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Abstract

A graph is called *subpancyclic* if it contains a cycle of length l for each l between 3 and the circumference of a graph. We show that if G is a connected graph on $n \geq 146$ vertices such that $d(u)+d(v)+d(x)+d(y) > \frac{n+10}{2}$ for all four u, v, x, y of a path $P = uvxy$ in G, then its line graph is subpancyclic unless G is isomorphic to an exceptional graph, and the result is best possible, even under the condition that $L(G)$ is hamiltonian.

Keywords: degree sums, line graph, subpancyclicity

AMS Subject Classifications (1991): 05C45, 05C35

1 Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. Let H be a subgraph of G. If S is a subgraph of H or a subset of $V(H)$, then the

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degree of S in H, denoted by $d_H(S)$, is defined to the degree sum of vertices in S, i.e., $d_H(S) = \sum_{u \in V(S)} d_H(u)$, or just $d(S)$ if $G = H$. $cr(G)$ will denote the circumference of G, i.e., the length of a longest cycle of G. G is called *pancyclic* if $\lambda(G) = [3, |V(G)|] = \{3, 4, \cdots, |V(G)|\}$. G is said *subpancyclic* if $\lambda(G) = [3, cr(G)] = \{3, 4, \cdots, cr(G)\}$.
Define

$$\rho_i(G) = min\{d(P) : P \ is \ a \ path \ of \ length \ i-1 \ in \ G \ \}.$$

Obviously $\delta(G) = \rho_1(G)$. As introduced in [1], let $f_i(n)$ be the smallest integer such that for any graph G of order n with $\rho_i(G) > f_i(n)$, the line graph $L(G)$ of G is pancyclic whenever $L(G)$ is hamiltonian. Van Blanken et al. [1] prove that $f_1(n)$ has the same order of magnitude: $O(n^{1/3})$. The following results are obtained.

Theorem 1. Let G be a connected graph of order n. If G satisfies one of the following conditions:

(i) [9] $\rho_2(G) > (\sqrt{8n+1}+1)/2$ and $n \geq 600$;

(ii) [10] $\rho_3(G) > (n+6)/2$ and $n \geq 76$;

(iii) [10] $\rho_4(G) > (2n+16)/3$ and $n \geq 76$,

then $L(G)$ is subpancyclic and the results are all best possible.

Trommel, et al. showed a consequence of Theorem 1 (i) for large line graphs.

Corollary 2. (Trommel *et al.* [7]) Let G be a line graph on at least 100577 vertices. If

$$\delta > (\sqrt{8n+1}-3)/2,$$

then G is subpancyclic.

Theorem 1 shows that the graphs in [4], [6] and [8] are pancyclic. Results related to Theorem 1 have appeared in [7].

Theorem 3. (Trommel *et al.* [7]). Let G be a claw-free graph on at least 5 vertices. If $\delta > \sqrt{3n+1}-2$, then G is subpancyclic.

In this paper, we will characterize those graphs G with $\rho_4(G) = f_4(G)$ such that $L(G)$ is not subpancyclic.

Theorem 4. Let G be a connected graph of order $n (n \geq 146)$. If

$$\rho_4(G) > (n+10)/2,$$

then its line graph $L(G)$ is subpancyclic unless G is isomorphic to an exceptional graph F showed in the following and the result is best possible, even under the condition that $L(G)$ is hamiltonian.

The exceptional graph F is defined as follows: Let $n \equiv 1(mod\ 3)$, and let C_1, $C_2, \cdots, C_{(n-1)/3}$ be $(n-1)/3$ edge-disjoint cycles of length 4. Now F is obtained from those cycles such that $C_1, C_2, \cdots, C_{(n-1)/3}$ have exactly one common vertex in F and $E(F) = E(C_1) \cup E(C_2) \cup \cdots \cup E(C_{(n-1)/3})$. Obviously $|V(F)| = n$ and $\rho_4(F) = f_4(n) = (2n+16)/3$, by Theorem 1.

In general, the condition involving the results on cycles will be slightly improved when we exclude an exceptional graph. But Theorem 4 shows that when we exclude an exceptional graph the condition involving degree sums of the vertices along a 4-path which ensure that its line graph is subpancyclic will be greatly improved (replace $\frac{2n}{3}$ with $\frac{n}{2}$) and they are almost the same as the condition involving degree sums of the vertices along a 3-path (comparing with Theorem 1).

