{\frac{1}{2}} - 2 \cdot DJ \cdot \overline{Q}_{\frac{1}{2}} \cdot \overline{D}Q_{\frac{1}{2}} + 2 \cdot DJ \cdot Q_{\frac{1}{2}} \cdot \overline{DQ}_{\frac{1}{2}} - DJ \cdot \overline{D}J + 2 \cdot J \cdot J_z + 2 \cdot D\overline{D}J_z + J_{zz}) + \theta\overline{\theta} \cdot (4 \cdot \overline{D}P_0 \cdot Q_{\frac{1}{2}} \cdot D\overline{D}J + 2 \cdot \overline{D}J_z \cdot Q_{\frac{1}{2}} + \overline{D}J \cdot D\overline{D}Q_{\frac{1}{2}} - 2 \cdot \overline{D}J \cdot Q_{\frac{1}{2}} \cdot J - 2 \cdot \overline{D}J \cdot Q_{\frac{1}{2}} \cdot D\overline{D}P_0 + 2 \cdot \overline{D}J \cdot \overline{D}P_0 \cdot DQ_{\frac{1}{2}} + 4 \cdot \overline{D}J \cdot DP_0 \cdot \overline{D}P_0 \cdot Q_{\frac{1}{2}} - 2 \cdot \overline{D}J \cdot DP_0 \cdot \overline{D}Q_{\frac{1}{2}} - 2 \cdot D\overline{D}J \cdot \overline{D}Q_{\frac{1}{2}}) + \theta \cdot e^{2(+)} \cdot (2 \cdot DJ \cdot DP_0 \cdot \overline{Q}_{\frac{1}{2}} + DJ \cdot D\overline{Q}_{\frac{1}{2}}) + \theta \cdot e^{(+)} \cdot (DP_0 \cdot J_z + DJ_z - 2 \cdot DJ \cdot Q_{\frac{1}{2}} \cdot \overline{Q}_{\frac{1}{2}} + DJ \cdot J - DJ \cdot D\overline{D}P_0) + \theta \cdot (2 \cdot Q_{\frac{1}{2}} \cdot D\overline{D}J + 2 \cdot \overline{D}J \cdot DP_0 \cdot Q_{\frac{1}{2}} - \overline{D}J \cdot DQ_{1\ over2}) + \overline{\theta} \cdot e^{2(+)} \cdot (2 \cdot \overline{Q}_{\frac{1}{2}} \cdot D\overline{D}J + 2 \cdot \overline{Q}_{\frac{1}{2}} \cdot J_z + 2 \cdot DJ \cdot \overline{D}P_0 \cdot \overline{Q}_{\frac{1}{2}} + DJ \cdot \overline{DQ}_{\frac{1}{2}}) + \overline{\theta} \cdot e^{(+)} \cdot (-\overline{D}P_0 \cdot J_z + \overline{D}J_z + 2 \cdot \overline{D}J \cdot Q_{\frac{1}{2}} \cdot \overline{Q}_{\frac{1}{2}} - 2 \cdot \overline{D}J \cdot J - \overline{D}J \cdot D\overline{D}P_0) + \overline{\theta} \cdot (2 \cdot \overline{D}J \cdot \overline{D}P_0 \cdot Q_{\frac{1}{2}} - \overline{D}J \cdot \overline{D}Q_{\frac{1}{2}}) + e^{2(+)} \cdot DJ \cdot \overline{Q}_{\frac{1}{2}} + e^{(+)} \cdot (\overline{D}J \cdot DP_0 - DJ \cdot \overline{D}P_0) - \overline{D}J \cdot Q_{\frac{1}{2}})/e^{(+)},$

