Faculty of Mathematical Sciences

University of Twente

University for Technical and Social Sciences

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl

Memorandum No. 1507

On minimum degree conditions for supereulerian graphs

H.J. Broersma and Liming Xiong[1]

December 1999

ISSN 0169-2690

[1]Department of Mathematics, Jiangxi Normal University, Nanchang, 330027, P.R. of China

On minimum degree conditions for supereulerian graphs

H.J. Broersma

Faculty of Mathematical Sciences
University of Twente
P.O. Box 217
7500AE Enschede, The Netherlands
e-mail: h.j.broersma@math.utwente.nl

*Liming Xiong**

Department of Mathematics
Jiangxi Normal University
Nanchang, 330027
P.R. of China

September 17, 1999

Abstract

A graph is called supereulerian if it has a spanning closed trail. Let G be a 2-edge-connected graph of order n such that each minimal edge cut $E \subseteq E(G)$ with $|E| \leq 3$ satisfies the property that each component of $G - E$ has order at least $(n-2)/5$. We prove that either G is supereulerian or G belongs to one of two classes of exceptional graphs. Our results slightly improve earlier results of Catlin and Li. Furthermore our main result implies the following strengthening of a theorem of Lai within the class of graphs with minimum degree $\delta \geq 4$: If G is a 2-edge-connected graph of order n with $\delta(G) \geq 4$ such that for every edge $xy \in E(G)$, we have $\max\{d(x), d(y)\} \geq \frac{n-2}{5} - 1$, then either G is supereulerian or G belongs to one of two classes of exceptional graphs. We show that the condition $\delta(G) \geq 4$ cannot be relaxed.

Keywords: supereulerian graph, spanning circuit, degree conditions, collapsible graph

AMS Subject Classifications (1991): 05C45, 05C35

---

*This research has been supported by the Natural Science Fund of Jiangxi Province, and was performed while the second author visited the University of Twente. He thanks prof.dr. C. Hoede for his kind hospitality.

1 Introduction

We use [2] for terminology and notation not defined here and consider finite loopless graphs only. Let G be a graph. We use $\lambda(G)$, $\Delta(G)$ and $\delta(G)$ to denote the edge-connectivity, the maximum degree and the minimum degree of G, respectively. If $E(G) \neq \emptyset$, then the edge degree of G, denoted by $\overline{\sigma}_2(G)$, is defined as $\min\{d(x) + d(y) \mid xy \in E(G)\}$. Let $O(G)$ denote the set of all vertices of G with odd degrees. An eulerian graph is a connected graph G with $O(G) = \emptyset$ (hence K_1 is an eulerian graph). A graph is called supereulerian if it has a spanning eulerian subgraph. A subgraph H of a graph G is dominating if $G - V(H)$ is edgeless, i.e. if every edge of G is incident with at least one vertex of H.

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. There is a close relationship between dominating eulerian subgraph in G and hamiltonian cycles in $L(G)$.

Theorem 1. [10] Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.

Various sufficient conditions for the existence of supereulerian graphs and dominating eulerian subgraph in terms of $\overline{\sigma}_2(G)$ have been derived (See, e.g. [1],[6]–[9]).
From Theorem 1 one easily sees that a supereulerian graph has a hamiltonian line graph. Simple examples show that not every graph with a hamiltonian line graph is supereulerian [5]. Veldman proved the following which is conjectured in [1]. Here $D_1(G)$ denotes the set of vertices of G with degree one.

Theorem 2. [13] If G is a simple graph of order n with $\lambda(G - D_1(G)) \geq 2$ and if

$$\overline{\sigma}(G) > \frac{2}{5}n - 2, \tag{1.1}$$

then for n sufficiently large, $L(G)$ is hamiltonian.

If (1.1) holds, then we have

$$\min\{\max\{d(x), d(y)\} \mid xy \in E(G)\} > \frac{1}{5}n - 1. \tag{1.2}$$

Therefore, it is natural to consider whether (1.1) can be replaced by (1.2). Lai investigated this problem. He obtained the following result with a slightly better lower bound.

Theorem 3. [12] If G is a simple graph of order n with $\lambda(G - D_1(G)) \geq 2$ and if $\min\{\max\{d(x), d(y) \mid xy \in E(G)\} \geq \frac{n}{5} - 1$, then for n sufficiently large, $L(G)$ is hamiltonian unless G is in a class of well-characterized graphs.