2 Proof of Theorem 4

Before we present our proof of main result, we introduce some additional terminology and notation, and state a number of preliminary results.

By a *circuit* of a graph G we will mean a eulerian subgraph of G, i.e., a connected subgraph in which every vertex has even degree. Note that by this definition (the trivial subgraph induced by) a single vertex is also a circuit. If C is a circuit of G, then $\overline{E}(C)$ denotes the set of edges of G incident with at least one vertex of C. The *distance* $d_H(G_1, G_2)$ between two subgraphs G_1 and G_2 of H is defined to be $\min\{d_H(v_1, v_2) : v_1 \in V(G_1) \text{ and } v_2 \in V(G_2)\}$. The *diameter* of a connected subgraph H, denoted by $dia(H)$, is defined to be max $\{d_H(u, v) : u, v \in V(H)\}$. By C_k we denote a cycle of length k. We write $\varepsilon(C)$ for $|E(C)|$ and $\overline{\varepsilon}(C)$ for $|\overline{E}(C)|$. For any subgraph H of G, let $N(H) = \bigcup_{u \in V(H)} N(u)$.

Harary and Nash-Williams [5] characterized those graphs with line graphs that are hamiltonian. One can easily prove a more general result (see, e.g., [3]).

Theorem 5. (Broersma [3]) The line graph $L(G)$ of a graph G contains a cycle of length $k \geq 3$ if and only if G contains a circuit C such that $\varepsilon(C) \leq k \leq \overline{\varepsilon}(C)$.

We now present the proof of Theorem 4.

Proof of Theorem 4.

We will complete the proof by contradiction.
Assuming G is a graph of order n which satisfies the conditions of Theorem 4 but its line graph $L(G)$ is not subpancyclic, we can define

$$k = max\{i : i \in [3, cr(L(G))] \backslash \lambda(L(G))\}.$$

Hence, by Theorem 5, we obtain

Claim 1. G does not contain a circuit C_0 with

$$\varepsilon(C_0) \leq k \leq \overline{\varepsilon}(C_0).$$

Obviously, $L(G)$ contains a cycle C_{k+1} of length $k+1$. Hence, by Theorem 5, we obtain that G contains a circuit C with $\varepsilon(C) \leq k+1 \leq \overline{\varepsilon}(C)$. By Claim 1, $\varepsilon(C) = k+1$. Since C is a circuit, there exist edge-disjoint cycles $D_1, D_2, ..., D_r$ such that $C = \bigcup_{i=1}^{r} D_i$ and r is maximized. Hence,

$$\text{If } r \geq 2, \text{then } |V(D_i) \cap V(D_j)| \leq 2 \text{ for } \{i,j\} \subseteq \{1,2,...,r\}. \tag{2.1}$$

Let $UP_i(C) = \{P : P \text{ is a path of length } i-1 \text{ in } C\}$. Since $\rho_4(G) > (n+10)/2 \geq 78$,

$$\varepsilon(C) = k+1 \geq \Delta(G) + 2 \geq \rho_4(G)/4 + 2 > (n+26)/8 \geq 21. \tag{2.2}$$

If $r = 1$, i.e., C is a cycle of length $k+1$, then we obtain the following claim.

Claim 2. G does not contain a cycle C' with $\varepsilon(C)/2 < \varepsilon(C') \leq k$.

Proof. Otherwise, in $\sum_{P \in UP_4(C)} d(P)$, every edge in $\overline{E}(C')$ is counted at most 8. Hence, by (2.2) and $\rho_4(G) \geq (n+10)/2 \geq 78$,

$$\begin{aligned}
\overline{\varepsilon}(C') &\geq \sum_{P \in UP_4(C')} (d(P)-8)/8 + \varepsilon(C') \\
&\geq (\rho_4 - 8)\varepsilon(C')/8 + \varepsilon(C') = \rho_4\varepsilon(C')/8 \\
&\geq \rho_4\varepsilon(C)/16 \\
&\geq k+1.
\end{aligned}$$

On the other hand, $\varepsilon(C') \leq k$. Thus $L(G)$ contains a C_k, a contradiction. This completes the proof of Claim 2.

