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Fluid dynamical systems as Hamiltonian boundary control systems

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Fluid dynamical systems as Hamiltonian boundary control systems

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Abstract

It is shown how the geometric framework for distributed-parameter port-controlled Hamiltonian systems as recently provided in [14, 15] can be adapted to formulate ideal adiabatic fluids with non-zero energy flow through the boundary of the spatial domain as Hamiltonian boundary control systems. The key ingredient is the modification of the Stokes-Dirac structure introduced in [14] to a Dirac structure defined on the space of mass density 3-forms and velocity 1-forms, incorporating three-dimensional convection. Some initial steps towards stabilization of these boundary control systems, based on the generation of Casimir functions for the closed-loop Hamiltonian system, are discussed.

Keywords

Fluid dynamical systems, boundary control, Stokes-Dirac structures, Hamiltonian systems.

1991 Mathematics Subject Classification: 70H05, 76N10, 93C20, 35B37, 35J55, 58A10

1 Introduction

In recent publications [13, 24, 25, 10, 3, 22, 23] a systematic framework has been provided for the geometric modelling of network models of lumped-parameter physical systems as port-controlled Hamiltonian (PCH) systems (with or without dissipation). The key notion in this framework is that of a power-conserving interconnection, formalized by the geometric concept of a Dirac structure. Furthermore ([26, 3, 23, 11, 21]) it has been shown how by interconnection with a controller system that is itself a PCH system, the system may be stabilized at a desired set-point by generating Casimir functions (conserved quantities) determined by the closed-loop interconnection structure, thus effectively shaping the energy of the system. Finally, this approach can be extended to the direct modification of the interconnection and dissipation structure of the system by feedback, leading to Interconnection-Damping Assignment (IDA) passivity based control ([21]).

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Recently [14, 15] we have started to expand this research program on finite-dimensional PCH systems to the distributed parameter case. The first idea for doing so is to try to extend the Hamiltonian formulation of distributed parameter systems, as e.g. exposed in [19] and in particular for fluid dynamics in [18, 8, 9, 17, 1], to distributed parameter systems. However, a fundamental difficulty which arises is the treatment of *boundary conditions*. Indeed, from a control and interconnection point of view it is essential to describe a distributed parameter system with varying boundary conditions inducing *energy exchange through the boundary*, since in many applications the interaction with the environment (e.g. actuation or measurement) will actually take place through the boundary of the system. On the other hand, the treatment of distributed parameter Hamiltonian systems in the literature seems mostly focussed on systems with infinite spatial domain, where the variables go to zero for the spatial variables tending to infinity, or on systems with boundary conditions such that the energy exchange through the boundary is *zero*. Furthermore, it is not obvious how to incorporate non-zero energy flow through the boundary in the existing framework. The problem is already illustrated by the Hamiltonian formulation of e.g. the Korteweg-de Vries equation. Here for zero boundary conditions a Poisson bracket can be formulated with the use of the differential operator $\frac{d}{dx}$, since by integration by parts this operator is obviously skew-symmetric. However, for boundary conditions corresponding to non-zero energy flow the differential operator is not skew-symmetric anymore (since the remainders are not zero when integrating by parts). Also the interesting paper [7] does not really solve the problem, since this latter paper is concerned with the modification of the Poisson bracket in case of a free boundary.

In [14, 15] we have proposed a framework to overcome this fundamental problem, by defining a *Dirac structure* on certain spaces of differential forms on the spatial domain and its boundary, based on the use of Stokes' theorem. This framework has been successfully applied to the port-controlled Hamiltonian formulation of the telegrapher's equations and Maxwell's equations on a bounded domain.

