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The $N=2$ supersymmetric unconstrained matrix GNLS hierarchies

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Abstract

The generalization of the $N = 2$ supersymmetric chiral matrix $(k|n, m)$ -GNLS hierarchy to the case when matrix entries are bosonic and fermionic unconstrained $N = 2$ superfields is proposed. This is done by exhibiting the corresponding matrix Lax-pair representation in terms of $N = 2$ unconstrained superfields. It is demonstrated that when matrix entries are chiral and antichiral $N = 2$ superfields, it reproduces the $N = 2$ chiral matrix $(k|n, m)$ -GNLS hierarchy, while in the scalar case, $k = 1$, it is equivalent to the $N = 2$ supersymmetric multicomponent hierarchy. The simplest example — the $N = 2$ unconstrained $(1|1, 0)$ -GNLS hierarchy — and its reduction to the $N = 2$ supersymmetric $\alpha = 1$ KdV hierarchy are discussed in more detail, and its rich symmetry structure is uncovered.

Keywords: Completely integrable systems; Supersymmetry; Discrete symmetries
AMS Subject Classification: 35Q51; 35Q53; 35Q58

1 Introduction

The $N = 2$ chiral matrix $(k|n, m)$ -Generalized Nonlinear Schrödinger (MGNLS) hierarchies were introduced in [1]. Their group-theoretical origin was clarified in [2] where it has been demonstrated that they are related to the $N = 2$ $sl(n|n-1)$ affine superalgebras via the coset construction. The $N = 2$ chiral $(k|n, m)$ -MGNLS hierarchies comprise rectangular matrix-valued *constrained* (chiral and antichiral) bosonic and fermionic $N = 2$ superfields. The aim of the present Letter is to construct their *integrable* generalizations in the case when matrix entries are *unconstrained* $N = 2$ superfields. It turns out that such generalizations indeed exist, and we call them the $N = 2$ supersymmetric unconstrained $(k|n, m)$ -MGNLS hierarchies in what follows.

The Letter is organized as follows. In Section 2 we introduce the $N = 2$ unconstrained $(k|n, m)$ -MGNLS hierarchy, discuss its properties and construct relevant quantities. In Section 3 we analyse its bosonic limit as well as a correspondence of some of its limited cases to known hierarchies. In particular, we discuss the reduction of the $N = 2$ unconstrained $(1|1, 0)$ -MGNLS hierarchy to the $N = 2$ supersymmetric $\alpha = 1$ KdV hierarchy [3, 4] and uncover a rich symmetry structure of the latter. In Section 4 we summarize our results.

2 The $N = 2$ unconstrained $(k|n, m)$ -MGNLS hierarchies

In this section we introduce the Lax-pair representation of the $N = 2$ unconstrained $(k|n, m)$ -MGNLS hierarchy, construct its conserved quantities and different complex conjugations.

2.1 Lax-pair representation

We propose the following Lax-pair representation for the bosonic flows of the $N = 2$ supersymmetric unconstrained $(k|n, m)$ -MGNLS hierarchies:

$$\frac{\partial}{\partial t_p} L = [A_p, L], \quad L = I\partial + \frac{1}{2}FDD\bar{\partial}^{-1}\bar{F}, \quad A_p = (L^p)_{\geq 0} + \text{res}(L^p), \quad p \in \mathbb{N} \quad (1)$$

generating the abelian algebra of the flows

$$\left[\frac{\partial}{\partial t_m}, \frac{\partial}{\partial t_n} \right] = 0 \quad (2)$$

where the subscript ≥ 0 denotes the sum of purely differential and constant parts of the operator L^p , and $\text{res}(L^p)$ is its $N = 2$ supersymmetric residue, i.e. the coefficient at $[D, \bar{D}]\partial^{-1}$. Here, $F \equiv F_{Aa}(Z)$ and $\bar{F} \equiv \bar{F}_{aA}(Z)$ ($A, B = 1, \dots, k$; $a, b = 1, \dots, n+m$) are rectangular matrices which entries are unconstrained $N = 2$ superfields, I is the unity matrix, $I \equiv \delta_{A,B}$, and the matrix product is understood, for example, as $(F\bar{F})_{AB} \equiv \sum_a F_{Aa}\bar{F}_{aB}$. The matrix entries are bosonic superfields for $a = 1, \dots, n$ and fermionic superfields for $a = n+1, \dots, n+m$, i.e., $F_{Aa}\bar{F}_{bB} = (-1)^{d_a \bar{d}_b} \bar{F}_{bB} F_{Aa}$ where d_a and \bar{d}_b are the Grassmann parities of the matrix elements F_{Aa} and \bar{F}_{bB} , respectively, $d_a = 1$ ($d_a = 0$) for fermionic (bosonic) entries; $Z = (z, \theta, \bar{\theta})$ is a coordinate in the $N = 2$ superspace, $dZ \equiv dzd\theta d\bar{\theta}$ and D, \bar{D} are the $N = 2$ supersymmetric fermionic covariant derivatives