So, C has no chord. Since $\rho_4 \geq 78, C$ cannot be a hamiltonian cycle of G. Let u be a vertex in $V(G) \backslash V(C)$. By Claim 2, u is adjacent to at most three vertices of C. Hence, by (2.2),

$$\overline{\varepsilon}(C) \leq 3|V(G) \backslash V(C)| + \varepsilon(C) = 3(n - \varepsilon(C)) + \varepsilon(C) < (11n-26)/4. \tag{2.3}$$

On the other hand, since C has no chord,

$$\begin{aligned}
\overline{\varepsilon}(C) &\geq \sum_{P \in UP_4(C)} (d(P)-8)/4 + \varepsilon(C) \\
&\geq (\rho_4 - 8)\varepsilon(C)/4 + \varepsilon(C) \\
&= (\rho_4 - 4)\varepsilon(C)/4 \\
&\geq (n^2 + 28n + 52)/64,
\end{aligned}$$

which contradicts (2.3) and $n \geq 146$. This implies that $r \neq 1$.

Hence it suffices to consider the case that $r \geq 2$.

Let H be the graph with $V(H) = \{D_1, D_2, ..., D_r\}$ and $D_iD_j \in E(H)$ if and only if $V(D_i) \cap V(D_j) \neq \emptyset$. Since C is a circuit, H is connected. Without loss of generality, we assume that D_1 and D_r are two vertices of H such that

$$d_H(D_1, D_r) = dia(H). \tag{2.4}$$

Hence, any element of $\{D_1, D_r\}$ is not cut vertex of H, so $C^1 = \bigcup_{i=2}^{r} D_i$ and $C^r = \bigcup_{i=1}^{r-1} D_i$ are two circuits of G. Let

$$E_1(D_i) = E(D_i) \cap \overline{E}(C^i) \;\text{ and } E_2(D_i) = E(D_i) \backslash E_1(D_i)$$

and

$$V_1(D_i) = V(D_i) \cap V(C^i) \text{ and } V_2(D_i) = \{u, v : uv \in E_2(D_i)\}$$

where $i \in \{1, r\}$.

For any path P of C, let $d_2(P) = d(P) - d_C(P)$. Since $\overline{\varepsilon}(C^i) \geq \varepsilon(C) - |E_2(D_i)| = k + 1 - |E_2(D_i)|$,

$$|V_2(D_i)| - 1 \geq |E_2(D_i)| \geq 2 \tag{2.5}$$

where $i \in \{1, r\}$. Otherwise $\varepsilon(C^i) \leq k \leq \overline{\varepsilon}(C^i)$ which contradicts Claim 1.

Since $\overline{\varepsilon}(C^t) \geq \varepsilon(C) - |E_2(D_t)| + |\overline{E}(D_s) \backslash E(C)|$,

$$|\overline{E}(D_s) \backslash E(C)| \leq |E_2(D_t)| - 2 \tag{2.6}$$

where $\{s, t\} = \{1, r\}$. Otherwise $\varepsilon(C^t) \leq k \leq \overline{\varepsilon}(C^t)$ which contradicts Claim 1.

We now prove the following claim.

Claim 3. Let P be a path of length 3 in D_s. We obtain

$$d_C(P) > (n + 14)/2 - |E_2(D_t)| \tag{2.7}$$

and

$$|E_2(D_t)| \leq 2|V_2(D_t)|/3 \text{ and } d_C(P) > (n + 14)/2 - 2|V_2(D_t)|/3, \tag{2.8}$$

where $\{s, t\} = \{1, r\}$.

Proof. Let P be a path of length 3 in D_s. Then

$$|\overline{E}(D_s) \backslash E(C)| \geq d(P) - d_C(P).$$

Hence be (2.6) and $\rho_3(G) > (n+10)/2$,

$$d_C(P) > (n+10)/2 - (|E_2(D_t)| - 2),$$

i.e., (2.7) is true.

In order to obtain (2.8), it suffices to prove the following claim.