It is in its turn natural to investigate whether the minimum degree condition in the above theorem (combined with other conditions, e.g. $\lambda(G) \geq 2$) guarantees a spanning eulerian subgraph in G instead of a dominating eulerian subgraph. This is indeed the case. We show that in fact a slightly weaker condition is sufficient for 2-edge-connected graphs with minimum degree at least four to be supereulerian, with again some exceptional classes.

Theorem 4. Let G be a simple graph with $\lambda(G) \geq 2$ and with $n > 12$ vertices. If $\delta(G) \geq 4$ and if

$$\min\{\max\{d(x), d(y)\} \mid xy \in E(G)\} \geq \frac{n-2}{5} - 1, \tag{1.3}$$

then exactly one of the following holds:

(a) G is supereulerian;

(b) The reduction G' of G is isomorphic to $K_{2,5}$ such that each pre-image of a vertex with degree 2 in G' has exactly order $(n-2)/5$ in G;

(c) The reduction G' of G is isomorphic to $K_{2,3}$ such that each vertex of G' corresponds to a pre-image in G of order at least $\frac{n-2}{5}$.

Before we present a proof of this result as well as related results, we have to define what we mean with the reduction of a graph G. For this purpose we give a short description of Catlin's reduction method in Section 2. We present our results and proofs in Section 3. Our main result (Theorem 8 in Section 3) implies several known and new results on dominating eulerian subgraphs and supereulerian graphs of minimum degree at least four. The proofs are similar to the proofs of Catlin and Li in [8]. In Section 4 we show that we cannot relax our lower bound four on the minimum degree in the above result.

2 Catlin's reduction method

Let G be a graph and let H be a connected subgraph of G. G/H denotes the graph obtained from G by contracting H, i.e. by replacing H by a vertex v_H such that the number of edges in G/H joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v in G to H. A graph G is contractible to a graph G' if G contains pairwise vertex-disjoint connected subgraph $H_1, \ldots, H_k$ with $\bigcup_{i=1}^{k} V(H_i) = V(G)$ such that G' is obtained from G by successively contracting $H_1, H_2, \ldots, H_k$. The subgraph H_i of G is called the pre-image of the vertex v_{H_i} of G'; the vertex v_{H_i} is called trivial if H_i contains precisely one vertex ($i = 1, 2, \ldots, k$). A graph is collapsible if for every even subset X of $V(G)$ there exists a spanning connected subgraph G_X of G such that $X = O(G_X)$. In particular, K_1 is collapsible. Note that any collapsible graph G is supereulerian since $\emptyset$ is an even subset of $V(G)$. Catlin [7] showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \dots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$. The reduction of G is the graph obtained from G by successively contracting $H_1, H_2, \dots, H_k$. A graph is reduced if it is the reduction of some graph.

The following results from [6] and [7] are necessary for the proofs of our results.

Theorem 5. [7] Let G be a connected graph and let G' be the reduction of G. Then G is supereulerian if and only if G' is supereulerian.

Theorem 6. [7] Let G be a connected graph. Then each of the following holds:

(a) G is reduced if and only if G contains no nontrivial collapsible subgraphs.

(b) If G is reduced, then every subgraph of G is reduced.

Theorem 7. [6] Let G be a nontrivial graph and let $V_3 = \{v \in V(G) \mid d(v) \leq 3\}$. If G is a reduced graph, then each of the following holds:

(a) G is a simple graph.

(b) G has no cycle of length less than four.

(c) If $\lambda(G) \geq 2$, then either $|V_3| = 4$ and G is eulerian or $|V_3| \geq 5$.

3 Main results and its consequences

Using Catlin's reduction method, we now prove our main result.
In the sequel a bond of a graph G is a minimal edge cut of G.

Theorem 8. Let G be a simple graph of order n with $\lambda(G) \geq 2$. If for every bond $E \subseteq E(G)$ with $|E| \leq 3$ we have that every component of $G - E$ has order at least $(n-2)/5 > 2$, then exactly one of the following holds:

(a) G is supereulerian.

(b) The reduction G' of G is isomorphic to $K_{2,5}$ such that each pre-image of a vertex with degree 2 in G' has exactly order $(n-2)/5$ in G.