$$D = \frac{\partial}{\partial \theta} - \frac{1}{2}\bar{\theta}\frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial z}, \quad D^2 = \bar{D}^2 = 0, \quad \{D, \bar{D}\} = -\frac{\partial}{\partial z} \equiv -\partial. \quad (3)$$

The chosen grading guarantees that the Lax operator L (1) is Grassman even [1].

We have verified for first few non-trivial flows that the Lax-pair representation (1) consistently provides their locality and leads to the following general flow equations for F and \bar{F} :

$$\frac{\partial}{\partial t_p} F = (A_p F)_0, \quad \frac{\partial}{\partial t_p} \bar{F}^T = -(A_p^T \bar{F}^T)_0 \quad (4)$$

which explicitly are¹:

$$\begin{aligned} \frac{\partial}{\partial t_0} F &= F, & \frac{\partial}{\partial t_0} \bar{F} &= -\bar{F}; & \frac{\partial}{\partial t_1} F &= F', & \frac{\partial}{\partial t_1} \bar{F} &= \bar{F}'; \\ \frac{\partial}{\partial t_2} F &= F'' + F(D\bar{D}\bar{F}F), & \frac{\partial}{\partial t_2} \bar{F} &= -\bar{F}'' + (\bar{D}D\bar{F}F)\bar{F}; \\ \frac{\partial}{\partial t_3} F &= F''' + \frac{3}{2}[F'(D\bar{D}\bar{F}F) + F(D\bar{D}\bar{F}F')] - \frac{1}{2}F(D\mathcal{I}^2\bar{F}F\bar{D}\mathcal{I}^2\bar{F}F), \\ \frac{\partial}{\partial t_3} \bar{F} &= \bar{F}''' - \frac{3}{2}[(\bar{D}D\bar{F}F)\bar{F}' + (\bar{D}D\bar{F}'F)\bar{F}] + \frac{1}{2}(\bar{D}(D\bar{F}F)\bar{F}F)\bar{F}. \end{aligned} \quad (5)$$

Here, we have introduced the matrix \mathcal{I} ,

$$\mathcal{I} \equiv (i)^{\bar{d}_a} \delta_{ab}, \quad \mathcal{I}^2 = (-1)^{\bar{d}_a} \delta_{ab}, \quad \mathcal{I}^4 = I, \quad (6)$$

' denotes the derivative with respect to z and we use the following standard convention regarding the operator conjugation (transposition) ^T

$$\begin{aligned} (\partial, D, \bar{D})^T &= -(\partial, D, \bar{D}), & (D\bar{D})^T &= -\bar{D}D, \\ \left(\sum_a F_{Aa}\bar{F}_{aB}\right)^T &= \sum_a (-1)^{d_a} (\bar{F}_{aB})^T (F_{Aa})^T. \end{aligned} \quad (7)$$

For completeness, let us also present expressions for the corresponding operators A_p , $res(L^p)$ and $(L^p)_0$,

$$\begin{aligned} A_0 &= 1, & A_1 &= \partial, & A_2 &= I\partial^2 + F D\bar{D}\bar{F}, \\ A_3 &= I\partial^3 + \frac{3}{2}F'D\bar{D}\bar{F} + \frac{3}{2}F D\bar{D}\bar{F}\partial - \frac{3}{4}F D\mathcal{I}^2\bar{F}F\bar{D}\mathcal{I}^2\bar{F} \end{aligned} \quad (8)$$