$Y_{P_1,\overline{\theta}} := (\overline{\theta} \cdot e^{2(+)} \cdot (4 \cdot DP_0 \cdot \overline{Q}_{\frac{1}{2}} \cdot D\overline{D}J + 4 \cdot DP_0 \cdot \overline{Q}_{\frac{1}{2}} \cdot J_z + 2 \cdot DJ_z \cdot \overline{Q}_{\frac{1}{2}} - DJ \cdot D\overline{DQ}_{\frac{1}{2}} - 2 \cdot DJ \cdot \overline{Q}_{\frac{1}{2}} \cdot D\overline{D}P_0 + 2 \cdot DJ \cdot \overline{D}P_0 \cdot D\overline{Q}_{\frac{1}{2}} - 4 \cdot DJ \cdot DP_0 \cdot \overline{D}P_0 \cdot \overline{Q}_{\frac{1}{2}} - 2 \cdot DJ \cdot DP_0 \cdot \overline{DQ}_{\frac{1}{2}} + 2 \cdot D\overline{D}J \cdot D\overline{Q}_{\frac{1}{2}} + 2 \cdot J_z \cdot D\overline{Q}_{\frac{1}{2}}) + \overline{\theta} \cdot e^{(+)} \cdot (-2 \cdot Q_{\frac{1}{2}} \cdot \overline{Q}_{\frac{1}{2}} \cdot J_z - 2 \cdot \overline{D}J \cdot \overline{Q}_{\frac{1}{2}} \cdot DQ_{\frac{1}{2}} + 2 \cdot \overline{D}J \cdot Q_{\frac{1}{2}} \cdot D\overline{Q}_{\frac{1}{2}} - 2 \cdot DJ \cdot \overline{Q}_{\frac{1}{2}} \cdot \overline{D}Q_{\frac{1}{2}} + 2 \cdot DJ \cdot Q_{\frac{1}{2}} \cdot \overline{DQ}_{\frac{1}{2}} - DJ \cdot \overline{D}J + 2 \cdot J \cdot J_z + 2 \cdot D\overline{D}J_z + J_{zz}) + \overline{\theta} \cdot (4 \cdot \overline{D}P_0 \cdot Q_{\frac{1}{2}} \cdot D\overline{D}J + 2 \cdot \overline{D}J_z \cdot Q_{\frac{1}{2}} + \overline{D}J \cdot D\overline{D}Q_{\frac{1}{2}} - 2 \cdot \overline{D}J \cdot Q_{\frac{1}{2}} \cdot J - 2 \cdot \overline{D}J \cdot Q_{\frac{1}{2}} \cdot D\overline{D}P_0 + 2 \cdot \overline{D}J \cdot \overline{D}P_0 \cdot DQ_{\frac{1}{2}} + 4 \cdot \overline{D}J \cdot DP_0 \cdot \overline{D}P_0 \cdot Q_{\frac{1}{2}} - 2 \cdot \overline{D}J \cdot DP_0 \cdot \overline{D}Q_{\frac{1}{2}} - 2 \cdot D\overline{D}J \cdot \overline{D}Q_{\frac{1}{2}}) + e^{2(+)} \cdot (2 \cdot DJ \cdot DP_0 \cdot \overline{Q}_{\frac{1}{2}} + DJ \cdot D\overline{Q}_{\frac{1}{2}}) + e^{(+)} \cdot (DP_0 \cdot J_z + DJ_z - 2 \cdot DJ \cdot Q_{\frac{1}{2}} \cdot \overline{Q}_{\frac{1}{2}} + DJ \cdot J - DJ \cdot D\overline{D}P_0) + 2 \cdot Q_{\frac{1}{2}} \cdot D\overline{D}J + 2 \cdot \overline{D}J \cdot DP_0 \cdot Q_{\frac{1}{2}} - \overline{D}J \cdot DQ_{\frac{1}{2}})/e^{(+)},$

