Faculty of Mathematical Sciences

University of Twente

University for Technical and Social Sciences

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl

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A uniform approach to semi-marginalistic values for set games

T.S.H. Driessen and H. Sun[1]

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[1]Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi, 710072, People's Tepublic of China

A UNIFORM APPROACH to SEMI–MARGINALISTIC VALUES for SET GAMES *

Theo DRIESSEN † **Hao SUN** ‡

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Abstract

Concerning the solution theory for set games, the paper focuses on a family of solutions, each of which allocates to any player some type of marginalistic contribution with respect to any coalition containing the player. Here the marginalistic contribution may be interpreted as an individual one, or a coalitionally one. For any value of the relevant family, an axiomatization is given by three properties, namely one type of an efficiency property, the substitution property and one type of a monotonocity property. We present two proof techniques, each of which is based on the decomposition of any arbitrary set game into a union of either simple set games or elementary set games, the solutions of which are much easier to determine. A simple respectively elementary set game is associated with an arbitrary, but fixed item of the universe respectively coalition.

Keywords: set game, solution theory, value, axiomatization
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1 Concepts and Introduction

Let $\mathcal{U}$, called the *universe*, denote an abstract set which is fixed throughout the remainder. Following the introductory papers [1] (chapter 7), [2], [3], [6], a *set game* is a pair $\langle N, v\rangle$, where N is a nonempty, finite set, called *player set*, and $v : 2^N \to 2^{\mathcal{U}}$ is a *characteristic mapping*, defined on the power set of N, satisfying $v(\emptyset) := \emptyset$. Let $\mathcal{G}$ denote the space of all set games with an arbitrary player set, whereas $\mathcal{G}^N$ denotes the space of all set games with reference to a player set N which is fixed beforehand. An element of N (notation: $i \in N$) and a nonempty subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) is called a *player* and *coalition* respectively, and the associated set $v(S) \subseteq \mathcal{U}$ is called the *worth* of coalition S, to be interpreted as the (sub)set of items from $\mathcal{U}$ that can be obtained (are needed, preferred, owned) by coalition S if its members cooperate. Given a set game $\langle N, v\rangle$ and a coalition S,

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†Theo S.H. Driessen, Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands. Author in charge of correspondence. E-mail: t.s.h.driessen@math.utwente.nl

‡Hao Sun, Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi, 710072, People's Republic of China. E-mail: hsun@nwpu.edu.cn iamlxl@nwpu.edu.cn xdliu@nwpu.edu.cn fax: 00-86-29-849-1000 phone: 00-86-29-849-5957

we write $\langle S, v_S\rangle$ for the *sub set game* obtained by restricting v to subsets of S only (i.e., to 2^S).
Concerning the solution theory for set games, a *solution* ψ on $\mathcal{G}$ (or on a particular subclass of $\mathcal{G}$) associates a so-called allocation $\psi(N,v) = (\psi_i(N,v))_{i\in N} \in (2^{\mathcal{U}})^N$ with every set game $\langle N, v\rangle$. The so-called *allocation* $\psi_i(N,v) \subseteq \mathcal{U}$ to player i in the set game $\langle N, v\rangle$ represents the items that are given, according to the solution ψ, to player i from participating in the game. Until further notice, no constraints are imposed upon a solution ψ on $\mathcal{G}$. The difference of two sets A, $B \subseteq \mathcal{U}$ is denoted by $A - B$ and defined to be $A - B := \{x \in A \mid x \notin B\}$.
In Section 2 we introduce a family of solutions called *semi–marginalistic values*. According to a semi–marginalistic value, any player's allocation in a set game is the overall union of appropriately chosen marginalistic contributions of the player with respect to coalitions containing the player. Here the player's marginalistic contribution may be interpreted in various ways to allow for a uniform treatment of semi–marginalistic values (see Definition 2.3). The goal of the paper is twofold: on the one hand, we provide an axiomatization of any semi–marginalistic value in terms of three basic properties (global efficiency, substitution, and marginalistic contributions monotonicity; see Theorem 3.3), and on the other, we present two proof techniques. Section 3 is devoted to the first proof technique which is based on the decomposition of any set game into a union of so-called *simple set games*. Each simple set game is associated with an arbitrary, but fixed item, and the worth of a coalition in a simple set game equals either the empty set or the singleton consisting of the underlying item. Section 4 deals with the second proof technique which is based on the decomposition of any set game into a union of so-called *elementary set games*. Each elementary set game is associated with an arbitrary, but fixed coalition, and the worth of all coalitions, except the fixed one, in an elementary set game equals the empty set. In fact, the decomposition technique is mainly applied to the marginalistic contribution, the concept of which is treated as a new set game arising from an initial set game. In the final Section 5 we discuss the similarities between the two fields of set game theory and cooperative game theory.

2 Semi-marginalistic values for set games

We review four different solutions, studied throughout the solution theory for set games, before introducing a family of set games solutions containing each one of them. The purpose of the paper is to present a uniform axiomatization of the new family of solutions.

Example 2.1. For every set game $\langle N, v\rangle \in \mathcal{G}$, we say, on the one hand, an item $x \in \mathcal{U}$ is *attainable* by player i through a certain coalition S containing i whenever the item belongs to the coalition's worth, that is $x \in v(S)$; on the other hand, we say a coalition T can not block an item x whenever the item does not belong to the coalition's worth, that is $x \notin v(T)$.

(i) The *individually marginalistic IM–value*, as introduced by [1], allocates those items that are attainable by player i, but can not be blocked by the coalition consisting of the remaining members (different from player i). To be exact,

$$IM_i(N,v) := \cup_{\substack{S\subseteq N,\\ S\ni i}} \Big[v(S) - v(S\backslash\{i\}) \Big] \qquad \text{for all } i \in N. \tag{2.1}$$

(ii) The *overall-individually marginalistic OIM–value*, as introduced by [1], allocates those items that are attainable by player i, but can not be blocked by any subcoalition with one player less. To be exact,

$$OIM_i(N,v) := \cup_{\substack{S\subseteq N,\\ S\ni i}} \Big[v(S) - \cup_{j\in S} v(S\backslash\{j\}) \Big] \qquad \text{for all } i\in N. \tag{2.2}$$

(iii) The *overall-coalitionally marginalistic OCM–value*, as introduced by [10], allocates those items that are attainable by player i, but can not be blocked by any strict subcoalition. To be exact,

$$OCM_i(N,v) := \cup_{\substack{S\subseteq N,\\ S\ni i}} \Big[v(S) - \cup_{T\subsetneq S} v(T) \Big] \qquad \text{for all } i\in N. \tag{2.3}$$

(iv) The *Driessen–Sun DS–value*, as introduced by [5], allocates those items that are attainable by player i, but can not be blocked by any coalition not containing i. To be exact,

$$DS_i(N,v) := \cup_{\substack{S\subseteq N,\\ S\ni i}} \Big[v(S) - \Big[\cup_{T\subseteq N\backslash\{i\}} v(T) \Big] \Big] \qquad \text{for all } i\in N. \tag{2.4}$$

Among these four solutions, the IM-value is the largest in that the inclusions $OCM_i(N,v)\subseteq OIM_i(N,v)\subseteq IM_i(N,v)$ and $DS_i(N,v)\subseteq IM_i(N,v)$ hold for any player i in the set game $\langle N,v\rangle$. In this setting, for any $S\subseteq N$, the underlying expressions $v(S)-v(S\backslash\{i\})$, $v(S)-\cup_{j\in S}v(S\backslash\{j\})$ and $v(S)-\cup_{T\subsetneq S}v(T)$ respectively, are called the *marginalistic contribution of coalition S* induced by either one particular member, all members, or all subcoalitions.

