Faculty of Mathematical Sciences

University of Twente

University for Technical and Social Sciences

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl

Memorandum No. 1508

On some intriguing problems in Hamiltonian graph theory – A survey

H.J. Broersma

December 1999

ISSN 0169-2690

On some intriguing problems in hamiltonian graph theory – a survey

H.J. Broersma

Faculty of Mathematical Sciences
University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands
email: broersma@math.utwente.nl

November 3, 1999

Abstract

We survey results and open problems in hamiltonian graph theory centred around three themes: regular graphs, t-tough graphs, and claw-free graphs.

Keywords: hamiltonian graph, traceable graph, regular graph, toughness, t-tough graph, claw-free graph, line graph, cubic graph, closure, hopping lemma, cyclically 4-edge-connected, essentially 4-edge-connected

AMS Subject Classifications (1991): 05C45, 05C38, 05C35

1 Introduction

As this survey paper is the outcome of an invited lecture at the eighth *Workshop* on Cycles and Colourings (Stará Lesná, Slovakia (1999)), the presentation of results is motivated by *open* problems in hamiltonian graph theory rather than the intention to write an exhaustive concise survey on this topic. Therefore the presented results and problems are centred around three themes: regular graphs, t-tough graphs, and claw-free graphs. Namely, for all these three graph classes there exist some intriguing 'long-standing' conjectures on hamiltonicity, as well as a number of recent developments towards proving or refuting these conjectures. It is our aim to stimulate and inspire the reader to continue the work in this fascinating area of graph theory.

We use Bondy and Murty's book [15] for terminology and notation not defined here, and consider finite simple graphs only. To avoid irrelevant technicalities we will assume throughout that all graphs have at least three vertices. This implies, e.g., that all complete graphs considered are hamiltonian.

A graph G is *hamiltonian* if it contains a Hamilton cycle (a cycle containing every vertex of G). The number of vertices of a graph will be denoted by n.

It is well-known that the problem of deciding whether a given graph is hamiltonian, is NP-complete, and that (up to now) there exists no easily verifiable necessary and sufficient condition for the existence of a Hamilton cycle. This fact gave rise to a growing number of conditions that are either necessary or sufficient. We refer to [7], [8], [11], [12], [25], and [34] for more background and general surveys.

Before we turn to our three graph classes, we mention a few results that inspired most of today's work, and give some recent developments that cannot be found in the most recent survey [34].

1.1 Early degree conditions and a closure operation

Most of the sufficient conditions for hamiltonicity are based on the intuitive idea that a Hamilton cycle is likely to exist if all vertices have many neighbors. The earliest degree condition is based on the minimum degree $\delta(G)$ of the graph G.

Theorem 1 (Dirac [26])
If $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian.

The lower bound in Theorem 1, often referred to as Dirac's Theorem, cannot be relaxed without destroying the conclusion of the theorem (unless we add an extra condition, e.g. that G is regular, G is t-tough, or G is claw-free, as we will see in the next sections). Nevertheless Dirac's Theorem has been generalized in several directions.

Denote by $d(v)$ the degree of a vertex v in the graph G. We will refer to the next generalization of Theorem 1 as Ore's Theorem.

Theorem 2 (Ore [64])
If $d(u) + d(v) \geq n$ for every pair of distinct nonadjacent vertices u and v of G, then G is hamiltonian.

For further generalizations of Ore's Theorem in terms of vertex degrees we refer to the aforementioned surveys.

As remarked in [25], the closure concept introduced by Bondy and Chvátal [13] was found in an attempt to find a constructive proof for a sufficient condition for hamiltonicity based on degree sequences. It exploits the following variation on Ore's Theorem.

Theorem 3 (Ore [64])
Let u and v be distinct nonadjacent vertices of a graph G such that $d(u) + d(v) \geq n$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.

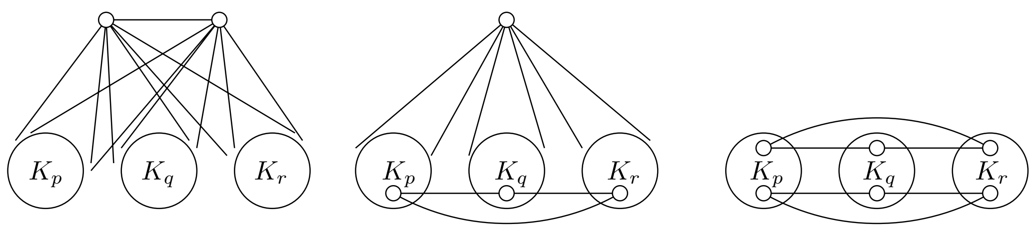
The closure technique based on this result generalized many known degree conditions, and opened a new horizon for the research on hamiltonian and related properties of graphs. We refer to [21] for a recent survey on closure concepts and their applications. We will consider a different kind of closure concept for claw-free graphs in the last section.

1.2 Recent generalizations of Ore's Theorem

Around ten years ago, sufficient conditions for hamiltonicity appeared in which certain vertex sets are required to have large neighborhood unions instead of large degree sums. Many of these new results do not generalize Ore's Theorem. The following more recent result in [18] uses a neighborhood type condition, and generalizes Ore's Theorem.

Denote by $N(v)$ the set of neighbors of a vertex v in the graph G.

Theorem 4 (Broersma, van den Heuvel, and Veldman [18])
If G is a 2-connected graph and $|N(u) \cup N(v)| \geq \frac{n}{2}$ for every pair of distinct nonadjacent vertices u and v of G, then either G is hamiltonian, or G is the Petersen graph, or G is in one of the three classes of exceptional graphs of connectivity 2 shown in Figure 1 .



$p+q+r = n-2$
$p, q, r \geq 1$

$p+q+r = n-1$
$p, q, r \geq 2$

$p+q+r = n$
$p, q, r \geq 3$

Figure 1. Three classes of exceptional graphs.

The three classes of exceptional graphs shown in Figure 1 will be described more formally in the next section, and play an important role in many recent developments in hamiltonian graph theory.

Theorem 4 has been further generalized in [57] and [58].

2 Hamiltonicity of regular graphs

It is likely that the minimum degree bound in Dirac's Theorem (Theorem 1) can be relaxed if we add the condition that the graph under consideration is regular (and 2-connected). This is indeed the case. In this section we will discuss several results, conjectures, and partial solutions on minimum degree conditions for regular graphs to be hamiltonian. As the graphs are regular, we formulate a degree condition as an upper bound on n in terms of the degree of regularity. We will also describe a variant of an important technique known as 'hopping', since it has been a key ingredient in most of the proofs of the results described in this section.

We start with a result of Jackson [42], and refer to [42] for earlier results.

Theorem 5 (Jackson [42])
Every 2-connected k-regular graph on at most $3k$ vertices is hamiltonian.