Each component of $C[E_2(D_1) \cup E_2(D_r)]$ is a path of length at most two. (2.9)

Otherwise, there must exist an $s \in \{1, r\}$ and a path $P_0 = u_0v_0x_0y_0$ of D_s such that $\{u_0, v_0, x_0, y_0\} \subseteq V_2(D_s)$. By (2.5) and (2.7),

$$d_C(P_0) > (n+16)/2 - |V_2(D_t)| \tag{2.10}$$

where $\{s, t\} = \{1, r\}$.

Since $d_C(P_0) = 8, |V_2(D_t)| > n/2 \geq 78$. Hence there exists a path $P_0' = u_0'v_0'x_0'y_0'$ in D_t such that $u_0'v_0' \in E_2(D_t)$ and $\{x_0', y_0'\} \cap V_1(D_s) = \emptyset$.

For any $x \in N_C(x_0') \cap N_C(y_0')$, $C - x$ has at least an nontrivial component, denoted by Q_x, which does not contain any vertex of D_t. Otherwise $\varepsilon(C') \leq k \leq \overline{\varepsilon}(C')$, where $C' = C - \{xx_0', xy_0', x_0'y_0'\}$, a contradiction.

It is easy to see that

$$|V(Q_x)| \geq 3. \tag{2.11}$$

Otherwise $\varepsilon(C') \leq k \leq \overline{\varepsilon}(C')$, where $C' = C - Q_x$.

Let B denote the cut-vertex set of $N_C(x_0') \cap N_C(y_0')$ such that for any $x \in B, C - x$ has a nontrivial component, denoted by Q_x, which does not contain any vertex of $V(D_1) \cup V(D_r)$. Set

$$\beta = |N_C(x_0') \cap N_C(y_0')|.$$

Obviously

$$|\{Q_x : x \in B\}| = |B| \geq \beta - 1. \tag{2.12}$$

Using (2.11) and (2.12), we obtain

$$\begin{aligned}
&d_C(x_0') + d_C(y_0') = |N_C(x_0') \cup N_C(y_0')| + |N_C(x_0') \cap N_C(y_0')| \\
&\leq \begin{cases} n - (|V_2(D_1)| + |V_2(D_r)| - 4) + 1, & \text{if } \beta \leq 1, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 2 + 3(\beta - 1)) + \beta, & \text{if } \beta \geq 2. \end{cases} \\
&\leq n - (|V_2(D_1)| + |V_2(D_r)|) + 5 \\
&\leq n - n/2 - |V_2(D_t)| + 5 \\
&= (n+10)/2 - |V_2(D_s)|,
\end{aligned}$$

which contradicts (2.10) and $d_C(u_0') = d_C(v_0') = 2$. This implies that (2.8) and (2.9) are true. This completes the proof of Claim 3.

We will consider the following three cases to obtain contradictions.

Case 1. $dia(H) \geq 2$.

This implies that $V(D_1) \cap V(D_r) = \emptyset$.

We can take two paths $P = uvxy$ and $P' = u'v'x'y'$ of length 3 in D_1 and D_r respectively with $\{uv, u'v'\} \subseteq E_2(D_1) \cup E_2(D_r)$ and $\{x, x'\} \subseteq V_1(D_1) \cup V_1(D_r)$ such that $V(P) \cap V(P') = \emptyset$. Let

$$\begin{aligned} S &= \{x, y, x', y'\}, \\ N_i &= \{u \in V(C) : |N_C(u) \cap S| = i\}, \\ M_1 &= ((N_C(x) \cap N_C(y)) \cup (N_C(x') \cap N_C(y'))) \cap N_2, \\ M_2 &= N_2 \backslash M_1, \\ n_i &= |N_i| \; and \; m_i = |M_i|. \end{aligned}$$

We now prove three claims.

Claim 4. $|N_3 \cup N_4| \leq 1$.

Proof. Otherwise, let $w, w' \in N_3 \cup N_4$. Obviously,

$$w, w' \in (N_C(x) \cap N_C(y)) \cup (N_C(x') \cap N_C(y')).$$

Without loss of genenality, we assume that $wx, wy \in E(C)$. Hence $C' = C - \{wx, wy, xy\}$ is a circuit with $\varepsilon(C') = \varepsilon(C) - 3 \leq k \leq \overline{\varepsilon}(C')$, a contradiction. This completes the proof of Claim 4.