and

$$\begin{aligned} res(L) &= \frac{1}{4}F\bar{F}, & res(L^2) &= \frac{1}{4}(F'\bar{F} - F\bar{F}' - \frac{1}{2}(F\bar{F})^2), \\ res(L^3) &= \frac{1}{4}(F\bar{F})'' + \frac{1}{8}[F\bar{F}, (F\bar{F})'] \\ &+ \frac{3}{4}(-F'\bar{F}' + \frac{1}{2}F(D\bar{D}\bar{F}F)_0\bar{F} + \frac{1}{2}F\bar{F}'F\bar{F} + \frac{1}{12}(F\bar{F})^3) \end{aligned} \quad (9)$$

as well as

$$\begin{aligned} L_0 &= -res(L), & (L^2)_0 &= F(D\bar{D}\bar{F})_0 - res(L^2), \\ (L^3)_0 &= \frac{3}{2}F'(D\bar{D}\bar{F})_0 - \frac{3}{4}F(D\mathcal{I}^2\bar{F}F\bar{D}\mathcal{I}^2\bar{F})_0 - res(L^3), \end{aligned} \quad (10)$$

¹Hereafter, the subscript 0 denotes the constant part of the corresponding operators, and we also use the notation $(\mathbb{O}f)$ for an operator \mathbb{O} acting only on a function f inside the brackets.

respectively, which will be useful in what follows.

Let us note that besides the global $N = 2$ supersymmetry the Lax operator L (1) and the flows (4–5) of the $N = 2$ unconstrained $(k|n, m)$ -MGNLS hierarchy are obviously invariant with respect to the direct product of the (super)groups $GL(k) \times GL(n|m)$, and the matrices L_{AB} , $(\overline{F}_{aB}) F_{Ab}$ realize their (anti)fundamental representations over the $GL(k)$ -indices (B) A and the $GL(n|m)$ -indices (b) a .

The Lax operator L (1) contains the constant part over the derivative ∂

$$L_0 = -\frac{1}{4}F\overline{F}. \quad (11)$$

Let us discuss another, gauge-related Lax-pair representation

$$\frac{\partial}{\partial t_p}\tilde{L} = [\tilde{A}_p, \tilde{L}], \quad \tilde{L} = GLG^{-1} \quad \tilde{A}_p = G\left(A_p + G^{-1}\frac{\partial}{\partial t_p}G\right)G^{-1} \quad (12)$$

which is fixed by a requirement that the gauge-transformed Lax operator \tilde{L} does not contain the constant part. The latter leads to the following equation for the matrix of the gauge transformation $G \equiv (G)_{AB}$:

$$G^{-1}G' = L_0. \quad (13)$$

In order to find an equation for the quantity $G^{-1}\frac{\partial}{\partial t_p}G$ entering into eq. (12) we differentiate eq. (13) over $\frac{\partial}{\partial t_p}$, then substitute

$$\frac{\partial}{\partial t_p}L_0 = -(res(L^p))' + 2[res(L^p), L_0] \quad (14)$$

which results from the constant part of the Lax-pair representation (1), and finally have

$$(G^{-1}\frac{\partial}{\partial t_p}G)' - [G^{-1}\frac{\partial}{\partial t_p}G, L_0] = -(res(L^p))' + 2[res(L^p), L_0]. \quad (15)$$

A simple inspection of this equation shows that it has an obvious local solution

$$G^{-1}\frac{\partial}{\partial t_p}G = -res(L^p) \quad (16)$$

in the commutative, scalar case, i.e. for the $N = 2$ supersymmetric unconstrained $(1|n, m)$ -MGNLS hierarchies, and, consequently, the operator \tilde{A}_p (12) becomes

$$\tilde{A}_p = (\tilde{L}^p)_{\geq 0} \quad (17)$$

using the identity

$$G(L^p)_{\geq 0}G^{-1} \equiv (GL^pG^{-1})_{\geq 0}. \quad (18)$$

Though for the noncommutative, matrix case, i.e. for the $N = 2$ unconstrained $(k \geq 2|n, m)$ -MGNLS hierarchies, the solution for $G^{-1}\frac{\partial}{\partial t_p}G$ is nonlocal in general, and as a consequence the Lax-pair representation (12) is nonlocal as well. In this respect the *local* Lax-pair representation (1) we started with is rather *exceptional*.