$Y_{P_1,\theta} := (\theta \cdot e^{2(+)} \cdot (4 \cdot DP_0 \cdot \overline{Q}_{\frac{1}{2}} \cdot D\overline{D}J + 4 \cdot DP_0 \cdot \overline{Q}_{\frac{1}{2}} \cdot J_z + 2 \cdot DJ_z \cdot \overline{Q}_{\frac{1}{2}} - DJ \cdot D\overline{DQ}_{\frac{1}{2}} - 2 \cdot DJ \cdot \overline{Q}_{\frac{1}{2}} \cdot D\overline{D}P_0 + 2 \cdot DJ \cdot \overline{D}P_0 \cdot D\overline{Q}_{\frac{1}{2}} - 4 \cdot DJ \cdot DP_0 \cdot \overline{D}P_0 \cdot \overline{Q}_{\frac{1}{2}} - 2 \cdot DJ \cdot DP_0 \cdot \overline{DQ}_{\frac{1}{2}} + 2 \cdot D\overline{D}J \cdot D\overline{Q}_{\frac{1}{2}} + 2 \cdot J_z \cdot D\overline{Q}_{\frac{1}{2}}) + \theta \cdot e^{(+)} \cdot (-2 \cdot Q_{\frac{1}{2}} \cdot \overline{Q}_{\frac{1}{2}} \cdot J_z - 2 \cdot \overline{D}J \cdot \overline{Q}_{\frac{1}{2}} \cdot DQ_{\frac{1}{2}} + 2 \cdot \overline{D}J \cdot Q_{\frac{1}{2}} \cdot D\overline{Q}_{\frac{1}{2}} - 2 \cdot DJ \cdot \overline{Q}_{\frac{1}{2}} \cdot \overline{D}Q_{\frac{1}{2}} + 2 \cdot DJ \cdot Q_{\frac{1}{2}} \cdot \overline{DQ}_{\frac{1}{2}} - DJ \cdot \overline{D}J + 2 \cdot J \cdot J_z + 2 \cdot D\overline{D}J_z + J_{zz}) + \theta \cdot (4 \cdot \overline{D}P_0 \cdot Q_{\frac{1}{2}} \cdot D\overline{D}J + 2 \cdot \overline{D}J_z \cdot Q_{\frac{1}{2}} + \overline{D}J \cdot D\overline{D}Q_{\frac{1}{2}} - 2 \cdot \overline{D}J \cdot Q_{\frac{1}{2}} \cdot J - 2 \cdot \overline{D}J \cdot Q_{\frac{1}{2}} \cdot D\overline{D}P_0 + 2 \cdot \overline{D}J \cdot \overline{D}P_0 \cdot DQ_{\frac{1}{2}} + 4 \cdot \overline{D}J \cdot DP_0 \cdot \overline{D}P_0 \cdot Q_{\frac{1}{2}} - 2 \cdot \overline{D}J \cdot DP_0 \cdot \overline{D}Q_{\frac{1}{2}} - 2 \cdot D\overline{D}J \cdot \overline{D}Q_{\frac{1}{2}}) + e^{2(+)} \cdot (-2 \cdot \overline{Q}_{\frac{1}{2}} \cdot D\overline{D}J - 2 \cdot \overline{Q}_{\frac{1}{2}} \cdot J_z - 2 \cdot DJ \cdot \overline{D}P_0 \cdot \overline{Q}_{\frac{1}{2}} - DJ \cdot \overline{DQ}_{\frac{1}{2}}) + e^{(+)} \cdot (\overline{D}P_0 \cdot J_z -$

$\overline{D}J_z - 2\cdot\overline{D}J\cdot Q_{\frac{1}{2}}\cdot\overline{Q}_{\frac{1}{2}} + 2\cdot\overline{D}J\cdot J + \overline{D}J\cdot D\overline{D}P_0) - 2\cdot\overline{D}J\cdot\overline{D}P_0\cdot Q_{\frac{1}{2}} + \overline{D}J\cdot\overline{D}Q_{\frac{1}{2}})/e^{(+)}$,