In the present paper we extend and generalize this differential-geometric framework to the Eulerian description of 3-dimensional ideal isentropic fluids (see Section 2). The basic set up is to represent the mass density as a 3-form and the Eulerian velocity as a 1-form (see also [8, 9] for a similar point of view), and to define a *modified* Stokes-Dirac structure on the space of these state variables according to mass and momentum balance ("modified" because of an additional term arising from 3-dimensional convection). For zero-boundary conditions our formulation reduces to the elegant Poisson bracket formulation given before in [18, 8, 9, 17]. The resulting infinite-dimensional system with boundary variables can be interpreted as a (nonlinear) *boundary control* system in the sense of [6].

Of course, the Hamiltonian formulation of boundary control fluid systems is only a first step towards their analysis and control. Nevertheless, the identification of the underlying Hamiltonian structure of fluid dynamics with zero boundary energy flow has proved to be instrumental in deriving all sorts of results on integrability, existence of soliton solutions, stability, reduction, etc., and in *unifying* existing results, see e.g. [5]; and we believe it will also be a fruitful starting point for the control of such systems. In Section 3 we shall already provide some initial ideas how the theory of interconnection and energy-shaping as developed for finite-dimensional port-controlled Hamiltonian systems might be extended to the fluid dynamics case.

2 Geometric boundary control formulation of fluid dynamics

2.1 Introduction

An ideal compressible isentropic fluid in three dimensions is described by the standard equations (in vector calculus notation)

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v) \tag{1}$$

$$\frac{\partial v}{\partial t} = -v \cdot \nabla v - \frac{1}{\rho}\nabla p \tag{2}$$

Here $\rho(x,t)$ denotes the *mass density* at the spatial position $x \in \mathbb{R}^3$ at time t, and $v(x,t)$ is the *Eulerian velocity*, that is, the velocity of the fluid at the (fixed) spatial position x at time t. Furthermore, $p(x,t)$ is the pressure, which is derivable from a potential energy density $U(\rho)$ as

$$p(x,t) = \rho^2(x,t)\frac{\partial U}{\partial \rho}(\rho(x,t)) \tag{3}$$

(In the non-isentropic case U will also depend on the non-constant *entropy*.) Finally, ∇ is the standard gradient operator $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$.

Both equations (1) and (2) are *conservation equations*, expressing respectively mass-balance and momentum-balance, and more generally can be expressed in an *integral* form. Indeed, let W be any fixed 3-dimensional subdomain of some given domain $\mathcal{D} \subset \mathbb{R}^3$, filled with the fluid. Then (1) expresses that the change of mass inside W is equal to minus the mass flow through the boundary of W, while (2) corresponds to Newton's second law (we could have also added external distributed forces to the right hand side of (2)).

It can be readily checked that the total stored energy in W (with dV the standard volume element in $\mathbb{R}^3$)

$$H_W := \int_W \left(\frac{1}{2}\rho \parallel v \parallel^2 + \rho U(\rho)\right) dV \tag{4}$$

satisfies the balance equation

$$\frac{d}{dt}H_W = -\int_{\partial W} \left[\frac{1}{2} \parallel v \parallel^2 + h(\rho)\right] \rho v \cdot n \, dA \tag{5}$$

(with dA denoting the standard area element) where n is the outward normal vector to the boundary ∂W, and

$$h(\rho) := U(\rho) + \rho\frac{\partial U}{\partial \rho}(\rho) \tag{6}$$

is the *enthalpy*. Alternatively, the energy balance (5) can be rewritten in "convective form" as

$$\begin{aligned} \frac{d}{dt}H_W &= -\int_{\partial W}\left[\frac{1}{2}\rho \parallel v \parallel^2 + \rho U(\rho)\right] v\cdot n\, dA \\ & -\int_{\partial W} pv\cdot n\, dA \end{aligned} \tag{7}$$

where we have used the definition (3) of the pressure p.