To close this subsection, we would like to remark that integrability conditions (2) for the equations (14) read

$$\frac{\partial}{\partial t_n}res(L^m) - \frac{\partial}{\partial t_m}res(L^n) + 2[res(L^m), res(L^n)] = 0, \quad (19)$$

therefore, $res(L^m)$ can consistently be represented in terms of the single matrix $X \equiv X_{AB}$

$$res(L^m) = -\frac{1}{2}X^{-1}\frac{\partial}{\partial t_m}X \quad (20)$$

which will be useful in what follows (see, subsection 2.3).

2.2 Hamiltonians

The infinite set of Hamiltonians can be defined as:

$$H_p = 4 \int dZ \mathcal{H}_p, \quad \mathcal{H}_p \equiv \text{tr}(\text{res}(L^p)) \quad (21)$$

where tr is the usual matrix trace. Their conservation is the obvious consequence of the Lax–pair representation (1). By construction, these Hamiltonians presumably correspond to the flows $\frac{\partial}{\partial t_p}$ (1) via the corresponding Hamiltonian structure (if any) as usually, and in this case they have to form an abelian algebra because of the well–known homomorphism between algebra of flows, which is the abelian algebra (2) in the case under consideration, and algebra of their Hamiltonians. Substituting expressions for $\text{res}(L^p)$ (9) into eq. (21) one can obtain few first Hamiltonians from the set (21)

$$\begin{aligned} H_1 &= \int dZ \text{tr}(F\bar{F}), \quad H_2 = -2 \int dZ \text{tr}(F\bar{F}' + \frac{1}{4}(F\bar{F})^2), \\ H_3 &= 3 \int dZ \text{tr}(F\bar{F}'' + \frac{1}{2}F(D\bar{D}\bar{F}F)\bar{F} + \frac{1}{2}F\bar{F}'F\bar{F} + \frac{1}{12}(F\bar{F})^3). \end{aligned} \quad (22)$$

Besides the Hamiltonians H_p (21) there exist other conserved quantities of the flows (5) which presumably form a non-abelian algebra. Thus, one can verify by direct calculation that the supermatrix functionals

$$H_{ab} \equiv \int dZ (\bar{F}F)_{ab} \quad (23)$$

and the superfield

$$H_0 \equiv \int dz \text{tr}(F\bar{F}) \quad (24)$$

are integrals of the flows (5) as well. Furthermore, at $n = 0$ or $m = 0$ the superfield

$$\bar{H}_0 \equiv \int dz \text{tr}(\bar{F}F) \quad (25)$$

is also the integral of the flows (5). The integrals H_{ab} (23) are fermionic (bosonic) ones when the indices a, b belong to the following ranges: $1 \leq a \leq n$ and $n + 1 \leq b \leq n + m$, or $n + 1 \leq a \leq n + m$ and $1 \leq b \leq n$ ($1 \leq a, b \leq n$, or $n + 1 \leq a, b \leq n + m$). Therefore, it is natural to suppose that besides the bosonic flows (4–5) and two fermionic flows of the $N = 2$ supersymmetry, originated from the superfield Hamiltonian H_0 (24), the $N = 2$ unconstrained $(k|n, m)$ –MGNLS hierarchy possesses additional series of fermionic and bosonic flows, related to the local fermionic and bosonic integrals H_{ab} (23) via the corresponding Hamiltonian structure, and that just the algebra of these integrals is the $gl(n|m)$ superalgebra (see the discussion at the paragraph after eq. (10)).

2.3 Involutions

We restrict our considerations to the case when iz , θ and $\bar{\theta}$ are coordinates of the real $N = 2$ superspace which satisfy the following standard complex conjugation properties:

$$(iz, \theta, \bar{\theta})^* = (iz, \bar{\theta}, \theta) \quad (26)$$

where i is the imaginary unity. We will also use the standard convention regarding complex conjugation $*$ of products involving odd operators and functions. In particular, if \mathbb{O} is some even differential operator acting on a superfield F , we define the complex conjugate of \mathbb{O} by $(\mathbb{O}F)^* = \mathbb{O}^*F^*$. Then, in the case under consideration one can derive, for example, the following relations

$$\begin{aligned} (F_{Aa}\bar{F}_{bB})^* &= \bar{F}_{bB}^*F_{Aa}^*, & (\epsilon, \bar{\epsilon})^* &= (\bar{\epsilon}, \epsilon), & (\epsilon\bar{\epsilon})^* &= \epsilon\bar{\epsilon}, \\ \partial^* &= -\partial, & (\epsilon D, \bar{\epsilon}\bar{D})^* &= (\bar{\epsilon}\bar{D}, \epsilon D), & (D\bar{D})^* &= -\bar{D}D \end{aligned} \quad (27)$$

which we use in what follows. Here, ϵ and $\bar{\epsilon}$ are constant odd parameters.