As noted in [42], Theorem 5 is best possible for $k = 3$ in view of the Petersen graph, and essentially best possible for $k \geq 4$. For future reference also, we define three classes $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{J}$ of graphs (See Figure 1) illustrating the latter assertion. For a positive integer t, let $\mathcal{K}_t$ denote the set of all graphs consisting of three disjoint complete graphs, where each of the components has order at least t. Now $\mathcal{G}$ is the class of all spanning subgraphs of graphs that can be obtained as the join of K_2 and a graph in $\mathcal{K}_1$. The class $\mathcal{H}$ is the set of all spanning subgraphs of graphs that can be obtained from the join of K_1 and a graph G in $\mathcal{K}_2$ by adding the edges of a triangle between three vertices from distinct components of G. The class $\mathcal{J}$ is the set of all spanning subgraphs of graphs that can be obtained from a graph G in $\mathcal{K}_3$ by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of G. It is easy to check that all graphs in $\mathcal{G} \cup \mathcal{H} \cup \mathcal{J}$ are nonhamiltonian. (Indeed, $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{J}$ were first obtained by Watkins and Mesner [73] in a characterization of the 2-connected graphs that have three vertices which are not contained in a common cycle.) Furthermore, each of the classes $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{J}$ contains 2-connected k-regular graphs on $3k + 4$ vertices for even $k \geq 4$, and $3k + 5$ vertices for all $k \geq 3$. (Note that $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{J}$ are not pairwise disjoint.) We set $\mathcal{F} = \mathcal{G} \cup \mathcal{H} \cup \mathcal{J}$.

Theorem 5 has been extended in several papers, e.g., in [14], [39] and [78]. The strongest among these extensions is due to Hilbig [39]. Let Π denote the Petersen graph and Π^{Δ} the 3-regular graph obtained from Π by replacing one vertex by a triangle.

Theorem 6 (Hilbig [39])
Let G be a 2-connected k-regular graph on at most $3k + 3$ vertices. Then G is hamiltonian if and only if $G \notin \{\Pi, \Pi^{\Delta}\}$.

In [46], the following improvement of Theorem 5 for 3-connected graphs is conjectured. (Note that no graph in $\mathcal{F}$ is 3-connected.)

Conjecture 7 (Jackson, Li & Zhu [46])
For $k \geq 4$, every 3-connected k-regular graph on at most $4k$ vertices is hamiltonian.

Conjecture 7 is a special case of Häggkvist's Conjecture, appearing in [42], that every m-connected k-regular graph ($k \geq 4$) on at most $(m + 1)k$ vertices is hamiltonian. However, for $k \equiv 0$ (mod 4), the graph $\overline{K_k} \vee (\overline{K_{k-1}} + 2\,K_{k+1})$ contains a nonhamiltonian $\frac{1}{2}k$-connected

k-regular spanning subgraph G_k, showing that Häggkvist's Conjecture is not true in general. The graphs G_k were independently found by Jung and Jackson. For a more detailed description we refer to [46] or [61]. The graphs G_k also show that Conjecture 7 would be best possible.

A first step towards proving Conjecture 7 was made in [46].

A cycle C of a graph G is called a *dominating cycle* if $V(G) \setminus V(C)$ is an independent set of G.

Theorem 8 (Jackson, Li & Zhu [46])
Let G be a 3-connected k-regular graph on at most $4k$ vertices. Then for $k \geq 63$, every longest cycle of G is a dominating cycle.

In the graph G_k, every longest cycle is dominating. Still Theorem 8 is essentially best possible: for even $k \geq 8$, the graph $K_3 \vee (2\,K_k + 2\,K_{k+1})$ of order $4k+5$ has a 3-connected k-regular spanning subgraph containing no dominating cycle.

In [77], Theorem 8 was used to obtain another result in the direction of Conjecture 7.

Theorem 9 (Zhu & Li [77])
For $k \geq 63$, every 3-connected k-regular graph on at most $\frac{22}{7}k$ vertices is hamiltonian.

This was improved in [16].

Theorem 10 (Broersma, Van den Heuvel, Jackson, and Veldman [16])
Let G be a 2-connected k-regular graph on at most $\frac{7}{2}k - 7$ vertices. Then G is hamiltonian if and only if $G \notin \mathcal{F}$.

Since no graph in $\mathcal{F}$ is 3-connected, the following improvement of Theorem 9 is an immediate consequence of Theorem 10.

Corollary 11 (Broersma, Van den Heuvel, Jackson, and Veldman [16])
Every 3-connected k-regular graph on at most $\frac{7}{2}k - 7$ vertices is hamiltonian.

The necessity of the condition for hamiltonicity in Theorem 10 is obvious. The sufficiency is an immediate consequence of the following two results that are proved in [16].

Theorem 12 (Broersma, Van den Heuvel, Jackson, and Veldman [16])
Let G be a k-regular graph on at most $\frac{7}{2}k - 7$ vertices. If G contains a dominating cycle, then G is hamiltonian.

Theorem 13 (Broersma, Van den Heuvel, Jackson, and Veldman [16])
Let G be a 2-connected k-regular graph on at most $4k - 3$ vertices. Then G contains a dominating cycle or $G \in \mathcal{F}$.

The proof of Theorem 13 is based on ideas from [18] and [71].

The proof of Theorem 10 (via Theorems 12 and 13) uses several ideas from [17], where a relatively short proof of (an extension of) Theorem 5 occurs. In particular the idea of breaking the proof into two parts in the way reflected by Theorems 12 and 13, stems from [17].

In view of the above results the following strengthening of Conjecture 7 was proposed in [16].

Conjecture 14 (Broersma, Van den Heuvel, Jackson, and Veldman [16])
Let G be a 2-connected k-regular graph on at most $4k$ vertices. Then for $k \geq 4$, G is hamiltonian if and only if $G \notin \mathcal{F}$.

The proof of Theorem 12 as well as most of the other results in this section, uses a variation of Woodall's Hopping Lemma [74]. To demonstrate the general idea of 'hopping', we will describe the variant that was used to prove Theorem 12. For other variants of this important lemma and their applications we refer to [6] and [37], Chapter 4.

2.1 A hopping lemma

In order to describe the 'hopping' technique and its premisses, we first develop some additional terminology and notation.

Let C be a cycle of a graph G. We call C *extendable* if there exists an *extension* of C, i.e., a cycle C' with $V(C) \subseteq V(C')$ and $V(C) \neq V(C')$. For $v \in V(G) \setminus V(C)$, the cycle C is *v-extendable* if there exists a *v-extension* of C, i.e., an extension with vertex set $V(C) \cup \{v\}$.

We denote by $\overrightarrow{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor. If $Z \subseteq V(C)$, then $Z^+ = \{ z^+ \mid z \in Z \}$ and $Z^- = \{ z^- \mid z \in Z \}$. Similar notation is used for paths. When more than one cycle or path is under consideration, we write u^{+C}, u^{-C} instead of just u^+, u^- in order to avoid ambiguity.

In this variation of Woodall's Hopping Lemma [74], Lemma 15 below, we use the following hypotheses and definitions.

Let G be a graph, $\overrightarrow{C}$ a cycle of G with $V(C) \neq V(G)$, and a a vertex in $V(G) \setminus V(C)$. Assume C is not a-extendable. Set

$$X_1 = N(a) \cap V(C),$$

and for $i \geq 1$,

$$\begin{aligned} Y_i &= X_i^+ \cap X_i^-, \\ U_i &= X_i^+ \setminus Y_i, \end{aligned}$$

$$\begin{aligned}
W_i &= X_i^- \setminus Y_i, \\
X'_{i+1} &= \{\, v \in V(C) \mid \text{there exist six neighbors } w_1, u_1, w_2, u_2, w_3, u_3 \text{ of } v \text{ such that} \\
&\qquad\qquad w_j \in W_i,\ u_j \in U_i \text{ and } w_j^+ \overrightarrow{C} u_j^- \subseteq X_i \cup Y_i \ (\, j = 1,2,3\,)\,\}, \\
X''_{i+1} &= N(Y_i) \cap V(C), \\
X_{i+1} &= X_i \cup X'_{i+1} \cup X''_{i+1}.
\end{aligned}$$

Then $X_1 \subseteq X_2 \subseteq \cdots$ and $Y_1 \subseteq Y_2 \subseteq \cdots$. Set

$$\begin{aligned}
X &= \bigcup_{i=1}^{\infty} X_i, \\
Y &= \bigcup_{i=1}^{\infty} Y_i.
\end{aligned}$$

Then

(1) $\qquad N(Y) \cap V(C) \subseteq X.$

The name 'hopping' reflects the fact that we obtain the sets X''_{i+1} from vertices in Y_i by considering their neighbors on the cycle, and by iterating this process in the way described above. In other variants the main differences are the assumptions on C and a, the choice of X_1, and the definition of X_{i+1}.

