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A MULTIPLICATIVE POTENTIAL APPROACH to SOLUTIONS for COOPERATIVE TU-GAMES *

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Abstract

Concerning the solution theory for cooperative games with transferable utility, it is well-known that the Shapley value is the most appealing representative of the family of (not necessarily efficient) game-theoretic solutions with an additive potential representation. This paper introduces a new solution concept, called Multiplicativily Proportional (MP) value, that can be regarded as the counterpart of the Shapley value if the additive potential approach to the solution theory is replaced by a multiplicative potential approach in that the difference of two potential evaluations is replaced by its quotient. One out of two main equivalence theorems states that every solution with a multiplicative potential representation is equivalent to this specifically chosen efficient value in that the solution of the initial game coincides with the MP value of an auxiliary game. The associated potential function turns out to be of a multiplicative form (instead of an additive form) with reference to the worth of all the coalitions. The second equivalence theorem presents four additional characterizations of solutions that admit a multiplicative potential representation, e.g., preservation of discrete ratios or path independence.
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1 Introduction

In physics the potential is a highly important concept, for instance a vector field u is said to be conservative if there exists a continuously differentiable function U called potential the gradient of which agrees with the vector field (notation: $\bigtriangledown U = u$). There exist several characterizations of conservative vector fields (e.g., $\bigtriangledown_j u_i = \bigtriangledown_i u_j$, or every contour integral with respect to the vector field u is zero). Surprisingly, the successful treatment of the potential in physics turned out to be reproducible, in the late eighties, in the mathematical field called cooperative game theory. Informally, a solution concept ψ on the universal game

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space $\mathcal{G}$ is said to possess a potential representation if it is the discrete gradient of a real-valued function P on $\mathcal{G}$ called potential (notation: $\bigtriangledown P = \psi$). In other words, if possible, each component of the game-theoretic solution (or each player's payoff) may be interpreted as the incremental return, determined by the difference of the potential function evaluation at the given cooperative game and one of its subgames in which the relevant player is not included. In their innovative paper, Hart and Mas-Colell (cf. [7]) showed that the well-known game-theoretic solution called Shapley value is the unique solution that has a potential representation and satisfies the efficiency principle as well. The role of the Shapley value has been strengthened later on by a second fundamental result concerning the family of game-theoretic solutions that possess a potential representation. This fundamental equivalence theorem (cf. [1]) states that every single-valued solution with a potential representation is equivalent to the Shapley value in that the solution of the initial game coincides with the Shapley value of an auxiliary game.

The main purpose of this paper is to introduce a new solution concept for cooperative games (see Section 2) that can be regarded as the counterpart of the Shapley value whenever the *additive potential approach* to the solution theory is replaced by a *multiplicative potential approach* in that the difference of two potential evaluations is replaced by its quotient. A first feature of the new solution concept called *Multiplicativily Proportional (MP) value* is its implicit description through a recursively defined sequence of real numbers (associated with the initial game). In spite of its unusual introduction, the MP value turns out to possess standard properties like efficiency (by convention), the dummy player as well as the substitution properties, but fails to satisfy the additivity property (through which the Shapley value is usually axiomatized). A second feature of the MP value is its proportionality solution in the setting of two-person games. In Section 3 we claim a third feature of the MP value, namely its equivalence to solution concepts with a multiplicative potential representation. In words, the main equivalence Theorem 3.3 states that every single-valued solution with a multiplicative potential representation is equivalent to the MP value in that the solution of the initial game coincides with the MP value of an auxiliary game. Moreover, the determination of the potential function itself, if it exists, results in a multiplicative form in that it is composed of a product (instead of a sum) of expressions induced by both the initial game and the solution (which do simplify to the worth of coalitions in the initial game whenever the solution is efficient). In Section 4, the main Theorem 4.1 treats four additional characterizations of solution concepts with a multiplicative potential representation. Three out of these six equivalent characterizations of solutions, stated in terms of the multiplicative potential approach applied in cooperative game theory, resemble similar ones stated in physical terminology. For instance, the characterization $\bigtriangledown_j u_i = \bigtriangledown_i u_j$ of a conservative vector field u is analogous to its discrete version $\bigtriangledown_j \psi_i = \bigtriangledown_i \psi_j$ with respect to a game-theoretic solution ψ, known as the law of preservation of discrete ratios. In the framework of the additive potential approach, it concerns the law of preservation of discrete differences (cf. [9]), otherwise called the balanced contributions principle (cf. [1], [8], [12]). Moreover, the game-theoretic counterpart of the path independence characterization of a conservative vector field is treated. The two remaining characterizations listed in Theorem 4.1 deal with two types of recursive formulae in order to determine the solution concept applied to an n-person game, the determination of which refers to the solution concept applied to various $(n-1)$-person subgames.

2 The introduction of the Multiplicativily Proportional value

A *cooperative game* with transferable utility (TU) is a pair $\langle N, v\rangle$, where N is a nonempty, finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) := 0$. Let $\mathcal{G}$ denote the set of all cooperative TU-games with an arbitrary player set. Throughout this paper we deal with *positive cooperative games* the characteristic function $v : 2^N \to \mathbb{R}_+$ of which satisfies $v(S) > 0$ for all $S \in 2^N$, $S \neq \emptyset$. An element of N (notation: $i \in N$) and a nonempty subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) is called a *player* and *coalition* respectively, and the associated positive real number $v(S)$ is called the *worth* of coalition S. The size (cardinality) of coalition S is denoted by $|S|$ or, if no ambiguity is possible, by s. Particularly, n denotes the size of the player set N. Given a (transferable utility) game $\langle N, v\rangle$ and a coalition S, we write $\langle S, v_S\rangle$ for the *subgame* obtained by restricting v to subsets of S only (i.e., to 2^S). Let $\mathcal{G}_+$ denote the set of all positive cooperative TU-games with an arbitrary player set, whereas $\mathcal{G}_+^N$ denotes the (vector) space of all positive games with reference to a player set N which is fixed beforehand.

Concerning the solution theory for cooperative TU-games, the paper is devoted to single-valued solution concepts. Formally, a *positive solution* ψ on $\mathcal{G}_+$ (or on a particular subclass of $\mathcal{G}_+$) associates a single payoff vector $\psi(N, v) = (\psi_i(N, v))_{i\in N} \in \mathbb{R}_+^N$ with every positive TU-game $\langle N, v\rangle$. The so-called *allocation* $\psi_i(N, v)$ to player i in the game $\langle N, v\rangle$ represents an assessment by i of his gains from participating in the game. Until further notice, no constraints are imposed upon a solution ψ on $\mathcal{G}_+$.

We say a solution ψ on $\mathcal{G}_+$ satisfies the well-known *efficiency* property whenever it holds

$$\sum_{i\in N} \psi_i(N, v) = v(N) \qquad \text{for all } \langle N, v\rangle \in \mathcal{G}_+. \tag{2.1}$$

An efficient solution is called a *value*. Moreover, we say a solution ψ on $\mathcal{G}_+$ satisfies the *proportionality* property for two-person games whenever the overall gains the amount of $v(\{i, j\})$ are divided among the two players i, j in accordance with equal ratios with respect to their individual worths $v(\{i\})$ and $v(\{j\})$ respectively, that is $\frac{\psi_i(\{i,j\},v)}{v(\{i\})} = \frac{\psi_j(\{i,j\},v)}{v(\{j\})}$ or equivalently (in the framework of efficiency for two-person games),

$$\psi_i(\{i, j\}, v) = \frac{v(\{i\})}{[v(\{i\}) + v(\{j\})]} \cdot v(\{i, j\}) \qquad \text{for any two-person game } \langle\{i, j\}, v\rangle. \tag{2.2}$$

In this section our main goal is to introduce a so-called *multiplicative* value which is supposed to be proportional for two-person games and the second feature being that a player's value in a multi-person game is measured by the product of the worths of all the coalitions containing the player. For reasons that will be explained in Section 3, the implicit description of the fundamental value involves a sequence of real numbers which are defined recursively (with respect to the inclusion of coalitions).

Definition 2.1. Let $\langle N, v\rangle \in \mathcal{G}_+$ be a positive cooperative game.

(i) Let the sequence of real numbers $(\alpha_T^v)_{T\subseteq N}$ be given by the following recursive formula:

$$\alpha_T^v := \frac{1}{v(T)} \cdot \sum_{i\in T} \Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i}} v(S) \Bigg] \cdot \alpha_{T\setminus\{i\}}^v \qquad \text{for all } T \subseteq N,\ T \neq \emptyset, \tag{2.3}$$

where $\alpha_\emptyset^v := 1$. Note that $\alpha_{\{i\}}^v = 1$ and $\alpha_{\{i,j\}}^v = v(\{i\}) + v(\{j\})$ for all $i,\ j \in N$, $i \neq j$.