Direct verification shows that the evolution equations (5) admit two different complex conjugations

$$(F^*, \bar{F})^* = (\bar{F}^T, F^T), \quad t_p^* = -t_p, \quad (iz, \theta, \bar{\theta})^* = (iz, \bar{\theta}, \theta) \quad (28)$$

and

$$(F, \bar{F})^* = (XF\mathcal{I}, \mathcal{I}\bar{F}X^{-1}), \quad t_p^* = (-1)^p t_p, \quad (iz, \theta, \bar{\theta})^* = (iz, \bar{\theta}, \theta) \quad (29)$$

where the matrices \mathcal{I} and X are defined in eqs. (6) and (20), respectively. Using eqs. (20) and (27–29) one can derive the following involution properties of the matrix X :

$$X^* = (X^T)^{-1}, \quad X^\star = X^{-1}. \quad (30)$$

3 Reductions and limited cases of the $N = 2$ unconstrained $(k|n, m)$ –MGNLS hierarchy

In this section we discuss different reductions and particular, limited cases of the $N = 2$ unconstrained $(k|n, m)$ –MGNLS hierarchy as well as their correspondence to known hierarchies.

3.1 Relations to the $N = 2$ chiral $(k|n, m)$ –MGNLS and $N = 2$ multi-component hierarchies

For the case, when F and \bar{F} are constrained to be chiral and antichiral rectangular matrix-valued $N = 2$ superfields, i.e.

$$DF = 0, \quad \bar{D}\bar{F} = 0, \quad (31)$$

respectively, the Lax operator L (1) of the $N = 2$ unconstrained $(k|n, m)$ –MGNLS hierarchy reproduces the Lax operator of the $N = 2$ chiral $(k|n, m)$ –MGNLS hierarchy [1] on the subspace of the chiral wave function Ψ ,

$$L\Psi \equiv \left(I\partial - F\bar{F} - F\bar{D}\partial^{-1}(D\bar{F})_0 \right)\Psi, \quad D\Psi = 0. \quad (32)$$

Therefore, at the reduction (31) our hierarchy is equivalent to the $N = 2$ chiral $(k|n, m)$ –MGNLS hierarchy.

In the very particular, scalar case, i.e. at $k = 1$, the $N = 2$ unconstrained $(1|n, m)$ -MGNLS hierarchy is equivalent to the $N = 2$ supersymmetric multicomponent hierarchy [5]. Indeed, in the new superfield basis $\{\tilde{F}, \tilde{\bar{F}}\}$, defined as

$$\tilde{F} \equiv -\frac{i}{2} F e^{-\frac{1}{4} \partial^{-1}(F\bar{F})}, \quad \tilde{\bar{F}} \equiv \frac{1}{2} \bar{F} e^{+\frac{1}{4} \partial^{-1}(F\bar{F})}, \quad (33)$$

the gauge-transformed Lax operator \tilde{L} (12)

$$\tilde{L} \equiv e^{-\frac{1}{4} \partial^{-1}(F\bar{F})} L e^{+\frac{1}{4} \partial^{-1}(F\bar{F})} = \partial + i\tilde{F}[D, \bar{D}]\partial^{-1}\tilde{\bar{F}} \quad (34)$$

reproduces the Lax operator of the $N = 2$ supersymmetric multicomponent hierarchy [5]. At this point we would like to especially underline that as concerns to the general, matrix case $k > 1$, the $N = 2$ unconstrained $(k|n, m)$ -MGNLS hierarchy to our best knowledge is introduced here for the first time.

As a byproduct of this consideration, we have also established the correspondence between the $N = 2$ GNLS ($N = 2$ chiral $(1|n, m)$ -MGNLS) hierarchy of ref. [6] and the $N = 2$ multicomponent hierarchy of ref. [5]: the former hierarchy is related to the latter by the reduction constraints (31) and the basis transformation (33) (see, also the corresponding discussion in ref. [7]).