Claim 5. Each element of M_1 is cutvertex of C.

Proof. Otherwise, there exists a vertex $w \in M_1$, say, $w \in N_C(x) \cap N_C(y) \cap N_2$, which is not cutvertex of C. Hence $C' = C - \{wx, wy, xy\}$ is a circuit with $\varepsilon(C) - 3 = \varepsilon(C') \leq k \leq \overline{\varepsilon}(C')$, a contradiction. This completes the proof of Claim 5.

Let W_1 denote the cutvertex set of C in M_1 such that for any $z \in W_1, C - z$ has a nontrivial component which does not contain any element of S. By Claim 2, we obtain

$$|W_1| \geq m_1 - 2 \tag{2.13}$$

and

$$\text{If either } N_3 \cup N_4 \neq \emptyset \text{ or } m_2 \neq 0, \text{ then } |W_1| = m_1. \tag{2.14}$$

Claim 6. If $m_2 \geq 3$, then for any pair of vertices $\{w, w'\}$ of M_2, a cycle of $C[\{w, w'\} \cup ((N_C(w) \cup N_C(w')) \cap S)]$ which does not contain ww', is nontrivial cutset of C.

Proof. Otherwise $C[\{w,w'\} \cup ((N_C(w) \cup N_C(w')) \cap S)]$ has a cycle C' which does not contain ww', such that $\varepsilon(C'') \le k \le \overline{\varepsilon}(C'')$ where $C'' = C - C'$, a contradiction. This completes the proof of Claim 6.

Let W_2 denote the cut-vertex set of C in M_2 such that for any $y \in W_2, C - y$ has a nontrivial component which does not contain any element of S. By Claim 6, we obtain

$$|W_2| \ge m_2 - 2, \tag{2.15}$$

$$\text{if } n_4 = 1, \text{ then } |W_2| = m_2 \tag{2.16}$$

and

$$\text{if } n_3 = 1, \text{ then } |W_2| = m_2 - 1. \tag{2.17}$$

For $y \in W_1 \cup W_2$, let Q_y denote the nontrivial component of $C - y$ which does not contain any element of S. Then it is easy to see that

$$|V(Q_y)| \ge 3. \tag{2.18}$$

Otherwise $\varepsilon(C') \le k \le \overline{\varepsilon}(C')$, where $C'_y = C - Q_y$. We also obtain

$$|\{Q_y : y \in W_1 \cup W_2\}| = |W_1 \cup W_2|. \tag{2.19}$$

If $N_3 \cup N_4 \ne \emptyset$ then, using Claims 3,4,5,6 and (2.14) up to (2.19), we obtain

$$\begin{aligned} d_C(S) &= |\bigcup_{i=1}^{4} N_i| + n_2 + 2n_3 + 3n_4 \\ &\le \begin{cases} n - (|V_2(D_1)| + |V_2(D_r)| - 4 + 3n_2) + n_2 + 3, & \text{if } n_4 = 1, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(n_2 - 1)) + n_2 + 2, & \text{if } n_2 \ge 2, n_3 = 1, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 5) + 3, & \text{if } n_2 \le 1, n_3 = 1, \end{cases} \\ &\le n - (|V_2(D_1)| + |V_2(D_r)|) + 8. \end{aligned}$$

If $N_3 \cup N_4 = \emptyset$ then, using Claims 5, 6 and (2.13), (2.15), (2.18), (2.19), we obtain

$$\begin{aligned} d_C(S) &= |\bigcup_{i=1}^{4} N_i| + n_2 + 2n_3 + 3n_4 \\ &\le \begin{cases} n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(m_1 - 2) + 3(m_2 - 2)) \\ \qquad + m_1 + m_2, \text{ if } m_1 \ge 2, m_2 \le 3, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(m_1 - 2)) + m_1 + 2, \\ \qquad \text{if } m_1 \ge 2, m_2 \ge 2, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 5 + 3(m_2 - 2)) + 1 + m_2, \\ \qquad \text{if } m_1 \le 1, m_2 \ge 3, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 5) + 3, \\ \qquad \text{if } m_1 = 1, m_2 \le 2, \\ n - (|V_2(D_1)| + |V_2(D_r)| - 6) + 2, \\ \qquad \text{if } m_1 = 0, m_2 \le 2, \end{cases} \\ &\le n - (|V_2(D_1)| + |V_2(D_r)|) + 8. \end{aligned}$$

