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On the convergence to stationarity of birth-death processes

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On the convergence to stationarity of birth-death processes

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Abstract. Taking up a recent proposal by Stadje and Parthasarathy in the setting of the many-server Poisson queue, we consider the integral $\int_0^\infty [\lim_{u\to\infty} E(X(u)) - E(X(t))]dt$ as a measure of the speed of convergence towards stationarity of the process $\{X(t),\ t \geq 0\}$, and evaluate the integral explicitly in terms of the parameters of the process in the case that $\{X(t),\ t \geq 0\}$ is an ergodic birth-death process on $\{0, 1, \ldots\}$ starting in 0. We also discuss the discrete-time counterpart of this result, and examine some specific examples.

Keywords and phrases: birth-death process, random walk, speed of convergence

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1 Introduction

Let $X(t)$ be the number of customers at time t in a stable $M/M/c$ queueing system and suppose that the system is initially empty. The process $\{X(t),\ t \geq 0\}$ is then stochastically increasing, and, as a consequence, $E(X(t))$ converges monotonically to its limiting value

$$M \equiv \lim_{t\to\infty} E(X(t)).$$

This has recently motivated Stadje and Parthasarathy [10] to propose the quantity

$$\int_0^\infty [M - E(X(t))]\, dt \qquad (1)$$

as a measure of the speed of convergence as $t \to \infty$ of the distribution of $X(t)$ to the stationary distribution of the number of customers in an $M/M/c$ system. They subsequently evaluate the integral (1) explicitly in terms of the number of servers c, and the arrival and service rates of the system.

Clearly, the process $\{X(t),\ t \geq 0\}$ constitutes a birth-death process. Moreover, any birth-death process on the nonnegative integers which starts in state 0 is stochastically increasing (see, for example, Kijima [9, Section 4.8]). It is therefore natural to ask whether the result of Stadje and Parthasarathy can be extended into the more general setting of birth-death processes. The purpose of this paper is to resolve this question in the affirmative. So in what follows $\mathcal{X} \equiv \{X(t),\ t \geq 0\}$ will be an ergodic birth-death process taking values in $\mathcal{N} \equiv \{0, 1, \ldots\}$ with birth rates $\{\lambda_j,\ j \in \mathcal{N}\}$ and death rates $\{\mu_j,\ j \in \mathcal{N}\}$, all strictly positive except $\mu_0 = 0$. Throughout we will assume $X(0) = 0$ and use the notation

$$p_j(t) \equiv \Pr\{X(t) = j \mid X(0) = 0\}, \quad j \in \mathcal{N},\ t \geq 0,$$

and

$$p_j \equiv \lim_{t\to\infty} p_j(t), \quad j \in \mathcal{N}.$$

The speed of convergence to stationarity of the process $\mathcal{X}$ is usually characterized by the *decay parameter*

$$\gamma(\mathcal{X}) \equiv \sup\{\gamma \geq 0 \mid p_j - p_j(t) = \mathcal{O}(\exp(-\gamma t)) \ \text{ as } \ t \to \infty\}$$

(which is independent of j), or its reciprocal $r(\mathcal{X}) \equiv 1/\gamma(\mathcal{X})$, the *relaxation time* (see, for example, [1] and [12]). If $M \equiv \lim_{t\to\infty} E(X(t)) < \infty$ we also have

$$r(\mathcal{X}) = \inf\{r > 0 \mid M - E(X(t)) = \mathcal{O}(\exp(-t/r)) \text{ as } t \to \infty\}, \tag{2}$$

the infimum of an empty set being infinity. The relaxation times of many specific birth-death processes are known, but there exists no general expression for $r(\mathcal{X})$ in terms of the birth and death rates of $\mathcal{X}$. Since, as we will show, the integral (1) *can* be evaluated explicitly in terms of the birth and death rates of $\mathcal{X}$ it may be an attractive alternative to $r(\mathcal{X})$ as a one-parameter characterization of the speed of convergence. Rather than (1), however, we propose its normalized value

$$m(\mathcal{X}) \equiv \int_0^\infty [1 - E(X(t))/M]\, dt \tag{3}$$

as an alternative to $r(\mathcal{X})$ as a measure of the speed of convergence towards stationarity of the process $\mathcal{X}$.