Definition 2.2. A *semi–marginalistic value* ψ on the set game space $\mathcal{G}$ is defined to be one member out of the family of set games solutions of the following form:

$$\psi_i(N,v) = \cup_{\substack{S\subseteq N,\\ S\ni i}} \Big[v(S) - \triangledown^v_{S,i} \Big] \qquad \text{for all } \langle N,v\rangle\in\mathcal{G} \text{ and all } i\in N, \tag{2.5}$$

or equivalently,

$$\psi_i(N,v) = \cup_{\substack{S\subseteq N,\\ S\ni i}} MC^v_{S,i} \qquad \text{where} \qquad MC^v_{S,i} := v(S) - \triangledown^v_{S,i} \tag{2.6}$$

In words, for every player i, the *marginalistic contribution* $MC^v_{S,i}$ of every coalition S is determined by the set difference of the coalition's worth $v(S)$ and some (yet unspecified) expression $\triangledown^v_{S,i}$ which is supposed to depend, to some weak or strong extent, upon the worths of a certain collection of coalitions, somehow determined by S and/or i (for instance, through the unions and/or intersections of a number of (sub)coalitions). By convention, $\triangledown^v_{S,i} := \emptyset$ in the framework of one-person set games.

By (2.1) and (2.4), the IM– and DS–values are semi–marginalistic values of the form (2.5) by choosing $\triangledown^v_{S,i} := v(S\backslash\{i\})$ and $\triangledown^v_{S,i} := \cup_{T\subseteq N\backslash\{i\}} v(T)$ respectively, the expression of which still depends upon player i. By (2.2) and (2.3), the overall OIM– and OCM–values are semi–marginalistic values by choosing $\triangledown^v_{S,i} := \cup_{j\in S} v(S\backslash\{j\})$ and $\triangledown^v_{S,i} := \cup_{T\subsetneq S} v(T)$ respectively, the expression of which satisfies the so-called players' contributions independence.

Definition 2.3. We say a semi–marginalistic value ψ of the form (2.5) satisfies *players' contributions independence* whenever, for any coalition S, the associated expression $\nabla_{S,i}^{v}$ does not depend upon player i, that is $\nabla_{S,i}^{v} := \nabla_{S}^{v}$ is the same for all $i \in N$. Shortly,

$$\psi_i(N,v) = \cup_{\substack{S\subseteq N,\\ S\ni i}} MC_S^v = \cup_{\substack{S\subseteq N,\\ S\ni i}} \left[v(S) - \nabla_S^v\right] \qquad \text{for all } \langle N,v\rangle \in \mathcal{G} \text{ and all } i \in N; \quad (2.7)$$

For reasons that will be explained later: $\qquad \nabla_N^v \subseteq \cup_{S\subsetneqq N} v(S) \qquad (2.8)$

For the class of *monotonic set games* $\langle N,v\rangle$ (i.e., $v(S) \subseteq v(T)$ for all $S \subseteq T \subseteq N$), it was shown in [1] that the IM– and OIM–values coincide. So, for every monotonic set game $\langle N,v\rangle$, it holds $IM(N,v) = OIM(N,v) = OCM(N,v)$, whereas, by (2.4), $DS_i(N,v) = v(N) - v(N\backslash\{i\})$ for all $i \in N$, and consequently, $\cup_{i\in N} DS_i(N,v) = v(N) - \cap_{i\in N} v(N\backslash\{i\})$ on the class of monotonic set games. According to the next lemma, the DS–value differs from the remaining solutions in that another type of efficiency applies.

Lemma 2.4. Let ψ be a semi–marginalistic value of the form (2.7) assuming players' contributions independence, such that (2.8) holds. Then ψ satisfies the *global efficiency principle*, that is the solution ψ allocates all the attainable items to the players in that

$$\cup_{i\in N}\psi_i(N,v) = \cup_{S\subseteq N} v(S) \qquad \text{for all } \langle N,v\rangle \in \mathcal{G}. \qquad (2.9)$$

Proof of Lemma 2.4.
Clearly, for the semi–marginalistic value ψ of the form (2.7), its global efficiency condition (2.9) is equivalent to the following condition:

$$\cup_{S\subseteq N} MC_S^v = \cup_{S\subseteq N} v(S) \qquad \text{for all } \langle N,v\rangle \in \mathcal{G}. \qquad (2.10)$$

We prove (2.10) by induction on the number of players. The case $n = 1$ is trivial due to $\nabla_{S,i}^{v} := \emptyset$ in the framework of one-person set games. Let $\langle N,v\rangle \in \mathcal{G}$ with $n \geq 2$. Then we obtain the following chain of equalities:

$$\begin{aligned}
\cup_{S\subseteq N} MC_S^v &= MC_N^v \cup \left[\cup_{S\subsetneqq N} MC_S^v\right] = MC_N^v \cup \left[\cup_{k\in N}\left[\cup_{S\subseteq N\backslash\{k\}} MC_S^v\right]\right] \\
&\overset{(2.10)}{=} MC_N^v \cup \left[\cup_{k\in N}\left[\cup_{S\subseteq N\backslash\{k\}} v(S)\right]\right] \qquad \text{(by the induction hypothesis)} \\
&= MC_N^v \cup \left[\cup_{S\subsetneqq N} v(S)\right] \\
&\overset{(2.6)}{=} \left[v(N) - \nabla_N^v\right] \cup \left[\cup_{S\subsetneqq N} v(S)\right] \\
&\overset{(2.8)}{=} \cup_{S\subseteq N} v(S).
\end{aligned}$$

This completes the proof of the global efficiency (2.10) for the semi–marginalistic value ψ. □

Due to their mutual inclusions, we derive from the global efficiency of the OCM–value (by Lemma 2.4), the global efficiency of the OIM– and IM–values as well, although the latter one is not of the form (2.7). In addition to the global efficiency axiom, we study another axiom, called substitution property, in order to be able to provide, in the next section, an axiomatization of any semi–marginalistic value satisfying appropriately chosen marginalistic contributions.

Definition 2.5. (Substitutes in a set game and substitution property for a solution)

(i) Two players $i \in N$, $j \in N$, $i \neq j$, are said to be *substitutes* in the set game $\langle N, v\rangle \in \mathcal{G}$ whenever it holds $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

(ii) We say a solution ψ on the set game space $\mathcal{G}$ possesses the *substitution property* if $\psi_i(N, v) = \psi_j(N, v)$ for any pair $i \in N$, $j \in N$, $i \neq j$, of substitutes in the set game $\langle N, v\rangle \in \mathcal{G}$. In words, two substitutes in a set game are allocated the same items.

Lemma 2.6. Let ψ be a semi–marginalistic value of the form (2.7) assuming players' contributions independence, such that the marginalistic contribution MC concept inherits the role of substitutes, that is, for any pair $i \in N$, $j \in N$, $i \neq j$, of substitutes in the set game $\langle N, v\rangle$

$$MC^v_{S\cup\{i\}} = MC^v_{S\cup\{j\}} \quad \text{or "equivalently"}, \quad \nabla^v_{S\cup\{i\}} = \nabla^v_{S\cup\{j\}} \quad \text{for all } S \subseteq N\backslash\{i, j\}. \tag{2.11}$$

Then ψ satisfies the substitution property.