3.2 Bosonic limit

Now we would like to discuss the bosonic limit of the $N = 2$ unconstrained $(k|0, m)$ -MGNLS hierarchy using its second flow equations $\frac{\partial}{\partial t_2}$ (5) with the pure fermionic matrices F, \bar{F} , and establish their relationship with the bosonic matrix $gl(2k+m)/(gl(2k) \times gl(m))$ -NLS equations introduced in [8].

To derive the bosonic limit of the $N = 2$ supersymmetric unconstrained $(k|0, m)$ -MGNLS hierarchy, let us define the matrix components of the fermionic superfield matrices as

$$\begin{aligned} f &\equiv \begin{pmatrix} D & F| \\ \bar{D} & F| \end{pmatrix}, & \bar{f} &\equiv \begin{pmatrix} -\bar{D} & \bar{F}| \\ D & \bar{F}| \end{pmatrix}, \\ \psi &\equiv \begin{pmatrix} F| \\ \bar{D}D & F| \end{pmatrix}, & \bar{\psi} &\equiv \begin{pmatrix} \bar{F}| \\ D\bar{D} & \bar{F}| \end{pmatrix} \end{aligned} \quad (35)$$

where $|$ means the $(\theta, \bar{\theta}) \rightarrow 0$ limit. So, ψ and $\bar{\psi}$ are fermionic matrix components, while f and \bar{f} are bosonic ones. To get the bosonic limit we have to put the fermionic matrices ψ and $\bar{\psi}$ equal to zero. This leaves us with the following set of matrix equations

$$\frac{\partial}{\partial t_2} f = f'' - f\bar{f}f, \quad \frac{\partial}{\partial t_2} \bar{f} = -\bar{f}'' + \bar{f}f\bar{f} \quad (36)$$

for the bosonic matrix components f and \bar{f} . The derived equations (36) reproduce the bosonic matrix NLS equations which can be elaborated via the $gl(2k+m)/(gl(2k) \times gl(m))$ -coset construction [8]. They can be viewed as the second flow of the bosonic matrix NLS hierarchies with the matrix Lax operators \mathcal{L}_1

$$\mathcal{L}_1 = I\partial - \frac{1}{2}f\partial^{-1}\bar{f} \quad (37)$$

which can easily be derived from the Lax operator (1) in the bosonic limit.

Thus we are led to the conclusion that the $N = 2$ supersymmetric unconstrained $(k|0, m)$ -MGNLS hierarchy is the $N = 2$ superextension of the bosonic matrix $gl(2k + m)/(gl(2k) \times gl(m))$ -NLS hierarchy. At this point let us remark the difference of the $N = 2$ unconstrained $(k|0, m)$ -MGNLS hierarchy comparing to the $N = 2$ chiral $(r|0, m)$ -MGNLS hierarchy: the latter corresponds to the $N = 2$ superextension of the $gl(r + m)/(gl(r) \times gl(m))$ -NLS hierarchy [1]. Therefore, at even value of r , $r = 2k$, the $N = 2$ chiral $(2k|0, m)$ -MGNLS hierarchy and $N = 2$ unconstrained $(k|0, m)$ -MGNLS hierarchy are two different $N = 2$ superextensions of the same bosonic matrix hierarchy — $gl(2k + m)/(gl(2k) \times gl(m))$ -NLS hierarchy. It seems these two $N = 2$ superextensions are not equivalent in general because of different length dimensions of their fermionic superfield components, but, this question requires a more careful analysis which is out of the scope of the present Letter.

3.3 Reduction of the $N = 2$ unconstrained $(1|1, 0)$ -MGNLS hierarchy

The $N = 2$ unconstrained $(1|1, 0)$ -MGNLS hierarchy involves two bosonic unconstrained $N = 2$ superfields $F(Z)$ and $\bar{F}(Z)$, and does not admit fermionic integrals (23) (see the paragraph after eqs. (24)). Nevertheless, there is some hidden possibility for generating other fermionic integrals at its reduction which we discuss in this subsection.