The rest of the paper is organised as follows. After presenting some preliminary results on birth-death processes in Section 2, we will obtain our main result – an explicit expression for the integral (1) in terms of the birth and death rates – in Section 3. The expression will be evaluated for some specific birth-death processes in Section 4. In particular, we will compare our findings with those of Stadje and Parthasarathy [10] (and find a discrepancy). Finally, in Section 5, we consider birth-death processes in discrete time, and show that a similar result may be obtained in this setting by performing a suitable transformation, provided the birth and death probabilities satisfy certain requirements.

2 Preliminaries

The *potential coefficients* of the birth-death process $\mathcal{X} \equiv \{X(t),\ t \geq 0\}$ are defined by

$$\pi_0 \equiv 1 \quad \text{and} \quad \pi_j \equiv \frac{\lambda_0\lambda_1 \dots \lambda_{j-1}}{\mu_1\mu_2 \dots \mu_j}, \quad j \geq 1. \tag{4}$$

Since $\mathcal{X}$ is assumed to be ergodic these constants must satisfy the condition

$$K \equiv \sum_{j=0}^{\infty} \pi_j < \infty. \tag{5}$$

We will additionally assume

$$\sum_{j=0}^{\infty}(\lambda_j \pi_j)^{-1} = \infty, \tag{6}$$

ensuring that $\mathcal{X}$ is uniquely determined by its birth and death rates (see [4]).

It is well known that

$$p_j \equiv \lim_{t\to\infty} p_j(t) = \frac{\pi_j}{K}, \quad j \in \mathcal{N}, \tag{7}$$

while (see, for example, Holewijn and Hordijk [3])

$$\lim_{t\to\infty} E(X(t)) = E(X),$$

X denoting a random variable with distribution $\{p_j,\ j \in \mathcal{N}\}$. Evidently, we will assume throughout that

$$E(X) = \sum_{j=0}^{\infty} j p_j < \infty. \tag{8}$$

It will be convenient to introduce the quantities

$$\tau_j \equiv p_j \sum_{k=0}^{j-1}(\lambda_k p_k)^{-1} \sum_{\ell=k+1}^{\infty} p_\ell, \quad j \geq 0, \tag{9}$$

and

$$T \equiv \sum_{j=0}^{\infty} \tau_j.$$

Here, and henceforth, the empty sum should be interpreted as zero (so that $\tau_0 \equiv 0$). By interchanging summations it is easily seen that

$$T = \sum_{k=0}^{\infty}(\lambda_k p_k)^{-1} \left(\sum_{\ell=k+1}^{\infty} p_\ell \right)^2, \tag{10}$$

which may be finite or infinite.

3 The main result

In this section we will first evaluate the integrals

$$I_j \equiv \int_0^{\infty} [p_j(t) - p_j]\, dt, \quad j \geq 0, \tag{11}$$

after which the value of the integral (1) will follow as a corollary. Since $p_j(t)$ is a unimodal function (see Keilson [8]) the integrals I_j exist, but may be infinite.

The integrals I_j have been evaluated explicitly by Whitt [14, Proposition 6] in the setting of a birth-death process with finite state space $\{0, 1, \ldots, n\}$. Letting n tend to infinity in the expression for I_j given by Whitt yields after a little algebra

$$I_j \equiv \int_0^\infty [p_j(t) - p_j]\, dt = Tp_j - \tau_j, \quad j \geq 0, \tag{12}$$

with the interpretation that $I_j = \infty$ whenever $T = \infty$. We have verified this result by substituting in (11) the spectral representation for $p_j(t)$ developed by Karlin and McGregor [4] and exploiting the technique suggested by Karlin and McGregor [5, p. 399] to evaluate the resulting integral.