Proof of Lemma 2.6.
For any arbitrary pair of players $i \in N$, $j \in N$, $i \neq j$, in a set game $\langle N, v\rangle \in \mathcal{G}$, it holds

$$\begin{aligned} \psi_i(N, v) &\stackrel{(2.7)}{=} \cup_{\substack{S\subseteq N,\\ S\ni i}} MC^v_S = \left[\cup_{\substack{S\subseteq N,\\ S\ni i,\, S\ni j}} MC^v_S\right] \cup \left[\cup_{\substack{S\subseteq N,\\ S\ni i,\, S\not\ni j}} MC^v_S\right] \\ &= \left[\cup_{\substack{S\subseteq N,\\ S\ni i,\, S\ni j}} MC^v_S\right] \cup \left[\cup_{S\subseteq N\backslash\{i,j\}} MC^v_{S\cup\{i\}}\right] \end{aligned}$$

¿From this decomposition it follows immediately that, for the equality $\psi_i(N, v) = \psi_j(N, v)$, it suffices to require that $MC^v_{S\cup\{i\}} = MC^v_{S\cup\{j\}}$ for all $S \subseteq N\backslash\{i, j\}$. So, the semi–marginalistic value ψ satisfies the substitution property whenever (2.11) holds. □

It is left to the reader to verify the right-hand side of (2.11) in the framework of the marginalistic contribution of every coalition S induced by either all members, or all subcoalitions. Thus, by Lemma 2.6, the OIM– and OCM–values possess the substitution property, as it holds for the DS–value too (cf. [5]) although this latter value is not of the form (2.7). As a minor contribution, we conclude this section with an alternative, but shortened proof of the coincidence of the IM– and OIM–values on the class of monotonic set games (the direct proof of which is much different from the inductive proof by [1], Theorem 2.2, pages 110-111).

Lemma 2.7. $IM(N, v) = OIM(N, v)$ for every monotonic set game $\langle N, v\rangle \in \mathcal{G}$.

Proof of Lemma 2.7.
Let $\langle N, v\rangle \in \mathcal{G}$ be a (monotonic) set game and $i \in N$. As noted earlier, by (2.1)–(2.2), the inclusion $OIM_i(N, v) \subseteq IM_i(N, v)$ is always valid since $v(S\backslash\{i\}) \subseteq \cup_{j\in S} v(S\backslash\{j\})$ for all $S \subseteq N$ satisfying $i \in S$. In order to prove the inverse inclusion $IM_i(N, v) \subseteq OIM_i(N, v)$, it suffices to show that $x \notin OIM_i(N, v)$ implies $x \notin IM_i(N, v)$.
Suppose $x \notin OIM_i(N, v)$. Let $S \subseteq N$ with $i \in S$. We show $x \notin v(S) - v(S\backslash\{i\})$. In case $x \notin v(S)$, then we are done. Without loss of generality, we may assume $x \in v(S)$. Under these circumstances we show $x \in v(S\backslash\{i\})$. Since $x \notin OIM_i(N, v)$, it holds $x \notin v(S) - \cup_{j\in S} v(S\backslash\{j\})$ and together with the assumption $x \in v(S)$, we arrive at $x \in \cup_{j\in S} v(S\backslash\{j\})$. In summary, so

far we conclude, from $x \in v(S)$ (where $i \in S$), that $x \in v(S \backslash \{j\})$ for some $j \in S$. By repeating the same procedure, step by step, there exists some $k \in S \backslash \{j\}$ such that $x \in v(S \backslash \{j, k\})$ and so on. Note that $x \notin v(\{i\})$ because of the assumption $x \notin OIM_i(N, v)$. By repeatedly applying the same procedure, there exists a coalition $R \subseteq S$ not containing player i such that $x \in v(R)$. Finally, from $x \in v(R)$, $R \subseteq S \backslash \{i\}$ and the (tacitly assumed) monotonicity of the set game $\langle N, v \rangle$, we deduce that $x \in v(S \backslash \{i\})$ as was to be shown. □

3 An axiomatization of semi-marginalistic values for set games

The purpose of this section is to present an axiomatic characterization of any semi-marginalistic value of the form (2.7). To be exact, we show that such a value is fully determined by the global efficiency (2.9) and the (tacitly assumed) substitution properties, as treated in Section 2, together with a type of monotonicity property. One out of two proof techniques is based on the decomposition of any set game into a union of a new type of set games, called *simple set games*. Concerning simple set games, the worth of any coalition equals either the empty set or a singleton consisting of one arbitrary, but fixed item.

Definition 3.1. Let ψ be a semi–marginalistic value of the form (2.7) assuming players' contributions independence. We say the solution ψ possesses the *marginalistic contributions monotonicity property* if

$$\psi_i(N, v) \subseteq \psi_i(N, w) \qquad \text{for all } \langle N, v \rangle \in \mathcal{G},\ \langle N, w \rangle \in \mathcal{G}, \text{ and all } i \in N, \tag{3.1}$$

satisfying $MC_S^v \subseteq MC_S^w$ for all $S \subseteq N$ with $i \in S$, where the marginalistic contribution MC is associated with ψ. In words, with respect to two different set games, the larger the player's marginalistic contributions in the game, the more items allocated to him.

Corollary 3.2. (cf. Lemmata 2.4 and 2.6).
Any semi–marginalistic value ψ of the form (2.7) assuming players' contributions independence, such that both (2.8) and (2.11) hold, satisfies the global efficiency, substitution, and the marginalistic contributions monotonicity properties.

Theorem 3.3. (Axiomatization) Consider the setting of Definitions 2.3, 2.5(ii) and 3.1(i). There exists a unique solution on the set game space $\mathcal{G}^N$ (with reference to a fixed player set N) satisfying the global efficiency, substitution, and marginalistic contributions monotonicity properties, and it is given by the semi–marginalistic value ψ of the form (2.7) based on players' contributions independence.

The proof of Theorem 3.3 proceeds in three steps. The first preliminary result provides another interpretation of any semi–marginalistic value in that the value represents the *maximal* solution satisfying the global efficiency and marginalistic contributions monotonicity properties.

Proposition 3.4. If a solution ρ on $\mathcal{G}^N$ satisfies the global efficiency and marginalistic contributions monotonicity properties, then the inclusion $\rho_i(N, v) \subseteq \psi_i(N, v)$ holds for all $\langle N, v \rangle$ and all $i \in N$.