With this aim let us introduce the new superfield basis $\{J, \bar{J}\}$, defined as²

$$\begin{aligned} J &\equiv \frac{1}{2}F\bar{F} - (\ln F) ', & \bar{J} &\equiv (\ln F) ', \\ \bar{F} &= 2(J + \bar{J})e^{-\partial^{-1}\bar{J}}, & F &= e^{\partial^{-1}\bar{J}}, \end{aligned} \quad (38)$$

in which the second and third bosonic flows (5) as well as the Hamiltonians (22) and (24) become

$$\frac{\partial}{\partial t_2} J = (-[D, \bar{D}]J - J^2 + 2\bar{D}D\bar{J}) ', \quad \frac{\partial}{\partial t_2} \bar{J} = (+[D, \bar{D}]\bar{J} + \bar{J}^2 + 2D\bar{D}J) ' \quad (39)$$

and

$$\begin{aligned} \frac{\partial}{\partial t_3} J &= [J'' + 3J[D, \bar{D}]J + J^3 + 3(DJ)(\bar{D}J) - 3(D\bar{J})\bar{D}\bar{J} + 3(\bar{J} - J)\bar{D}D\bar{J}] ', \\ \frac{\partial}{\partial t_3} \bar{J} &= [\bar{J}'' + 3\bar{J}[D, \bar{D}]\bar{J} + \bar{J}^3 - 3(DJ)(\bar{D}J) + 3(D\bar{J})\bar{D}\bar{J} + 3(\bar{J} - J)D\bar{D}J] ', \end{aligned} \quad (40)$$

as well as

$$\begin{aligned} H_0 &= 2 \int dz J, & H_2 &= -2 \int dZ (J^2 - \bar{J}^2), \\ H_3 &= 6 \int dZ [J\bar{J}' + \frac{1}{2}(J + \bar{J})[D, \bar{D}](J + \bar{J}) + \frac{1}{3}(J + \bar{J})(J^2 - J\bar{J} + \bar{J}^2)], \end{aligned} \quad (41)$$

respectively, and admit the following complex conjugations

$$(J, \bar{J})^* = (J + (\ln(J + \bar{J}))', \bar{J} - (\ln(J + \bar{J}))'), \quad t_p^* = -t_p, \quad (iz, \theta, \bar{\theta})^* = (iz, \bar{\theta}, \theta), \quad (42)$$

$$(J, \bar{J})^* = (\bar{J}, J), \quad t_p^* = (-1)^p t_p, \quad (iz, \theta, \bar{\theta})^* = (iz, \bar{\theta}, \theta) \quad (43)$$

²We assume the following inverse length dimensions of the involved superfields: $[J] = [\bar{J}] = [\bar{F}] = 1$ and $[F] = 0$.

where at deriving eqs. (42–43) we have used eqs. (28–29) and (38) as well as

$$X = e^{-\frac{1}{2}\partial^{-1}(F\bar{F})} \quad (44)$$

which results from eqs. (9), (16) and (20) for the hierarchy under consideration.

The odd bosonic flows $\frac{\partial}{\partial t_{2p+1}}$ of the hierarchy under consideration and the involution (43) are consistent with the following reduction constraint:

$$J = \bar{J} \quad (45)$$

which has the following form:

$$\bar{F} = -\left(\frac{4}{F}\right)' \quad (46)$$

in terms of the original superfields F and \bar{F} (38). Let us remark that in another superfield basis this reduction was discussed in [5], and it is equivalent to the requirement that even Hamiltonians H_{2p} (21) of the hierarchy are subjected equal to zero (see, e.g. the Hamiltonian H_2 (41)).

Now, one can easily verify that the composite fermionic superfield

$$\mathcal{I}_{\frac{3}{2}} \equiv FDF' - F'DF \quad (47)$$

satisfies the following important relation

$$\frac{\partial}{\partial t_3}\mathcal{I}_{\frac{3}{2}} = \left[F^2(DJ'' - 6(DJ)\bar{D}DJ + JDJ' - J'DJ + J^2DJ) \right]' \quad (48)$$

on the constraint shell (46), therefore, both the third flow $\frac{\partial}{\partial t_3}$ (5) and the whole reduced hierarchy possess the fermionic superfield integral

$$I_{\frac{1}{2}} = \int dz \mathcal{I}_{\frac{3}{2}} \quad (49)$$

as well as its complex conjugated quantity

$$I_{\frac{1}{2}}^* = - \int dz \mathcal{I}_{\frac{3}{2}}^*, \quad \mathcal{I}_{\frac{3}{2}}^* \equiv \frac{1}{F^4}(F\bar{D}F' - F'\bar{D}F), \quad (50)$$

$$\frac{\partial}{\partial t_3}\mathcal{I}_{\frac{3}{2}}^* = \left[\frac{1}{F^2}(\bar{D}J'' + 6(\bar{D}J)D\bar{D}J - J\bar{D}J' + J'\bar{D}J + J^2\bar{D}J) \right]' \quad (51)$$

which are nonlocal and even nonpolynomial in the basis $\{J, \bar{J}\}$ (38) where their densities are

$$\mathcal{I}_{\frac{3}{2}} = e^{+2\partial^{-1}J}DJ, \quad \mathcal{I}_{\frac{3}{2}}^* = e^{-2\partial^{-1}J}\bar{D}J. \quad (52)$$