(ii) The *Multiplicativily Proportional value* $MP_i(N,v)$ of player $i \in N$ in the positive game $\langle N, v\rangle \in \mathcal{G}_+$ is defined as follows:

$$MP_i(N,v) := \Big[\prod_{\substack{S\subseteq N,\\ S\ni i}} v(S)\Big]\cdot \frac{\alpha^v_{N\backslash\{i\}}}{\alpha^v_N} \qquad \text{for all } \langle N,v\rangle \in \mathcal{G}_+ \text{ and all } i\in N. \tag{2.4}$$

Obviously, by (2.3) and (2.4), the Multiplicativily Proportional MP value satisfies the proportionality property (2.2) for two-person games as well as the efficiency principle (2.1). As a matter of fact, the denominator α^v_N in (2.4) is due to a normalization in order to guarantee the efficiency property for the solution under consideration.
Given two games $\langle N, v\rangle$ and $\langle N, w\rangle$, let the *product game* $\langle N, v\cdot w\rangle$ be defined in a natural way in that $(v\cdot w)(S) := v(S)\cdot w(S)$ for all $S\subseteq N$. The Multiplicativily Proportional MP value behaves in a *semi-multiplicative* manner since, by (2.4), the player's value of the product game equals the product of the player's values of the two initial games, by taking into account the relevant normalization coefficients. Formally, it follows immediately from (2.4) that it holds

$$MP_i(N, v\cdot w) = \frac{\alpha^v_N\cdot\alpha^w_N}{\alpha^{(v\cdot w)}_N}\cdot\frac{\alpha^{(v\cdot w)}_{N\backslash\{i\}}}{\alpha^v_{N\backslash\{i\}}\cdot\alpha^w_{N\backslash\{i\}}}\cdot MP_i(N,v)\cdot MP_i(N,w) \tag{2.5}$$

for all $\langle N,v\rangle \in \mathcal{G}_+$, $\langle N,w\rangle \in \mathcal{G}_+$ and all $i\in N$. Particularly, in the setting of two-person games, (2.5) simplifies as follows:

$$MP_i(\{i,j\}, v\cdot w) = \frac{[v(\{i\})+v(\{j\})]\cdot[w(\{i\})+w(\{j\})]}{[v(\{i\})\cdot w(\{i\})+v(\{j\})\cdot w(\{j\})]}\cdot MP_i(\{i,j\},v)\cdot MP_i(\{i,j\},w)$$

In view of (2.5), the prefix "multiplicative" (in the weak sense) is justifiable. Note that, for the n-person *unit game* $\langle N, v\rangle$ defined by $v(S) := 1$ for all $S\subseteq N$, $S\neq\emptyset$, its Multiplicativily Proportional value is given by $MP_i(N,v) = \frac{1}{n}$ for all $i\in N$ since the corresponding sequence $(\alpha^v_T)_{T\subseteq N}$ is determined by $\alpha^v_T = |T|!$ for all $T\subseteq N$, $T\neq\emptyset$ (to be shown by a simple inductive proof). In order to have a better understanding of the Multiplicativily Proportional value as well as the underlying sequence of real numbers, we establish in the remaining part of this section three standard properties for these concepts, namely properties with reference to additive games, substitutes and dummy players. Firstly, it is shown that additive games are invariant under the MP value in the sense that a player's value of an additive game agrees with the player's individual worth.

Proposition 2.2. Let $\langle N,v\rangle \in \mathcal{G}_+$ be an *additive game*, that is $v(S) := \sum_{j\in S} v(\{j\})$ for all $S\subseteq N$, $S\neq\emptyset$. Then the corresponding sequence of real numbers $(\alpha^v_T)_{T\subseteq N}$ (see (2.3)) and the Multiplicativily Proportional value $MP(N,v)$ are given as follows:

$$\alpha^v_T = \prod_{\substack{S\subseteq T,\\ |S|\geq 2}} v(S) \qquad \text{and} \qquad MP_i(N,v) = v(\{i\}) \qquad \text{for all } T\subseteq N \text{ and all } i\in N. \tag{2.6}$$

Proof of Proposition 2.2.
Let $\langle N,v\rangle \in \mathcal{G}_+$ be an additive game. The inductive proof concerning the sequence $(\alpha^v_T)_{T\subseteq N}$ is as follows. For $|T| = 2$, say $T = \{i,j\}$, it holds that $\alpha^v_{\{i,j\}} = v(\{i\}) + v(\{j\}) = v(\{i,j\})$ where the latter equality is due to the additivity property for the game $\langle N,v\rangle$. Let $T\subseteq N$

with $|T| \geq 3$. By applying the induction hypothesis to the real numbers of the sequence with reference to coalitions of size $|T|-1$, it holds that $\alpha^v_{T\backslash\{i\}} = \prod_{\substack{S\subseteq T\backslash\{i\},\\ |S|\geq 2}} v(S)$ for all $i \in T$. From this we deduce the following chain of equalities:

$$
\begin{aligned}
\alpha^v_T &\overset{(2.3)}{=} \frac{1}{v(T)} \cdot \sum_{i\in T} \Big[\prod_{\substack{S\subseteq T,\\ S\ni i}} v(S) \Big] \cdot \alpha^v_{T\backslash\{i\}} \\
&\overset{(IH)}{=} \frac{1}{v(T)} \cdot \sum_{i\in T} v(\{i\}) \cdot \Big[\prod_{\substack{S\subseteq T,\\ S\ni i,\ |S|\geq 2}} v(S) \Big] \cdot \Big[\prod_{\substack{S\subseteq T\backslash\{i\},\\ |S|\geq 2}} v(S) \Big] \\
&= \frac{1}{v(T)} \cdot \sum_{i\in T} v(\{i\}) \cdot \Big[\prod_{\substack{S\subseteq T,\\ |S|\geq 2}} v(S) \Big] \\
&= \frac{1}{v(T)} \cdot \Big[\prod_{\substack{S\subseteq T,\\ |S|\geq 2}} v(S) \Big] \cdot \Big[\sum_{i\in T} v(\{i\}) \Big] \\
&= \prod_{\substack{S\subseteq T,\\ |S|\geq 2}} v(S) \qquad \text{(due to the additivity property for the game } \langle N, v\rangle).
\end{aligned}
$$

This completes the inductive proof of the statement (2.6) concerning the sequence $(\alpha^v_T)_{T\subseteq N}$. Now we are in a position to conclude that, for every additive game $\langle N, v\rangle$, the Multiplicativily Proportional value $MP_i(N, v)$ of any player $i \in N$ is determined by

$$
MP_i(N,v) \overset{(2.4)}{=} \Big[\prod_{\substack{S\subseteq N,\\ S\ni i}} v(S) \Big] \cdot \frac{\alpha^v_{N\backslash\{i\}}}{\alpha^v_N} \overset{(2.6)}{=} v(\{i\}) \cdot \Big[\prod_{\substack{S\subseteq N,\\ S\ni i,\ |S|\geq 2}} v(S) \Big] \cdot \frac{\Big[\prod_{\substack{S\subseteq N\backslash\{i\},\\ |S|\geq 2}} v(S) \Big]}{\Big[\prod_{\substack{S\subseteq N,\\ |S|\geq 2}} v(S) \Big]} = v(\{i\})
$$

□

Proposition 2.3. The Multiplicativily Proportional MP value possesses the *substitution property.* To be exact, it holds

$$
\alpha^v_{T\backslash\{i\}} = \alpha^v_{T\backslash\{j\}} \qquad \text{for all } T \subseteq N \text{ with } i,\ j \in T, \qquad \text{and} \qquad MP_i(N,v) = MP_j(N,v) \quad (2.7)
$$

whenever players $i,\ j \in N$ are *substitutes* in the game $\langle N, v\rangle \in \mathcal{G}_+$, i.e., $v(S\cup\{i\}) = v(S\cup\{j\})$ for all $S \subseteq N\backslash\{i, j\}$.