Hence

$$d_C(S) \le n - |V_2(D_1)| - |V_2(D_r)| + 8. \tag{2.20}$$

On the other hand, by (2.8),

$$d_C(S) > n + 6 - 2(|V_2(D_1)| + |V_2(D_r)|)/3. \tag{2.21}$$

Using (2.20) and (2.21), we obtain

$$|V_2(D_1)| + |V_2(D_r)| < 6$$

which contradicts (2.5).

Case 2. $dia(H) = 1$ and $|V(D_1) \cap V(D_r)| = 1$.

Hence H is a complete graph.

Let $V(D_1) \cap V(D_r) = \{y\}$. We will consider the following two subcases.

Subcase 2.1. $|V(D_i)| \geq 5$ for $i \in \{1, r\}$.

Hence, we can take two paths $P = u'v'x'y'$ and $P' = u''v''x''y''$ of length 3 in D_1 and D_r respectively such that $\{uv, u'v'\} \subseteq E_2(D_1) \cup E_2(D_r)$ and $V(P) \cap V(P') = \emptyset$. By (2.9), $|V(P) \cap V_1(D_1)| \geq 1$ and $|V(P') \cap V_1(D_r)| \geq 1$. In a way similar to the proof of Case 1, we derive contradictions.

Subcase 2.2. There exists a $D_i (i \in \{1, r\})$, say, D_1, such that $|V(D_1)| = 4$.

Hence we can take a path $P = uvxy$ in D_1 such that $\{uv, vx\} = E_2(D_1)$.

Claim 7. $|V(D_r)| = 4$.

Proof. Otherwise $|V(D_r)| \geq 5$. Hence by (2.9), we can take a path $P' = u'v'x'y'$ in D_r such that $u'v' \in E_2(D_r), y \notin \{x', y'\}$ and $|\{x', y'\} \cap V_2(D_r)| \leq 1$.

In a way similar to Claims 4, 5, we obtain

Claim 8. $n_3 = n_4 = 0$ and $n_2 \leq 1$.

If $|E_2(D_r)| = 2$ or 3, then by (2.7),

$$d_C(S) > n + 6 - |E_2(D_1)| - |E_2(D_r)| = n + 1. \tag{2.22}$$

On the other hand, by Claim 8,

$$d_C(S) = |\bigcup_{i=1}^{4} N_i| + n_2 + 2n_3 + 3n_4 \leq n + 1,$$

which contradicts (2.22).

If $|E_2(D_r)| \geq 4$, then using (2.8), we obtain $|V_2(D_r)| \geq 3|E_2(D_r)|/2 \geq 6$. This implies that by (2.5),

$$|V_2(D_1)| + |V_2(D_r)| \geq 9. \tag{2.23}$$

On the other hand, using Claim 8, we obtain

$$\begin{aligned} d_C(S) &= |\bigcup_{i=1}^{4} N_i| + n_2 + 2n_3 + 3n_4 \\ &\leq n - (|V_2(D_1)| + |V_2(D_r)| - 8) + 1 \\ &= n - |V_2(D_1)| - |V_2(D_r)| + 9, \end{aligned}$$

implying that by (2.21),

$$|V_2(D_1)| + |V_2(D_r)| < 3(9 - 6) = 9,$$

which contradicts (2.23). This completes the proof of Claim 7.

If there exists a $D_i (i \in \{2, 3, \ldots, r-1\})$, say, D_2, such that $|V(D_2)| \geq 5$, then D_2 plays the same role as D_r in above subcase. In a way similar to the proof of above subcase, we derive contradictions.

So $|V(D_i)| = 4$ for $i \in \{1, 2, \ldots, r\}$.

Next we will prove that $G \cong F$.

If there exists a vertex $x \in V(C)$ such that $d_C(x) < d(x)$, then $C_i = C - D_i (x \notin V(D_i)$ unless $x = y)$ is a circuit with $\varepsilon(C_i) \leq k \leq \overline{\varepsilon}(C_i)$, a contradiction.