Proof of Proposition 3.4.
Suppose a solution ρ on $\mathcal{G}^N$ satisfies the global efficiency and marginalistic contributions monotonicity properties. Let $\langle N, v\rangle$ be a set game and $i \in N$. In order to show the inclusion $\rho_i(N,v) \subseteq \psi_i(N,v)$, let $x \in \rho_i(N,v)$, but assume, on the contrary, $x \notin \psi_i(N,v)$. Define a new set game $\langle N, w\rangle$ as follows:

$$w(S) := \begin{cases} v(S) - \{x\} & \text{for all } S \subseteq N \text{ with } x \in v(S); \\ v(S) & \text{for all } S \subseteq N \text{ with } x \in \mathcal{U} - v(S). \end{cases}$$

Notice that $x \notin w(S)$ for all $S \subseteq N$. From this observation, together with the global efficiency (2.9) of ρ applied to the set game $\langle N, w\rangle$, we derive the following chain of inclusions:

$$\rho_i(N,w) \subseteq \cup_{j\in N}\rho_j(N,w) \overset{(2.9)}{=} \cup_{S\subseteq N} w(S) \subseteq \mathcal{U} - \{x\} \qquad \text{Particularly, } x \notin \rho_i(N,w).$$

Next we claim $MC^w_S = MC^v_S$ for all $S \subseteq N$ with $i \in S$ (where $MC^v_S := v(S) - \bigtriangledown^v_S$). Consequently, $\rho_i(N,w) = \rho_i(N,v)$ by the marginalistic contributions monotonicity (3.1) of ρ, but this equality contradicts the facts $x \in \rho_i(N,v)$ and $x \notin \rho_i(N,w)$. This contradiction completes the proof, provided we establish the claim above-mentioned.
Let $S \subseteq N$ with $i \in S$. We distinguish two cases. If $x \notin v(S)$, then $w(S) = v(S)$ and it holds

$$MC^w_S = w(S) - \bigtriangledown^w_S = v(S) - \bigtriangledown^w_S = v(S) - \bigtriangledown^v_S = MC^v_S$$

If $x \in v(S)$, then $w(S) = v(S) - \{x\}$ as well as $x \in \bigtriangledown^v_S$ (because of the assumption $x \notin \psi_i(N,v)$) and thus, it holds

$$MC^w_S = w(S) - \bigtriangledown^w_S = \Big[v(S) - \{x\}\Big] - \bigtriangledown^w_S = v(S) - \bigtriangledown^v_S = MC^v_S$$

This completes the proof of the remaining claim. Further, this proof indicates that the global efficiency may be replaced by any weak form of global efficiency, that is $\cup_{j\in N}\psi_j(N,w) \subseteq \cup_{S\subseteq N} w(S)$ for every set game $\langle N, w\rangle$. In addition, the definition of the expression $\bigtriangledown^w_S$ does not matter so much. □

The final part of the preliminary results, for the sake of a first proof technique of Theorem 3.3, deals with a particular type of set games, called simple set games, which will be treated as the components of a decomposition for any arbitrary set game.

Definition 3.5. With every set game $\langle N, v\rangle \in \mathcal{G}$ and every $x \in \mathcal{U}$, there is associated the *simple set game* $\langle N, v_x\rangle \in \mathcal{G}$ defined to be

$$v_x(S) := \begin{cases} \{x\} & \text{for all } S \subseteq N \text{ with } x \in v(S); \\ \emptyset & \text{for all } S \subseteq N \text{ with } x \in \mathcal{U} - v(S). \end{cases} \tag{3.2}$$

The coalition $S \subseteq N$ is said to be *winning* in the simple set game $\langle N, v_x\rangle$ if $v_x(S) = \{x\}$ or equivalently, $x \in v(S)$.

Proposition 3.6. (Decomposition results for set games and semi–marginalistic values)
Let $\langle N, v\rangle$ be a set game, $x \in \mathcal{U}$, and $S \subseteq N$. Recall $MC^v_S := v(S) - \bigtriangledown^v_S$.

(i) $\quad v = \cup_{y\in\mathcal{U}} v_y \qquad$ that is, $\qquad v(T) = \cup_{y\in\mathcal{U}} v_y(T) \qquad$ for all $T \subseteq N$. (3.3)

(ii) The following equivalence holds: $MC_S^{v_x} = \{x\} \Longleftrightarrow x \in MC_S^v$ (3.4)

(iii) $\psi_i(N,v) = \cup_{y\in\mathcal{U}}\psi_i(N,v_y)$ for all $i \in N$ and every semi–marginalistic (3.5) value ψ of the form (2.7) assuming players' contributions independence.

(iv) If a solution ρ on $\mathcal{G}^N$ possesses the marginalistic contributions monotonicity property, then it holds $\rho_i(N,v_x) \subseteq \rho_i(N,v)$ for all $i \in N$ and all $x \in \mathcal{U}$.

Proof of Proposition 3.6.

The decomposition statement (3.3) of the set game $\langle N,v\rangle$ is trivial since $\mathcal{U} = v(T) \cup \left[\mathcal{U} - v(T)\right]$ for all $T \subseteq N$. The decomposition statement (3.5) of the semi–marginalistic value ψ of the set game $\langle N,v\rangle$ is a direct consequence of the equivalence (3.4) because, for all $i \in N$, it holds

$$\cup_{y\in\mathcal{U}}\psi_i(N,v_y) \overset{(2.7)}{=} \cup_{y\in\mathcal{U}} \cup_{\substack{S\subseteq N,\\ S\ni i}} MC_S^{v_y} = \cup_{\substack{S\subseteq N,\\ S\ni i}} \cup_{y\in\mathcal{U}} MC_S^{v_y} \overset{(3.4)}{=} \cup_{\substack{S\subseteq N,\\ S\ni i}} MC_S^{v} \overset{(2.7)}{=} \psi_i(N,v).$$

The statement in part (iv) is a direct consequence of the equivalence (3.4) too due to the inclusion $MC_S^{v_x} \subseteq MC_S^v$ for all $S \subseteq N$ with $i \in S$, and all $x \in \mathcal{U}$. It remains to prove, for every $S \subseteq N$, the equivalence (3.4) as follows.

$$\begin{aligned}
MC_S^{v_x} = \{x\} &\Longleftrightarrow v_x(S) - \bigtriangledown_S^{v_x} = \{x\} \\
&\Longleftrightarrow v_x(S) = \{x\} \quad \text{and} \quad \bigtriangledown_S^{v_x} = \emptyset \\
&\Longleftrightarrow x \in v(S) \quad \text{and} \quad x \notin \bigtriangledown_S^{v_x} \\
&\Longleftrightarrow x \in v(S) \quad \text{and} \quad x \notin \bigtriangledown_S^{v_y} \quad \text{for all } y \in \mathcal{U} \\
&\Longleftrightarrow x \in v(S) \quad \text{and} \quad x \notin \bigtriangledown_S^{v} \\
&\Longleftrightarrow x \in v(S) - \bigtriangledown_S^{v} \\
&\Longleftrightarrow x \in MC_S^v
\end{aligned}$$

Concerning the fourth and fifth equivalence in the above chain, we make use of the following relationships: $\bigtriangledown_S^v = \bigtriangledown_S^{(\cup_{y\in\mathcal{U}}v_y)} = \cup_{y\in\mathcal{U}}\bigtriangledown_S^{v_y}$, while $v_y(S) \cap v_z(S) = \emptyset$ whenever $y \neq z$. □

Proof of the uniqueness part of Theorem 3.3.

