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Toughness and hamiltonicity in k-trees

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Abstract

We consider toughness conditions that guarantee the existence of a hamiltonian cycle in k-trees, a subclass of the class of chordal graphs. By a result of Chen et al. 18-tough chordal graphs are hamiltonian, and by a result of Bauer et al. there exist nontraceable chordal graphs with toughness arbitrarily close to $\frac{7}{4}$. It is believed that the best possible value of the toughness guaranteeing hamiltonicity of chordal graphs is less than 18, but the proof of Chen et al. indicates that proving a better result could be very complicated. We show that every 1-tough 2-tree on at least three vertices is hamiltonian, a best possible result since 1-toughness is a necessary condition for hamiltonicity. We generalize the result to k-trees for $k \geq 2$: Let G be a k-tree. If G has toughness at least $\frac{k+1}{3}$, then G is hamiltonian. Moreover, we present infinite classes of nonhamiltonian 1-tough k-trees for each $k \geq 3$.

Key words: toughness, t-tough graph, hamiltonian graph, traceable graph, chordal graph, k-tree, complexity
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1 Introduction

We begin with a brief section on terminology and notation and then motivate our results by a number of recent papers. A good reference for any undefined terms in

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graph theory is [7] and in complexity theory is [13]. We consider only undirected graphs with no loops and no multiple edges.

1.1 Basic terminology and notation

Let $\omega(G)$ denote the number of components of a graph G. A graph G is ***t*-tough** if $|S| \geq t\omega(G-S)$ for every subset S of the vertex set $V(G)$ with $\omega(G-S) > 1$. The **toughness** of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). Hence if G is not complete, $\tau(G) = \min\{|S|/\omega(G-S)\}$, where the minimum is taken over all cutsets S of vertices in G. In [18], Plummer defined a set $S \subseteq V(G)$ to be a **tough set** if $\tau(G) = |S|/\omega(G-S)$. A graph G is **hamiltonian** if G contains a **hamiltonian cycle** (a cycle containing every vertex of G); G is **traceable** if it admits a path containing every vertex. A ***k*-factor** of a graph is a k-regular spanning subgraph. Of course, a hamiltonian cycle is a (connected) 2-factor. Let S be a nonempty subset of $V(G)$. The subgraph of G with vertex set S and edge set consisting of all edges in G with both ends in S is called the **subgraph of** G **induced by** S and is denoted by $G[S]$. For a proper subset $S \subset V(G)$, we let $G-S$ denote the subgraph of G induced by $V(G) \setminus S$. If $S = \{x\}$, then we use $G-x$ instead of $G-\{x\}$. We say a graph G is **chordal** if G contains no chordless cycle of length at least four. It is well-known that chordal graphs have a nice elimination property: a chordal graph G on at least two vertices contains a **simplicial vertex** v, i.e. all neighbors of v are mutually adjacent, such that $G-v$ is again a chordal graph. A subclass of chordal graphs that plays a central role in this paper is the class of k-trees. We define it according to the elimination property. The only difference with chordal graphs is that at each step in the elimination, the simplicial vertex has the same degree in the present graph. Let k be a positive integer. Then we define a ***k*-tree** as follows: K_k is the smallest k-tree, and a graph G on at least $k+1$ vertices is a k-tree if and only if it contains a simplicial vertex v with degree k such that $G-v$ is a k-tree; for convenience, we say that v is ***k*-simplicial** in this case. Clearly, 1-trees are just trees.

1.2 Motivation

We begin our motivation with the 1973 paper in which Chvátal [10] introduced the definition of toughness. From the definition it is clear that being 1-tough is a necessary condition for a graph to be hamiltonian. In [10] Chvátal conjectured that there exists a finite constant t_0 such that every t_0-tough graph is hamiltonian. For many years, however, the focus was on determining whether all 2-tough graphs are hamiltonian. We now know that not all 2-tough graphs are hamiltonian, as indicated by the result below.