It immediately follows that if v is such that $v \cdot n = 0$ at the boundary ∂W (no fluid flow through the boundary), then the total energy H_W is conserved. In fact, not only the energy H_W is conserved in this case, but the dynamical equations (1), (2) of the fluid on W can be formulated as an *infinite-dimensional Hamiltonian system* on the infinite-dimensional space of mass densities ρ and Eulerian velocities v on W. This is done via the introduction of an infinite-dimensional Poisson bracket, see e.g. [18, 8, 9, 5, 17] for clear expositions and further ramifications.

¿From a control point of view, however, we would like to consider the fluid dynamical system as a *boundary control* system, with time-varying boundary conditions different from $v.n|_{\partial W} = 0$, since the interaction of the system with its environment will often take place through the boundary. On the other hand, it remains advantageous to represent the boundary control fluid dynamical system as a (controlled) Hamiltonian system, certainly for stabilizability and regulation purposes. For lumped parameter systems this program has been initiated in e.g. [12, 13, 24, 26, 10, 11, 16, 21] by introducing the notion of a port-controlled Hamiltonian (PCH) system. Later on in [25, 3, 22, 23] this has been generalized to *implicit* port-controlled Hamiltonian systems, by replacing Poisson structures with the more general geometric notion of a Dirac structure. Furthermore, in [14, 15] it has been shown how the finite-dimensional PCH framework can be extended to the distributed parameter case by defining a certain type of infinite-dimensional Dirac structures based on Stokes' theorem. This framework has been shown in [14] to cover the ideal transmission line, Maxwell's equations on a bounded domain, as well as the vibrating string with boundary forces. In the present paper we further extend this framework in order to cover the boundary control fluid dynamical system system (1), (2) on an arbitrary 3-dimensional subdomain W with (smooth) boundary ∂W.

2.2 Stokes-Dirac structure

The basic concept we need is that of a Dirac structure, as introduced by Courant [3] and Dorfman [6] as a generalization of symplectic and Poisson structures, and employed in e.g. [25, 3, 23] as the geometric notion formalizing general *power-conserving interconnections.*

Definition 2.1. Let V be a linear space (possibly infinite-dimensional). There exists on $V \times V^*$ the canonically defined symmetric bilinear form

$$\ll (f_1, e_1), (f_2, e_2) \gg := < e_1|f_2 > + < e_2|f_1 > \tag{8}$$

with $f_i \in V, e_i \in V^*, i = 1, 2$, and $< | >$ denoting the duality product between V and its dual space V^*. A constant Dirac structure on V is a linear subspace $D \subset V \times V^*$

such that

$$D = D^{\perp} \tag{9}$$

where $\perp$ denotes orthogonal complement with respect to the bilinear form $\ll, \gg$.

Let now $(f, e) \in D = D^{\perp}$. Then as an immediate consequence of (8)

$$0 = \ll (f, e), (f, e) \gg = 2 < e|f > \tag{10}$$

Thus for all $(f, e) \in D$ we obtain $< e|f > = 0$, expressing power conservation with respect to the dual power variables $f \in V$ (e.g., currents or forces) and $e \in V$ (e.g., voltages or velocities).

The Stokes-Dirac structure corresponding to 3-dimensional fluid dynamics is now defined as follows. Let $W \subset \mathcal{D} \subset \mathbb{R}^3$ be a 3-dimensional manifold with smooth 2-dimensional boundary ∂W. Let $\Omega^k(W)$ denote the space of differential k-forms on $W, k = 0, 1, 2, 3$, and let $\Omega^k(\partial W)$ denote the k-forms on $\partial W, k = 0, 1, 2$. We identify the mass density ρ with a 3-form on W (see e.g. [8, 9]), that is, with an element in $\Omega^3(W)$. Furthermore, we assume the existence of a *Riemannian metric* $<,>$ on W (usually the standard Euclidean metric). Then we can identify (by "index raising" w.r.t. this Riemannian metric) the Eulerian vector field v on W with a 1-form v, that is, with an element of $\Omega^1(W)$. This leads to the consideration of the (linear) space of energy variables