$Y_{P_1} := (e^{2(+)}\cdot(4\cdot DP_0\cdot\overline{Q}_{\frac{1}{2}}\cdot D\overline{D}J + 4\cdot DP_0\cdot\overline{Q}_{\frac{1}{2}}\cdot J_z + 2\cdot DJ_z\cdot\overline{Q}_{\frac{1}{2}} - DJ\cdot D\overline{DQ}_{\frac{1}{2}} - 2\cdot DJ\cdot\overline{Q}_{\frac{1}{2}}\cdot D\overline{D}P_0 + 2\cdot DJ\cdot\overline{D}P_0\cdot D\overline{Q}_{\frac{1}{2}} - 4\cdot DJ\cdot DP_0\cdot\overline{D}P_0\cdot\overline{Q}_{\frac{1}{2}} - 2\cdot DJ\cdot DP_0\cdot\overline{DQ}_{\frac{1}{2}} + 2\cdot D\overline{D}J\cdot D\overline{Q}_{\frac{1}{2}} + 2\cdot J_z\cdot D\overline{Q}_{\frac{1}{2}}) + e^{(+)}\cdot(-2\cdot Q_{\frac{1}{2}}\cdot\overline{Q}_{\frac{1}{2}}\cdot J_z - 2\cdot\overline{D}J\cdot\overline{Q}_{\frac{1}{2}}\cdot DQ_{\frac{1}{2}} + 2\cdot\overline{D}J\cdot Q_{\frac{1}{2}}\cdot D\overline{Q}_{\frac{1}{2}} - 2\cdot DJ\cdot\overline{Q}_{\frac{1}{2}}\cdot\overline{D}Q_{\frac{1}{2}} + 2\cdot DJ\cdot Q_{\frac{1}{2}}\cdot\overline{DQ}_{\frac{1}{2}} - DJ\cdot\overline{D}J + 2\cdot J\cdot J_z + 2\cdot D\overline{D}J_z + J_{zz}) + 4\cdot\overline{D}P_0\cdot Q_{\frac{1}{2}}\cdot D\overline{D}J + 2\cdot\overline{D}J_z\cdot Q_{\frac{1}{2}} + \overline{D}J\cdot D\overline{D}Q_{\frac{1}{2}} - 2\cdot\overline{D}J\cdot Q_{\frac{1}{2}}\cdot J - 2\cdot\overline{D}J\cdot Q_{\frac{1}{2}}\cdot D\overline{D}P_0 + 2\cdot\overline{D}J\cdot\overline{D}P_0\cdot DQ_{\frac{1}{2}} + 4\cdot\overline{D}J\cdot DP_0\cdot\overline{D}P_0\cdot Q_{\frac{1}{2}} - 2\cdot\overline{D}J\cdot DP_0\cdot\overline{D}Q_{\frac{1}{2}} - 2\cdot D\overline{D}J\cdot\overline{D}Q_{\frac{1}{2}})/e^{(+)}$.

Note that in previous formulas $e^{2(+)}$ refers to e^{4P_0}.

3.3. Recursion operator. Here we present the recursion operator $\mathcal{R}$ for symmetries for this case obtained as a higher symmetry in the Cartan covering of equation (31) augmented by equations governing the nonlocal variables (36–37). As explained in the previous section, the recursion operator is in effect a deformation of the equation structure.
As demonstrated there, this deformation is a form-valued vector field and has to satisfy

$$\bar{\ell}_F^C(\mathcal{R}) = 0. \tag{38}$$

In order to arrive at a nontrivial result as was explained for the KdV equation, we have to introduce associated to the nonlocal variables

$$P_0,\ DP_0,\ \overline{D}P_0,\ D\overline{D}P_0,\ Q_{\frac{1}{2}},\ \dots$$

their Cartan forms

$$\omega_{P_0},\ \omega_{DP_0},\ \omega_{\overline{D}P_0},\ \omega_{D\overline{D}P_0},\ \omega_{Q_{\frac{1}{2}}},\ \omega_{DQ_{\frac{1}{2}}},\ \omega_{\overline{D}Q_{\frac{1}{2}}},\ \omega_{D\overline{D}Q_{\frac{1}{2}}},$$

$$\omega_{\overline{Q}_{\frac{1}{2}}},\ \omega_{D\overline{Q}_{\frac{1}{2}}},\ \omega_{\overline{DQ}_{\frac{1}{2}}},\ \omega_{D\overline{DQ}_{\frac{1}{2}}},\ \omega_{P_1},\ \omega_{DP_1},\ \omega_{\overline{D}P_1},\ \omega_{D\overline{D}P_1}.$$

Motivated by the results of the previous subsections and the grading of the equation our search is for a one-form-valued vector field whose defining function is of degree 3.
So besides the Cartan forms associated to the nonlocal variables $P_0, \dots$ which will account for the pseudo differential part of the recursion operator, we also have to introduce the local Cartan forms

$$\omega_J,\ \omega_{DJ},\ \omega_{\overline{D}J},\ \omega_{D\overline{D}J},\ \omega_{J_z},\ \omega_{DJ_z},\ \omega_{\overline{D}J_z},\ \omega_{D\overline{D}J_z},\ \omega_{J_{zz}},\ \omega_{DJ_{zz}},\ \omega_{\overline{D}J_{zz}},\ \omega_{D\overline{D}J_{zz}}$$