(c) The reduction G' of G is isomorphic to $K_{2,3}$ such that each vertex of G' corresponds to a pre-image in G with order at least $(n-2)/5$.

Proof. Let G' be the reduction of G. If $G' = K_1$, then G is supereulerian. Next suppose $G' \neq K_1$. Then G' is 2-edge-connected and nontrivial. By (c) of Theorem 7, it is sufficient to consider the case that $|V_3| \geq 5$. Let $v_1, v_2, \dots, v_5$ be vertices of $V(G')$ in V_3, i.e., $d(v_i) \leq 3$ for each i. The corresponding pre-images are $H_1, H_2, \dots, H_5$. Each H_i is joined to the rest of G by a bond consisting of $d(v_i) \leq 3$ edges. By the hypothesis of Theorem 8, $|V(H_i)| \geq (n-2)/5$ and so

$$n = |V(G)| \geq \sum_{i=1}^{5} |V(H_i)| \geq n - 2. \tag{3.1}$$

Hence

$$5 \leq |V(G')| \leq 7.$$

If $|V_3| \geq 6$, we would similarly obtain $n \geq \frac{6}{5}(n-2)$, hence $n \leq 10$, contradicting $\frac{6}{5}(n-2) > 12$. Hence $|V_3| = 5$.
We distinguish three cases to complete the proof.

Case 1. $|V(G')| = 5$.
By (b) of Theorem 7 and since $\lambda(G') \geq 2$, $\Delta(G') \leq 3$ and there exist at most two vertices of G' with degree three. G' cannot have exactly one vertex of degree three. Hence G' has exactly two vertices of degree three, and, by (b) of Theorem 7 and since $\lambda(G') \geq 2$, $G' = K_{2,3}$. In this case, G' satisfies (c) of Theorem 8.

Case 2. $|V(G')| = 6$.
Let $u \in V(G') \backslash \{v_1, v_2, \ldots, v_5\}$.
By (b) of Theorem 7 and since G' is 2-edge-connected, $d_{G'}(u) = 4$. Let $N_{G'}(u) = \{v_1, v_2, v_3, v_4\}$. Since $\delta \geq \lambda \geq 2$, we obtain that $v_1v_5, v_2v_5, v_3v_5, v_4v_5 \in E(G')$. Thus G' is eulerian. So G is supereulerian by Theorem 6.

Case 3. $|V(G')| = 7$.
Let $\{u, v\} = V(G') \backslash \{v_1, v_2, \ldots, v_5\}$. Clearly $d_{G'}(u) \geq 4$ and $d_{G'}(v) \geq 4$.

By (b) of Theorem 7 $uv \notin E(G')$. Hence

$$|N_{G'}(u) \cap N_{G'}(u)| \geq 3.$$

Since $\lambda(G') \geq 2$ and G' contains no 3-cycle,

$$|N_{G'}(u) \cap N_{G'}(v)| \neq 4.$$

We distinguish the following two subcases.

Subcase 3.1. $|N_{G'}(u) \cap N_{G'}(v)| = 3$.
Without loss of generality we assume that

$$N_{G'}(u) \cap N_{G'}(v) = \{v_1, v_2, v_3\}, \quad v_4 \in N_{G'}(u) \quad \text{and} \quad v_5 \in N_{G'}(v).$$

By (b) of Theorem 7, $v_4v_5 \in E(G')$. Hence G' is eulerian, implying that G is supereulerian by Theorem 6.

Subcase 3.2. $|N_{G'}(u) \cap N_{G'}(v)| = 5$.
By (b) of Theorem 7, $G' = K_{2,5}$. Now G' satisfies (b) of Theorem 8. This completes the proof of Theorem 8. □

Corollary 9. Let G be a simple graph of order n with $\lambda(G) \geq 2$. If for every bond $E \subseteq E(G)$ with $|E| \leq 3$ we have that every component of $G - E$ has order greater than $\frac{n}{5}$, then G is supereulerian.