Proof of Proposition 2.3.
The inductive proof concerning the real numbers $\alpha^v_{T\backslash\{k\}}$, $k \in \{i, j\}$, for a fixed pair $i,\ j \in N$ of substitutes in the game $\langle N, v\rangle \in \mathcal{G}_+$, is as follows. For $|T| = 2$, say $T = \{i, j\}$, it holds that $\alpha^v_{T\backslash\{i\}} = \alpha^v_{\{j\}} = 1 = \alpha^v_{\{i\}} = \alpha^v_{T\backslash\{j\}}$. For $|T| = 3$, say $T = \{i, j, k\}$, it holds that $\alpha^v_{T\backslash\{i\}} = \alpha^v_{\{j,k\}} = v(\{j\}) + v(\{k\}) = v(\{i\}) + v(\{k\}) = \alpha^v_{\{i,k\}} = \alpha^v_{T\backslash\{j\}}$. Let $T \subseteq N$ with $i \in T$, $j \in T$, and $|T| \geq 4$. By applying the induction hypothesis to the real numbers with

reference to the coalitions $T\backslash\{k\}$, $k\in T\backslash\{i,j\}$, of size $|T|-1$, it holds that $\alpha^v_{T\backslash\{i,k\}}=\alpha^v_{T\backslash\{j,k\}}$ for all $k\in T\backslash\{i,j\}$. Moreover, the following chain of equalities holds:

$$\begin{aligned}\alpha^v_{T\backslash\{i\}} &\overset{(2.3)}{=} \frac{1}{v(T\backslash\{i\})}\cdot\sum_{k\in T\backslash\{i\}}\Bigg[\prod_{\substack{S\subseteq T\backslash\{i\},\\ S\ni k}} v(S)\Bigg]\cdot\alpha^v_{T\backslash\{i,k\}}\\ &= \frac{1}{v(T\backslash\{i\})}\cdot\Bigg[\Bigg[\prod_{\substack{S\subseteq T\backslash\{i\},\\ S\ni j}} v(S)\Bigg]\cdot\alpha^v_{T\backslash\{i,j\}}+\sum_{k\in T\backslash\{i,j\}}\Bigg[\prod_{\substack{S\subseteq T\backslash\{i\},\\ S\ni k}} v(S)\Bigg]\cdot\alpha^v_{T\backslash\{i,k\}}\Bigg]\end{aligned}$$

Since players $i,\ j$ are substitutes in the game $\langle N,v\rangle\in\mathcal{G}_+$, it holds that $v(T\backslash\{i\})=v(T\backslash\{j\})$ as well as

$$\prod_{\substack{S\subseteq T\backslash\{i\},\\ S\ni j}} v(S)=\prod_{\substack{S\subseteq T\backslash\{j\},\\ S\ni i}} v(S)\qquad\text{and}\qquad\prod_{\substack{S\subseteq T\backslash\{i\},\\ S\ni k}} v(S)=\prod_{\substack{S\subseteq T\backslash\{j\},\\ S\ni k}} v(S)\qquad\text{for all } k\in T\backslash\{i,j\}.$$

This completes the inductive proof of the statement $\alpha^v_{T\backslash\{i\}}=\alpha^v_{T\backslash\{j\}}$ for all $T\subseteq N$ with $i\in T$, $j\in T$ (if $i,\ j$ are substitutes). Particularly, $\alpha^v_{N\backslash\{i\}}=\alpha^v_{N\backslash\{j\}}$ and together with the equality $\prod_{\substack{S\subseteq N,\\ S\ni i}} v(S)=\prod_{\substack{S\subseteq N,\\ S\ni j}} v(S)$, it follows immediately from the description (2.4) of the MP value that $MP_i(N,v)=MP_j(N,v)$ for any substitutes $i,\ j\in N$ in the game $\langle N,v\rangle\in\mathcal{G}_+$. $\square$

Proposition 2.4. The Multiplicativily Proportional MP value possesses the *dummy player property.* To be exact, it holds

$$\alpha^v_T=\Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i,\ |S|\geq 2}} v(S)\Bigg]\cdot\alpha^v_{T\backslash\{i\}}\qquad\text{for all } T\subseteq N \text{ with } i\in T \text{ and}\quad MP_i(N,v)=v(\{i\})\qquad(2.8)$$

whenever player $i\in N$ is a *dummy* in the game $\langle N,v\rangle\in\mathcal{G}_+$, i.e., $v(S\cup\{i\})-v(S)=v(\{i\})$ for all $S\subseteq N\backslash\{i\}$.

Proof of Proposition 2.4.
The inductive proof concerning the real numbers α^v_T, $T\subseteq N$ with $i\in T$, for a fixed dummy player $i\in N$ in the game $\langle N,v\rangle\in\mathcal{G}_+$, is as follows. For $|T|=2$, say $T=\{i,j\}$, it holds that $\alpha^v_{\{i,j\}}=v(\{i\})+v(\{j\})=v(\{i,j\})=v(\{i,j\})\cdot\alpha^v_{\{j\}}$. Let $T\subseteq N$ with $i\in T$ and $|T|\geq 3$. By applying the induction hypothesis to the real numbers with reference to the coalitions $T\backslash\{k\}$, $k\in T\backslash\{i\}$, of size $|T|-1$, it holds that $\alpha^v_{T\backslash\{k\}}=\Big[\prod_{\substack{S\subseteq T\backslash\{k\},\\ S\ni i,\ |S|\geq 2}} v(S)\Big]\cdot\alpha^v_{T\backslash\{i,k\}}$ for all $k\in T\backslash\{i\}$. Moreover, the following chain of equalities holds:

$$\begin{aligned}\alpha^v_T &\overset{(2.3)}{=} \frac{1}{v(T)}\cdot\sum_{k\in T}\Bigg[\prod_{\substack{S\subseteq T,\\ S\ni k}} v(S)\Bigg]\cdot\alpha^v_{T\backslash\{k\}}\\ &\overset{(IH)}{=} \frac{1}{v(T)}\cdot\Bigg[\Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i}} v(S)\Bigg]\cdot\alpha^v_{T\backslash\{i\}}+\sum_{k\in T\backslash\{i\}}\Bigg[\prod_{\substack{S\subseteq T,\\ S\ni k}} v(S)\Bigg]\cdot\Bigg[\prod_{\substack{S\subseteq T\backslash\{k\},\\ S\ni i,\ |S|\geq 2}} v(S)\Bigg]\cdot\alpha^v_{T\backslash\{i,k\}}\Bigg]\\ &= \frac{1}{v(T)}\cdot\Bigg[\Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i}} v(S)\Bigg]\cdot\alpha^v_{T\backslash\{i\}}+\sum_{k\in T\backslash\{i\}}\Bigg[\prod_{\substack{S\subseteq T,\\ S\ni k,\ S\not\ni i}} v(S)\Bigg]\cdot\Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i,\ |S|\geq 2}} v(S)\Bigg]\cdot\alpha^v_{T\backslash\{i,k\}}\Bigg]\end{aligned}$$

$$= \frac{1}{v(T)} \cdot \Bigg[\Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i}} v(S)\Bigg] \cdot \alpha^v_{T\backslash\{i\}} + \Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i,\ |S|\geq 2}} v(S)\Bigg] \cdot \sum_{k\in T\backslash\{i\}} \Bigg[\prod_{\substack{S\subseteq T\backslash\{i\},\\ S\ni k}} v(S)\Bigg] \cdot \alpha^v_{T\backslash\{i,k\}}\Bigg]$$

$$\overset{(2.3)}{=} \frac{1}{v(T)} \cdot \Bigg[\Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i}} v(S)\Bigg] \cdot \alpha^v_{T\backslash\{i\}} + \Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i,\ |S|\geq 2}} v(S)\Bigg] \cdot v(T\backslash\{i\}) \cdot \alpha^v_{T\backslash\{i\}}\Bigg]$$

$$= \frac{1}{v(T)} \cdot \Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i,\ |S|\geq 2}} v(S)\Bigg] \cdot \alpha^v_{T\backslash\{i\}} \cdot \Big[v(\{i\}) + v(T\backslash\{i\})\Big]$$

$$= \Bigg[\prod_{\substack{S\subseteq T,\\ S\ni i,\ |S|\geq 2}} v(S)\Bigg] \cdot \alpha^v_{T\backslash\{i\}} \qquad \text{(since player } i \text{ is a dummy in the game } \langle N, v\rangle).$$

This completes the inductive proof of the statement $\alpha^v_T = \Big[\prod_{\substack{S\subseteq T,\\ S\ni i,\ |S|\geq 2}} v(S)\Big] \cdot \alpha^v_{T\backslash\{i\}}$ for all $T \subseteq N$ with $i \in T$ (if player i is a dummy). Particularly, $\alpha^v_N = \Big[\prod_{\substack{S\subseteq N,\\ S\ni i,\ |S|\geq 2}} v(S)\Big] \cdot \alpha^v_{N\backslash\{i\}}$ and it follows immediately from the description (2.4) of the MP value that $MP_i(N, v) = v(\{i\})$ for any dummy player $i \in N$ in the game $\langle N, v\rangle \in \mathcal{G}_+$. □

To conclude with, the Multiplicativily Proportional MP value possesses the *monotonicity property* in that $MP_i(N, v) \geq MP_i(N, w)$ for all $i \in N$ and all $\langle N, v\rangle \in \mathcal{G}_+$, $\langle N, w\rangle \in \mathcal{G}_+$, satisfying $v(N) \geq w(N)$ and $v(S) = w(S)$ for all $S \subsetneq N$ (notation: $v \geq w$). The monotonicity property for the MP value follows immediately from its description (2.4), by noting that the underlying sequence of real numbers, recursively given by (2.3), does not change at all, that is we claim $\alpha^v_T = \alpha^w_T$ for all $T \subseteq N$, $T \neq \emptyset$, whenever $v \geq w$. The inductive proof of this claim is rather simple and left to the reader. Due to this claim, the increase in the players' payoffs is proportional to the increase in the worth of the grand coalition. To be exact, $\frac{MP_i(N,v)}{MP_i(N,w)} = \frac{v(N)}{w(N)}$ for all $i \in N$ whenever $v \geq w$.
It is still an open problem to axiomatize the Multiplicativily Proportional MP value on the game space with a fixed player set by means of its efficiency, dummy player property, substitution (or anonymity) property, and another "multiplicativity" property (that replaces the known additivity property in the axiomatization of the Shapley value).