Suppose a solution ρ on $\mathcal{G}^N$ satisfies the global efficiency, substitution, and marginalistic contributions monotonicity properties. Let $\langle N,v\rangle$ be a set game and $i \in N$. We show $\rho_i(N,v) = \psi_i(N,v)$. By Propositions 3.4 and 3.6 (iii)-(iv), the following relationships hold:

$$\psi_i(N,v) = \cup_{y\in\mathcal{U}}\psi_i(N,v_y) \quad \text{as well as} \quad \cup_{y\in\mathcal{U}}\rho_i(N,v_y) \subseteq \rho_i(N,v) \subseteq \psi_i(N,v)$$

Fixing the set game $\langle N,v\rangle$, player i and item $x \in \mathcal{U}$ at beforehand, it suffices to show

$$\psi_i(N,v_x) = \rho_i(N,v_x) \quad \text{for every simple set game } \langle N,v_x\rangle. \tag{3.6}$$

The proof of (3.6) proceeds by induction on the number of winning coalitions in the marginalistic contributions set game $\langle N, MC^{v_x}\rangle$, defined to be $MC^{v_x}(S) := MC_S^{v_x}$ for all $S \subseteq N$. Coalition S is said to be winning in the set game $\langle N, MC^{v_x}\rangle$ if it holds $MC^{v_x}(S) = \{x\}$ or

equivalently, $x \in MC_S^v$ (see (3.4)). We distinguish two cases, whether or not there exists a unique winning coalition.
Case one. Suppose there exists a unique winning coalition S_1 in the set game $\langle N, MC^{v_x}\rangle$, that is $MC^{v_x}(S_1) = \{x\}$ and $MC^{v_x}(S) = \emptyset$ for all $S \neq S_1$. Our first claim is the following:

$$\rho_j(N, v_x) = \psi_j(N, v_x) = \emptyset \qquad \text{for all } j \in N\backslash S_1. \tag{3.7}$$

Indeed, for all $j \in N\backslash S_1$, it holds, by definition of the set game, $MC_S^{v_x} = \emptyset$ for all $S \subseteq N$ with $j \in S$. From this, together with Proposition 3.4 applied to the simple set game $\langle N, v_x\rangle$, we deduce the following chain of inclusions:

$$\rho_j(N, v_x) \subseteq \psi_j(N, v_x) \overset{(2.7)}{=} \cup_{\substack{S\subseteq N,\\ S\ni j}} MC_S^{v_x} = \emptyset \qquad \text{for all } j \in N\backslash S_1, \text{ and so, (3.7) holds.}$$

Our second claim is the following: $\quad MC_S^{(MC^{v_x})} = MC_S^{v_x} \quad$ for all $S \subseteq N$ and thus,

$$\rho_j(N, MC^{v_x}) = \rho_j(N, v_x) \quad \text{and} \quad \psi_j(N, MC^{v_x}) = \psi_j(N, v_x) \quad \text{for all } j \in N. \tag{3.8}$$

Indeed, if $S \neq S_1$, then $MC^{v_x}(S) = \emptyset$ and so, $MC_S^{(MC^{v_x})} = \emptyset$. Otherwise, $MC_{S_1}^{(MC^{v_x})} = MC_{S_1}^{v_x} - \nabla_{S_1}^{(MC^{v_x})} = MC_{S_1}^{v_x}$ since $\nabla_{S_1}^{(MC^{v_x})} = \emptyset$ due to $MC^{v_x}(T) = \emptyset$ for all $T \subsetneqq S_1$. ¿From $MC_S^{(MC^{v_x})} = MC_S^{v_x}$ for all $S \subseteq N$, together with the marginalistic contributions monotonicity property for both ρ and ψ, it follows immediately that (3.8) holds.
The global efficiency (2.9) for both ρ and ψ, applied to the set game $\langle N, MC^{v_x}\rangle$, yields

$$\begin{aligned} \cup_{k\in N}\rho_k(N, MC^{v_x}) &= \cup_{k\in N}\psi_k(N, MC^{v_x}) \qquad \text{which equals } \{x\}, \text{ or equivalently,} \\ \cup_{k\in S_1}\rho_k(N, MC^{v_x}) &= \cup_{k\in S_1}\psi_k(N, MC^{v_x}) \qquad \text{which equals } \{x\}, \end{aligned}$$

since $\rho_j(N, MC^{v_x}) = \psi_j(N, MC^{v_x}) = \emptyset$ for all $j \in N\backslash S_1$. Note that any pair of players in S_1 are substitutes in the set game $\langle N, MC^{v_x}\rangle$ (since S_1 is the unique winning coalition). From the substitution property for both ρ and ψ, applied to the game $\langle N, MC^{v_x}\rangle$, we derive $\rho_j(N, MC^{v_x}) = \rho_k(N, MC^{v_x})$ as well as $\psi_j(N, MC^{v_x}) = \psi_k(N, MC^{v_x})$ for all $j,\ k \in S_1$. In summary, the latter efficiency equality simplifies to $\rho_j(N, MC^{v_x}) = \psi_j(N, MC^{v_x}) = \{x\}$ for all $j \in S_1$. From this and (3.7)–(3.8), we conclude $\rho_i(N, v_x) = \psi_i(N, v_x) = \emptyset$ if $i \in N\backslash S_1$; and, if $i \in S_1$, it holds $\rho_i(N, v_x) = \rho_i(N, MC^{v_x}) = \psi_i(N, MC^{v_x}) = \psi_i(N, v_x)$. This completes the proof of (3.6) if there exists one winning coalition in the game $\langle N, MC^{v_x}\rangle$.
Case two. Suppose there are at least two winning coalitions in the set game $\langle N, MC^{v_x}\rangle$, say, among others, coalition S_1. Particularly, it holds $MC^{v_x}(S_1) = \{x\}$ or equivalently, $x \in MC_{S_1}^v$. Define two new set games $\langle N, v_1\rangle$ and $\langle N, v_2\rangle$, arising from the marginalistic contributions game $\langle N, MC^v\rangle$ such that v_1 is almost the marginalistic contributions set game MC^v and v_2 almost the empty set game. To be exact,

$$v_1(S) := \begin{cases} MC_S^v & \text{for all } S \neq S_1; \\ \emptyset & \text{for } S = S_1; \end{cases} \tag{3.9}$$

$$v_2(S) := \begin{cases} \emptyset & \text{for all } S \neq S_1; \\ MC_S^v & \text{for } S = S_1. \end{cases} \tag{3.10}$$

¿From the descriptions (3.9)–(3.10) of both set games, together with the equivalence (3.4), we obtain that their associated simple set games $\langle N, (v_1)_x\rangle$ and $\langle N, (v_2)_x\rangle$ are given by

$$(v_1)_x(S) := \begin{cases} MC_S^{v_x} & \text{for all } S \neq S_1; \\ \emptyset & \text{for } S = S_1; \end{cases} \tag{3.11}$$

$$(v_2)_x(S) := \begin{cases} \emptyset & \text{for all } S \neq S_1; \\ MC_S^{v_x} & \text{for } S = S_1. \end{cases} \tag{3.12}$$

Note that, for all $S \subseteq N$, the inclusions $(v_1)_x(S) \subseteq v_x(S)$ and $(v_2)_x(S) \subseteq v_x(S)$ hold. Concerning the marginalistic contributions in both simple set games, as given by (3.11)–(3.12), we claim the following:

$$MC_{S_1}^{(v_1)_x} = \emptyset \qquad \text{and} \qquad MC_S^{(v_1)_x} = MC_S^{v_x} \qquad \text{for all } S \neq S_1; \tag{3.13}$$

$$MC_{S_1}^{(v_2)_x} = MC_{S_1}^{v_x} \qquad \text{and} \qquad MC_S^{(v_2)_x} = \emptyset \qquad \text{for all } S \neq S_1. \tag{3.14}$$