Proof. Let G satisfy the hypothesis of Corollary 9. Then G satisfies the hypothesis of Theorem 8 and satisfies neither (b) nor (c) of Theorem 8. So G is supereulerian. □

Corollary 10. Let G be a simple graph with $\lambda(G) \geq 2$ and with $n > 12$ vertices. If $\delta(G) \geq 4$ and if

$$\min\{\max\{d(x), d(y)\} \mid xy \in E(G)\} \geq \frac{n-2}{5} - 1, \tag{3.2}$$

then G satisfies the conclusion of Theorem 8.

Proof. Let G be a graph satisfying the hypothesis of Corollary 10. It is sufficient to show that G satisfies the hypothesis of Theorem 8.
Let E be a bond of G with $|E| \leq 3$, and let G_1 and G_2 be the two components of $G - E$ with $|V(G_1)| \leq |V(G_2)|$. It is sufficient to prove that $|V(G_1)| \geq (n-2)/5$. Since $\delta(G) \geq 4$, G_1 has at least an edge, say uv, such that both of u, v are not incident with any of E. By (3.2),

$$|V(G_1)| \geq \max\{d(u), d(v)\} + 1 \geq \frac{n-2}{5} - 1 + 1 = \frac{n-2}{5}.$$

Thus G satisfies the hypothesis of Theorem 8. This completes the proof of Corollary 10. □

Obviously, Corollary 10 improves the following result (for graphs on more than 12 vertices).

Theorem 11. [8] Let G be a simple graph of order n with $\lambda(G) \geq 2$. If $\delta(G) \geq 4$ and if

$$\overline{\sigma}_2(G) \geq \frac{2n}{5} - 2,$$

then exactly one of the following holds:

(a) G is supereulerian.

(b) The reduction G' of G is isomorphic to $K_{2,3}$ such that each vertex of G' corresponds to a pre-image in G with order exactly $n/5$.

We present some other consequences of Theorem 8 and Corollary 10.

Corollary 12. Let G be a simple graph of order $n > 12$ with $\delta(G) \geq 4$. If $L(G)$ is 4-connected, then G is supereulerian.

Proof. One easily checks that G satisfies the hypothesis of Theorem 8 and it neither satisfies (b) nor (c) of Theorem 8. So G is supereulerian. □

Corollary 12 improves the next result by Jaeger since any line graph of a 4-edge-connected graph is 4-connected.

Corollary 13. [11] Every 4-edge-connected graph is supereulerian.

Corollary 14. [4] Let G be a 2-edge-connected simple graph of order $n > 20$. If $\delta(G) > \frac{n}{5} - 1$, then either G is supereulerian or the reduction of G is isomorphic to $K_{2,3}$, where every pre-image of the vertices of $K_{2,3}$ is either $K_{\frac{n}{5}}$ or $K_{\frac{n}{5}} - e$.

Proof. It follows directly from Corollary 10. □

4 Remarks

From our main results and its consequences, one may wonder whether the minimum degree condition $\delta(G) \geq 4$ is crucial or not for our conclusions. One might expect that the same conclusions hold without this restriction on the minimum degree. However there exist graphs with a large minimum degree that are not supereulerian. $K_{2,3}$ with the vertices replaced by large complete subgraphs is such an example that appears in Corollary 14. Corollary 12 supports the conjecture due to Thomassen that every 4-connected line graph is hamiltonian. More recently, Broersma, Kriesell and Ryjáček [3] have shown that this conjecture is equivalent to seemingly weaker conjectures in which the conclusion is replaced by the conclusion that there exists a spanning subgraph consisting of a bounded number of paths. From Corollary 12, one might expect that a stronger conjecture holds, namely that if the line graph of a graph G is 4-connected, then G is supereulerian. But this is not true: $K_{2,n-2}$ (with $n \geq 7$ odd) is an exception. Similarly, we have examples showing that the minimum degree restriction $\delta(G) \geq 4$ is necessary for results of the above type for supereulerian graphs (and hamiltonian line graphs).

Minimum degree three

The following result on graphs with minimum degree at least three has been obtained by Veldman.

Theorem 15. [13] Let G be a 2-edge-connected simple graph of order n such that $\delta(G) \geq 3$ and

$$\overline{\sigma}_2(G) \geq 2(\frac{1}{7}n - 1).$$

If n is sufficiently large, then either G is supereulerian or G is contractible to $K_{2,3}$.