We are now in a position to state our main result.

Theorem 1 *If* $\sum_0^\infty j\tau_j < \infty$, *then*

$$\int_0^\infty [E(X) - E(X(t))]\, dt = \sum_{j=0}^\infty j\tau_j - TE(X), \tag{13}$$

whereas the integral is infinite otherwise.

Proof. Since $\mathcal{X}$ is stochastically increasing, we have

$$\sum_{j=0}^k (p_j(t) - p_j) > 0, \quad k \geq 0. \tag{14}$$

We also observe

$$E(X) - E(X(t)) = \sum_{k=1}^\infty \sum_{j=k}^\infty (p_j - p_j(t)) = \sum_{k=1}^\infty \sum_{j=0}^{k-1} (p_j(t) - p_j). \tag{15}$$

It follows that

$$E(X) - E(X(t)) > p_0(t) - p_0,$$

and hence, by (12), the integral is infinite if $T = \infty$. Now assuming $T < \infty$, and using (15) and the fact that $\sum I_j = 0$, we can write

$$\int_0^\infty [E(X) - E(X(t))]\, dt = \sum_{k=1}^\infty \sum_{j=0}^{k-1} I_j = -\sum_{k=1}^\infty \sum_{j=k}^\infty I_j = -\sum_{j=1}^\infty jI_j,$$

the interchange of integration and summation being justified by (14). In view of (8) and (12) the theorem follows. □

4 Examples

To check the theorem we first look at a process for which the value of the integral (1) is available. Namely, we let $\mathcal{X} \equiv \{X(t),\ t \geq 0\}$ be the number of customers in the $M/M/\infty$ queue, which is a birth-death process with rates

$$\lambda_j = \lambda \ \text{ and } \ \mu_j = j\mu, \quad j \in \mathcal{N}.$$

It is well known (see, for example, Feller [2, p. 461]) that when the system starts empty the mean number of customers in the system at time t is given by

$$E(X(t)) = \frac{\lambda}{\mu}\left(1 - e^{-\mu t}\right), \quad t \geq 0,$$

so that

$$\int_0^\infty [E(X) - E(X(t))]\, dt = \frac{\lambda}{\mu^2}. \tag{16}$$

This result can indeed be recovered – albeit somewhat tediously – by evaluating the right-hand side of (13). For completeness' sake we note that the convergence measures (2) and (3) for this process are given by

$$m(\mathcal{X}) = r(\mathcal{X}) = \frac{1}{\mu}. \tag{17}$$

Our second example is the birth-death process $\mathcal{X}$ with rates

$$\lambda_j = \lambda/(j+1) \ \text{ and } \ \mu_{j+1} = \mu, \quad j \geq 0,$$

which may be interpreted as the process of the number of customers in a queueing system in which customers are discouraged by queue length (see, for example, [11]). In this case no simple expression for $E(X(t))$ is available. To evaluate the right-hand side of (13) we write

$$a \equiv \lambda/\mu \tag{18}$$

and note that

$$K = e^a \ \text{ and } \ E(X) = a.$$

Moreover, letting

$$f_j(a) \equiv \sum_{\ell=1}^{\infty} \frac{(j+1)!}{(j+\ell)!} a^\ell, \quad j \geq 0, \tag{19}$$

we readily obtain

$$\tau_j = \frac{1}{\lambda} e^{-a} \frac{a^j}{j!} \sum_{k=0}^{j-1} f_k(a), \quad j \geq 0,$$

so that

$$T = \frac{1}{\lambda} e^{-a} \sum_{j=1}^{\infty} \frac{a^j}{j!} \sum_{k=0}^{j-1} f_k(a) \ \text{ and } \ \sum_{j=1}^{\infty} j\tau_j = \frac{a}{\lambda} e^{-a} \sum_{j=0}^{\infty} \frac{a^j}{j!} \sum_{k=0}^{j} f_k(a).$$