In order to verify (3.13), for all $S \neq S_1$, the following chain of equalities holds:

$$\begin{aligned} MC_S^{(v_1)_x} &= (v_1)_x(S) - \triangledown_S^{(v_1)_x} \overset{(3.11)}{=} MC_S^{v_x} - \triangledown_S^{(v_1)_x} = \Big[v_x(S) - \triangledown_S^{v_x} \Big] - \triangledown_S^{(v_1)_x} \\ &= v_x(S) - \triangledown_S^{v_x} = MC_S^{v_x} \end{aligned}$$

due to the inclusion $\triangledown_S^{(v_1)_x} \subseteq \triangledown_S^{v_x}$ because of the inclusions $(v_1)_x(T) \subseteq v_x(T)$ for all $T \subsetneqq S$. So, (3.13) holds. In order to verify (3.14), the following chain of equalities holds:

$$MC_{S_1}^{(v_2)_x} = (v_2)_x(S_1) - \triangledown_{S_1}^{(v_2)_x} \overset{(3.12)}{=} MC_{S_1}^{v_x} - \triangledown_{S_1}^{(v_2)_x} = MC_{S_1}^{v_x}$$

due to the equality $\triangledown_{S_1}^{(v_2)_x} = \emptyset$ because of $(v_2)_x(T) = \emptyset$ for all $T \subsetneqq S_1$. So, (3.14) holds too. Clearly, it concerns a disjoint union in that $MC_S^{v_x} = MC_S^{(v_1)_x} \cup MC_S^{(v_2)_x}$ for all $S \subseteq N$. From this we deduce the following chain of equalities:

$$\begin{aligned} \psi_i(N, v_x) &\overset{(2.7)}{=} \cup_{\substack{S \subseteq N, \\ S \ni i}} MC_S^{v_x} = \cup_{\substack{S \subseteq N, \\ S \ni i}} \Big[MC_S^{(v_1)_x} \cup MC_S^{(v_2)_x} \Big] \\ &= \Big[\cup_{\substack{S \subseteq N, \\ S \ni i}} MC_S^{(v_1)_x} \Big] \cup \Big[\cup_{\substack{S \subseteq N, \\ S \ni i}} MC_S^{(v_2)_x} \Big] \\ &\overset{(2.7)}{=} \psi_i(N, (v_1)_x) \cup \psi_i(N, (v_2)_x) \end{aligned}$$

By (3.14), the marginalistic contributions set game $\langle N, MC^{(v_2)_x}\rangle$ has a unique winning coalition S_1, whereas by (3.13), the collection of winning coalitions in the marginalistic contributions set game $\langle N, MC^{(v_1)_x}\rangle$ is identical to the one in the initial marginalistic contributions set game $\langle N, MC^{v_x}\rangle$, except for coalition S_1. The induction hypothesis (3.6) applied to both set games $\langle N, (v_1)_x\rangle$ and $\langle N, (v_2)_x\rangle$ yields

$$\rho_i(N, (v_1)_x) = \psi_i(N, (v_1)_x) \qquad \text{as well as} \qquad \rho_i(N, (v_2)_x) = \psi_i(N, (v_2)_x)$$

Further, from the inclusion $MC_S^{(v_1)_x} \subseteq MC_S^{v_x}$ for all $S \subseteq N$, together with the marginalistic contributions monotonicity property for ρ, we derive the inclusion $\rho_i(N, (v_1)_x) \subseteq \rho_i(N, v_x)$

and similarly, $\rho_i(N,(v_2)_x) \subseteq \rho_i(N,v_x)$. Finally, we conclude that the following chain of inclusions holds:

$$\begin{aligned}\psi_i(N,v_x) &= \psi_i(N,(v_1)_x) \cup \psi_i(N,(v_2)_x) \\ &= \rho_i(N,(v_1)_x) \cup \rho_i(N,(v_2)_x) \qquad \text{(by the induction hypothesis)} \\ &\subseteq \rho_i(N,v_x) \qquad \text{(by the contributions monotonicity property of } \rho) \\ &\subseteq \psi_i(N,v_x) \qquad \text{(by Proposition 3.4).}\end{aligned}$$

We arrive at the equality $\rho_i(N,v_x) = \psi_i(N,v_x)$. This completes both the inductive proof of (3.6) and the full proof of Theorem 3.3. □

Remark 3.7. Throughout the above proof of Theorem 3.3, for any set game $\langle N,v\rangle \in \mathcal{G}$ and any coalition $S \subseteq N$, the associated expression $\bigtriangledown_S^v$ is supposed to possess the following minor property:

$$\bigtriangledown_S^w \subseteq \bigtriangledown_S^v \qquad \text{whenever } w(T) \subseteq v(T) \text{ for all } T \subsetneqq S. \tag{3.15}$$

In the context of the empty set game, (3.15) is meant to be read as $\bigtriangledown_S^w = \emptyset$ whenever $w(T) = \emptyset$ for all $T \subsetneqq S$.

4 A second proof of Theorem 3.3.

The second proof technique is based on the decomposition of any set game into a union of a new type of set games, called *elementary set games*. Concerning elementary set games, the worth of all coalitions, except one, equals the empty set. In fact, the decomposition technique is mainly applied to the marginalistic contribution MC, as given by (2.7), the concept of which is treated as a new set game arising from an initial set game.

Definition 4.1. (The marginalistic contributions set game and elementary set games)

(i) With every set game $\langle N,v\rangle \in \mathcal{G}$, there is associated the *marginalistic contributions set game* $\langle N, MC^v\rangle \in \mathcal{G}$ defined to be

$$MC^v(S) := MC_S^v = v(S) - \bigtriangledown_S^v \qquad \text{for all } S \subseteq N. \tag{4.1}$$

(ii) With every set game $\langle N,v\rangle \in \mathcal{G}$, and every coalition $T \subseteq N$, $T \neq \emptyset$, there is associated the *elementary set game* $\langle N, MC_T^v \cdot E_T\rangle \in \mathcal{G}$ defined to be

$$(MC_T^v \cdot E_T)(T) := MC_T^v; \qquad (MC_T^v \cdot E_T)(S) := \emptyset \quad \text{for all } S \subseteq N,\ S \neq T. \tag{4.2}$$

Proposition 4.2. Consider the setting of Definition 4.1. It is tacitly assumed that, for all $S \subseteq N$, (3.15) holds, that is $\bigtriangledown_S^w \subseteq \bigtriangledown_S^v$ whenever $w(T) \subseteq v(T)$ for all $T \subsetneqq S$.

(i) $MC^v = \cup_{T\subseteq N} MC_T^v \cdot E_T$ that is, $MC^v(S) = \cup_{T\subseteq N}(MC_T^v \cdot E_T)(S)$ for all $S \subseteq N$. (4.3)

(ii) $$MC_S^{(MC^v)} = MC_S^v \qquad \text{for all } S \subseteq N. \tag{4.4}$$

(iii) If a solution ρ on $\mathcal{G}^N$ possesses the marginalistic contributions monotonicity property, then it holds $\rho_i(N, v) = \rho_i(N, MC^v)$ for all $i \in N$.