3 One Characterization of solutions that admit a mulitiplicative potential

The main goal of this section is to show that the Multiplicativily Proportional MP value, as given by (2.4), is the most fundamental solution among all the solutions with a so-called multiplicative potential representation (as introduced in [10]). The main result states that any solution with a multiplicative potential representation is somehow equivalent to the MP value, by taking into account a particular transformation on positive games.

Definition 3.1. (cf. [10]) Let ψ be a positive solution on $\mathcal{G}_+$.

(i) We say the solution ψ admits a *multiplicative potential* if there exists a function $P_\psi : \mathcal{G}_+ \to \mathbb{R}_+$ satisfying

$$\frac{P_\psi(N,v)}{P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}})} = \psi_i(N,v) \qquad \text{for all } \langle N,v\rangle \in \mathcal{G}_+ \text{ and all } i \in N. \tag{3.1}$$

(ii) The mapping $F_\psi : \mathcal{G}_+ \to \mathcal{G}_+$ associates with every game $\langle N,v\rangle \in \mathcal{G}_+$ its *solution game* $\langle N, F_\psi^v\rangle \in \mathcal{G}_+$ defined to be

$$F_\psi^v(S) := \sum_{j\in S} \psi_j(S, v_S) \qquad \text{for all } S \subseteq N,\ S \neq \emptyset. \tag{3.2}$$

In words, the (multiplicative) potential function P_ψ represents a scalar evaluation for cooperative TU-games, of which any player's marginal ratio agrees with the player's allocation according to the relevant solution ψ (notation: $\bigtriangledown P_\psi = \psi$). If the potential exists, it is uniquely determined up to a (multiplicative) constant and the potential's formula (see (3.6)) to be listed later on in our fundamental Equivalence Theorem 3.3 justifies the prefix "multiplicative". Usually, it is tacitly assumed that the potential function is unit-normalized (i.e., $P_\psi(\emptyset, v) := 1$).
By (3.2), the worth $F_\psi^v(S)$ of coalition S in the solution game $\langle N, F_\psi^v\rangle$ represents the overall gains (according to the solution ψ) to the members of S from participating in the induced subgame $\langle S, v_S\rangle$ (on the understanding that players outside S are supposed not to cooperate). Generally speaking, the solution game differs from the initial game. Notice that both games are the same if and only if the solution ψ satisfies the efficiency principle (2.1).

Example 3.2. Semi-values (including the Shapley and Banzhaf values) for cooperative games have been introduced by [6]. Analogous to these additive semi-values, we say a positive solution ψ on $\mathcal{G}_+$ is a *multiplicative semi-value* if it is the exponential form of some additive semi-value, that is of the following multiplicative form:

$$\psi_i(N,v) = \prod_{S\subseteq N\backslash\{i\}} e^{p_n(s+1)\cdot[v(S\cup\{i\})-v(S)]} \qquad \text{for all } \langle N,v\rangle \in \mathcal{G}_+ \text{ and all } i \in N, \tag{3.3}$$

where the collection of real numbers called *weights* $\{p_n(t) \in \mathbb{R} \mid n \in \{1,2,\ldots\}, t \in \{1,2,\ldots,n\}\}$ is supposed to satisfy the *upwards triangle property*, i.e.,

$$p_n(s) + p_n(s+1) = p_{n-1}(s) \qquad \text{for all } n \geq 2 \text{ and all } 1 \leq s \leq n-1. \tag{3.4}$$

Two well-known examples of such a collection of weights are given by $p_n(s) := \frac{1}{n\cdot\binom{n-1}{s-1}}$ and $p_n(s) := (\frac{1}{2})^{n-1}$ (yielding the multiplicative versions of the Shapley value and the Banzhaf value respectively). We claim the multiplicative semi-value ψ, arising from any collection of weights satisfying the upwards triangle property, admits a multiplicative potential the function $P_\psi : \mathcal{G}_+ \to \mathbb{R}_+$ of which is given by

$$P_\psi(N,v) = \prod_{S\subseteq N} e^{p_n(s)\cdot v(S)} \qquad \text{for all } \langle N,v\rangle \in \mathcal{G}_+. \tag{3.5}$$

Indeed, for all $\langle N, v\rangle \in \mathcal{G}_+$ and all $i \in N$, straightforward computations yield the following chain of equalities:

$$\frac{P_\psi(N,v)}{P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}})} \overset{(3.5)}{=} \frac{\prod\limits_{S\subseteq N} e^{p_n(s)\cdot v(S)}}{\prod\limits_{S\subseteq N\backslash\{i\}} e^{p_{n-1}(s)\cdot v(S)}}$$

$$= \left[\prod_{\substack{S\subseteq N,\\ S\ni i}} e^{p_n(s)\cdot v(S)}\right]\cdot\left[\prod_{S\subseteq N\backslash\{i\}} \frac{e^{p_n(s)\cdot v(S)}}{e^{p_{n-1}(s)\cdot v(S)}}\right]$$

$$= \left[\prod_{S\subseteq N\backslash\{i\}} e^{p_n(s+1)\cdot v(S\cup\{i\})}\right]\cdot\left[\prod_{S\subseteq N\backslash\{i\}} e^{[p_n(s)-p_{n-1}(s)]\cdot v(S)}\right]$$

$$\overset{(3.4)}{=} \left[\prod_{S\subseteq N\backslash\{i\}} e^{p_n(s+1)\cdot v(S\cup\{i\})}\right]\cdot\left[\prod_{S\subseteq N\backslash\{i\}} e^{-p_n(s+1)\cdot v(S)}\right]$$

$$= \prod_{S\subseteq N\backslash\{i\}} \left[e^{p_n(s+1)\cdot v(S\cup\{i\})}\cdot e^{-p_n(s+1)\cdot v(S)}\right]$$

$$= \prod_{S\subseteq N\backslash\{i\}} e^{p_n(s+1)\cdot[v(S\cup\{i\})-v(S)]} \overset{(3.3)}{=} \psi_i(N,v).$$

In the framework of multiplicative semi-values, the underlying weights $p_n(s)$ are not necessarily non-negative.

Above all, we treat an equivalence theorem concerning solutions that admit a multiplicative potential; the main result of which is referring to the Multiplicativily Proportional MP value, as given by (2.4).

Theorem 3.3. (Equivalence Theorem) Consider the setting of Definitions 2.1 and 3.1.

(i) If a solution ψ on $\mathcal{G}_+$ admits a multiplicative potential function $P_\psi : \mathcal{G}_+ \to \mathbb{R}_+$, then the following holds:

$$P_\psi(N,v) = \left[\alpha_N^{(F_\psi^v)}\right]^{-1}\cdot\prod_{\substack{S\subseteq N,\\ S\neq\emptyset}} F_\psi^v(S) \qquad \text{and} \qquad \psi(N,v) = MP(N, F_\psi^v) \tag{3.6}$$

for all $\langle N, v\rangle \in \mathcal{G}_+$. In words, the solution of any positive game agrees with the Multiplicativily Proportional value of the associated solution game.

(ii) If $\psi(N,v) = MP(N, F_\psi^v)$ for all $\langle N, v\rangle \in \mathcal{G}_+$, then the solution ψ admits a multiplicative potential.