The *height* $h(x)$ of $x \in X$ is defined by

$$h(x) = \min\{\, i \mid x \in X_i \,\}.$$

A path $P = x_1 \overrightarrow{P} x_2$ is called a *hopping path* if each of the following conditions is satisfied:

(2) $\qquad x_1, x_2 \in X$;

(3) $\qquad V(P) = V(C)$;

(4) $\qquad$ if $1 \le i < \max\{\, h(x_1), h(x_2)\,\}$ and $y \in Y_i \setminus \{x_1, x_2\}$, then $\{y^{-P}, y^{+P}\} = \{y^{-C}, y^{+C}\}$;

(5) $\qquad$ if $1 \le i < \max\{\, h(x_1), h(x_2)\,\}$, then $X_i \setminus \{x_1, x_2\}$ contains at most one vertex x for which $\{x^{-P}, x^{+P}\} \neq \{x^{-C}, x^{+C}\}$.

The *height* $h(P)$ of a hopping path $P = x_1 \overrightarrow{P} x_2$ is defined by

$$h(P) = \max\{\, h(x_1), h(x_2)\,\}.$$

These definitions differ from those given in [74] in that:

– we do not require $N(a) \subseteq V(C)$;

– we add the sets X'_i to X;

– the conditions (4) and (5) for a hopping path are more restrictive.

The following lemma is crucial for the proof of Theorem 12 in [16].

Lemma 15 (Broersma, Van den Heuvel, Jackson, and Veldman [16])
There exists no hopping path.

2.2 How the hopping lemma is used

To conclude the section on regular graphs, we shall now briefly indicate how Lemma 15 has been used in [16] to prove Theorem 12. This also reflects the way variants of the hopping technique have been applied in other proofs.

First Lemma 15 has been applied in [16] to obtain several other lemmas concerning the (non)existence of edges with one end in $X^+ \cup X^-$.

In the proof of Theorem 12 given there, G is supposed to be a nonhamiltonian graph satisfying the hypotheses of Theorem 12. Then the lemmas derived from Lemma 15 have been applied to a nonextendable dominating cycle in G, in order to restrict the number of edges between $X^+ \cup X^-$ and $V(G) \setminus X$. The regularity condition then implies that there must be 'many' edges between $X^+ \cup X^-$ and X. Finally a contradiction to k-regularity is obtained by showing that this number of edges is greater than $k\,|X|$.

3 Hamiltonicity of t-tough graphs

The number of components of a graph G is denoted by $\omega(G)$. The graph G is *t-tough* ($t \in \mathbb{R}$, $t \geq 0$) if $|S| \geq t \cdot \omega(G - S)$ for every subset S of $V(G)$ with $\omega(G - S) > 1$. The *toughness* of G, denoted by $\tau(G)$, is the maximum value of t for which G is t-tough (For K_n we define $\tau(K_n) = \infty$).

The concept of toughness of a graph was introduced by Chvátal [24]. It is an easy exercise to show that 1-toughness is a necessary condition for hamiltonicity, but that it is not sufficient. Jung [49] proved that in the degree bounds in Dirac's Theorem and Ore's Theorem (Theorems 1 and 2) n can be replaced by $n - 4$ if the graphs are assumed to be 1-tough and n is large enough, and this is essentially best possible. In [2] it is shown that the bound $n/2$ in Theorem 1 can be replaced by roughly $n/(t + 1)$ if the graphs are assumed to be t-tough. We refer to [2] for the details. It is a natural question whether we need a degree bound at all if we require a high toughness. In fact, in [24] the following conjecture is stated.

Conjecture 16 (Chvátal [24])
There exists t_0 such that every t_0-tough graph is hamiltonian.

The stronger conjecture that every t-tough graph with $t > \frac{3}{2}$ is hamiltonian, also occurring in [24], was first disproved by Thomassen (see [7]). Enomoto, Jackson, Katerinis and Saito [27] showed that every 2-tough graph contains a 2-factor (a 2-regular spanning subgraph), while for arbitrary $\varepsilon > 0$ there exist $(2 - \varepsilon)$-tough graphs without a 2-factor, and hence without a Hamilton cycle. Therefore the following conjecture, usually attributed to Chvátal, appeared to be both reasonable and best possible.

Conjecture 17
Every 2-tough graph is hamiltonian.

Since every 2-tough graph is 4-connected, the conjecture is true for planar graphs by a result of Tutte [69]. By a result of Fleischner [32] the conjecture also holds for squares of 2-connected graphs. We refer to [69] and [32] for additional terminology and details.

A graph G is *traceable* if G contains a Hamilton path (a path containing every vertex of G); G is *hamiltonian-connected* if for every pair of distinct vertices x and y of G there is a Hamilton path with endvertices x and y.

In [1] a construction of a nontraceable graph from non-hamiltonian-connected building blocks was used to show that Conjecture 17 is equivalent to several other statements, some (seemingly) weaker, some (seemingly) stronger than Conjecture 17. This construction was inspired by examples of graphs of high toughness without 2-factors occurring in [5].

In [3] the same construction was used to obtain $\left(\frac{9}{4} - \varepsilon\right)$-tough nontraceable graphs for arbitrary $\varepsilon > 0$, thereby refuting Conjecture 17. We will give a brief outline of the construction of these counterexamples in the next section.

Conjecture 16 remains open, but we do not believe that $\frac{9}{4}$-tough graphs are hamiltonian. In fact, we hope that constructions will be found yielding counterexamples to Conjecture 16 for arbitrary t_0.

3.1 Counterexamples to Conjecture 17

For a given graph H and two vertices x and y of H we define the graph $G(H, x, y, \ell, m)$ ($\ell, m \in \mathbb{N}$) as follows. Take m disjoint copies $H_1, \ldots, H_m$ of H, with x_i, y_i the vertices in H_i corresponding to the vertices x and y in H ($i = 1, \ldots, m$). Let F_m be the graph obtained from $H_1 \cup \ldots \cup H_m$ by adding all possible edges between pairs of vertices in $\{x_1, \ldots, x_m, y_1 \ldots, y_m\}$. Let $T = K_\ell$ and let $G(H, x, y, \ell, m)$ be the join $T \vee F_m$ of T and F_m.

The proof of the following theorem occurs in [3] and almost literally also in [1].

Theorem 18 (Bauer, Broersma, and Veldman [3])
Let H be a graph and x, y two vertices of H which are not connected by a Hamilton path of H. If $m \geq 2\ell + 3$, then $G(H, x, y, \ell, m)$ is nontraceable.

Consider the graph L of Figure 2. There is obviously no Hamilton path in L between u and v. Hence $G(L, u, v, \ell, m)$ is nontraceable for every $m \geq 2\ell + 3$. The toughness of these graphs has been established in [3].