Proof of Proposition 4.2.
By (4.1)–(4.2), the decomposition statement (4.3) of the marginalistic contributions set game $\langle N, MC^v \rangle$ is trivial. To prove (4.4), we claim, for all $S \subseteq N$, the following chain of equalities:

$$MC_S^{(MC^v)} = MC^v(S) - \triangledown_S^{(MC^v)} = \left[v(S) - \triangledown_S^v\right] - \triangledown_S^{(MC^v)} = v(S) - \triangledown_S^v = MC_S^v,$$

where the third equality holds because of the inclusion $\triangledown_S^{(MC^v)} \subseteq \triangledown_S^v$. The latter inclusion is due to the fact that $MC^v(T) \subseteq v(T)$ for all $T \subsetneqq S$ and all $S \subseteq N$. So, (4.4) holds. The statement in part (iii) is a direct consequence of (4.4) and the marginalistic contributions monotonicity property of the solution ρ. □

Lemma 4.3. Let $\mathcal{C}$ be an arbitrary, non-empty sub-collection of 2^N not including the empty set. With every set game $\langle N, v \rangle \in \mathcal{G}$, there is associated a partially marginalistic contributions set game $\langle N, w_{\mathcal{C}} \rangle \in \mathcal{G}$ defined to be $w_{\mathcal{C}} := \cup_{T \in \mathcal{C}} MC_T^v \cdot E_T$, that is

$$w_{\mathcal{C}}(S) := \begin{cases} MC_S^v & \text{for all } S \subseteq N \text{ with } S \in \mathcal{C}; \\ \emptyset & \text{for all } S \subseteq N \text{ with } S \notin \mathcal{C}. \end{cases} \tag{4.5}$$

(i) Then the game $\langle N, w_{\mathcal{C}} \rangle$ is invariant under the MC–concept, that is $MC_S^{(w_{\mathcal{C}})} = w_{\mathcal{C}}(S)$ for all $S \subseteq N$ or equivalently,

$$MC_S^{(w_{\mathcal{C}})} = \begin{cases} MC_S^v & \text{for all } S \subseteq N \text{ with } S \in \mathcal{C}; \\ \emptyset & \text{for all } S \subseteq N \text{ with } S \notin \mathcal{C}. \end{cases} \tag{4.6}$$

(ii) If a solution ρ on $\mathcal{G}^N$ possesses the global efficiency, substitution, and marginalistic contributions monotonicity properties, then it holds $\rho_i(N, w_{\mathcal{C}}) = \cup_{\substack{T \in \mathcal{C}, \\ T \ni i}} MC_T^v$ for all $\langle N, v \rangle \in \mathcal{G}$ and all $i \in N$.

Before we prove Lemma 4.3, we claim that both Proposition 4.2(iii) and Lemma 4.3(ii), applied to the trivial collection $\mathcal{C} = 2^N$, complete the alternative proof of the main Theorem 3.3. Indeed, by (4.1) and (4.5), the choice $\mathcal{C} = 2^N$ yields $w_{\mathcal{C}} = MC^v$ and so, for every set game $\langle N, v \rangle \in \mathcal{G}$, it holds

$$\rho_i(N, v) = \rho_i(N, MC^v) = \rho_i(N, w_{\mathcal{C}}) = \cup_{\substack{T \in \mathcal{C}, \\ T \ni i}} MC_T^v = \cup_{\substack{T \subseteq N, \\ T \ni i}} MC_T^v = \psi_i(N, v) \qquad \text{for all } i \in N.$$

We conclude $\rho = \psi$ whenever the solution ρ on $\mathcal{G}^N$ possesses the global efficiency, substitution, and marginalistic contributions monotonicity properties.

We say a player i is a *destructive player* in the set game $\langle N, v \rangle \in \mathcal{G}$ if $v(S) = \emptyset$ for all $S \subseteq N$ with $i \in S$. A solution ρ on the set game space $\mathcal{G}$ is said to possess the *destructive player property* if $\rho_i(N, v) = \emptyset$ for every destructive player i in the set game $\langle N, v \rangle$. In words, a destructive player receives no items. Obviously, any semi–marginalistic value ψ of the form (2.5) satisfies the destructive player property, whereas, for any solution ρ on $\mathcal{G}$ the destructive player property arises from the marginalistic contributions monotonicity property of ρ (with reference to the MC concept induced by ψ) and the global efficiency of ρ (applied to the empty set game).

Proof of Lemma 4.3.
(i) Let $S \subseteq N$. If $S \notin \mathcal{C}$, then $w_{\mathcal{C}}(S) = \emptyset$ and so, $MC_S^{(w_{\mathcal{C}})} = \emptyset$. In case $S \in \mathcal{C}$, then we claim the following chain of equalities:

$$MC_S^{(w_{\mathcal{C}})} = w_{\mathcal{C}}(S) - \bigtriangledown_S^{(w_{\mathcal{C}})} = MC_S^v - \bigtriangledown_S^{(w_{\mathcal{C}})} = \left[v(S) - \bigtriangledown_S^v\right] - \bigtriangledown_S^{(w_{\mathcal{C}})} = v(S) - \bigtriangledown_S^v = MC_S^v,$$

where the fourth equality holds because of the inclusion $\bigtriangledown_S^{(w_{\mathcal{C}})} \subseteq \bigtriangledown_S^v$. The latter inclusion is due to the fact that $w_{\mathcal{C}}(T) \subseteq v(T)$ for all $T \subsetneqq S$ and all $S \subseteq N$. So, (4.6) holds.
(ii) Suppose a solution ρ on $\mathcal{G}^N$ satisfies the global efficiency (GEF), substitution ($SUBS$), and marginalistic contributions monotonicity ($MCMON$) properties. Fix the set game $\langle N, v\rangle$ and player $i \in N$. We show, by induction on the number $|\mathcal{C}|$ of coalitions in the collection $\mathcal{C}$, that it holds

$$\rho_i(N, w_{\mathcal{C}}) = \cup_{\substack{T\in\mathcal{C},\\ T\ni i}} MC_T^v \qquad \text{for all sub-collections } \mathcal{C} \text{ of } 2^N. \tag{4.7}$$

Case one. Suppose, for the moment, $|\mathcal{C}| = 1$, say $\mathcal{C} = \{T\}$. By (4.5), $w_{\mathcal{C}}(T) = MC_T^v$ and $w_{\mathcal{C}}(S) = \emptyset$ for all $S \subseteq N$ with $S \neq T$. The global efficiency of ρ yields $\cup_{j\in N}\rho_j(N, w_{\mathcal{C}}) = \cup_{S\subseteq N} w_{\mathcal{C}}(S) = MC_T^v$. On the one hand, any pair of members of T are substitutes in the game $\langle N, w_{\mathcal{C}}\rangle$ and consequently, $SUBS$ of ρ yields $\rho_j(N, w_{\mathcal{C}}) = \rho_k(N, w_{\mathcal{C}})$ for all j, $k \in T$. On the other, non-members of T are destructive players in the game $\langle N, w_{\mathcal{C}}\rangle$ and consequently, the destructive player property $DESP$ of ρ yields $\rho_\ell(N, w_{\mathcal{C}}) = \emptyset$ for all $\ell \in N\backslash T$. Notice that $DESP$ follows immediately from $MCMON$ together with GEF (applied to the empty set game). So far, in case $\mathcal{C} = \{T\}$, we conclude $\rho_j(N, w_{\mathcal{C}}) = \emptyset$ for all $j \in N\backslash T$ and $\rho_j(N, w_{\mathcal{C}}) = MC_T^v$ for all $j \in T$. So, (4.7) holds.
Case two. From now on, we may suppose $|\mathcal{C}| \geq 2$. We distinguish three subcases.
Subcase one. Suppose $i \in \left[\cup_{S\in\mathcal{C}} S\right] - \left[\cap_{S\in\mathcal{C}} S\right]$. Define the collection $\mathcal{C}_i := \{S \in \mathcal{C} \mid i \in S\}$. By assumption, the strict inclusion $\mathcal{C}_i \subsetneqq \mathcal{C}$ holds and so, the induction hypothesis applies to the new collection $\mathcal{C}_i$, yielding

$$\rho_i(N, w_{\mathcal{C}_i}) = \cup_{\substack{T\in\mathcal{C}_i,\\ T\ni i}} MC_T^v = \cup_{\substack{T\in\mathcal{C},\\ T\ni i}} MC_T^v$$