Proof of Theorem 3.3. (A direct and computational proof)
(i) Suppose the solution ψ on $\mathcal{G}_+$ admits a multiplicative potential function $P_\psi : \mathcal{G}_+ \to \mathbb{R}_+$. The proof of the formula (3.6) for the multiplicative potential function proceeds by induction

on the size n, $n \geq 1$, of the player set N. For $n = 1$, say $\langle \{i\}, v\rangle \in \mathcal{G}_+$, the (tacitly assumed) unit-normalization of the potential function P_ψ (in that $P_\psi(\emptyset, v) = 1$) yields

$$\frac{F_\psi^v(\{i\})}{\alpha_{\{i\}}^{(F_\psi^v)}} = F_\psi^v(\{i\}) \overset{(3.2)}{=} \psi_i(\{i\}, v) \overset{(3.1)}{=} \frac{P_\psi(\{i\}, v)}{P_\psi(\emptyset, v)} = P_\psi(\{i\}, v).$$

So, (3.6) holds whenever $n = 1$. Let $\langle N, v\rangle \in \mathcal{G}_+$ ($n \geq 2$) and suppose (3.6) holds for all games the player set of which contains less than n players. By adding (3.1) over all $i \in N$, we derive a recursive formula for the multiplicative potential function P_ψ as follows:

$$\sum_{i\in N} \frac{P_\psi(N, v)}{P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}})} = \sum_{i\in N} \psi_i(N, v) \overset{(3.2)}{=} F_\psi^v(N) \qquad \text{or equivalently,}$$

$$P_\psi(N, v) = F_\psi^v(N) \cdot \left[\sum_{i\in N} \frac{1}{P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}})}\right]^{-1} \tag{3.7}$$

At this stage, notice that, for all $i \in N$, the solution game $\langle N\backslash\{i\}, F_\psi^{(v_{N\backslash\{i\}})}\rangle$ and the subgame $\langle N\backslash\{i\}, (F_\psi^v)_{N\backslash\{i\}}\rangle$ are equal since, for all $S \subseteq N\backslash\{i\}$, $S \neq \emptyset$, it holds

$$F_\psi^{(v_{N\backslash\{i\}})}(S) = \sum_{j\in S} \psi_j(S, (v_{N\backslash\{i\}})_S) = \sum_{j\in S} \psi_j(S, v_S) = F_\psi^v(S) = (F_\psi^v)_{N\backslash\{i\}}(S)$$

Moreover, it holds that $\alpha_{N\backslash\{i\}}^{(F_\psi^{(v_{N\backslash\{i\}})})} = \alpha_{N\backslash\{i\}}^{(F_\psi^v)_{N\backslash\{i\}}} = \alpha_{N\backslash\{i\}}^{(F_\psi^v)}$ for all $i \in N$. ¿From both these observations and by applying the potential's induction hypothesis (3.6) to all subgames $\langle N\backslash\{i\}, v_{N\backslash\{i\}}\rangle$, $i \in N$, with $n - 1$ players , we obtain the following:

$$P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}}) \overset{(3.6)}{=} \left[\alpha_{N\backslash\{i\}}^{(F_\psi^v)}\right]^{-1} \cdot \prod_{\substack{S\subseteq N\backslash\{i\},\\ S\neq\emptyset}} F_\psi^v(S) \qquad \text{for all } i \in N. \tag{3.8}$$

For notational convenience, write $w := F_\psi^v$. From the recursive formula (3.7), together with the induction hypothesis result (3.8), the description (2.4) of the Multiplicativily Proportional value and its efficiency property applied to the game $\langle N, w\rangle$ respectively, we obtain the following chain of equalities:

$$\begin{aligned}
P_\psi(N, v) &\overset{(3.7)}{=} F_\psi^v(N) \cdot \left[\sum_{i\in N} \frac{1}{P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}})}\right]^{-1} \\
&\overset{(3.8)}{=} w(N) \cdot \left[\sum_{i\in N} \alpha_{N\backslash\{i\}}^w \cdot \left[\prod_{\substack{S\subseteq N\backslash\{i\},\\ S\neq\emptyset}} w(S)\right]^{-1}\right]^{-1} \\
&= w(N) \cdot \left[\sum_{i\in N} \alpha_{N\backslash\{i\}}^w \cdot \left[\prod_{\substack{S\subseteq N,\\ S\ni i}} w(S)\right] \cdot \left[\prod_{\substack{S\subseteq N,\\ S\neq\emptyset}} w(S)\right]^{-1}\right]^{-1} \\
&= w(N) \cdot \left[\prod_{\substack{S\subseteq N,\\ S\neq\emptyset}} w(S)\right] \cdot \left[\sum_{i\in N} \alpha_{N\backslash\{i\}}^w \cdot \left[\prod_{\substack{S\subseteq N,\\ S\ni i}} w(S)\right]\right]^{-1}
\end{aligned}$$

$$\overset{(2.4)}{=} \quad w(N) \cdot \Bigg[\prod_{\substack{S \subseteq N, \\ S \neq \emptyset}} w(S) \Bigg] \cdot \Bigg[\sum_{i \in N} \alpha_N^w \cdot MP_i(N, w) \Bigg]^{-1}$$

$$= \quad w(N) \cdot \Bigg[\prod_{\substack{S \subseteq N, \\ S \neq \emptyset}} w(S) \Bigg] \cdot \Bigg[\alpha_N^w \cdot w(N) \Bigg]^{-1}$$

$$= \quad \Bigg[\alpha_N^w \Bigg]^{-1} \cdot \prod_{\substack{S \subseteq N, \\ S \neq \emptyset}} w(S)$$

This completes the inductive proof of the formula (3.6) for the multiplicative potential function P_ψ. In order to prove the remaining statement $\psi(N,v) = MP(N, F_\psi^v)$, let $\langle N, v \rangle \in \mathcal{G}_+$. From the multiplicative potential representation (3.1) for the solution ψ, the formula (3.6) for the multiplicative potential function P_ψ and the description (2.4) of the Multiplicativily Proportional value applied to the game $\langle N, w \rangle$, we deduce that, for all $i \in N$, it holds

$$\psi_i(N,v) \quad \overset{(3.1)}{=} \quad \frac{P_\psi(N,v)}{P_\psi(N \backslash \{i\}, v_{N \backslash \{i\}})}$$

$$\overset{(3.6)}{=} \quad \Bigg[\alpha_N^w \Bigg]^{-1} \cdot \Bigg[\prod_{\substack{S \subseteq N, \\ S \neq \emptyset}} w(S) \Bigg] \cdot \alpha_{N \backslash \{i\}}^w \cdot \Bigg[\prod_{\substack{S \subseteq N \backslash \{i\}, \\ S \neq \emptyset}} w(S) \Bigg]^{-1}$$

$$= \quad \Bigg[\prod_{\substack{S \subseteq N, \\ S \ni i}} w(S) \Bigg] \cdot \frac{\alpha_{N \backslash \{i\}}^w}{\alpha_N^w}$$

$$\overset{(2.4)}{=} \quad MP_i(N, w) = MP_i(N, F_\psi^v) \qquad \text{(as was to be shown).}$$

(ii) Suppose the solution ψ satisfies $\psi(N,v) = MP(N, F_\psi^v)$ for all $\langle N, v \rangle \in \mathcal{G}_+$. Define the function $P_\psi : \mathcal{G}_+ \to \mathbb{R}_+$ as given by (3.6). Since the latter computations carried out in the proof of part (i) are still valid, it follows immediately that $\frac{P_\psi(N,v)}{P_\psi(N \backslash \{i\}, v_{N \backslash \{i\}})} = MP_i(N, F_\psi^v) = \psi_i(N,v)$ for all $\langle N, v \rangle \in \mathcal{G}_+$ and all $i \in N$. Thus, ψ admits a multiplicative potential. □

4 Additional Characterizations of solutions that admit a mulitiplicative potential

In this section we generalize the fundamental equivalence Theorem 3.3 in that we put together five different, but equivalent characterizations of solutions that admit a multiplicative potential. Three of these characterizations have clear affinities with the potential approach in physics (so that the former additive representations are to be replaced by multiplicative representations).

Theorem 4.1. Consider the setting of Definitions 2.1 and 3.1. Let ψ be a positive solution on $\mathcal{G}_+$. Then the following six statements are equivalent.

(i) ψ admits a multiplicative potential, i.e., there exists a function $P_\psi : \mathcal{G}_+ \to \mathbb{R}_+$ satisfying

$$\frac{P_\psi(N,v)}{P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}})} = \psi_i(N,v) \qquad \text{for all } \langle N,v\rangle \in \mathcal{G}_+ \text{ and all } i \in N. \tag{4.1}$$

We say ψ is a *discrete rationator field* (in physics notation: $\psi_i = \triangledown_i P_\psi$ for all $i \in N$ or briefly, $\psi = \triangledown P_\psi$).

(ii) ψ satisfies the *balanced ratios axiom*, i.e., for all $\langle N,v\rangle \in \mathcal{G}_+$ and all $i \in N$, $j \in N$, $i \neq j$, it holds

$$\frac{\psi_i(N,v)}{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})} = \frac{\psi_j(N,v)}{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})} \tag{4.2}$$

We say ψ *preserves ratios* (in physics notation: $\triangledown_j \psi_i = \triangledown_i \psi_j$ for all $i \in N$, $j \in N$).

(iii) ψ satisfies the next recursive formula: for all $\langle N,v\rangle \in \mathcal{G}_+$ and all $i \in N$, it holds

$$\psi_i(N,v) = F_\psi^v(N) \cdot \left[1 + \sum_{j \in N\backslash\{i\}} \frac{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})}{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})}\right]^{-1} \tag{4.3}$$

(iv) $\psi(N,v) = MP(N, F_\psi^v)$ for all $\langle N,v\rangle \in \mathcal{G}_+$. That is, the solution of any game agrees with the Multiplicativily Proportional value of the associated solution game (see (3.2)).