$$X := \Omega^3(W) \times \Omega^1(W) \tag{11}$$

Next we consider the *boundary external variables* (or boundary input and output variables). First we consider the space $\Omega^0(\partial W)$ of 0-forms, that is, the functions on ∂W. They will represent the "dynamic pressure" at the boundary. Secondly, we consider the space $\Omega^2(\partial W)$ of 2-forms on ∂W, representing the "boundary mass flow". Thus we consider the space of boundary variables

$$\Omega^0(\partial W) \times \Omega^2(\partial W) \tag{12}$$

Note that (see also [8, 9]) there is a pairing (,) between $\Omega^0(\partial W)$ and $\Omega^2(\partial W)$, given by

$$(f, \alpha) := \int_{\partial W} f\alpha, \quad f \in \Omega^0(\partial W),\ \alpha \in \Omega^2(\partial W). \tag{13}$$

This pairing is weakly non-degenerate, that is, if $(f, \alpha) = 0$ for all $\alpha \in \Omega^2(\partial W)$ then $f = 0$, and if $(f, \alpha) = 0$ for all f, then $\alpha = 0$. Thus we can regard $\Omega^0(\partial W)$ as a *dual* space of $\Omega^2(\partial W)$, that is,

$$\Omega^0(\partial W) = (\Omega^2(\partial W))^* \tag{14}$$

(Note that in this way $\Omega^0(\partial W)$ is a subspace of the functional analytic dual of $\Omega^2(\partial W)$.) The pairing (13) will represent the power flowing into the system through the boundary ∂W.

In a similar way we define

$$\begin{array}{rcl}(\Omega^3(W))^* & = & \Omega^0(W)\\ (\Omega^1(W))^* & = & \Omega^2(W)\end{array} \tag{15}$$

using the weakly non-degenerate pairing

$$(\alpha, \beta) = \int_W \alpha \wedge \beta \tag{16}$$

with $\alpha \in \Omega^0(W), \beta \in \Omega^3(W)$, respectively $\alpha \in \Omega^2(W), \beta \in \Omega^1(W)$.

The Stokes-Dirac structure will now be defined on $V := X \times \Omega^0(\partial W)$ (i.e., the space of energy variables and "half" of the space of boundary variables), in the following way.

Theorem 2.2. *(Stokes-Dirac structure) Let $W \subset \mathbb{R}^3$ be a 3-dimensional manifold with boundary ∂W. Consider $V = X \times \Omega^0(\partial W) = \Omega^3(W) \times \Omega^1(W) \times \Omega^0(\partial W)$, and $V^* = \Omega^0(W) \times \Omega^2(W) \times \Omega^2(\partial W)$, together with the bilinear form induced by the pairing (13) and (16)*

$$\begin{aligned}&\ll \left(f_p^1, f_v^1, f_b^1, e_\rho^1, e_v^1, e_b^1\right), \left(f_\rho^2, f_v^2, f_b^2, e_\rho^2, e_v^2, e_b^2\right) \gg\\ &:= \textstyle\int_W \left(e_\rho^1 \wedge f_\rho^2 + e_\rho^2 \wedge f_\rho^1 + e_v^1 \wedge f_v^2 + e_v^2 \wedge f_v^1\right)\\ &+ \textstyle\int_{\partial W} \left(e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1\right)\end{aligned} \tag{17}$$

where

$$\begin{aligned}&f_\rho^i \in \Omega^3(W), f_v^i \in \Omega^1(W), f_b^i \in \Omega^0(\partial W)\\ &e_\rho^i \in \Omega^0(W), e_v^i \in \Omega^2(W), e_b^i \in \Omega^2(\partial W)\end{aligned} \tag{18}$$

Then $D \subset V \times V^*$ defined as

$$\begin{array}{rcl} D & = & \{(f_\rho, f_v, f_b, e_\rho, e_v, e_b) \in V \times V^* |\\ & & f_\rho = de_v, f_v = de_\rho, f_b = e_{\rho|\partial W}, e_b = -e_{v|\partial W}\}\end{array} \tag{19}$$

where d is the exterior derivative (mapping k-forms into $(k+1)$-forms), and where $|_{\partial W}$ denotes the restriction of k-forms on W to k-forms on the boundary ∂W, is a Dirac structure with respect to the bilinear form $\ll, \gg$ defined in (17).