So $d_C(u) = d_G(u)$ for any $u \in V(C)$. Since G is connected, $G \cong F$.

Case 3. $dia(H) = 1$ and $|V(D_1) \cap V(D_r)| = 2$.

So H is a complete graph.
Let $V(D_1) \cap V(D_r) = \{u, v\}$. Hence there exist four paths P_1, P_2, P_3, P_4 such that $D_1 = P_1 \cup P_2$, $D_r = P_3 \cup P_4$, and $V(P_i) \cap V(P_j) = \{u, v\}$ for $\{i, j\} \subseteq \{1, 2, 3, 4\}$.

If there exists a pair of $\{P_s, P_t\}$ and a D_i where $i \in \{2, 3, \ldots, r-1\}$, say, D_2, such that $|(V(P_s) \cup V(P_t)) \cap V(D_2)| \leq 1$ where $\{s, t\} \subseteq \{1, 2, 3, 4\}$, then let $D'_1 = P_s \cup P_t$, $D'_r = (D_1 \cup D_r) - (E(P_s) \cup E(P_t))$ and $D'_i = D_j$ for $j \in \{2, 3, \ldots, r-1\}$. Let H' be a graph with vertex set $V(H') = \{D'_1, D'_2, \cdots, D'_r\}$, $D'_i D'_j \in E(H')$ if and only if $V(D'_i) \cap V(D'_j) \neq \emptyset$. Obviously H' is a complete graph. Note that D'_1 and D'_2 in H' play the same role as D_1 and D_r in H respectively. Since $|V(D'_1) \cap V(D'_2)| \leq 1$, we derive contradictions in a way similar to the proof of Case 1 or 2.

Hence, using (2.1), we obtain the following claim.

Claim 9. $|(V(P_s)\cup V(P_t))\cap V(D_i)| = 2$ for any pair of $\{s,t\}\subseteq\{1,2,3,4\}$ and $i\in\{2,3,\cdots,r-1\}$.

Furthermore, we can prove the following claim.

Claim 10. $\{u,v\}\subseteq V(D_i)$ for $i\in\{2,3,\cdots,r-1\}$.

Proof. Otherwise, there exists a D_i, say, D_2, such that $|\{u,v\}\cap V(D_2)|\le 1$.

If $|\{u,v\}\cap V(D_2)| = 1$, say, $u\in V(D_2)$, then by (2.1), there exist two paths P_s and P_t such that $V(P_s)\cap V(D_2) = V(P_t)\cap V(D_2) = \{u\}$ where $s\in\{1,2\}$ and $t\in\{3,4\}$ which contradicts Claim 9. So $\{u,v\}\cap V(D_2) = \emptyset$.

If $|V(P_j)\cap V(D_2)| = 1$ for $j\in\{1,2,3,4\}$, then there exist four edge-disjoint cycles C_1, C_2, C_3, C_4 in $D_1\cup D_2\cup D_r$ such that $D_1\cup D_2\cup D_r = C_1\cup C_2\cup C_3\cup C_4$ which contradicts the maximum of r. So there exists a P_i, say P_1, such that $|V(P_1)\cap V(D_2)| = 0$ or 2. By (2.1) and $\{u,v\}\cap V(D_2) = \emptyset$, there exists an $s\in\{3,4\}$ such that $|V(P_s)\cap V(D_2)|\le 1$.

If $|V(P_1)\cap V(D_2)| = 2$ then, by (2.1), $|(V(P_s)\cup V(P_2))\cap V(D_2)|\le 1$ which contradicts Claim 9.

If $|V(P_1)\cap V(D_2)| = 0$ then, $|(V(P_s)\cup V(P_1))\cap V(D_2)|\le 1$ which contradicts Claim 9. This completes the proof of Claim 10.