Substitution of these results in (13) gives us

$$\int_0^\infty [E(X) - E(X(t))]\, dt = \frac{a}{\lambda} e^{-a} \sum_{j=0}^{\infty} \frac{a^j}{j!} f_j(a),$$

which, after substitution of (18) and (19) and some algebra, reduces to

$$\int_0^\infty [E(X) - E(X(t))]\, dt = \frac{\lambda(\lambda + 2\mu)}{2\mu^3}. \tag{20}$$

It now follows that

$$m(\mathcal{X}) = \frac{\lambda + 2\mu}{2\mu^2}, \tag{21}$$

while we know from [11] that the relaxation time of the process is given by

$$r(\mathcal{X}) = \frac{\lambda + 2\mu + \sqrt{\lambda^2 + 4\lambda\mu}}{2\mu^2}. \tag{22}$$

We will finally apply our results to the process of the number of customers in an $M/M/c$ queueing system – the setting in which Stadje and Parthasarathy [10] proposed the integral (1) as a measure of the speed of convergence to stationarity – and compare our findings with those in [10]. The process at hand is a birth-death process $\mathcal{X}$ with rates

$$\lambda_j = \lambda \ \text{ and } \ \mu_j = \min\{j, c\}\mu, \quad j \in \mathcal{N}.$$

Writing

$$\rho \equiv \frac{\lambda}{c\mu}, \tag{23}$$

we must have $\rho < 1$ for the system to be stable. The potential coefficients of the process are given by

$$\pi_j = \begin{cases} \dfrac{(c\rho)^j}{j!}, & 0 \leq j \leq c, \\ \dfrac{c^c \rho^j}{c!}, & j \geq c, \end{cases}$$

so, with

$$K_c \equiv \sum_{j=c}^{\infty} \pi_j = \frac{1}{1-\rho} \frac{(c\rho)^c}{c!}, \tag{24}$$

we have

$$K = \sum_{j=0}^{c-1} \frac{(c\rho)^j}{j!} + K_c \quad \text{and} \quad E(X) = c\rho + \frac{\rho}{1-\rho} \frac{K_c}{K}. \tag{25}$$

It is convenient to let

$$A_j \equiv \sum_{k=0}^{j-1} (\lambda_k p_k)^{-1} \sum_{\ell=k+1}^{\infty} p_\ell, \quad j \geq 0$$

(so that $A_0 \equiv 0$), which is readily seen to imply

$$A_j = \frac{1}{\lambda} \sum_{k=0}^{j-1} \frac{k!}{(c\rho)^k} \left(\sum_{\ell=k+1}^{c-1} \frac{(c\rho)^\ell}{\ell!} + K_c \right), \quad 0 \leq j \leq c. \tag{26}$$

The quantities τ_j of (9) can now be expressed as

$$\tau_j = \begin{cases} \dfrac{A_j}{K} \dfrac{(c\rho)^j}{j!}, & 0 \leq j \leq c, \\ \dfrac{1}{K} \left(A_c + (j-c) \dfrac{1}{\lambda} \dfrac{\rho}{1-\rho} \right) \dfrac{c^c \rho^j}{c!}, & j \geq c, \end{cases}$$

from which it follows after some algebra that

$$T = \frac{1}{K} \sum_{j=1}^{c-1} A_j \frac{(c\rho)^j}{j!} + T_c \tag{27}$$

and

$$\sum_{j=0}^{\infty} j\tau_j = \frac{c\rho}{K} \sum_{j=0}^{c-2} A_{j+1} \frac{(c\rho)^j}{j!} + T_c \left(c + \frac{\rho}{1-\rho} \right) + \frac{K_c}{K} \frac{1}{\lambda} \frac{\rho^2}{(1-\rho)^3}, \tag{28}$$

where

$$T_c \equiv \sum_{j=c}^{\infty} \tau_j = \frac{K_c}{K} \left(A_c + \frac{1}{\lambda} \frac{\rho^2}{(1-\rho)^2} \right). \tag{29}$$