Proof This can be proved along the same lines as in [14], making use of Stokes' theorem $\int_W d\alpha = \int_{\partial W} \alpha$ for any 2-form α. (In [14] the "symmetric" case was considered with $V = \Omega^2(W) \times \Omega^2(W) \times \Omega^1(\partial W)$ on a 3-dimensional domain $W \subset \mathbb{R}^3$, which turns out to be the appropriate setting for Maxwell's equations.) □

2.3 The Hamiltonian formulation

The idea is now to regard the Stokes-Dirac structure of Theorem 2.2 as the power-conserving interconnection relating the boundary external variables f_b, e_b to the internal variables f_ρ, f_v, e_ρ, e_v. Furthermore, following the framework in [14, 15] the internal variables f_ρ, f_v are equated with (minus) the time-derivatives $\frac{\partial \rho}{\partial t}, \frac{\partial v}{\partial t}$ of the energy variables ρ, v, while the internal variables e_ρ, e_v are equated with the co-energy variables $\delta_\rho H, \delta_v H$. However, contrary to the case of the telegrapher's equations or Maxwell's equations as treated in [14, 15], we still need to introduce an additional term to the Stokes-Dirac structure given above, which is due to the 3-dimensional geometry associated with convection. The problem thus concerns the geometric formulation of the term $v.\nabla v$ in (2).

¿From a general differential-geometric point of view this can be done as follows. Let $<,>$ be any Riemannian metric on W, with ∇ denoting its unique symmetric covariant derivative. (If $<,>$ is the Euclidean metric then ∇ is just the ordinary derivative operator $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ as above.) Let u be a vector field on W, and let $u^\flat$ denote the corresponding 1-form, defined as $u^\flat = i_u <,>$ (" index raising" via the metric). Then the following formula holds, relating the covariant derivative to the Lie-derivative:

$$L_u u^\flat = (\nabla_u u)^\flat + \frac{1}{2} d < u, u > \tag{20}$$

(see for a proof [1, p. 202]). Since by Cartan's magical formula

$$L_u u^\flat = i_u du^\flat + d i_u u^\flat = i_u du^\flat + d < u, u > \tag{21}$$

we also obtain

$$(\nabla_u u)^\flat = i_u du^\flat + \frac{1}{2} d < u, u >, \tag{22}$$

which is the coordinate-free analog of the classical vector calculus formula (using the standard Euclidean metric)

$$u.\nabla u = \text{ curl } \ u \times u + \frac{1}{2}\nabla |u|^2 \tag{23}$$

Let us now consider (2), and let us consider v to be a 1-form. By "index lowering" with respect to the Riemannian metric the 1-form v defines a vectorfield $v^\sharp$ (such that $(v^\sharp)^\flat = v$). Hence, we may represent (2) with respect to any Riemannian metric $<,>$ as

$$\frac{\partial v}{\partial t} = -i_{v^\sharp} dv - d\left(\frac{1}{2} < v^\sharp, v^\sharp >\right) - \frac{1}{\tilde{\rho}} dp \tag{24}$$

with $\tilde{\rho}$ the mass density *function*, formally defined as $\tilde{\rho} = *\rho$, with $*$ denoting the Hodge star operator determined by $<,>$; converting the 3-form ρ into the 0-form (function) $*\rho$.