Theorem 1 [2]. *For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$-tough nontraceable graph.*

1.2.1 Special graph classes

Chvátal [10] obtained $\left(\frac{3}{2} - \epsilon\right)$-tough graphs without a 2-factor for arbitrary $\epsilon > 0$. These examples are all chordal. Recently it was shown in [4] that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Based on this, Kratsch [15] raised the question whether every $\frac{3}{2}$-tough chordal graph is hamiltonian. In [2] it has been shown there exists an infinite class of chordal graphs with toughness close to $\frac{7}{4}$ having no hamiltonian path. Hence $\frac{3}{2}$-tough chordal graphs need not be hamiltonian. However for other classes of perfect graphs (for definitions, see [6]), being 1-tough is already sufficient to ensure hamiltonicity. For example, in [14] it was shown (implicitly) that 1-tough interval graphs are hamiltonian, and in [11] it was shown that 1-tough cocomparability graphs are hamiltonian. However in [5] it was proven that for chordal planar graphs, 1-toughness does not ensure hamiltonicity. The following result was established, however.

Theorem 2 [5]. *Let G be a chordal, planar graph with $\tau > 1$. Then G is hamiltonian.*

Furthermore, all 1-tough $K_{1,3}$-free chordal graphs are hamiltonian. This follows from the well-known result of Matthews and Sumner [17] relating toughness and vertex connectivity in $K_{1,3}$-free graphs, and a result of Balakrishnan and Paulraja [1] showing that 2-connected $K_{1,3}$-free chordal graphs are hamiltonian.

Let us now consider $\frac{3}{2}$-tough chordal graphs. We have already seen that such graphs need not be hamiltonian. However for a certain subclass of chordal graphs, namely split graphs, we have a different result. A graph G is called a **split graph** if $V(G)$ can be partitioned into an independent set and a clique. We have the following.

Theorem 3 [16]. *Every $\frac{3}{2}$-tough split graph is hamiltonian.*

Theorem 4 [16]. *There is a sequence $\{G_n\}_{n=1}^{\infty}$ of non-2-factorable split graphs with $\tau(G_n) \rightarrow \frac{3}{2}$.*

Even though $\frac{3}{2}$-tough chordal graphs need not be hamiltonian, it was shown in [4] that they will have a 2-factor.

The previous results on tough chordal graphs lead to a very natural question. This question was answered by Chen et al. in the title of their paper “Tough enough chordal graphs are hamiltonian” [8]. Using an algorithmic proof they were able to prove the result below.

Theorem 5 *Every 18-tough chordal graph is hamiltonian.*

The authors did not claim that 18 is best possible. The natural question, in light of the disproof of the 2-tough conjecture for general graphs, is what level of toughness will ensure that a chordal graph is hamiltonian. More specifically, are 2-tough chordal graphs hamiltonian?

Here we study the related problem for the subclass of chordal graphs the members of which are k-trees.

1.2.2 Some basic properties of k-trees

We present some basic facts on k-trees that will be used throughout the paper without references.

Lemma 6 *Let $G \neq K_k$ be a k-tree ($k \geq 2$) and let $S_1(G)$ denote the set of k-simplicial vertices of G if $G \neq K_{k+1}$ and a set of one arbitrary vertex of G if $G = K_{k+1}$. Then*

(i) $S_1(G) \neq \emptyset$;
(ii) $S_1(G)$ *is an independent set;*
(iii) Every k-simplicial vertex (if any) of $G - S_1(G)$ is adjacent in G to at least one vertex of $S_1(G)$.
(iv) $\tau(G - S_1(G)) \geq \tau(G)$.

PROOF.

(i) This follows immediately from the definition;
(ii) If not, then for some adjacent vertices $u, v \in S_1(G)$, u is a k-simplicial vertex of $G - v$ with degree $d(u) < k$, a contradiction;
(iii) If u is a k-simplicial vertex of $G - S_1(G)$, i.e. with $d_{G-S_1(G)}(u) = k$, then $d(u) > k$, since $u \notin S_1(G)$. Hence the claim follows;
(iv) It is sufficient to show that $\tau(G-v) \geq \tau(G)$ for a k-simplicial vertex $v \in S_1(G)$. Suppose, to the contrary, that S is a tough set of $G - v$ such that $\tau(G - v) = \frac{|S|}{\omega((G-v)-S)} < \tau(G)$. Then v is adjacent to vertices in at least two components of $(G - v) - S$, contradicting the fact that all neighbors of v are mutually adjacent (in G and hence in $G - v$). This completes the proof.