which will represent the pure differential part of the recursion operator.
We have to remark that the coefficients of the one-form-valued vector field are just functions dependent on all local and nonlocal variables.
Moreover it should be noted that the representation of the vector field with respect to its form part is to be understood as to be equipped with a right-module structure. This will account for the correct action of contraction.
Since the coordinate P_0 is of degree zero, this procedure requires the introduction of approximately 2400, yet free, functions dependent on this variable .
Now this quite extensive one-form-valued vector field has to satisfy eq. (38). Although this is just the deformation condition which should be satisfied, we decided, in order to reduce the already extremely extensive computations, first to require that the resulting recursion operator performs its "duties", sending symmetries to

symmetries.
In order to achieve these goals we required the operator to satifies the following requirements

$$\mathcal{R}(Y_1) = Y_3, \quad \mathcal{R}(Y_3) = Y_5, \quad \mathcal{R}(Y_5) = Y_7. \tag{39}$$

Due to these conditions it has been possible to fix all 2400 free functions in the defining function R of the one-form-valued vector field $\mathcal{R}$.
In effect the condition $\mathcal{R}(Y_5) = Y_7$ was itself sufficient to fix all coefficients.
After this, the result has been substituted into (38), satisfying it completely.
So, the final result is the following:
Starting form the defining function R for the form-valued vector field $\mathcal{R}$, given by

$$R = \sum_\alpha \omega_\alpha \cdot \Phi_\alpha \tag{40}$$

where α runs over $D\overline{D}J_{zz}, \ldots, DJ, J, P_0, DP_0, \ldots, D\,\overline{D}P_1$, the coefficients are given by

$$
\begin{aligned}
\Phi_{D\overline{D}P_1} &:= 0,\\
\Phi_{P_1} &:= J_z,\\
\Phi_{\overline{DQ}_{\frac{1}{2}}} &:= e^{(+)} \cdot DJ \cdot DP_0 - DJ \cdot Q_{\frac{1}{2}},\\
\Phi_{\overline{D}Q_{\frac{1}{2}}} &:= -(\overline{D}J \cdot DP_0 + D\overline{D}J) \cdot e^{(-)},\\
\Phi_{D\overline{Q}_{\frac{1}{2}}} &:= -(e^{(+)} \cdot DJ \cdot \overline{D}P_0 + e^{(+)} \cdot D\overline{D}J + e^{(+)} \cdot J_z + \overline{D}J \cdot Q_{\frac{1}{2}}),\\
\Phi_{DQ_{\frac{1}{2}}} &:= e^{(-)} \cdot \overline{D}J \cdot \overline{D}P_0,\\
\Phi_{D\overline{D}P_0} &:= -\overline{D}J \cdot DP_0 + 3 \cdot J_z,\\
\Phi_{P_0} &:= -DJ \cdot \overline{D}J + 2 \cdot J \cdot J_z + 2 \cdot D\overline{D}J_z + J_{zz},\\
\Phi_{\overline{D}P_1} &:= -DJ,\\
\Phi_{DP_1} &:= -\overline{D}J,\\
\Phi_{D\overline{DQ}_{\frac{1}{2}}} &:= -e^{(+)} \cdot DJ/2,\\
\Phi_{D\overline{D}Q_{\frac{1}{2}}} &:= -e^{(-)} \cdot \overline{D}J/2,\\
\Phi_{\overline{Q}_{\frac{1}{2}}} &:= 2 \cdot e^{(+)} \cdot DP_0 \cdot D\overline{D}J + 2 \cdot e^{(+)} \cdot DP_0 \cdot J_z + e^{(+)} \cdot DJ_z - 2 \cdot e^{(+)} \cdot DJ \cdot DP_0 \cdot \overline{D}P_0\\
&\quad - e^{(+)} \cdot DJ \cdot D\overline{D}P_0 - Q_{\frac{1}{2}} \cdot J_z - \overline{D}J \cdot DQ_{\frac{1}{2}} - DJ \cdot \overline{D}Q_{\frac{1}{2}},\\
\Phi_{Q_{\frac{1}{2}}} &:= e^{(-)} \cdot (-2 \cdot \overline{D}P_0 \cdot D\overline{D}J - \overline{D}J_z - 2 \cdot \overline{D}J \cdot DP_0 \cdot \overline{D}P_0 + \overline{D}J \cdot J + \overline{D}J \cdot D\overline{D}P_0),\\
\Phi_{\overline{D}P_0} &:= -DP_0 \cdot J_z - DJ_z - DJ \cdot J + DJ \cdot D\overline{D}P_0,\\
\Phi_{DP_0} &:= \overline{D}J_z - 2 \cdot \overline{D}J \cdot J,\\
\Phi_{J_{zz}} &:= 1,\\
\Phi_{D\overline{D}J_z} &:= 0,\\
\Phi_{J_z} &:= 2 \cdot J,\\
\Phi_{D\overline{D}J} &:= 4 \cdot J,\\
\Phi_J &:= J^2 + 3 \cdot D\overline{D}J + 3 \cdot J_z,
\end{aligned}
$$