(v) ψ satisfies the *multiplicative path independence*, i.e., for all $\langle N,v\rangle \in \mathcal{G}_+$ it holds

$$\prod_{i \in N} \psi_i(R_i^\omega, v_{R_i^\omega}) \qquad \text{is the same for every } \textit{order} \text{ (one-to-one function) } \omega : N \to N \tag{4.4}$$

where, for every order $\omega : N \to N$ of the player set N and every player $i \in N$, the associated coalition $R_i^\omega \subseteq N$ of *predecessors* is given by $R_i^\omega := \{j \in N \mid \omega(j) \leq \omega(i)\}$.

(vi) ψ satisfies the next formula: for all $\langle N,v\rangle \in \mathcal{G}_+$ and all $i \in N$, it holds

$$\Big[\psi_i(N,v)\Big]^n \cdot \Big[G_\psi^v(N)\Big]^{-1} = \Big[\prod_{j \in N\backslash\{i\}} \psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})\Big] \cdot \Big[G_\psi^v(N\backslash\{i\})\Big]^{-1} \tag{4.5}$$

$$\text{where} \quad G_\psi^v(S) := \prod_{j \in S} \psi_j(S, v_S) \quad \text{for all } \langle N,v\rangle \in \mathcal{G}_+ \text{ and all } S \subseteq N,\ S \neq \emptyset. \tag{4.6}$$

Remark 4.2. We claim the Multiplicativily Proportional MP value preserves ratios since, by (2.4), for all $\langle N,v\rangle \in \mathcal{G}_+$ and all $i \in N$, $j \in N$, it holds

$$\frac{MP_i(N,v)}{MP_i(N\backslash\{j\}, v_{N\backslash\{j\}})} \overset{(2.4)}{=} \frac{\Big[\prod_{\substack{S \subseteq N,\\ S \ni i}} v(S)\Big] \cdot \alpha_{N\backslash\{i\}}^v \cdot [\alpha_N^v]^{-1}}{\Big[\prod_{\substack{S \subseteq N\backslash\{j\},\\ S \ni i}} v(S)\Big] \cdot \alpha_{N\backslash\{i,j\}}^v \cdot \Big[\alpha_{N\backslash\{j\}}^v\Big]^{-1}}$$

$$= \Big[\prod_{\substack{S \subseteq N,\\ S \ni i,\ S \ni j}} v(S)\Big] \cdot \frac{\alpha_{N\backslash\{i\}}^v \cdot \alpha_{N\backslash\{j\}}^v}{\alpha_N^v \cdot \alpha_{N\backslash\{i,j\}}^v}$$

Obviously, the latter expression remains unaltered when interchanging the roles of both players i and j and consequently, the MP value preserves ratios. As a matter of fact, the preservation of ratios for a solution is the very reason to take into account the recursively defined real numbers $\alpha^v_{N\backslash\{i\}}$, $i \in N$, (and consequently, its normalization coefficient α^v_N due to efficiency) when proposing a semi-multiplicative solution of the form (2.4). Because the MP value is efficient, the solution game $\langle N, F^v_{MP}\rangle$ agrees with the initial game $\langle N, v\rangle$ and thus, by Theorem 3.3, its multiplicative potential representation arises from the potential function $P_{MP} : \mathcal{G}_+ \to \mathbb{R}_+$ given by $P_{MP}(N,v) = \left[\alpha^v_N\right]^{-1} \cdot \prod_{S\subseteq N,\, S\neq\emptyset} v(S)$ for all $\langle N, v\rangle \in \mathcal{G}_+$. In words, the potential evaluation of a positive cooperative game equals, up to a normalization, the product of the worths of all the coalitions (and thus, the prefix "multiplicative" is justifiable). In the framework of additive games $\langle N, v\rangle$ as treated in Proposition 2.2, we claim the potential function is given by $P_{MP}(N,v) = \prod_{i\in N} v(\{i\})$ since it holds

$$P_{MP}(N,v) \overset{(3.6)}{=} \left[\alpha^v_N\right]^{-1} \cdot \prod_{\substack{S\subseteq N,\\ S\neq\emptyset}} v(S) \overset{(2.6)}{=} \left[\prod_{\substack{S\subseteq N,\\ |S|\geq 2}} v(S)\right]^{-1} \cdot \left[\prod_{\substack{S\subseteq N,\\ S\neq\emptyset}} v(S)\right] = \prod_{i\in N} v(\{i\})$$

Ortmann (cf. [10], Theorem 2.4, page 238) already proved the equivalence $(i) \iff (ii)$ stated in Theorem 4.1. For the sake of completeness, we repeat its proof and subsequently, we prove the chain of implications $(ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (i)$ as well as the two equivalences $(i) \iff (v)$ and $(ii) \iff (vi)$.

Proof of the equivalence $(i) \iff (ii)$ stated in Theorem 4.1.
First suppose (i) holds. From the multiplicative potential representation (4.1) for the solution ψ we derive that, for all $\langle N, v\rangle \in \mathcal{G}_+$ and all $i \in N$, $j \in N$, it holds

$$\begin{aligned}(\triangledown_j \psi_i)(N,v) &= \frac{\psi_i(N,v)}{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})} \overset{(4.1)}{=} \left[\frac{P_\psi(N,v)}{P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}})}\right] \cdot \left[\frac{P_\psi(N\backslash\{i,j\}, v_{N\backslash\{i,j\}})}{P_\psi(N\backslash\{j\}, v_{N\backslash\{j\}})}\right] \\ &= \frac{P_\psi(N,v) \cdot P_\psi(N\backslash\{i,j\}, v_{N\backslash\{i,j\}})}{P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}}) \cdot P_\psi(N\backslash\{j\}, v_{N\backslash\{j\}})}\end{aligned}$$

The very last formula remains unaltered when interchanging the roles of both players i and j. Thus, $(\triangledown_j \psi_i)(N,v) = (\triangledown_i \psi_j)(N,v)$. Briefly, $\triangledown_j \psi_i = \triangledown_j(\triangledown_i P_\psi) = \triangledown_i(\triangledown_j P_\psi) = \triangledown_i \psi_j$. This proves the implication $(i) \Longrightarrow (ii)$.
In order to prove the converse implication, suppose (ii) holds. Under these circumstances, it is necessary and sufficient to show that $P_\psi(N,v) := \psi_i(N,v) \cdot P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}})$ is well-defined for all $\langle N, v\rangle \in \mathcal{G}_+$, in spite of its dependence on player $i \in N$. Equivalently, we show by induction on the size n, $n \geq 2$, of the player set N that the following holds:

$$\psi_i(N,v) \cdot P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}}) = \psi_j(N,v) \cdot P_\psi(N\backslash\{j\}, v_{N\backslash\{j\}}) \tag{4.7}$$

for all $\langle N, v\rangle \in \mathcal{G}_+$ and all $i \in N$, $j \in N$. For $n = 2$, say $\langle \{i,j\}, v\rangle \in \mathcal{G}_+$, (4.7) holds because of

$$\begin{aligned}\psi_i(\{i,j\}, v) \cdot P_\psi(\{j\}, v_{\{j\}}) &= \psi_i(\{i,j\}, v) \cdot \psi_j(\{j\}, v_{\{j\}}) \cdot P_\psi(\emptyset, v) \\ &\overset{(4.2)}{=} \psi_j(\{i,j\}, v) \cdot \psi_i(\{i\}, v_{\{i\}}) \cdot P_\psi(\emptyset, v) \\ &= \psi_j(\{i,j\}, v) \cdot P_\psi(\{i\}, v_{\{i\}})\end{aligned}$$

So, (4.7) holds whenever $n = 2$. Let $\langle N, v\rangle \in \mathcal{G}_+$ $(n \geq 3)$ and $i \in N$, $j \in N$. Now we deduce from the balanced ratios property (4.2) for the solution ψ and the induction hypothesis (4.7) applied to subgames with $n-1$ players respectively, that the following holds:

$$\begin{aligned}
\bullet \quad & \psi_i(N,v) \cdot P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}}) \\
\overset{(4.2)}{=} \quad & \psi_j(N,v) \cdot \left[\frac{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})}{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})}\right] \cdot P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}}) \\
\overset{(4.7)}{=} \quad & \psi_j(N,v) \cdot \left[\frac{P_\psi(N\backslash\{j\}, v_{N\backslash\{j\}})}{P_\psi(N\backslash\{i,j\}, v_{N\backslash\{i,j\}})}\right] \cdot \left[\frac{P_\psi(N\backslash\{i,j\}, v_{N\backslash\{i,j\}})}{P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}})}\right] \cdot P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}}) \\
= \quad & \psi_j(N,v) \cdot P_\psi(N\backslash\{j\}, v_{N\backslash\{j\}})
\end{aligned}$$

This completes the inductive proof of (4.7). Therefore, the function $P_\psi(N,v) := \psi_i(N,v) \cdot P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}})$ is well-defined for all $\langle N,v\rangle \in \mathcal{G}_+$. In words, ψ admits a multiplicative potential and this proves the implication $(ii) \Longrightarrow (i)$. □