Theorem 19 (Bauer, Broersma, and Veldman [3])
For $\ell \geq 2$ and $m \geq 1$,

$$\tau(G(L, u, v, \ell, m)) = \frac{\ell + 4m}{2m + 1}.$$

Combining Theorems 18 and 19 for sufficiently large values of m and ℓ, one obtains the next result.

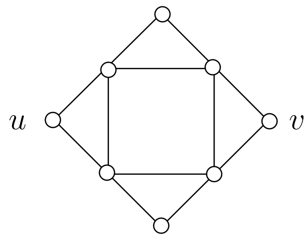


Figure 2. The graph L.

Corollary 20 (Bauer, Broersma, and Veldman [3])
For every $\varepsilon > 0$ *there exists a* $\left(\frac{9}{4} - \varepsilon\right)$*-tough nontraceable graph.*

It is easily seen from the proof in [3] that Theorem 18 remains valid if "$m \geq 2\ell + 3$" and "nontraceable" are replaced by "$m \geq 2\ell + 1$" and "nonhamiltonian", respectively. Thus the graph $G(L, u, v, 2, 5)$ is a nonhamiltonian graph, which by Theorem 19 has toughness 2. This graph is sketched in Figure 3. It follows that a smallest counterexample to Conjecture 17 has at most 42 vertices. Similarly, a smallest nontraceable 2-tough graph has at most 58 $(|V(G(L, u, v, 2, 7))|)$ vertices.

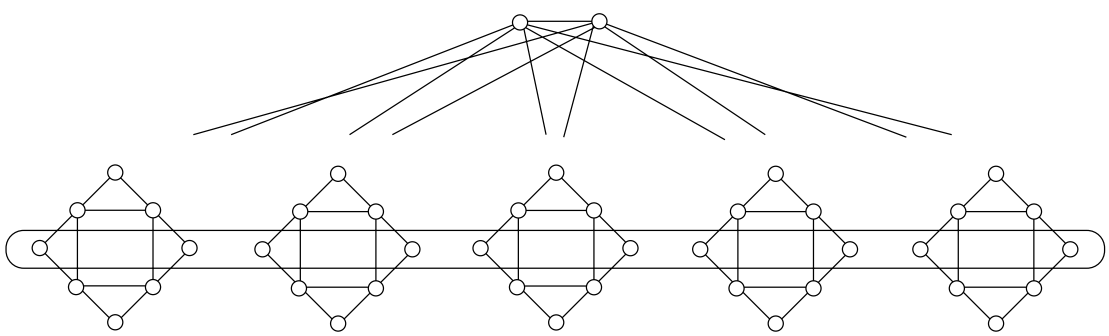


Figure 3. The graph $G(L, u, v, 2, 5)$.

A graph G is *neighborhood-connected* if the neighborhood of each vertex of G induces a connected subgraph of G. In [24] Chvátal also states the following weaker version of Conjecture 17: every 2-tough neighborhood-connected graph is hamiltonian. Since all counterexamples to Conjecture 17 described above are neighborhood-connected, this weaker conjecture is also false.

Most of the ingredients used in the above counterexamples to Conjecture 17 were already present in [1]. It only remained to observe that using the specific graph L as a "building block" produced a graph with toughness at least 2. We hope that other building blocks and/or smarter constructions will lead to counterexamples with a higher toughness.

3.2 Chordal graphs

A graph G is *chordal* if it contains no induced cycles of length at least 4. Chvátal [24] obtained $\left(\frac{3}{2} - \varepsilon\right)$-tough graphs without a 2-factor for arbitrary $\varepsilon > 0$. These examples are all chordal. Recently it was shown in [4] that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Based on this, Kratsch [51] raised the question whether every $\frac{3}{2}$-tough chordal graph is hamiltonian. Using Theorem 18 in [3] it has been shown that this conjecture, too, is false. A key observation in this context is that the graphs $G(H, x, y, \ell, m)$ are chordal whenever H is chordal, as is easily shown.

Consider the graph M of Figure 4.

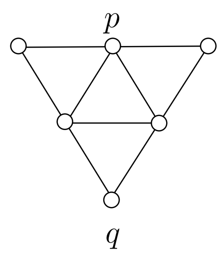


Figure 4. The graph M.

The graph M is chordal and has no Hamilton path with endvertices p and q. Hence by Theorem 18 the chordal graph $G(M, p, q, \ell, m)$ is nontraceable whenever $m \geq 2\ell + 3$. By arguments as used in the proof of Theorem 19 (in [3]) its toughness is $\frac{\ell+3m}{2m+1}$ if $\ell \geq 2$. Hence for $\ell \geq 2$ the graph $G(M, p, q, \ell, 2\ell + 3)$ is a chordal nontraceable graph with toughness $\frac{7\ell+9}{4\ell+7}$. This gives the following result.

Theorem 21 (Bauer, Broersma, and Veldman [3])
For every $\varepsilon > 0$ there exists a $\left(\frac{7}{4} - \varepsilon\right)$-tough chordal nontraceable graph.

On the other hand Chen, Jacobson, Kézdy, and Lehel [23] recently proved that every 18-tough chordal graph is hamiltonian, which means that Conjecture 16 is true when restricted to chordal graphs. We expect that the lower bound 18 on the toughness can be considerably decreased. In fact, for chordal planar graphs it has been proved by Böhme, Harant, and Tkáč [10] that a toughness strictly larger than one implies hamiltonicity, and this is best possible. For the class of split graphs toughness at least $\frac{3}{2}$ suffices and is best possible, as shown by Kratsch, Lehel, and Müller [52]. We refer to the sources for additional terminology and details.

4 Hamiltonicity of claw-free graphs

During the last two decades many results on hamiltonian properties of claw-free graphs (i.e. graphs that do not contain $K_{1,3}$ as an induced subgraph) have appeared. We refer the reader to [29] for a recent survey. Most of these results involve sufficient conditions in terms of degrees, neighborhoods, forbidden subgraphs or (local) connectivity. In this section we will discuss several recent developments on hamiltonicity of claw-free graphs. We start with the earliest minimum degree condition. As shown in [60] the minimum degree bound in Theorem 1 can be relaxed if we add the condition that the graph under consideration is claw-free (and 2-connected).

Theorem 22 (Matthews and Sumner [60])
Every 2-connected claw-free graph G with $\delta(G) \geq \frac{1}{3}(n-2)$ is hamiltonian.

Theorem 22 has been generalized in several directions. We refer to [29] for a survey, and come back with the most recent developments on minimum degree conditions later.

4.1 On two conjectures and a closure technique

Most of the results in this section are motivated by the following two conjectures.

Conjecture 23 (Thomassen [67])
Every 4-connected line graph is hamiltonian.

Conjecture 24 (Matthews and Sumner [59])
Every 4-connected claw-free graph is hamiltonian.

A smallest 3-connected claw-free graph was obtained by Matthews and Sumner [59]. It is the line graph of the graph obtained from the Petersen graph by subdividing each edge of a perfect matching, and has twenty vertices.

Both conjectures are special cases of Conjecture 17, since every line graph is claw-free and the toughness of a (noncomplete) claw-free graph is half its connectivity (an easy exercise, see [59]).

A recent result on closures due to Ryjáček [65] (Theorem 25 below) implies that Conjecture 23 and Conjecture 24 are equivalent.

We first introduce some terminology and notation. The *neighborhood* of a vertex v of a graph G is the subgraph of G induced by the set $N(v)$ of neighbors of v in G. The *local completion of a graph G at a vertex v* is the operation of joining all pairs of nonadjacent vertices in $N(v)$, i.e. replacing the neighborhood of v by the complete graph on $N(v)$.