The integral (1) can now easily be evaluated for specific values of c, λ and μ from (13) and the expressions (23) – (29). In particular, for $c = 1$ we obtain

$$\int_0^\infty [E(X) - E(X(t))]\, dt = \frac{1}{\mu} \frac{\rho}{(1-\rho)^3}. \tag{30}$$

As a consequence the measure (3) for the $M/M/1$ queue is given by

$$m(\mathcal{X}) = \frac{1}{\mu} \frac{1}{(1-\rho)^2}, \tag{31}$$

while it is well known that the relaxation time of the $M/M/1$ queue satisfies

$$r(\mathcal{X}) = \frac{1}{\mu}\frac{(1+\sqrt{\rho})^2}{(1-\rho)^2}. \tag{32}$$

Evaluating (12) for $c = 2$ leads to

$$\int_0^\infty [E(X) - E(X(t))]\,dt = \frac{1}{\mu}\frac{2\rho(1-\rho+\rho^2)}{(1-\rho)^3(1+\rho)^2}, \tag{33}$$

so in this case we have

$$m(\mathcal{X}) = \frac{1}{\mu}\frac{1-\rho+\rho^2}{(1-\rho)^2(1+\rho)}, \tag{34}$$

while the relaxation time of the $M/M/2$ queue is given in [1] as

$$r(\mathcal{X}) = \begin{cases} \dfrac{1}{\mu}\dfrac{2}{1+4\rho+\sqrt{1-8\rho}}, & 0 < \rho < \frac{1}{9}, \\ \dfrac{1}{2\mu}\dfrac{(1+\sqrt{\rho})^2}{(1-\rho)^2}, & \frac{1}{9} \le \rho < 1. \end{cases} \tag{35}$$

Comparing our results with those of Stadje and Parthasarathy [10], we find agreement for $c = 1$, but a discrepancy for $c = 2$. As a check, we evaluated the integral I_j of (11) directly by using the representation for $p_j(t)$ derived in Karlin and McGregor [6] for $j = \lambda = \mu = 1$, and found that it equals 0, which is consistent with (12), but *not* with Theorem 3 of Stadje and Parthasarathy [10].

5 Discrete-time birth-death processes

A discrete-time birth-death process or *random walk* $\tilde{\mathcal{X}} \equiv \{\tilde{X}(n),\ n = 0, 1, \ldots\}$ on the state space $\mathcal{N} \equiv \{0, 1, \ldots\}$ is a Markov chain with stationary one-step transition probabilities p_{ij} satisfying $p_{ij} = 0$ for $|i - j| > 1$. We shall only consider honest random walks in which $p_j \equiv p_{j,j+1} > 0$, $q_{j+1} \equiv p_{j+1,j} > 0$, and $r_j \equiv p_{jj} \ge 0$ for all $j \in \mathcal{N}$, but $r_j > 0$ for at least one $j \in \mathcal{N}$ (the latter to avoid periodicity). We assume throughout that $\tilde{X}(0) = 0$ and let

$$\tilde{p}_j(n) \equiv \Pr(\tilde{X}(n) = j \mid \tilde{X}(0) = 0), \quad j \in \mathcal{N},\ n \ge 0.$$

Defining

$$\tilde{\pi}_0 = 1 \ \text{ and } \ \tilde{\pi}_j = \frac{p_0 p_1 \cdots p_{j-1}}{q_1 q_2 \cdots q_j}, \quad j \ge 1, \tag{36}$$

it is well known that the process is ergodic if

$$\tilde{K} \equiv \sum_{j=0}^{\infty} \tilde{\pi}_j < \infty, \tag{37}$$

in which case

$$\tilde{p}_j \equiv \lim_{n\to\infty} \tilde{p}_j(n) = \frac{\tilde{\pi}_j}{\tilde{K}}, \quad j \in \mathcal{N}, \tag{38}$$

and

$$\lim_{n\to\infty} E(\tilde{X}(n)) = E(\tilde{X}) = \sum_{j=0}^{\infty} j\tilde{p}_j, \tag{39}$$

$\tilde{X}$ denoting a random variable with distribution $\{\tilde{p}_j,\ j \in \mathcal{N}\}$ (see, for example, Karlin and McGregor [7]).