Furthermore, by (3) it follows that

$$\frac{1}{\tilde{\rho}}dp = d(U(\tilde{\rho}) + \tilde{\rho}\frac{\partial U}{\partial \tilde{\rho}}(\tilde{\rho})) = d(\frac{\partial}{\partial \tilde{\rho}}(\tilde{\rho}U(\tilde{\rho}))) \ (= d(h(\tilde{\rho})) \tag{25}$$

Hence we may rewrite (24) as

$$\frac{\partial v}{\partial t} = -i_{v^\sharp}dv - d\left(\frac{\partial}{\partial \tilde{\rho}}\left[\frac{1}{2}\tilde{\rho} < v^\sharp, v^\sharp > + \tilde{\rho}U(\tilde{\rho})\right]\right) \tag{26}$$

where in the second term on the right-hand side we recognize (see (4)) the total energy density.

Finally, consider the total energy H_W given in (4) which formally can be rewritten as a function of the 3-form ρ and the 1-form v as

$$H_W = \int_W \left[\frac{1}{2} < v^\sharp, v^\sharp > + U(*\rho)\right]\rho. \tag{27}$$

The partial derivative $\delta_\rho H_W$ is an element of $(\Omega^3(W))^*$, and thus can be identified with an element of $\Omega^0(W)$ (namely, with the function $\frac{\partial}{\partial \rho}\left[\frac{1}{2}\tilde{\rho} < v^\sharp, v^\sharp > + \tilde{\rho}U(\tilde{\rho}\right] = \frac{1}{2} < v^\sharp, v^\sharp > + h(*\rho)$ in (26)), while the other partial derivative $\delta_v H_W$ is an element of $(\Omega^1(W))^*$, and thus can be equated with an element of $\Omega^2(W)$ (in fact, with the 2-form $i_{v^\sharp}\rho$). It also follows immediately that $\delta_\rho H_W$ and $\delta_v H_W$ only depend on the *energy density* (the integrand in (4) or (27)), and thus we simply write $\delta_\rho H$ and $\delta_v H$. Finally, we note the equality (most easily checked in a basis)

$$i_{v^\#}dv = \frac{1}{*\rho} * ((*dv) \wedge (*\delta_v H)) \tag{28}$$

with dv, $\delta_\rho H$ denoting 2-forms, and $*$ again denoting the Hodge star operator, converting 2-forms into 1-forms.

Summarizing, we can rewite (1) into the following form

$$\frac{\partial \rho}{\partial t} = -d(\delta_v H) \tag{29}$$

$$\frac{\partial v}{\partial t} = -d(\delta_\rho H) - \frac{1}{*\rho} * ((*dv) \wedge (*\delta_v H)). \tag{30}$$

Comparing with the Stokes-Dirac structure given in Theorem 2.2 we notice the additional term in the right-hand side of (30). This is incorporated into the following definition of a *modified* Stokes-Dirac structure

Proposition 2.3. (Modified Stokes-Dirac structure) Consider the same setting as in Theorem 2.2. Then $D \subset V \times V^*$ defined as

$$\begin{aligned} D^m = \ & \{(f_\rho, f_v, f_b, e_\rho, e_v, e_b) \in V \times V^* | \\ & f_\rho = de_v, f_v = de_\rho + \tfrac{1}{*\rho} * ((*dv) \wedge (*e_v)), \\ & f_b = e_{\rho|\partial W}, \ e_b = -e_{v|\partial W}\} \end{aligned} \tag{31}$$

is a Dirac structure.

Proof This is based on the fact that $e_v^2 * ((*dv) \wedge (*e_v^1))$ is skew-symmetric in $e_v^1, e_v^2 \in \Omega^2(W)$, and hence does not contribute to the bilinear form (17). (In fact, in vector calculus notation $e_v^2 * ((*dv) \wedge (*e_v^1)) = (*dv) \cdot (e_v^1 \times e_v^2)$.) □

Remark 2.4. Note however that D as given in (31) *not* anymore a *constant* Dirac structure, since it depends on the energy variables ρ and v.