In [65] the following has been proved.

Theorem 25 (Ryjáček [65])
Let G be a claw-free graph, v a vertex of G whose neighborhood is connected, and G' the graph obtained from G by local completion at v. Then

(i) G' is claw-free, and

(ii) for every cycle C' of G' there exists a cycle C of G such that $V(C') \subseteq V(C)$.

For a claw-free graph G, we define the *closure* $cl(G)$ of G as the graph obtained from G by iteratively performing local completions at vertices with connected neighborhoods until no more edges can be added. As shown in [65], $cl(G)$ is uniquely determined by G, and $cl(G)$ is the line graph of a triangle-free graph. Moreover, in [65] it is shown that Theorem 25 has the following consequences. Let $c(G)$ denote the *circumference* of G, i.e. the length of a longest cycle of G. A *factor* of G is a spanning subgraph of G.

Theorem 26 (Ryjáček [65])
Let G be a claw-free graph. Then

(i) $c(cl(G)) = c(G)$.

(ii) If $cl(G)$ is complete, then G is hamiltonian.

(iii) Every nonhamiltonian claw-free graph is a factor of a nonhamiltonian line graph.

Theorem 26(ii) implies the result of Oberly and Sumner [63] that every 2-connected locally-connected claw-free graph is hamiltonian. Theorem 26(iii) together with a result of Zhan [76] and, independently, Jackson [43] implies that every 7-connected claw-free graph is hamiltonian, showing that Conjecture 16 is true for claw-free graphs. Slightly more general results on 6-connected claw-free graphs with some additional conditions were obtained by Fan [28] and Li [56]. Moreover Theorem 26(iii) yields the mentioned equivalence of Conjecture 23 and Conjecture 24.

4.2 On factors in 4-connected claw-free graphs

In this section we give several recent results concerning the existence of certain factors in 4-connected claw-free graphs that were obtained in [19].

First of all it has been shown there that Conjecture 24 holds within the subclass of *hourglass-free* graphs, i.e. graphs that do not contain an induced subgraph isomorphic to the *hourglass*, a graph consisting of two triangles meeting in exactly one vertex. This result also follows from a recent result due to Kriesell [53]. To obtain this result, in [19] the following observation made by several graph theorists is proved. The *inflation* of a graph G

is the graph obtained from G by replacing all vertices $v_1, v_2, \ldots, v_n$ of G by disjoint complete graphs on $d(v_i)$ vertices $v_{i,1}, v_{i,2}, \ldots v_{i,d(v_i)}$, and all edges v_iv_j by disjoint edges $v_{i,p}v_{j,q}$ $(i,j \in \{1,\ldots,n\}; p \in \{1,\ldots,d(v_i)\}; q \in \{1,\ldots,d(v_j)\})$. We use the term *inflation* for a graph that is isomorphic to the inflation of some graph. It is obvious that inflations are claw-free and hourglass-free.

Lemma 27 (Broersma, Kriesell, and Ryjáček [19])
Every 4-connected inflation is hamiltonian.

The connectivity bound in Lemma 27 cannot be decreased, since there are nonhamiltonian 3-connected inflations, e.g. the inflation of the Petersen graph. These graphs also show that the connectivity bound in the next result is best possible.

Theorem 28 (Broersma, Kriesell, and Ryjáček [19])
Every 4-connected claw-free hourglass-free graph is hamiltonian.

Furthermore the validity of a weaker form of Conjecture 24 has been proved in [19].

By Theorem 3.1 in [47], every connected claw-free graph has a 2-*walk*, i.e. a (closed) walk which passes every vertex at most twice. Clearly, the edges of a 2-walk induce a connected factor of maximum degree at most 4.

In [19] the following related result is proved.

Theorem 29 (Broersma, Kriesell, and Ryjáček [19])
Every 4-connected claw-free graph contains a connected factor which has degree two or four at each vertex.

By the results of [53] it is also possible to prove the related result that between every pair of distinct vertices in a 4-connected line graph there exists a spanning trail which passes every vertex at most twice.

Finally, it has been shown in [19] that Conjectures 23 and 24 are equivalent to seemingly weaker conjectures in which the conclusion is replaced by the conclusion that there exists a factor consisting of a bounded number of paths.

For convenience we use the term r-*path-factor* for a factor consisting of at most r paths. A *path-factor* is an r-path factor for some r, and its *endvertices* are the vertices of degree less than two of its components. Recall that n denotes the number of vertices of a graph.

Theorem 30 (Broersma, Kriesell, and Ryjáček [19])
Let $k \geq 2$ be an integer, and let $f(n)$ be a function of n with the property that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. Then the following statements are equivalent.

(1) Every k-connected claw-free graph is hamiltonian.

(2) *Every k-connected claw-free graph has an $f(n)$-path-factor.*

(3) *Every k-connected claw-free graph has a 2-factor with at most $f(n)$ components.*

(4) *Every k-connected claw-free graph has a spanning tree with at most $f(n)$ vertices of degree one.*

(5) *Every k-connected claw-free graph on n vertices has a path of length at least $n - f(n)$.*

In particular Theorem 30 shows that Conjecture 24 is true if one could show that, e.g., every 4-connected claw-free graph admits a factor consisting of a number of paths which is sublinear in n.

Recently, in [41] it has been shown that a claw-free graph G has an r-path-factor if and only if $cl(G)$ has an r-path-factor. Similarly, in [66] it has been shown that a claw-free graph G has a 2-factor with at most r components if and only if $cl(G)$ has such a 2-factor. These results immediately imply the equivalence of the following statements related to Conjecture 23.

Theorem 31 (Broersma, Kriesell, and Ryjáček [19])
Let $k \geq 2$ be an integer, and let $f(n)$ be a function of n with the property that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. Then the following statements are equivalent.

(1) *Every k-connected line graph is hamiltonian.*

(2) *Every k-connected line graph has an $f(n)$-path-factor.*

(3) *Every k-connected line graph has a 2-factor with at most $f(n)$ components.*

In particular Theorem 31 shows that Conjecture 23 is true if one could show that, e.g., every 4-connected line graph admits a 2-factor consisting of a number of components which is sublinear in n. The equivalences between (1) and (2) of Theorem 30 and of Theorem 31 appear also in a sequence of equivalences in [50].

4.3 Back to degree conditions for hamiltonicity

If $G = cl(G)$, then we say that the graph G is *closed* (thus, G is closed if and only if G is the line graph of a triangle-free graph). Using the structural properties of closed claw-free graphs, it is possible to prove that a nonhamiltonian closed claw-free graph with large degrees can be covered by relatively few cliques ([31]). Denote by $\theta(G)$ the clique covering number of the graph G, and denote by $\sigma_k(G)$ the minimum degree sum of a set of k distinct mutually nonadjacent vertices of G (or ∞ if such a set does not exist).

Theorem 32 (Favaron, Flandrin, Li, and Ryjáček [31])
Let $k \geq 4$ be an integer and let G be a 2-connected claw-free graph with $n \geq 3k^2 - k - 4$, $\delta(G) \geq 3k - 1$ and $\sigma_k(G) > n + (k-2)^2$. Then either $\theta(cl(G)) \leq k - 1$ or G is hamiltonian.