$$\Phi_{\overline{D}J_z} := 0,$$
$$\Phi_{DJ_z} := 0,$$
$$\Phi_{\overline{D}J} := 0,$$
$$\Phi_{DJ} := (-\overline{D}J)/2.$$

This now finishes the longstanding problem of the existence of the recursion operator for the $N = 2$ supersymmetric $\alpha = 1$ KdV-equation.
Transformation of the formvaluedness to the Fréchet derivatives leads to the presentation of the recursion operator in classical form ([**17**]). We have as mentioned before checked that the form-valued vector field satisfies (38) and so indeed gives the proper recursion operator for symmetries of the $N = 2$ $\alpha = 1$ KdV-hierarchy.

3.4. Factorization of the Recursion Operator and the Bi-Hamiltonian structure. Here we shall present the factorization of the recursion operator obtained in last section.
Factorization in this respect means

$$R = J_2 \cdot J_0^{-1} \tag{41}$$

where J_2 is the second Hamiltonian structure

$$J_2 \equiv \frac{1}{2}[D, \overline{D}\,]\partial + \overline{D}JD + DJ\overline{D} + \partial J + J\partial \tag{42}$$

and J_0 will be the zero Hamiltonian structure.
We assume J_0^{-1} to be a one-form-valued function, i.e., in effect a pseudo differential operator, the pseudo part of which is realized through the Frechet derivatives of the nonlocal variables.
So we describe J_0^{-1} in a similar way as the defining function of the deformation structure

$$J_0^{-1} = \sum_\alpha \omega_\alpha \cdot \Phi_\alpha^0 \tag{43}$$

where α runs over local and nonlocal variables and Φ_α^0 being function of appropriate degree.
A rather straightforward computation does lead to the following result:
α and the associated Φ_α^0 are given by

$$\Phi^0_{P_1} := 1,$$
$$\Phi^0_{\overline{D}Q_{\frac{1}{2}}} := -e^{(-)})/2,$$
$$\Phi^0_{D\overline{Q}_{\frac{1}{2}}} := e^{(+)}/2,$$
$$\Phi^0_{D\overline{D}P_0} := 3,$$
$$\Phi^0_{P_0} := J,$$
$$\Phi^0_{\overline{Q}_{\frac{1}{2}}} := -(e^{(+)} * DP_0 + Q_{\frac{1}{2}}),$$
$$\Phi^0_{Q_{\frac{1}{2}}} := -e^{(-)} * \overline{D}P_0,$$
$$\Phi^0_{\overline{D}P_0} := -DP_0,$$

$$\Phi_J^0 := 3/2.$$

If we use now Frechet derivatives associated to the occurring nonlocal variables in the presentation, we arrive at

$$J_0^{-1} = [D, \overline{D}\,]\partial^{-1} + \partial^{-1} J_2 \partial^{-1} + \frac{1}{2}\overline{f}_{\frac{1}{2}}{}^T \partial^{-1} f_{\frac{1}{2}} - \frac{1}{2} f_{\frac{1}{2}}^T \partial^{-1} \overline{f}_{\frac{1}{2}} \tag{44}$$

where $f_{\frac{1}{2}}, \overline{f}_{\frac{1}{2}}$ are the Fréchet derivatives of $Q_{\frac{1}{2}}, \overline{Q}_{\frac{1}{2}}$ respectively.
The factorization of the recursion operator for symmetries has wider applicability in the construction of Hamiltonian operators and will be discussed more deeply elsewhere.