Proof of the implication $(ii) \Longrightarrow (iii)$ **stated in Theorem 4.1.**
Suppose (ii) holds. Let $\langle N,v\rangle \in \mathcal{G}_+$ and $i \in N$. Since ψ preserves ratios, (4.2) implies that it holds

$$\psi_j(N,v) = \psi_i(N,v) \cdot \left[\frac{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})}{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})}\right] \qquad \text{for all } j \in N\backslash\{i\}. \tag{4.8}$$

Adding (4.8) over all $j \in N\backslash\{i\}$ and recalling (3.2) concerning the solution game, we deduce that the following chain of equalities holds:

$$\begin{aligned}
F^v_\psi(N) \quad & \overset{(3.2)}{=} \quad \sum_{j\in N} \psi_j(N,v) = \psi_i(N,v) + \sum_{j\in N\backslash\{i\}} \psi_j(N,v) \\
& \overset{(4.8)}{=} \quad \psi_i(N,v) + \psi_i(N,v) \cdot \sum_{j\in N\backslash\{i\}} \frac{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})}{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})} \\
& = \quad \psi_i(N,v) \cdot \left[1 + \sum_{j\in N\backslash\{i\}} \frac{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})}{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})}\right]
\end{aligned}$$

So, the recursive formula (4.3) holds. This proves the implication $(ii) \Longrightarrow (iii)$. □

Proof of the implication $(iii) \Longrightarrow (iv)$ **stated in Theorem 4.1.**
Suppose (iii) holds. The proof of the statement $\psi(N,v) = MP(N, F^v_\psi)$ proceeds by induction on the size n, $n \geq 1$, of the player set N. For $n = 1$, say $\langle\{i\}, v\rangle \in \mathcal{G}_+$, the equality $\psi(N,v) = MP(N, F^v_\psi)$ holds because of $\psi_i(\{i\}, v) \overset{(3.2)}{=} F^v_\psi(\{i\}) = MP_i(\{i\}, F^v_\psi)$. Let $\langle N,v\rangle \in \mathcal{G}_+$ $(n \geq 2)$ and suppose $\psi(N^*, u) = MP(N^*, F^u_\psi)$ holds for all games $\langle N^*, u\rangle$ the player set N^* of which contains less than n players. Particularly, by applying the induction hypothesis to all subgames $\langle N\backslash\{k\}, v_{N\backslash\{k\}}\rangle$, $k \in N$, with $n-1$ players, we obtain the following: for all $j \in N$, $k \in N$, $j \neq k$, it holds that

$$\psi_j(N\backslash\{k\}, v_{N\backslash\{k\}}) = MP_j(N\backslash\{k\}, F_\psi^{(v_{N\backslash\{k\}})}) = MP_j(N\backslash\{k\}, (F^v_\psi)_{N\backslash\{k\}})$$

where the latter equality follows from the fact that, for all $k \in N$, the solution game $\langle N\backslash\{k\}, F_{\psi}^{(v_{N\backslash\{k\}})}\rangle$ and the subgame $\langle N\backslash\{k\}, (F_{\psi}^{v})_{N\backslash\{k\}}\rangle$ are equal (as shown in the proof of Theorem 3.3(i)). ¿From the recursive formula (4.3) for ψ, the induction hypothesis applied to almost all subgames with $n-1$ players, the balanced ratios property (4.2) for the MP value as well as its efficiency property respectively, we deduce that, for all $i \in N$, it holds,

$$\begin{aligned}
\psi_i(N,v) &\overset{(4.3)}{=} F_{\psi}^{v}(N) \cdot \left[1 + \sum_{j \in N\backslash\{i\}} \frac{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})}{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})}\right]^{-1} \\
&\overset{(IH)}{=} F_{\psi}^{v}(N) \cdot \left[1 + \sum_{j \in N\backslash\{i\}} \frac{MP_j(N\backslash\{i\}, (F_{\psi}^{v})_{N\backslash\{i\}})}{MP_i(N\backslash\{j\}, (F_{\psi}^{v})_{N\backslash\{j\}})}\right]^{-1} \\
&\overset{(4.2)}{=} F_{\psi}^{v}(N) \cdot \left[1 + \sum_{j \in N\backslash\{i\}} \frac{MP_j(N, F_{\psi}^{v})}{MP_i(N, F_{\psi}^{v})}\right]^{-1} \\
&= MP_i(N, F_{\psi}^{v}) \cdot F_{\psi}^{v}(N) \cdot \left[\sum_{j \in N} MP_j(N, F_{\psi}^{v})\right]^{-1} \\
&= MP_i(N, F_{\psi}^{v})
\end{aligned}$$

This completes the inductive proof of the statement $\psi(N,v) = MP(N, F_{\psi}^{v})$. This proves the implication $(iii) \Longrightarrow (iv)$. □

Proof of the equivalence $(i) \Longleftrightarrow (v)$ stated in Theorem 4.1.
Suppose (i) holds. Let $\langle N, v\rangle \in \mathcal{G}_+$ and $\omega : N \to N$ be an order of N. Write $N = \{i_1, i_2, \ldots, i_n\}$ such that $\omega(i_k) = k$ for all $1 \le k \le n$. Clearly, $R_{i_k}^{\omega} = \{i_1, i_2, \ldots, i_k\}$ for all $1 \le k \le n$ and particularly, $R_{i_k}^{\omega}\backslash\{i_k\} = R_{i_{k-1}}^{\omega}$ for all $1 \le k \le n$, where $R_{i_0}^{\omega} := \emptyset$. ¿From this, together with the potential representation (4.1), we deduce that the following holds:

$$\psi_{i_k}(R_{i_k}^{\omega}, v_{R_{i_k}^{\omega}}) = \frac{P_{\psi}(R_{i_k}^{\omega}, v_{R_{i_k}^{\omega}})}{P_{\psi}(R_{i_{k-1}}^{\omega}, v_{R_{i_{k-1}}^{\omega}})} \qquad \text{for all } 1 \le k \le n, \qquad \text{and consequently,}$$

$$\begin{aligned}
\prod_{i \in N} \psi_i(R_i^{\omega}, v_{R_i^{\omega}}) &= \prod_{k=1}^{n} \psi_{i_k}(R_{i_k}^{\omega}, v_{R_{i_k}^{\omega}}) = \prod_{k=1}^{n} \left[\frac{P_{\psi}(R_{i_k}^{\omega}, v_{R_{i_k}^{\omega}})}{P_{\psi}(R_{i_{k-1}}^{\omega}, v_{R_{i_{k-1}}^{\omega}})}\right] = \frac{P_{\psi}(R_{i_n}^{\omega}, v_{R_{i_n}^{\omega}})}{P_{\psi}(\emptyset, v)} \\
&= \frac{P_{\psi}(N,v)}{1} = P_{\psi}(N,v)
\end{aligned}$$

So, the multiplicative path independence (4.4) holds. This proves the implication $(i) \Longrightarrow (v)$. In order to prove the converse implication, suppose (v) holds. Let $\langle N, v\rangle \in \mathcal{G}_+$. By (4.4), the potential $P_{\psi}(N,v)$ is well-defined by $\prod_{k \in N} \psi_k(R_k^{\omega}, v_{R_k^{\omega}})$ for any order ω on the relevant player set N. For any $i \in N$, choose some order ω on N satisfying $\omega(i) = n$ and thus, the order $\bar{\omega}$ on $N\backslash\{i\}$, defined to be the restriction of ω to $N\backslash\{i\}$, satisfies $R_k^{\bar{\omega}} = R_k^{\omega}$ for all $k \in N\backslash\{i\}$. From this, together with the definition of the potential function P_{ψ}, we derive that the following chain of equalities holds:

$$\begin{aligned}
P_\psi(N\backslash\{i\}, v_{N\backslash\{i\}}) &= \prod_{k\in N\backslash\{i\}} \psi_k(R_k^{\bar\omega}, v_{R_k^{\bar\omega}}) = \prod_{k\in N\backslash\{i\}} \psi_k(R_k^{\omega}, v_{R_k^{\omega}}) = \frac{\prod_{k\in N} \psi_k(R_k^{\omega}, v_{R_k^{\omega}})}{\psi_i(R_i^{\omega}, v_{R_i^{\omega}})} \\
&= \frac{P_\psi(N,v)}{\psi_i(N,v)}
\end{aligned}$$

So, the potential representation (4.1) holds. This proves the implication $(v) \Longrightarrow (i)$. □

Proof of the equivalence $(ii) \Longleftrightarrow (vi)$ **stated in Theorem 4.1.**
First suppose (ii) holds. Let $\langle N,v\rangle \in \mathcal{G}_+$ and $i \in N$. Multiplying the balanced ratios condition (4.2) over all $j \in N\backslash\{i\}$ yields

$$\frac{\Big[\psi_i(N,v)\Big]^{n-1}}{\prod_{j\in N\backslash\{i\}} \psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})} = \frac{\prod_{j\in N}\psi_j(N,v)}{\psi_i(N,v)} \cdot \Big[\prod_{j\in N\backslash\{i\}} \psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})\Big]^{-1} \qquad \text{or equivalently,}$$

$$\Big[\psi_i(N,v)\Big]^{n} \cdot \Big[\prod_{j\in N}\psi_j(N,v)\Big]^{-1} = \Big[\prod_{j\in N\backslash\{i\}} \psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})\Big] \cdot \Big[\prod_{j\in N\backslash\{i\}} \psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})\Big]^{-1}$$