Specifically, Theorem 32 implies that, for any integer $k \geq 4$, every nonhamiltonian claw-free graph G with $n \geq 3k^2 - k - 4$ and $\delta(G) > \frac{n+(k-2)^2}{k}$ can be covered by at most $k-1$ cliques. This implies that for proving a minimum degree condition for hamiltonicity of type $\delta(G) > \frac{n}{k} + c$ for any given $k \geq 4$, it is enough to list all nonhamiltonian closed claw-free graphs with $\theta(G) \leq k - 1$.

A characterization of closed nonhamiltonian claw-free graphs with small clique covering number can be achieved by using the correspondence between the graphs and their line graph preimages. The following was proved for $\theta \leq 5$ in [31] and independently by Kuipers and Veldman in [54]. We refer to [31] for a definition of the classes of graphs contained in $\mathcal{F}'$.

Theorem 33 (Favaron, Flandrin, Li, and Ryjáček [31], Kuipers and Veldman [54])
Let G be a 2-connected closed claw-free graph.

(i) If $\theta(G) \leq 2$, then G is hamiltonian.

(ii) If $3 \leq \theta(G) \leq 5$, then either G is hamiltonian or G is a spanning subgraph of a graph from $\mathcal{F}'$.

Combining Theorems 32 and 33 one can obtain the following result.

Corollary 34 (Favaron, Flandrin, Li, and Ryjáček [31])
Let G be a 2-connected claw-free graph with $n \geq 77$ vertices such that $\delta(G) \geq 14$ and $\sigma_6(G) > n + 19$. Then either G is hamiltonian or G is a spanning subgraph of a graph from $\mathcal{F}'$.

All the nonhamiltonian exceptional graphs have connectivity 2 and hence, under the assumptions of Corollary 34, 3-connectedness implies hamiltonicity.

Presently, the best sufficient minimum degree condition for hamiltonicity of 3-connected claw-free graphs we are aware of is due to Favaron and Fraisse [30]; using the claw-free closure and a relationship between properties of cubic graphs and line graphs that will be explained in the next section, they proved that $\delta(G) \geq (n+38)/10$ suffices. This is essentially best possible.

Kuipers and Veldman [54] further exploited the fact that the basic idea of finding the exceptional classes of $\mathcal{F}'$ yields a general method for listing these classes for any fixed upper bound on $\theta(G)$. This was a starting point for the proof of the following result. Consider the following two problems.

HAM(c)
Instance: A graph G with $\delta(G) \geq cn$.
Question: Is G hamiltonian?

HAMCL(c)
Instance: A claw-free graph G with $\delta(G) \geq cn$.
Question: Is G hamiltonian?

Häggkvist [35] proved that HAM($\frac{1}{2} - \varepsilon$) is NP-complete for any fixed $\varepsilon > 0$ (while HAM($\frac{1}{2}$) is trivial by Dirac's Theorem). In claw-free graphs, hamiltonicity is known to be NP-complete [9]. In contrast to these results, the surprizing result in [54] says that HAMCL(c) is polynomial for any $c > 0$.

Theorem 35 (Kuipers and Veldman [54])
HAMCL(c) is solvable in polynomial time for any constant $c > 0$.

The proof of this result in [54] is a clever combination of reduction techniques. Apart from the claw-free closure which opens the possibility to turn to line graphs and their preimages, the key ingredient is a variant of a powerful reduction technique introduced by Catlin in [22] and refined by Veldman in [72]. These techniques are extremely useful if one is interested in the existence of spanning eulerian subgraphs or eulerian subgraphs that contain at least one endvertex of every edge of the graph, respectively. We show in the next section why such subgraphs are relevant in this context.

4.4 A relationship with properties of cubic graphs

We return to regular graphs in this section, so we are back at the start of our exposition. Despite this, the results and conjectures mentioned below have nothing in common with the former section on regular graphs because, in contrast to the high degrees assumed there, the degree of regularity in this section is just three.

In the sequel we will focus on cyclic and other properties of cubic (i.e. 3-regular) graphs, and show their close relationship with results and conjectures on line graphs and claw-free graphs.

4.4.1 Cyclically – and essentially k-edge-connected graphs

We start with some additional terminology and basic facts.

A graph G is *cyclically k-edge-connected* if there exists no subset E' of $E(G)$ such that $|E'| < k$ and $G - E'$ has at least two components containing cycles. A graph G is *essentially k-edge-connected* if $|E(G)| \geq k + 1$ and there exists no subset E' of $E(G)$ such that $|E'| < k$ and $G - E'$ has at least two components containing edges.

It is easy to check that the line graph $L(G)$ of a graph G is k-connected if and only if G is essentially k-edge-connected, and that a cubic graph is cyclically 4-edge-connected if and only if it is essentially 4-edge-connected.

A subgraph H of a graph G is *dominating* if every edge of G has at least one end in H.

The following basic result relates the hamiltonicity of a line graph to the existence of a dominating closed trail in its preimage.

Theorem 36 (Harary and Nash-Williams [36])
Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.

Some other auxiliary results are related to the existence of edge-disjoint spanning trees, and to their implication for the existence of spanning eulerian subgraphs.

Theorem 37 (Nash-Williams [62], Tutte [70])
A graph G has k edge-disjoint spanning trees if and only if for every partition $\mathcal{P}$ of $V(G)$ we have $\varepsilon(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$, where $\varepsilon(\mathcal{P})$ counts the number of edges of G joining distinct parts of $\mathcal{P}$.

Theorem 38 (Kundu [55])
Every 4-edge-connected graph has two edge-disjoint spanning trees.

Theorem 39 (Jaeger [48])
Every graph with two edge-disjoint spanning trees has a spanning eulerian subgraph.

Combining the above results, we immediately obtain the next corollary.

Corollary 40

(i) Every 4-edge-connected graph has a spanning eulerian subgraph.

(ii) Every 4-edge-connected graph has a hamiltonian line graph.

On the other hand, it is not difficult to show that Conjecture 23 is equivalent to the following conjecture.

Conjecture 41
Every essentially 4-edge-connected graph has a hamiltonian line graph.

At first sight the gap between Corollary 40(ii) and Conjecture 41 does not look that large. Moreover in Corollary 40(i) we obtain a spanning eulerian subgraph, whereas we would only need a dominating eulerian subgraph in order to prove Conjecture 41. Nevertheless Conjecture 41 seems to be very hard.

The next conjecture, that would clearly imply Conjecture 41, was put up by Jackson in [44]. It resembles the way one can prove that 4-connected planar graphs are hamiltonian by proving assertions on the existence of certain cycles in 2-connected planar graphs.

Conjecture 42 (Jackson [44])
Every 2-edge-connected graph G has an eulerian subgraph H with at least three edges such that each component of $G - V(H)$ is linked by at most three edges to H.

4.4.2 Cubic graphs

We now turn our attention to related conjectures and results for cubic graphs. The first conjecture is due to Fleischner and Jackson [33] who showed that this conjecture is equivalent to Conjecture 23.

Conjecture 43 (Fleischner and Jackson [33])
Every cyclically 4-edge-connected cubic graph has a dominating cycle.

The main ingredients and observations used to prove the equivalence are sketched here. First use Theorem 36 and the correspondence between 4-connected line graphs and their essentially 4-edge-connected preimages. Secondly, note that one can transform an essentially 4-edge-connected graph into such a graph with minimum degree at least three by deleting the vertices of degree one and suppressing the vertices of degree two. Another transformation can be used to turn the new graph into a cyclically 4-edge-connected cubic graph: replace a vertex v of degree $d(v) \geq 4$ by a cycle $C_{d(v)}$ and the edges incident with v by edges incident with one vertex of $C_{d(v)}$ each; repeat this for all vertices of degree at least four. We omit the details.