4. Conclusion

We gave an outline of the theory of deformations of the equation structure of differential equations, leading to the construction of recursion operators for symmetries of such equations. The extension of this theory to the nonlocal setting of differential equations is essential for getting nontrivial results. The theory has been applied to the construction of the recursion operator for symmetries for a coupled KdV–mKdV system, leading to a highly nonlocal result for this system. Moreover the appearance of nonpolynomial nonlocal terms in all results, e.g., conservation laws, symmetries and recursion operator is striking and reveals some unknown and intriguing underlying structure of the equations.

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5. Appendix: The Second layer of Nonlocalities

Here we present the second set of sixteen nonlocal variables, conservation laws and Hamiltonians.
We shall present here the results for

$$P_2,\ DP_2,\ \overline{D}P_2,\ D\overline{D}P_2;$$
$$Q_{\frac{5}{2}},\ DQ_{\frac{5}{2}},\ \overline{D}Q_{\frac{5}{2}},\ D\overline{D}Q_{\frac{5}{2}};$$
$$\overline{Q}_{\frac{5}{2}},\ D\overline{Q}_{\frac{5}{2}},\ \overline{DQ}_{\frac{5}{2}},\ D\overline{DQ}_{\frac{5}{2}};$$
$$P_3,\ DP_3,\ \overline{D}P_3,\ D\overline{D}P_3.$$

The explicit formulae for $(P_2)_z,\ (Q_{\frac{5}{2}})_z,\ (\overline{Q}_{\frac{5}{2}})_z$ are

$(P_2)_z := ((-2)\cdot(-e^{2(+)}\cdot DJ\cdot\overline{Q}_{\frac{1}{2}}\cdot J - e^{2(+)}\cdot DJ\cdot\overline{Q}_{\frac{1}{2}}\cdot D\overline{D}P_0 - e^{2(+)}\cdot DJ\cdot \overline{D}P_0\cdot D\overline{Q}_{\frac{1}{2}} - 2\cdot e^{2(+)}\cdot DJ\cdot DP_0\cdot\overline{DQ}_{\frac{1}{2}} + e^{(+)} dot\overline{D}J\cdot Q_{\frac{1}{2}}\cdot D\overline{Q}_{\frac{1}{2}} + e^{(+)}\cdot\overline{D}J\cdot DP_0\cdot J + e^{(+)}\cdot\overline{D}J\cdot DP_0\cdot D\overline{D}P_0 + e^{(+)}\cdot DJ\cdot\overline{Q}_{\frac{1}{2}}\cdot\overline{D}Q_{\frac{1}{2}} - e^{(+)}\cdot DJ\cdot\overline{D}P_0\cdot D\overline{D}P_0 - 2\cdot e^{(+)}\cdot DJ\cdot\overline{D}J + \overline{D}J\cdot Q_{\frac{1}{2}}\cdot D\overline{D}P_0 - \overline{D}J\cdot DP_0\cdot\overline{D}Q_{\frac{1}{2}}))/e^{(+)},$