Remark 2.5. For a *1-dimensional* fluid, that is $W = [0, L]$ for some $L > 0$, the extra convection term in (31) is automatically zero. Indeed, in this case the mass-density ρ becomes a 1-form on $[0, L]$, while v remains a 1-form on $[0, L]$, and the Dirac structures defined in Theorem 2.2 and Proposition 2.3 both reduce to

$$\begin{aligned} D \quad = \quad & \{(f_\rho, f_v, f_b, e_\rho, e_v, e_b) \,|\, f_\rho, f_v \in \Omega^1(W), \\ & e_\rho, e_v \in \Omega^0(W), f_b, e_b \in \Omega^0(\partial W), \\ & f_\rho = de_v, f_v = de_\rho, f_b = e_{\rho|\partial W}, e_b = -e_{v|\partial W}\} \end{aligned} \tag{32}$$

As announced before, the dynamics corresponding to the modified Stokes-Dirac structure (31) and the Hamiltonian (4) is now defined by setting

$$\begin{aligned} f_\rho &= -\tfrac{\partial \rho}{\partial t}, \quad e_\rho = \delta_\rho H \\ f_v &= -\tfrac{\partial v}{\partial t}, \quad e_v = \delta_v H \end{aligned} \tag{33}$$

leading immediately to the port-controlled Hamiltonian system whose dynamics is given by (29), (30), with boundary external variables

$$\begin{aligned} f_b &= \delta_\rho H|_{\partial W} \left(= \left[\tfrac{1}{2} < v^\sharp, v^\sharp > + h(*\rho)\right] |_{\partial W}\right) \\ e_b &= -\delta_v H|_{\partial W} \left(= -i_{v^\sharp}\rho|_{\partial W}\right) \end{aligned} \tag{34}$$

The resulting system can be regarded as a *boundary control system* in the sense of e.g. [6]. Indeed, we can either regard f_b as the *boundary control* variable (with e_b being the boundary *output*), or the other way around.

Energy exchange through the boundary is not the only way a distributed-parameter system may interact with its environment. Instead of boundary external variables we may also incorporate *distributed* external variables, leading to *distributed* control problems; see [14] for some developments. Also, energy *dissipation* can be incorporated in the framework by *terminating* some of the ports (boundary or distributed) by a resistive relation (given by a Rayleigh dissipation functional). In this way we can for example represent the Navier-Stokes equations.

2.4 Energy-balance

It immediately follows from the power-conservation properly (10) of any Dirac structure that the modified Stokes-Dirac structure D^m defined in Proposition 2.3 has the property

$$\int_W (e_\rho \wedge f_\rho + e_v \wedge f_v) + \int_{\partial W} e_b \wedge f_b = 0. \tag{35}$$

Hence by substituting (33) we immediately obtain

$$\frac{d}{dt}H_W = \int_{\partial W} e_b \wedge f_b = -\int_{\partial W} \delta_v H \wedge \delta_\rho H \tag{36}$$

where $\delta_\rho H = \frac{1}{2} < v^\sharp, v^\sharp > + h(*\rho)$ is a function, and $\delta_v H$ is the 2-form $i_{v^\sharp}\rho$. This is exactly the coordinate-free version of (5) (with $\delta_\rho H$ in standard Euclidean metric given as $\frac{1}{2} \parallel v \parallel^2 + h(\rho)$ and $\delta_v H$ by $\rho v.n\ dA$). The 2-form $\delta_v H$ represents the mass-flow and $\delta_\rho H$ is the dynamic pressure. Note that alternatively we can write

$$\int_{\partial W} \delta_\rho H \wedge \delta_v H = \int_{\partial W} i_{v^\sharp} \left[\frac{1}{2} < v^\sharp, v^\sharp > \rho + U(*\rho)\rho\right] + \int_{\partial W} i_{v^\sharp}(*p) \tag{37}$$

where the 3-form $*p$ is given as

$$*p := h(*\rho)\rho - U(*\rho)\rho \tag{38}$$

with $h(*\rho) = \delta_\rho \left(\int_W U(*\rho)\rho\right)$ the enthalpy. The 3-form $*p$ is the (static) pressure form. This is the coordinate-free version of (7).