Thus, it remains to show the equality $\rho_i(N, w_{\mathcal{C}}) = \rho_i(N, w_{\mathcal{C}_i})$ and for that purpose, it suffices, by $MCMON$ of ρ, to show $MC_S^{(w_{\mathcal{C}})} = MC_S^{(w_{\mathcal{C}_i})}$ for all $S \subseteq N$ with $i \in S$. The latter equality is a direct consequence of Lemma 4.3(i) (applied to both collections $\mathcal{C}$ and $\mathcal{C}_i$ respectively), by taking into account that, for every $S \subseteq N$ with $i \in S$, the equivalence $S \in \mathcal{C}_i \iff S \in \mathcal{C}$ is valid. So, (4.7) holds.
Subcase two. Suppose $i \in \cap_{S\in\mathcal{C}} S$, that is $i \in S$ for all $S \in \mathcal{C}$. Take any $S^* \in \mathcal{C}$. By applying Lemma 4.3(i) to both collections $\mathcal{C}$ and the one-element collection $\mathcal{C}^* := \{S^*\}$, we obtain the inclusion $MC_S^{(w_{\mathcal{C}^*})} \subseteq MC_S^{(w_{\mathcal{C}})}$ for all $S \subseteq N$. Now $MCMON$ of ρ yields $\rho_i(N, w_{\mathcal{C}^*}) \subseteq \rho_i(N, w_{\mathcal{C}})$ for all $\mathcal{C}^* = \{S^*\}$, where $S^* \in \mathcal{C}$. On the one hand, for the one-element collection $\mathcal{C}^* = \{S^*\}$, it holds, as shown in case one, $\rho_i(N, w_{\mathcal{C}^*}) = MC_{S^*}^v$ (since $i \in S^*$ for all $S^* \in \mathcal{C}$). So far, we conclude the following chain of inclusions:

$$\cup_{S\in\mathcal{C}} MC_S^v = \cup_{S^*\in\mathcal{C}} MC_{S^*}^v = \cup_{S^*\in\mathcal{C}} \rho_i(N, w_{\mathcal{C}^*}) \subseteq \rho_i(N, w_{\mathcal{C}})$$

On the other hand, the global efficiency of ρ yields another chain of inclusions:

$$\rho_i(N, w_{\mathcal{C}}) \subseteq \cup_{j\in N}\rho_j(N, w_{\mathcal{C}}) = \cup_{S\subseteq N} w_{\mathcal{C}}(S) = \cup_{S\in\mathcal{C}} MC_S^v$$

All together, we conclude $\rho_i(N, w_{\mathcal{C}}) = \cup_{S\in\mathcal{C}} MC_S^v = \cup_{\substack{S\in\mathcal{C},\\ S\ni i}} MC_S^v$, where the latter equality is due to the assumption $i \in \cap_{S\in\mathcal{C}} S$. So, (4.7) holds.

Subcase three. Suppose $i \notin \cup_{S\in\mathcal{C}} S$, that is $i \notin S$ for all $S \in \mathcal{C}$. We claim $\rho_i(N, w_{\mathcal{C}}) = \emptyset$ since player i turns out to be a destructive player in the set game $\langle N, w_{\mathcal{C}}\rangle$. Indeed, for all $S \subseteq N$ with $i \in S$, it holds $S \notin \mathcal{C}$ and so, $w_{\mathcal{C}}(S) = \emptyset$. As in case one, recall that $DESP$ follows immediately from $MCMON$ together with GEF (applied to the empty set game). This completes the inductive proof of (4.7). □

Remark 4.4. The proof technique used throughout this section is similar to the one used by [1] to establish the very same axiomatization of the IM–value on the class of monotonic set games, with the understanding that the role of their so-called unanimity set games is replaced by our elementary set games. For the sake of a uniform treatment of a family of solutions for set games, the unanimity set games turn out to be much less applicable than the elementary set games.

5 Concluding Remarks.

The axiomatization of semi–marginalistic values for set games, as stated in Theorem 3.3, can be considered, more or less, as the counterpart of Young's axiomatization of the Shapley value for cooperative games. In order to elucidate these similarities between the two fields of set game theory and cooperative game theory, let us briefly summarize the basic concepts from the latter field.

A *cooperative game* with transferable utility (TU) is a pair $\langle N, v\rangle$, where N is a nonempty, finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) := 0$. Let $\mathcal{CG}$ denote the space of all cooperative TU-games with an arbitrary player set. An element of N (notation: $i \in N$) and a nonempty subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) is called a *player* and *coalition* respectively, and the associated *real number* $v(S)$ is called the *worth* of coalition S, to be interpreted as the earnings (in the utility of money) its members can attain by mutual cooperation among themselves. Concerning the solution theory for cooperative TU-games, a single-valued *solution* ψ on $\mathcal{CG}$ associates a single payoff vector $\psi(N, v) = (\psi_i(N, v))_{i\in N} \in \mathbb{R}^N$ with every cooperative game $\langle N, v\rangle \in \mathcal{CG}$. The payoff $\psi_i(N, v)$ to player i in the cooperative game $\langle N, v\rangle$ represents an assessment by i of his gains from participating in the game. We say a single-valued cooperative game solution ψ satisfies the *efficiency principle* if it holds $\sum_{i\in N} \psi_i(N, v) = v(N)$ for all $\langle N, v\rangle \in \mathcal{CG}$. The *substitution property* for ψ on $\mathcal{CG}$ is fully in accordance with Definition 2.5. Further, a single-valued cooperative game solution ψ on $\mathcal{CG}$ is said to satisfy the *strong monotonicity property* if it holds $\psi_i(N, v) \leq \psi_i(N, w)$ for all $\langle N, v\rangle \in \mathcal{CG}$, $\langle N, w\rangle \in \mathcal{CG}$, and all $i \in N$, satisfying $v(S) - v(S\backslash\{i\}) \leq w(S) - w(S\backslash\{i\})$ for all $S \subseteq N$ with $i \in S$. In [12], it is shown that there exists a unique solution on the cooperative game space $\mathcal{CG}^N$ (with reference to a fixed player set N) satisfying the efficiency, and strong monotonicity properties, and it is given by the well-known *Shapley value* $Sh(N, v) = (Sh_i(N, v))_{i\in N} \in \mathbb{R}^N$ as follows (cf. [9], [8]):

$$Sh_i(N, v) = \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{(|S| - 1)! \cdot (|N| - |S|)!}{|N|!} \cdot \Big[v(S) - v(S\backslash\{i\})\Big] \qquad \text{for all } i \in N,$$

where $|S|$ denotes the size (cardinality) of coalition S. For a detailed introduction about cooperative game theory, we refer to [4]. In summary, the main Theorem 3.3 concerning semi–marginalistic values for set games has been inspired by Young's axiomatization for the Shapley value, although their proofs differ very much. The counterpart of the Shapley value may be stated, at first glance, to be the individually marginalistic IM–value, as given by (2.1), but, from the viewpoint of the potential approach to the solution theory, it is justified to be the Driessen–Sun DS–value as given by (2.4) (cf. [5]). In addition, the semi–marginalistic value ψ of the form (2.7) by choosing $\bigtriangledown_S^v := \cap_{j \in S} v(S \backslash \{j\})$ (cf. [11]), may be interpreted as the counterpart of the solidarity value for cooperative games (cf. [7]).

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