In [33] the following related conjectures are presented.

Conjecture 44 (Jaeger; see [33])
Every cyclically 4-edge-connected cubic graph G has a cycle C such that $G - V(C)$ is acyclic.

Conjecture 45 (Bondy; see [33])
Every cyclically 4-edge-connected cubic graph has a cycle of length at least cn, for some constant c.

It is not difficult to show that Conjecture 43 implies Conjecture 44, while the latter one implies Conjecture 45.

The following related problem is mentioned in [68]: Does there exist a natural number m such that every cyclically m-edge-connected cubic graph contains a Hamilton cycle? The Coxeter graph shows that m must be at least 8.

The arguments used to prove the equivalence of Conjecture 23 and Conjecture 43 can be combined with the claw-free closure operation to obtain results on cycles in claw-free graphs from results on cycles in cubic graphs. As an example consider the following two results.

A graph G is *k-cyclable* if every set of k vertices of G is contained in a cycle of G.

Theorem 46 (Holton, McKay, Plummer, and Thomassen [40])
Every 3-connected cubic graph is 9-cyclable.

Theorem 47 (Jackson [45])
Every 3-connected claw-free graph is 9-cyclable.

Theorem 46 and Theorem 47 are both best possible, as shown by the inflation of the Petersen graph, which is not 10-cyclable.

The result of Favaron and Fraisse [30] mentioned in Section 4.3 is another example of applying these techniques. Combining these techniques with the reduction methods of Catlin [22] and Veldman [72] gives an opportunity to obtain long cycle results for claw-free graphs, as is done by Broersma and Van der Laag [20].

4.4.3 A possible approach to solving the conjectures

We finish this part on claw-free graphs with an approach to solving the main conjectures proposed by Jackson [45], and some remarks.

Conjecture 48 (Jackson [45])
Every essentially 6(or 5 or 4)-edge-connected graph has an eulerian subgraph containing all vertices of degree at least 4.

By Corollary 40(i) the conclusion of Conjecture 48 holds for 4-edge-connected graphs.

If Conjecture 48 is true, then every 6(or 5 or 4)-connected line graph is hamiltonian. It suffices to prove Conjecture 48 for graphs with minimum degree at least 3.

If a graph G contains two edge-disjoint trees T_1 and T_2 such that T_1 is spanning and T_2 contains all vertices of degree at least 4 in G, then G has an eulerian subgraph containing all vertices of degree at least 4 (similarly proved as Theorem 39).

We close this section with the following remark that has been made by Van den Heuvel [38]. Let $\mathcal{B}$ be the class of all connected bipartite graphs such that the vertices in one color class all have degree 3 and the vertices in the other color class degree 4. The graphs in $\mathcal{B}$ do not satisfy the above hypothesis concerning the two edge-disjoint trees. The class $\mathcal{B}$ may contain essentially 5-edge-connected graphs. If so, an approach of Conjecture 48 via this method could be successful only for essentially 6-edge-connected graphs.

5 Conclusion

We tried to give a flavour of the many open problems, conjectures and recent developments around three themes in hamiltonian graph theory. We hope this inspires and motivates graph theorists to work on these intriguing problems.

References

[1] D. Bauer, H.J. Broersma, J. van den Heuvel, and H.J. Veldman, *On hamiltonian properties of 2-tough graphs.* J. Graph Theory 18 (1994) 539–543.

[2] D. Bauer, H.J. Broersma, J. van den Heuvel, and H.J. Veldman, *Long cycles in graphs with prescribed toughness and minimum degree*. Discrete Math. 141 (1995) 1–10.

[3] D. Bauer, H.J. Broersma, and H.J. Veldman, *Not every 2-tough graph is hamiltonian*. Discrete Applied Math., to appear.

[4] D. Bauer, G.Y. Katona, D. Kratsch, and H.J. Veldman, *Chordality and 2-factors in tough graphs*. Discrete Applied Math., to appear.

[5] D. Bauer and E. Schmeichel, *Toughness, minimum degree, and the existence of 2-factors*. J. Graph Theory 18 (1994) 241–256.

[6] K. Baxter, *The Hopping Lemma*. MSc. Thesis, University of Waterloo (1992).

[7] J.C. Bermond, *Hamiltonian graphs*. In: Selected Topics in Graph Theory (L. Beineke and R.J. Wilson, Eds.). Academic Press, London and New York (1978) 127–167.

[8] J.C. Bermond and C. Thomassen, *Cycles in digraphs – a survey*. J. Graph Theory 5 (1981) 1–43.

[9] A.A. Bertossi, *The edge hamiltonian path problem is NP-complete*. Inform. Process. Lett. 13 (1981) 157–159.

[10] T. Böhme, J. Harant, and M. Tkáč, *More than one tough chordal planar graphs are hamiltonian*. J. Graph Theory, to appear.

[11] J.A. Bondy, *Hamilton cycles in graphs and digraphs*. Congr. Numer. 21 (1978) 3–28.

[12] J.A. Bondy, *Basic graph theory*. In: Handbook of Combinatorics (M. Grötschel, L. Lovász, and R.L. Graham, Eds.). North-Holland, Amsterdam (1995) 3–110.

[13] J.A. Bondy and V. Chvátal, *A method in graph theory*. Discrete Math. 15 (1976) 111–135.

[14] J.A. Bondy and M. Kouider, *Hamilton cycles in regular 2-connected graphs*. J. Combin. Theory B 44 (1988) 177–186.

[15] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*. Macmillan, London and Elsevier, New York (1976).

[16] H.J. Broersma, J. van den Heuvel, B. Jackson, and H.J. Veldman, *Hamiltonicity of regular 2-connected graphs*. J. Graph Theory 22 (1996) 105–124.

[17] H.J. Broersma, J. van den Heuvel, H.A. Jung, and H.J. Veldman, *Cycles containing all vertices of maximum degree*. J. Graph Theory 17 (1993) 373–385.

[18] H.J. Broersma, J. van den Heuvel, and H.J. Veldman, *A generalization of Ore's Theorem involving neighborhood unions*. Discrete Math. 122 (1993) 37–49.

[19] H.J. Broersma, M. Kriesell, and Z. Ryjáček, *On factors of 4-connected claw-free graphs*. Preprint (1999).

[20] H.J. Broersma and S. van der Laag, *Long cycles in 2-connected and 3-connected claw-free graphs*. Working paper.

[21] H.J. Broersma, Z. Ryjáček, and I. Schiermeyer, *Closure concepts – a survey*. Graphs and Combinatorics, to appear.

[22] P.A. Catlin, *A reduction method to find spanning eulerian subgraphs*. J. Graph Theory 12 (1988) 29–44.

[23] G. Chen, M.S. Jacobson, A. Kézdy, and J. Lehel, *Tough enough chordal graphs are hamiltonian*. Networks 31 (1998) 29–38.

[24] V. Chvátal, *Tough graphs and hamiltonian circuits*. Discrete Math. 5 (1973) 215–228.

[25] V. Chvátal, *Hamiltonian cycles*. Chapter 11 in: The Traveling Salesman Problem (E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, and D.B. Shmoys, Eds.). John Wiley & Sons Ltd (1985).

[26] G.A. Dirac, *Some theorems on abstract graphs*. Proc. London Math. Soc. (3) 2 (1952) 69–81.

[27] H. Enomoto, B. Jackson, P. Katerinis, and A. Saito, *Toughness and the existence of k-factors*. J. Graph Theory 9 (1985) 87–95.