Comparing the above result with Theorem 2, the condition of Theorem 15 is considerably weaker, but one has to exclude all graphs that are contractible to $K_{2,3}$. One might expect that the condition in (1.2) can be used instead, with the same exceptional graphs related to $K_{2,3}$. This is not the case: we can construct many other exceptional graphs. See Figure 1 for a class of examples. Here the black vertices represent large complete subgraphs, e.g. all isomorphic to $K_{\frac{n-2}{4}}$.

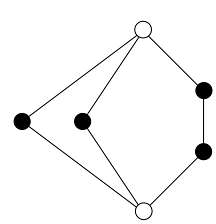

Figure 1: Not supereulerian, with minimum degree three.

Minimum degree two

Within the class of graphs with minimum degree at least four, Corollary 10 improves the following best possible results of Catlin [6].

Theorem 16. [6] Let G be a 2-edge-connected simple graph of order n such that

$$\overline{\sigma}_2(G) \geq \frac{2}{3}(n+1).$$

Then either G is supereulerian or $G = K_{2,n-2}$ and n is odd.

Without a restriction on the minimum degree, we can construct many graphs G with a large lower bound on $\min\{\max\{d(x), d(y)\} \mid xy \in E(G)\}$, but such that G is not supereulerian. In Figure 2 we give a class of examples and leave the others to the reader.

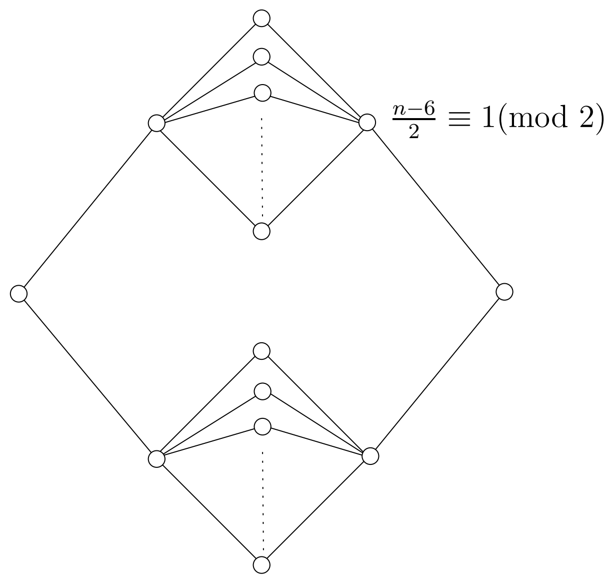

Figure 2: Not supereulerian, with minimum degree two.

References

[1] A. Benhocine, L. Clark, N. Köhler and H.J. Veldman, *On circuits and pancyclic line graphs*, J. Graph Theory 10 (1986) 411–425.

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, MacMillan, London and Elsevier, Amsterdam, 1976.

[3] H.J. Broersma, M. Kriesell and Z. Ryjáček, *On factors of 4-connected claw-free graphs*, preprint 1999.

[4] X.T. Cai, *A sufficient condition of minimum degree for a graph to have an s-circuit*, Chin. Ann. Math., 4A (1989), 117–122.

[5] P.A. Catlin, *Supereulerian graphs, a survey*, J. Graph Theory 16 (1992) 177–196.

[6] P.A. Catlin, *Spanning trails*, J. Graph Theory 11 (1987) 161–167.

[7] P.A. Catlin, *A reduction method to find spanning eulerian subgraphs*, J. Graph Theory 12 (1988) 29–45.

[8] P.A. Catlin and X.W. Li, *Supereulerian graphs of minimum degree at least 4*, Advances in Math. 29 (1999) 65–68.

[9] Z.-H. Chen, *Supereulerian graphs and the Petersen graph*, J. of Comb. Math. and Comb. Computing 9 (1991) 70–89.

[10] F. Harary and C.S.J.A. Nash-Williams, *On eulerian and hamiltonian graphs and line graphs*, Canad. Math. Bull. 8 (1965) 701–710.

[11] F. Jaeger, *A note on subeulerian graphs*, J. Graph Theory 3 (1979) 91–93.

[12] H.-J. Lai, *Eulerian subgraphs containing given vertices and hamiltonian line graphs*, Discrete Math. 178 (1998) 93–107.

[13] H.J. Veldman, *On dominating and spanning circuits in graphs*, Discrete Math. 124 (1994) 229–239.