2 Main results

Our first result gives a useful characterization of hamiltonian k-trees.

Theorem 7 *Let $G \neq K_2$ be a k-tree. Then G is hamiltonian if and only if G contains a 1-tough spanning 2-tree.*

PROOF. We first assume that G contains a 1-tough spanning 2-tree G'. We prove that G' is hamiltonian. In fact, we will prove that G' has a hamiltonian cycle containing all edges xy of G' with $\omega(G' - \{x, y\}) = 1$. We proceed by induction on $n = |V(G')|$.

If $G' = K_3$, then the conclusion clearly holds. Suppose $n \geq 4$ and suppose the claim holds for all 1-tough 2-trees on fewer than n vertices. Then G' has a 2-simplicial vertex v such that the neighbors p and q of v are adjacent. $G' - v$ is also a 1-tough 2-tree such that $\omega((G' - v) - \{p, q\}) = 1$ and $\omega(G' - \{p, q\}) = 2$.

By the induction hypothesis, $G' - v$ has a hamiltonian cycle C containing pq and all other edges xy of G' with $\omega((G' - v) - \{x, y\}) = 1$. Now replace pq in G' by the path pvq of G'. The new cycle is a hamiltonian cycle in G' containing all edges xy of G' with $\omega(G' - \{x, y\}) = 1$.

We now prove the converse, also by induction on $n = |V(G)|$. Let C be a hamiltonian cycle of G and let v be a k-simplicial vertex of G. In fact, we will prove by induction on n that G has a 1-tough spanning 2-tree containing every edge of C. Since $N_G(v)$ is a clique, the two neighbors x and y of v in C are adjacent in G. Replacing xvy by xy, the resulting cycle C' is a hamiltonian cycle of $G - v$. By the induction hypothesis, $G - v$ has a 1-tough spanning 2-tree F containing every edge of C'. It is easily seen that $\omega(F - \{x, y\}) = 1$. Thus $F + \{xv, yv\}$ is a 1-tough spanning 2-tree of G containing every edge of C.

Theorem 7 has the nice consequence for 2-trees that every 2-tree (except K_2) is hamiltonian if and only if it is 1-tough. We now turn to k-trees with $k \geq 3$. We use a number of easy lemmas and auxiliary results to prove our main result, Theorem 12 below. For a k-tree $G \neq K_k$, let $S_i(G)$ and G_i be defined as follows: $G_1 = G$, $S_1(G)$ is defined as in Lemma 6, $G_i = G_{i-1} - S_1(G_{i-1})$ and $S_i(G) = S_1(G_{i-1})$ for $i = 2, 3, \ldots$ as long as $S_i \neq \emptyset$ (i.e. $G_{i-1} \neq K_k$). We denote by $N_i(v)$ the set of neighbors of v in G_i.

Lemma 8 *For any vertex $u \in S_2(G)$ (if any), there exists a vertex $v \in S_1(G)$ such that $uv \in E(G)$, and $N_1(u) \setminus N_2(u) \subset S_1(G)$.*

PROOF. The proof is similar to the proof of Lemma 6(iii). Since $u \in S_2(G)$, $d_{G_2}(u) = k$. But $u \notin S_1(G)$. This implies that $d_{G_1}(u) > k$. Thus $N_1(u) \setminus N_2(u) \neq \emptyset$ and $N_1(u) \setminus N_2(u) \subset S_1(G)$.

Lemma 9 *If $u \in S_2(G)$, then $N_1(w) \subseteq N_2(u) \cup \{u\}$ for any $w \in N_1(u) \setminus N_2(u)$.*

PROOF. If there exists a vertex $x \in N_1(w) \setminus (N_2(u) \cup \{u\})$, then $ux \in E(G)$ since $N_1(w)$ is a clique. Thus $x \in N_1(u)$, but $x \notin N_2(u) \cup \{u\}$, i.e. $x \in N_1(u) \setminus N_2(u)$, so

$x \in S_1(G)$ by Lemma 8. Hence $\{x, w\} \subseteq S_1(G)$, contradicting that $xw \in E(G)$.