3 Conservation laws and passivity-based control of fluid dynamical systems

¿From the Dirac structure given in Proposition 2.3 one infers Casimir functions or conservation laws. A physically obvious conservation law corresponds to the total mass

$$\int_W \rho \tag{39}$$

Indeed, one immediately verifies that

$$\frac{d}{dt}\int_W \rho = \int_W \frac{\partial \rho}{\partial t} = -\int_W d(\delta_v H) = -\int_{\partial W} \delta_v H = \int_{\partial W} e_b \tag{40}$$

(which is nothing else than the mass-balance (1)). Nevertheless, this suggests some interesting possibilities for passivity-based control (see e.g. [20, 23]) based on interconnection and energy shaping, following the framework exposed e.g. in [26, 16, 21, 23]. Indeed, let us consider an additional controller system, also of port-controlled Hamiltonian form, but now with internally *distributed* control u_c and output y_c

$$\begin{aligned} \frac{\partial x_c}{\partial t} &= u_c \\ y_c &= \delta_{x_c} H_c \end{aligned} \tag{41}$$

with x_c a 2-form on ∂W, and $H_c = \int_{\partial W} \mathcal{H}_c(x_c)$ the controller Hamiltonian for a certain density 2-form $\mathcal{H}_c(x_c)$. Interconnect this controller to the fluid dynamic system via the power-conserving interconnection

$$\begin{aligned} u_c &= e_b \\ f_b &= -y_c \end{aligned} \tag{42}$$

(note that y_c is a function on ∂W). Then the closed-loop system is again a Hamiltonian system with total Hamiltonian

$$H_W + H_c \tag{43}$$

Furthermore, because of (40), the function

$$\int_W \rho - \int_{\partial W} x_c \tag{44}$$

is a Casimir function (conserved quantity). Therefore, by the Energy-Casimir (or Arnold) method, any other function

$$V := H_W + H_c + P\left(\int_W \rho - \int_{\partial W} x_c\right) \tag{45}$$

with $P : \mathbb{R} \to \mathbb{R}$ still to be assigned, can be used as an energy function for the closed-loop system, and therefore as a candidate Lyapunov function. In [16, 23] it has been shown that this approach covers the well-known energy-shaping method, which has proved to be quite powerful, e.g., for the control of mechanical systems, starting with the innovative paper by Takegaki & Arimoto [27]. Its potential for the control of fluid dynamical systems has to be investigated.

Next conservation law to be considered derives from the *helicity* of the fluid, defined as

$$\int_W v \wedge dv \tag{46}$$

This quantity measures the "knottedness" of the fluid, see e.g. [1]. Upon time-differentiation of (46) one obtains

$$\begin{aligned}
&\tfrac{d}{dt} \textstyle\int_W v \wedge dv = \int_W \left(\frac{\partial v}{\partial t} \wedge dv + v \wedge d\frac{\partial v}{\partial t}\right) = \\
&= - \textstyle\int_W d(\delta_\rho H) \wedge dv = - \int_W d(\delta_\rho H \wedge dv) = - \int_{\partial W} \delta_\rho H \wedge dv \\
&= - \textstyle\int_{\partial W} f_b \wedge dv
\end{aligned} \tag{47}$$

showing the boundary variable f_b which can be interconnected to a controller Hamiltonian system as before, leading again to new candidate Lyapunov functions.

4 Conclusions

We have shown how 3-dimensional ideal isentropic fluids in a fixed spatial domain can be modelled as a Hamiltonian boundary control system, using the notion of a Stokes-Dirac structure. Among others, this opens up the way for the application of passivity-based control techniques, which have been proven to be very effective for the control of lumped parameter physical systems modelled as port-controlled Hamiltonian systems.

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