If $E(\tilde{X})$ is finite it seems natural to propose – in analogy to (1) – the sum

$$\sum_{n=0}^{\infty} \left[E(\tilde{X}) - E(\tilde{X}(n))\right] \tag{40}$$

as a measure of the speed of convergence of $\tilde{X}(n)$ to $\tilde{X}$, provided $E(\tilde{X}(n))$ converges monotonically to $E(\tilde{X})$. However, it is easy to construct examples of random walks starting in 0 in which the latter does not happen, so that (40) is less attractive than its continuous-time counterpart as a measure of the speed of convergence to stationarity. For completeness' sake we shall nevertheless evaluate the sum (40) explicitly, under the condition that

$$E(\tilde{X}(n)) < E(\tilde{X}), \quad n \geq 0. \tag{41}$$

We note that a sufficient condition for $E(\tilde{X}(n))$ to converge monotonically to its limit $E(\tilde{X})$ as $n \to \infty$ (and hence for (41)), is stochastic monotonicity of $\tilde{\mathcal{X}}$, which prevails if and only if

$$p_j + q_{j+1} \leq 1, \quad j \in \mathcal{N} \tag{42}$$

(see Kijima [9, Example 3.12]).

To evaluate the sum (40) we associate with $\tilde{\mathcal{X}}$ a continuous-time birth-death process $\mathcal{X} \equiv \{X(t),\ t \geq 0\}$ with rates

$$\lambda_j = p_j \ \text{ and } \ \mu_j = q_j, \quad j \in \mathcal{N}. \tag{43}$$

Since $\lambda_j + \mu_j = p_j + q_j \leq 1$ for all j, the process $\mathcal{X}$ is uniformizable with uniformization parameter 1 and we get $\tilde{\mathcal{X}}$ back as the uniformized process. Moreover, with $\{N(t),\ t \geq 0\}$ denoting a Poisson process with intensity 1, we have

$$\{X(t),\ t \geq 0\} \stackrel{\mathrm{d}}{=} \{\tilde{X}(N(t)),\ t \geq 0\} \tag{44}$$

(see, for example, [9, Section 4.4] for these results on uniformization). The next theorem shows that the problem of evaluating (40) can now be reduced to that of evaluating the integral (1) for the continuous-time process $\mathcal{X}$.

Theorem 2 *If $E(\tilde{X}(n)) < E(\tilde{X})$ for all $n \geq 0$, then*

$$\sum_{n=0}^{\infty} \left[E(\tilde{X}) - E(\tilde{X}(n))\right] = \int_0^{\infty} \left[E(X) - E(X(t))\right] dt,$$

where $\{X(t),\ t \geq 0\}$ is the birth-death process with rates (43).

Proof. It is obvious from (44) that $E(\tilde{X}) = E(X)$. Moreover, by conditioning on the value of $N(t)$ we get

$$\int_0^{\infty} \left[E(X) - E(X(t))\right] dt = \int_0^{\infty} \left[E(\tilde{X}) - E(\tilde{X}(N(t)))\right] dt$$
$$= \int_0^{\infty} \left\{\sum_{n=0}^{\infty} \left[E(\tilde{X}) - E(\tilde{X}(n))\right] e^{-t} \frac{t^n}{n!}\right\} dt = \sum_{n=0}^{\infty} \left[E(\tilde{X}) - E(\tilde{X}(n))\right],$$

where the interchange of integration and summation is allowed by Fubini's theorem. □

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