In view of (4.6), this proves the implication $(ii) \Longrightarrow (vi)$.
In order to prove the converse implication, suppose (vi) holds. We claim the balanced ratios property (4.2) for the solution ψ holds, i.e.,

$$\frac{\psi_i(N,v)}{\psi_j(N,v)} = \frac{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})}{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})} \qquad \text{for all } \langle N,v\rangle \in \mathcal{G}_+ \text{ and all } i\in N,\ j\in N. \tag{4.9}$$

The proof of (4.9) proceeds by induction on the size n, $n \geq 2$, of the player set N. For $n = 2$, say $\langle\{i,j\},v\rangle \in \mathcal{G}_+$, we deduce from the recursive formula (4.5) and (4.6) respectively, that the following holds:

$$\begin{aligned}
\left[\frac{\psi_i(\{i,j\},v)}{\psi_j(\{i,j\},v)}\right]^2 &\overset{(4.5)}{=} \left[\frac{G_\psi^v(\{i,j\})}{G_\psi^v(\{j\})} \cdot \psi_i(\{i\}, v_{\{i\}})\right] \cdot \left[\frac{G_\psi^v(\{i\})}{G_\psi^v(\{i,j\})} \cdot \frac{1}{\psi_j(\{j\}, v_{\{j\}})}\right] \\
&\overset{(4.6)}{=} \left[\frac{\psi_i(\{i\}, v_{\{i\}})}{\psi_j(\{j\}, v_{\{j\}})}\right] \cdot \left[\frac{\psi_i(\{i\}, v_{\{i\}})}{\psi_j(\{j\}, v_{\{j\}})}\right] = \left[\frac{\psi_i(\{i\}, v_{\{i\}})}{\psi_j(\{j\}, v_{\{j\}})}\right]^2
\end{aligned}$$

So, (4.9) holds whenever $n = 2$. Let $\langle N,v\rangle \in \mathcal{G}_+$ $(n \geq 3)$, and $i \in N$, $j \in N$, and suppose (4.9) holds for all games the player set of which contains less than n players. ¿From the recursive formula (4.5) and the induction hypothesis (4.9) applied to almost all subgames with $n-1$ players, we deduce that the following holds:

$$\begin{aligned}
\left[\frac{\psi_i(N,v)}{\psi_j(N,v)}\right]^n &= \frac{\Big[\psi_i(N,v)\Big]^n}{\Big[\psi_j(N,v)\Big]^n} \overset{(4.5)}{=} \frac{G_\psi^v(N)}{G_\psi^v(N\backslash\{i\})} \cdot \frac{G_\psi^v(N\backslash\{j\})}{G_\psi^v(N)} \cdot \frac{\prod_{k\in N\backslash\{i\}} \psi_i(N\backslash\{k\}, v_{N\backslash\{k\}})}{\prod_{k\in N\backslash\{j\}} \psi_j(N\backslash\{k\}, v_{N\backslash\{k\}})} \\
&= \frac{G_\psi^v(N\backslash\{j\})}{G_\psi^v(N\backslash\{i\})} \cdot \frac{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})}{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})} \cdot \left[\prod_{k\in N\backslash\{i,j\}} \left[\frac{\psi_i(N\backslash\{k\}, v_{N\backslash\{k\}})}{\psi_j(N\backslash\{k\}, v_{N\backslash\{k\}})}\right]\right]
\end{aligned}$$

$$\overset{(4.9)}{=} \frac{G^v_\psi(N\backslash\{j\})}{G^v_\psi(N\backslash\{i\})} \cdot \frac{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})}{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})} \cdot \left[\prod_{k\in N\backslash\{i,j\}} \left[\frac{\psi_i(N\backslash\{j,k\}, v_{N\backslash\{j,k\}})}{\psi_j(N\backslash\{i,k\}, v_{N\backslash\{i,k\}})}\right]\right]$$

$$\overset{(4.5)}{=} \frac{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})}{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})} \cdot \frac{\left[\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})\right]^{n-1}}{\left[\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})\right]^{n-1}} = \left[\frac{\psi_i(N\backslash\{j\}, v_{N\backslash\{j\}})}{\psi_j(N\backslash\{i\}, v_{N\backslash\{i\}})}\right]^n$$

This completes the inductive proof of (4.9). In words, the solution ψ satisfies the balanced ratios property and this proves the implication $(vi) \Longrightarrow (ii)$. □

5 Summary and conclusion

In summary, we introduced a new solution concept, called MP value, from the viewpoint that the MP value turns out to be the most appealing representative of the family of (not necessarily efficient) game-theoretic solutions with a multiplicative potential representation. Due to some equivalent characterizations, the MP value is the unique solution that satisfies the efficiency property as well as the law of preservation of discrete ratios (see Remark 4.2). Consequently, the MP value agrees with the so-called *proportional value* introduced by Ortmann through both the efficiency axiom and the preservation of ratios (cf. [10], Theorem 2.6, page 239). Besides this implicit definition of the proportional value, Ortmann proves the equivalence (4.1) $\Longleftrightarrow$ (4.2) stated in our Theorem 4.1 (cf. [10], Theorem 2.4, page 238) and proves elementary properties like monotonicity, dummy player property, anonymity, and inessential (additive) game property for the proportional value, the proofs of which are based on the recursive formula (4.3), provided efficiency applies (cf. [10], Propositions 4.2, 4.3, 4.5, pages 245-247). In fact, in spite of its implicit description, Ortmann's paper axiomatizes the proportional value by means of the consistency (with reference to Hart and Mas-Colell's reduced game) as well as the proportionality property for two-person games (cf. [10], Theorems 2.10–2.11, pages 241-243). Generally speaking, Ortmann's paper introduces the very same value from a much different viewpoint, leaves out a lot of information about the value itself and its potential function. Hence, our introduction of the MP value by Definition 2.1, and, most of all, the two equivalence Theorems 3.3 and 4.1 contribute very much to a better understanding of an appealing value. To conclude with, the MP value deserves a lot of further research.

Example 5.1. Consider the (numerical) bankruptcy situation with three creditors, the claims of which are given by $d_1 = d_2 = 2$, $d_3 = 4$, whereas the estate $E = 7$. Generally speaking, the associated bankruptcy game $\langle N, v\rangle$ is defined to be $v(S) := \max\left[0,\ E - \sum_{i\in N\backslash S} d_i\right]$ for all $S \subseteq N$, $S \neq \emptyset$ (cf. [2], [3]). The numerical bankruptcy game $\langle N, v\rangle$ is given by $v(\{1\}) = v(\{2\}) = 1$, $v(\{3\}) = v(\{1,2\}) = 3$, $v(\{1,3\}) = v(\{2,3\}) = 5$ and $v(N) = 7$. Note that players 1 and 2 are substitutes (since $d_1 = d_2$). Concerning the MP value for the three two-person subgames, the proportionality solution (2.2) yields $MP(\{1,2\}, v_{\{1,2\}}) = (\frac{3}{2}, \frac{3}{2})$ and $MP(\{1,3\}, v_{\{1,3\}}) = MP(\{2,3\}, v_{\{2,3\}}) = (\frac{5}{4}, \frac{15}{4})$. By the recursive formula (4.3), we get

$$MP_1(N,v) = v(N) \cdot \left[1 + \frac{MP_2(\{2,3\}, v_{\{2,3\}})}{MP_1(\{1,3\}, v_{\{1,3\}})} + \frac{MP_3(\{2,3\}, v_{\{2,3\}})}{MP_1(\{1,2\}, v_{\{1,2\}})}\right]^{-1} = 7 \cdot \frac{2}{9} = \frac{14}{9}$$

By the substitution and efficiency properties for the MP value, we conclude that the MP value is given by $MP(N, v) = \frac{1}{9} \cdot (14, 14, 35)$, whereas the Shapley value (cf. [11], [13]) is given by $Sh(N, v) = \frac{1}{9} \cdot (15, 15, 33)$. Clearly, the creditor with the largest claim prefers the MP value to the Shapley value. For the sake of completeness, by (2.3), the corresponding sequence of real numbers $(\alpha^v_T)_{T \subseteq N}$ is given by $\alpha^v_{\{1\}} = \alpha^v_{\{2\}} = \alpha^v_{\{3\}} = 1$, $\alpha^v_{\{1,2\}} = 2$, $\alpha^v_{\{1,3\}} = \alpha^v_{\{2,3\}} = 4$, $\alpha^v_{\{1,2,3\}} = 270$. By its description (2.4), the MP value for any player can be computed directly.

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