Lemma 10 *Let $G \neq K_1, K_2$ be a 1-tough k-tree. If $S_2(G) = \emptyset$, then G is hamiltonian.*

PROOF. Let $G \neq K_1, K_2$ be a 1-tough k-tree with $S_2(G) = \emptyset$. By the definition of k-trees, $G - S_1(G)$ is a K_k, and 1-toughness implies $|S_1(G)| \leq k$. We can find a hamiltonian cycle C of $G - S_1(G)$. Now we replace $|S_1(G)|$ edges in C one by one by disjoint paths of length 2 containing the end vertices of these edges and exactly one vertex of $S_1(G)$. The resulting cycle is a hamiltonian cycle of G.

For the smallest cases in our proof of Theorem 12 below, we will use a well-known result of Dirac [12].

Theorem 11 [12]. *If G is a graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian.*

We now have all the ingredients to prove the following generalization of the consequence of Theorem 7 for 2-trees.

Theorem 12 *If $G \neq K_2$ is a $\frac{k+1}{3}$-tough k-tree ($k \geq 2$), then G is hamiltonian.*

PROOF. By Theorem 7 or its consequence for 2-trees, we only need to consider the case that $k \geq 3$. We proceed by induction on $n = |V(G)|$.

Obviously, $\delta(G) = k$. Hence using Theorem 11, we obtain that if either $4 \leq k \leq n \leq k + 4$ or $3 = k \leq n \leq k + 3 = 6$, then G is hamiltonian.

Suppose next that either $n \geq k + 5$ or $n = k + 4 = 7$, and that G is hamiltonian for any $\frac{k+1}{3}$-tough k-tree G with fewer than n vertices.

By Lemma 10, it suffices to consider the case that $S_2(G) \neq \emptyset$. For any $u \in S_2(G)$, by Lemma 8, there exists a vertex $v \in S_1(G)$ such that $uv \in E(G)$.

Since $u \in S_2(G)$ and the clique $N_1(v)$ contains u,

$$|N_2(u) \cap N_1(v)| = k - 1. \tag{1}$$

Hence

$$|N_2(u) \setminus N_1(v)| = 1. \tag{2}$$

Let v' be the vertex in $N_2(u) \setminus N_1(v)$.

We distinguish the following cases.

Case 1. u has no neighbor in $S_1(G) \setminus \{v\}$.

By the induction hypothesis, there is a hamiltonian cycle C in $G - v$. By (2), there exists at least one edge $ux \in E(C) \cap E(G[N_1(v)])$. Now replacing ux in C by the path uvx, the resulting cycle is a hamiltonian cycle of G.

Case 2. u has a neighbor in $S_1(G) \setminus \{v\}$.

By Lemma 9, $N_1(w) \subseteq N_2(u) \cup \{u\}$ for every $w \in (S_1(G) \setminus \{v\}) \cap N_1(u)$. If u has at least two neighbors in $S_1(G) \setminus \{v\}$, then when we delete all $k+1$ vertices of $N_2(u) \cup \{u\}$, we will obtain four components except for the unique case that $n = k + 4 = 7$. In the former case we obtain a contradiction, since $\tau(G) \geq \frac{k+1}{3}$. Hence u has exactly one neighbor in $S_1(G) \setminus \{v\}$ except for the unique case that $n = k + 4 = 7$ and u has exactly two neighbors in $S_1(G) \setminus \{v\}$. In the latter exceptional case, G is a K_4 with three 3-simplicial vertices attached to different 3-cliques, and one can easily find a hamiltonian cycle of G. Hence we now suppose $n \geq k + 5$, and we let $N_1(u) \setminus N_2(u) = \{v, w\}$.

Using that G is a $\frac{k+1}{3}$-tough graph, by Lemma 9, $v'w \in E(G)$; otherwise $N_1(w) = N_1(v)$, and if we delete all k vertices of $N_1(w)$, we obtain at least three components, contradicting that G is $\frac{k+1}{3}$-tough.