[28] G. Fan, Personal communication.

[29] R.J. Faudree, E. Flandrin, and Z. Ryjáček, *Claw-free graphs - a survey*. Discrete Math. 164 (1997) 87–147.

[30] O. Favaron and P. Fraisse, *Hamiltonicity and minimum degree in 3-connected claw-free graphs*. Preprint (1999).

[31] O. Favaron, E. Flandrin, H. Li, and Z. Ryjáček, *Clique covering and degree conditions for hamiltonicity in claw-free graphs*. Preprint (1997).

[32] H. Fleischner, *The square of every 2-connected graph is hamiltonian*. J. Combin. Theory (B) 16 (1974) 29–34.

[33] H. Fleischner and B. Jackson, *A note concerning some conjectures on cyclically 4-edge connected 3-regular graphs*. Annals of Discrete Math. 41 (1989) 171–178.

[34] R.J. Gould, *Updating the hamiltonian problem – a survey.* J. Graph Theory 15 (1991) 121–157.

[35] R. Häggkvist, *On the structure of non-hamiltonian graphs I.* Combin. Probab. Comput. 1 (1992) 27–34.

[36] F. Harary and C.St.J.A. Nash-Williams, *On eulerian and hamiltonian graphs and line graphs.* Canad. Math. Bull. 8 (1965) 701–710.

[37] J. van den Heuvel, *Degree and toughness conditions for cycles in graphs.* Ph.D. Thesis, University of Twente (1993).

[38] J. van den Heuvel, Personal communication.

[39] F. Hilbig, *Kantenstrukturen in nichthamiltonschen Graphen.* Ph.D. Thesis, Technische Universität Berlin (1986).

[40] D.A. Holton, B.D. McKay, M.D. Plummer, and C. Thomassen, *A nine point theorem for 3-connected graphs.* Combinatorica 2 (1982) 53–62.

[41] S. Ishizuka, *Closure, path factors and path coverings in claw-free graphs.* Preprint (1998).

[42] B. Jackson, *Hamilton cycles in regular 2-connected graphs.* J. Combin. Theory B 29 (1980) 27–46.

[43] B. Jackson, *Hamilton cycles in 7-connected line graphs.* Preprint (1989).

[44] B. Jackson, *Concerning the circumference of certain families of graphs.* In: Updated contributions to the Twente Workshop on Hamiltonian Graph Theory (H.J. Broersma, J. van den Heuvel, and H.J. Veldman, Eds.). Memorandum No. 1076 (University of Twente, Enschede, 1992) 87–94.

[45] B. Jackson, Personal communication.

[46] B. Jackson, H. Li, and Y. Zhu, *Dominating cycles in regular 3-connected graphs.* Discrete Math. 102 (1991) 163–176.

[47] B. Jackson and N.C. Wormald, *k-Walks of graphs.* Australasian Journal of Combinatorics 2 (1990) 135–146.

[48] F. Jaeger, *A note on subeulerian graphs.* J. Graph Theory 3 (1979) 91–93.

[49] H.A. Jung, *On maximal circuits in finite graphs.* Annals of Discrete Math. 3 (1987) 129–144.

[50] M. Kochol, *Sublinear defect principle in graph theory.* Manuscript (1999).

[51] D. Kratsch, Personal communication.

[52] D. Kratsch, J. Lehel, and H. Müller, *Toughness, hamiltonicity and split graphs.* Discrete Math. 150 (1996) 231–245.

[53] M. Kriesell, *All 4-connected line graphs of claw-free graphs are hamiltonian-connected.* Preprint (1998).

[54] E.J. Kuipers and H.J. Veldman, *Recognizing claw-free hamiltonian graphs with large minimum degree.* Preprint (1998).

[55] S. Kundu, *Bounds on the number of disjoint spanning trees.* J. Combin. Theory B 17 (1974) 199–203.

[56] H. Li, *A note on hamiltonian claw-free graphs.* Rapport de Recherche 1022, Univ. Paris-Sud, Orsay, France, 1996.

[57] X. Liu and D. Wang, *A new generalization of Ore's Theorem involving neighborhood unions.* Systems Sci. Math. Sci. 9 (1996) 182–192.

[58] X. Liu, L. Zhang, and Y. Zhu, *Distance, neighborhood unions and hamiltonian properties in graphs.* In: Combinatorics, Graph Theory, Algorithms and Applications (Y. Alavi, D.R. Lick, and J. Liu, Eds.). Proceedings of the Third China-USA Int. Conf. (Beijing, June 1–5, 1993). World Scientific Publ. Co., Inc., River Edge, NJ (1994) 255–268.

[59] M.M. Matthews and D.P. Sumner, *Hamiltonian results in $K_{1,3}$-free graphs.* J. Graph Theory 8 (1984) 139–146.

[60] M.M. Matthews and D.P. Sumner, *Longest paths and cycles in $K_{1,3}$-free graphs.* J. Graph Theory 9 (1985) 269–277.

[61] Min Aung, *Circumference of a regular graph.* J. Graph Theory 13 (1989) 149–155.

[62] C.St.J.A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs.* J. London Math. Soc. 36 (1961) 445–450.

[63] D.J. Oberly and D.P. Sumner, *Every connected, locally connected nontrivial graph with no induced claw is hamiltonian.* J. Graph Theory 3 (1979) 351–356.

[64] O. Ore, *Note on hamiltonian circuits.* Amer. Math. Monthly 67 (1960), 55.

[65] Z. Ryjáček, *On a closure concept in claw-free graphs.* J. Combin. Theory B 70 (1997) 217–224.

[66] Z. Ryjáček, A. Saito, and R.H. Schelp, *Closure, 2-factors and cycle coverings in claw-free graphs.* J. Graph Theory, to appear.

[67] C. Thomassen, *Reflections on graph theory.* J. Graph Theory 10 (1986) 309–324.

[68] C. Thomassen, *On the number of hamiltonian cycles in bipartite graphs.* Combin. Probab. Comput. 5 (1996) 437–442.

[69] W.T. Tutte, *A theorem on planar graphs.* Trans. Amer. Math. Soc. 82 (1956) 99–116.

[70] W.T. Tutte, *On the problem of decomposing a graph into n connected factors.* J. London Math. Soc. 36 (1961) 221–230.

[71] H.J. Veldman, *Existence of D_λ-cycles and D_λ-paths.* Discrete Math. 44 (1983) 309–316.

[72] H.J. Veldman, *On dominating and spanning circuits in graphs.* Discrete Math. 124 (1994) 229–239.

[73] M.E. Watkins and D.M. Mesner, *Cycles and connectivity in graphs.* Can. J. Math. 19 (1967) 1319–1328.

[74] D.R. Woodall, *The binding number of a graph and its Anderson number.* J. Combin. Theory B 15 (1973) 225–255.

[75] S. Zhan, *Hamiltonian connectedness of line graphs.* Ars Combinatoria 22 (1986) 89–95.

[76] S. Zhan, *On hamiltonian line graphs and connectivity.* Discrete Math. 89 (1991) 89–95.

[77] Y. Zhu and H. Li, *Hamilton cycles in regular 3-connected graphs.* Discrete Math. 110 (1992) 229–249.

[78] Y. Zhu, Z. Liu, and Z. Yu, *2-Connected k-regular graphs on at most $3k+3$ vertices to be hamiltonian.* J. Sys. Sci. & Math. Sci. 6 (1) (1986) 36–49 and (2) (1986) 136–145.