Here, the subscripts denote inverse length dimensions, and when calculating eqs. (50–51) we have applied the complex conjugation (29) with the function X (44) (restricted to the constraint shell (46)) to eqs. (47–49),

$$F^* = \frac{1}{F}, \quad J^* = J, \quad t_p^* = (-1)^p t_p, \quad (iz, \theta, \bar{\theta})^* = (iz, \bar{\theta}, \theta). \quad (53)$$

We would like to remark that in another field basis the superfield components of the integrals (52) were derived recently in [9, 10] by a tedious symmetry analysis using computer calculations, but their $N = 2$ superfield structure and origin were not clarified there.

On the constraint shell (45) the third flow (40) and the third Hamiltonian H_3 (41) become

$$\frac{\partial}{\partial t_3} J = [J'' + 3J[D, \bar{D}]J + J^3]' \quad (54)$$

and

$$H_3 = 12 \int dZ (J[D, \bar{D}]J + \frac{1}{3}J^3), \quad (55)$$

respectively, and one can easily recognize that they reproduce the third flow and the third Hamiltonian of the $N = 2$ supersymmetric $\alpha = 1$ KdV hierarchy possessing the $N = 2$ superconformal algebra as the second Hamiltonian structure [3]. Therefore, we are led to the conclusion that the $N = 2$ $\alpha = 1$ KdV hierarchy possesses hidden fermionic superfield integrals (52) as well as corresponding flows. Actually, the integrals (49–50) are only the first representatives of the series of nonlocal fermionic and bosonic superfield integrals arising at the reduction (46). Their more detailed consideration as well as their role for more deep understanding and more detail description of the $N = 2$ $\alpha = 1$ KdV hierarchy is out of the scope of the present Letter and will be discussed in [11].

To close this section we would like to remark that the established relation between the $N = 2$ unconstrained $(1|1, 0)$ -MG-NLS and $N = 2$ $\alpha = 1$ KdV hierarchies allows to derive the following nice formula for the flows $\frac{\partial}{\partial t_{2p+1}}$ and the corresponding Hamiltonian densities \mathcal{H}_{2p+1} of the latter hierarchy

$$\begin{aligned} \frac{\partial}{\partial t_{2p+1}} J &= \mathcal{H}'_{2p+1} = (\text{res}(L^{2p+1}))' \equiv (\text{res}(\tilde{L}^{2p+1}))', \\ \tilde{L} &\equiv e^{-(\partial^{-1}J)} L e^{(\partial^{-1}J)} = \partial + [D, \bar{D}]\partial^{-1}J \end{aligned} \quad (56)$$

where when deriving we have used the equations (14), (21), (38), (45–46) as well as the identity

$$\text{res}(f^{-1}\circ f) \equiv \text{res}(\circ) \quad (57)$$

which is valid for any $N = 2$ pseudo-differential operator \circ and superfield f .

4 Summary

In this Letter we have proposed a wide class of new $N = 2$ supersymmetric hierarchies — the $N = 2$ supersymmetric unconstrained $(k|n, m)$ -MG-NLS hierarchy — by exhibiting the corresponding super Lax-pair representation (1) in terms of matrix-valued $N = 2$ unconstrained superfields. Then we have explicitly calculated its first nontrivial flows (4–5) and conserved quantities (21–25). Furthermore we have constructed its two different admissible involutions (28–29). Then we have discussed its different limited cases and reductions as well as the correspondence to already known hierarchies. Finally we have analysed its simplest representative — the $N = 2$ unconstrained $(1|1, 0)$ -GNLS hierarchy — and the reduction to the $N = 2$ supersymmetric $\alpha = 1$ KdV hierarchy, and a rich symmetry structure of the latter is uncovered.

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