By the induction hypothesis, $G - \{v, w\}$ has a hamiltonian cycle C, implying that u has two neighbors x, y in C. If $v' \in \{x, y\}$, then $v'' \in (\{x, y\} \setminus \{v'\})$ is a vertex contained in C with $v''v \in E(G)$, and we replace uv' and uv'' by two paths uwv' and uvv'', respectively; if $v' \notin \{x, y\}$, then there exists at most one vertex in $\{x, y\} \setminus N_1(w)$, say $y \in N_1(w)$, and we replace ux and uy by two paths uvx and uwy, respectively. In both cases the resulting cycle is a hamiltonian cycle of G.

3 Nonhamiltonian k-trees with toughness one

We will present infinite classes of nonhamiltonian k-trees with toughness 1 for all $k \geq 3$. To check the toughness we make a number of observations collected in the following lemmas.

Recall the definition of a tough set: Let G be a k-tree with toughness $\tau(G)$. If $S \subseteq V(G)$ is a set such that $\tau(G) = \frac{|S|}{\omega(G-S)}$, then we call S a tough set.

Lemma 13 *If v is a k-simplicial vertex of a k-tree G, then v is not contained in a tough set of G.*

PROOF. Suppose S is a tough set and $v \in S$ is a k-simplicial vertex of G. Then it is clear that $N(v) \not\subseteq S$. Since $G[N(v) \cup \{v\}]$ is a clique, $\omega(G-S) = \omega(G-(S \setminus \{v\}))$ and $\frac{|S\setminus\{v\}|}{\omega(G-(S\setminus\{v\}))} < \tau(G)$, a contradiction.

Lemma 14 *Let G' be obtained from a k-tree G by adding a new vertex w and joining it to a k-clique containing exactly one k-simplicial vertex of G. If $\tau(G) \geq 1$, then $\tau(G') \geq 1$.*

PROOF. Consider a tough set S of G'. By Lemma 13, $w \notin S$. If some vertex $u \in N(w)$ is not contained in S, then $\omega(G'-S) = \omega(G-S) \leq |S|$. If $N(w) \subseteq S$, then S is not a tough set of G because of Lemma 13, so $\omega(G-S) \leq |S|-1$, and $\omega(G'-S) \leq |S|$. Thus in both cases $\tau(G') = \frac{|S|}{\omega(G'-S)} \geq 1$.

Lemma 15 *Let G be a k-tree such that $S_{k-1}(G) \neq \emptyset$, and suppose K is a k-clique of G with the property that for some $x_i \in K \cap S_i(G)$, $G - \{x_1, \ldots, x_i\}$ contains a k-simplicial vertex in K ($i = 1, 2, \ldots, k-2$). Let G' be obtained from G by adding $k-1$ new vertices $w_1, w_2, \cdots, w_{k-1}$ and joining them to all vertices of K. If $\tau(G) \geq 1$, then $\tau(G') \geq 1$.*

PROOF. Consider a tough set S of G'. By Lemma 13, $w_i \notin S$. If some vertex $u \in N(w_1)$ is not contained in S, then $\omega(G'-S) = \omega(G-S) \leq |S|$. If $N(w_1) \subseteq S$, then let $S^* = S \setminus \{x_1, x_2, \ldots, x_{k-2}\}$. Clearly, S^* is not a tough set of $G^* = G - \{x_1, x_2, \ldots, x_{k-2}\}$ because of Lemma 13. Since $\tau(G^*) \geq \tau(G)$, we obtain $\omega(G^*-S^*) \leq |S^*|-1$. Thus $\omega(G'-S) \leq \omega(G^*-S^*)+k-1 \leq |S^*|+k-2 = |S|$. In both cases we obtain that $\tau(G') \geq 1$.

For $k \geq 4$ we construct the following k-trees which are sketched in Figure 1.