It follows from Claim 10 that there exist $d_C(u) = d_C(v) = 2r$ edge-disjoint paths $P_1, P_2, \cdots, P_{2r}$ such that $C = \bigcup_{i=1}^{2r} P_i$ and $V(P_i)\cap V(P_j) = \{u,v\}(i\neq j)$. Hence by (2.9), we obtain the following.

$$|V(P_i)|\le 5 \text{ for } i\in\{1,2,\cdots,2r\}. \tag{2.24}$$

Hence, it is easy to see that

$$|E_2(D_1)| + |E_2(D_r)| \le 4 + min\{4, |\{i : V(P_i)| = 5\}|\}. \tag{2.25}$$

By (2.5), there exist two paths P_1' and P_2' of length 3 in D_1 and D_r respectively such that each path contains exactly one vertex in $V_1(D_1)\cup V_1(D_r)$, and $u\in V(P_1')$ and $v\in V(P_2')$. So by (2.7),

$$d_C(u) + d_C(v) > n + 2 - |E_2(D_1)| - |E_2(D_r)|. \tag{2.26}$$

If there exists a pair of $\{i,j\}$ such that $|V(P_i)|\le 3$ and $|V(P_j)|\le 4$, then $C' = C-(P_i\cup P_j)$ is a circuit with $\varepsilon(C')\le k\le\overline{\varepsilon}(C')$, contradicts Claim 1. Hence we obtain

$$\{(i,j) : |V(P_i)|\le 3 \text{ and } |V(P_j)|\le 4\} = \emptyset. \tag{2.27}$$

Without loss of generality, we can assume that $|V(P_1)|, |V(P_2)|, \cdots, |V(P_{2r})|$ is an increase sequence and $D_i = P_{2i-1} \cup P_{2i}$ for $i \in \{1, 2, \cdots, r\}$.

We can prove the following claim.

Claim 11. If $|V(P_1)| = 2$, i.e., $uv \in E(C)$, then $r \geq 4$.

Proof. Otherwise $r = 2$ or 3. By (2.24) and (2.27), $|V(P_i)| = 5$ for $i \in \{2, 3, \cdots, 2r\}$. Since $n \geq 146$, there exists a vertex x of C with $d_C(x) \leq d(x) - 1$. Hence there exists a circuit C' such that $\varepsilon(C') \leq k \leq \overline{\varepsilon}(C')$, where $C' = C - (P_i \cup P_1)(P_i \neq P_1$, and $x \notin V(P_i)$ unless $x \in \{u, v\})$, this contradicts Claim 1. This completes the proof of Claim 11.

Next, we will obtain some inequalities which contradict (2.25).

If $3 \leq |V(P_1)| \leq 5$ then, using (2.27), we obtain

$$d_C(u) + d_C(v) \leq n - 2 - |\{i : |V(P_i)| = 5\}|. \tag{2.28}$$

Using (2.26) and (2.28), we obtain an inequality that contradicts (2.25).

If $|V(P_1)| = 2$, i.e., $uv \in E(C)$ then, using (2.24) and (2.27), we obtain that $|V(P_i)| = 5$ for $i \geq 2$, and

$$d_C(u) + d_C(v) \leq n - |\{i : |V(P_i)| = 5\}|. \tag{2.29}$$

Using (2.26), (2.27), (2.29) and Claim 11, we obtain

$$|E_2(D_1)| + |E_2(D_r)| > 2 + |\{i : |V(P_i)| = 5\}| = 2 + (2r - 1) \geq 9,$$

which contradicts (2.25).

The results in Theorem 4 is best possible in the sense that the condition $\rho_4(G) > (n + 10)/2$ can not be relaxed, even under the condition that $L(G)$ is hamiltonian. To see this, we will construct a graph G_0 as follows:

Let $s = (n - 2)/2(n \equiv 2(mod\ 4))$. Define a graph G_0 of order n as ollows: vertex set $V(G_0) = \{u_1, v_1, u_2, v_2, \cdots, u_s, v_s, x, y\}$ and edge set $E(G) = \bigcup_{i=1}^{s} \{xu_i, u_iv_i, v_iy\}$. Clearly G_0 is a graph such that $\rho_4(G_0) = s + 6 = (m + 10)/2$. Theorem 5 implies that $L(G)$ is hamiltonian and $3s - 1 \in [3, \varepsilon(G_0)] \backslash \lambda(L(G_0))$, which implies that $L(G_0)$ is not (sub)pancyclic. This completes the proof of Theorem 4. □

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