$(Q_{\frac{5}{2}})_z := (2\cdot(-e^{2(+)}\cdot DP_1\cdot D\overline{D}J - e^{2(+)}\cdot DP_1\cdot J_z + e^{2(+)}\cdot DP_0\cdot D\overline{D}J\cdot D\overline{D}P_0 + e^{2(+)}\cdot DP_0\cdot J_z\cdot D\overline{D}P_0 - 2\cdot e^{2(+)}\cdot DJ\cdot\overline{D}P_0\cdot DP_1 - e^{2(+)}\cdot DJ\cdot J\cdot\cdot 2 - e^{2(+)}\cdot DJ\cdot J\cdot D\overline{D}P_0 - 2\cdot e^{2(+)}\cdot DJ\cdot D\overline{D}J - 2\cdot e^{2(+)}\cdot DJ\cdot J_z + e^{(+)}\cdot Q_{\frac{1}{2}}\cdot J\cdot D\overline{D}J + e^{(+)}\cdot Q_{\frac{1}{2}}\cdot D\overline{D}J\cdot D\overline{D}P_0 - e^{(+)}\cdot\overline{D}P_0\cdot D\overline{D}J\cdot DQ_{\frac{1}{2}} + 2\cdot e^{(+)}\cdot\overline{D}J\cdot DP_0\cdot Q_{\frac{1}{2}}\cdot J + 2\cdot e^{(+)}\cdot\overline{D}J\cdot DP_0\cdot Q_{\frac{1}{2}}\cdot D\overline{D}P_0 - e^{(+)}\cdot\overline{D}J\cdot J\cdot DQ_{\frac{1}{2}} - 2\cdot e^{(+)}\cdot\overline{D}J\cdot D\overline{D}P_0\cdot DQ_{\frac{1}{2}} - 2\cdot e^{(+)}\cdot DJ\cdot\overline{D}P_0\cdot Q_{\frac{1}{2}}\cdot D\overline{D}P_0 - 2\cdot e^{(+)}\cdot DJ\cdot\overline{D}J\cdot Q_{\frac{1}{2}} - 2\cdot\overline{D}J\cdot\overline{D}P_0\cdot Q_{\frac{1}{2}}\cdot DQ_{\frac{1}{2}}))/e^{(+)},$

$(\overline{Q}_{\frac{5}{2}})_z := (2\cdot(-e^{(+)}\cdot\overline{Q}_{\frac{1}{2}}\cdot D\overline{D}J\cdot D\overline{D}P_0 - e^{(+)}\cdot\overline{Q}_{\frac{1}{2}}\cdot J_z\cdot D\overline{D}P_0 + 2\cdot e^{(+)}\cdot DP_0\cdot\overline{D}P_0\cdot\overline{Q}_{\frac{1}{2}}\cdot J_z + 2\cdot e^{(+)}\cdot\overline{D}J\cdot DP_0\cdot\overline{Q}_{\frac{1}{2}}\cdot D\overline{D}P_0 + 2\cdot e^{(+)}\cdot DJ\cdot\overline{D}P_0\cdot\overline{Q}_{\frac{1}{2}}\cdot J - 2\cdot e^{(+)}\cdot DJ\cdot\overline{D}P_0\cdot\overline{Q}_{\frac{1}{2}}\cdot D\overline{D}P_0 - 2\cdot e^{(+)}\cdot DJ\cdot\overline{D}J\cdot\overline{Q}_{\frac{1}{2}} - e^{(+)}\cdot DJ\cdot D\overline{D}P_0\cdot\overline{DQ}_{\frac{1}{2}} - \overline{D}P_1\cdot D\overline{D}J - \overline{Q}_{\frac{1}{2}}\cdot D\overline{D}J\cdot\overline{D}Q_{\frac{1}{2}} + Q_{\frac{1}{2}}\cdot D\overline{D}J\cdot\overline{DQ}_{\frac{1}{2}} - 2\cdot\overline{D}J_z\cdot D\overline{D}P_0 - 2\cdot\overline{D}J\cdot\overline{D}P_0\cdot\overline{Q}_{\frac{1}{2}}\cdot DQ_{\frac{1}{2}} - 2\cdot\overline{D}J\cdot DP_0\cdot\overline{D}P_1 - 2\cdot\overline{D}J\cdot DP_0\cdot\overline{Q}_{\frac{1}{2}}\cdot\overline{D}Q_{\frac{1}{2}} + 2\cdot\overline{D}J\cdot DP_0\cdot Q_{\frac{1}{2}}\cdot\overline{DQ}_{\frac{1}{2}} + 2\cdot\overline{D}J\cdot DP_0\cdot\overline{D}P_0\cdot J + 3\cdot\overline{D}J\cdot J\cdot D\overline{D}P_0 - \overline{D}J\cdot DQ_{\frac{1}{2}}\cdot\overline{DQ}_{\frac{1}{2}}))/e^{(+)}.$

Other quantities can be obtained by action of D and $\overline{D}$.

The nonlocal variable P_3 results from the Poisson bracket of other Hamiltonians, i.e.,

$$\{Q_{\frac{1}{2}},\overline{Q}_{\frac{5}{2}}\}.$$

The general Poisson algebra structure of the Hamiltonians will be discussed elsewhere ([**18**]).

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