Let K be a complete graph with $k+1$ vertices labeled $x_0, x_1, \ldots, x_k$. Let Q^1, Q^2 and Q^3 denote three disjoint complete graphs with $k-1$ vertices (also disjoint from K) which are labeled $u_1^i, u_2^i, \cdots, u_{k-1}^i$ for $i = 1, 2, 3$. We add edges between u_j^i and x_l for all $u_j^i \in V(Q^i)$ and $l \geq j$. Let $W^i = \{w_1^i, w_2^i, \cdots, w_{k-1}^i\}$ be a set of additional vertices for $i = 1, 2, 3$, and let $u_0^i = x_k$. For each $w_j^i \in W^i$, we add edges joining w_j^i and u_l^i for all $l \leq k-1$. Using Lemmas 14 and 15 it is not difficult to check that these graphs have toughness 1. Moreover, these graphs are not hamiltonian, since to include all sets W_i in a possible hamiltonian cycle, we would have to pass x_k at least three times. We can extend each of the obtained graphs to an infinite family with the same properties by attaching a path $v_0 v_1 \ldots v_r$ with $v_0 = x_0$ and new vertices $v_1, \ldots, v_r$ for any integer r, and joining all v_i ($i = 1, \ldots, r$) to $x_1, \ldots, x_{k-1}$.

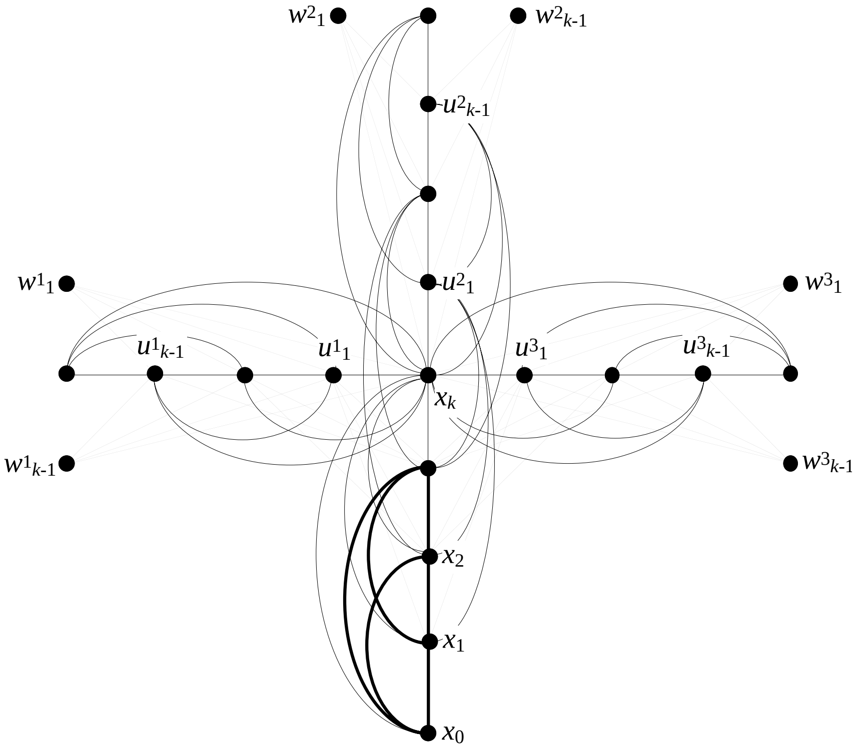

Fig. 1.

The above construction does not work for $k = 3$, since the set $\{x_1, x_2, x_3\}$ would disconnect the graph into four components. The example in Figure 2 is a nonhamiltonian 3-tree with toughness 1, as can be checked easily using Lemmas 14 and 15. As in the case $k \geq 4$, we can extend the example to an infinite class by attaching a path $v_0 v_1 \ldots v_r$ with $v_0 = c$ and joining the new vertices $v_1, \ldots, v_r$ to a and b.

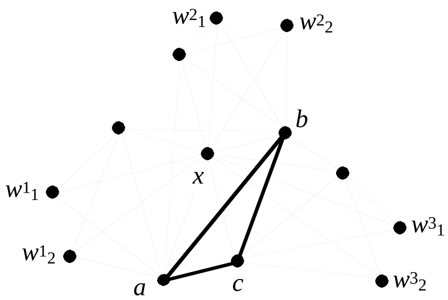

Fig. 2.

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