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Abstract

For the Wiener class of matrix-valued functions we provide necessary and sufficient conditions for the existence of a J-spectral factorization. One of these conditions is in terms of equalizing vectors. A second one states that the existence of a J-spectral factorization is equivalent to the invertibility of the Toeplitz operator associated to the matrix to be factorized. Our proofs are simple and only use standard results of general factorization theory. Note that we do not use a state space representation of the system. However, we make the connection with the known results for the Pritchard-Salamon class of systems where an equivalent condition with the solvability of an algebraic Riccati equation is given. For Riesz-spectral systems another necessary and sufficient conditions for the existence of a J-spectral factorization in terms of the Hamiltonian is added.

Keywor ds: J-spectral factorization, H_∞-control, Wiener class, Toeplitz operator, Pritchard Salamon system, Popov function, Algebraic Riccati equation, Riesz-spectral system, Hamiltonian.

Mathematics Subject Classification: 93B36, 93C25, 47A68, 47B35.

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1 Introduction

Given a matrix-valued function Z defined on the imaginary axis, the J-spectral factorization problem is to find a stable invertible matrix-valued function V with a stable inverse such that

$$Z(s) = V^T(-s) \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} V(s) \text{ for all } s \text{ on the imaginary axis.}$$

J-spectral factorization is one approach for solving H_∞-control problems. For finite-dimensional systems this can be found in [11], [12], [15] and [16]. For the Wiener class of transfer matrices Curtain and Green [6] proved that the full H_∞-synthesis problem is equivalent to the solution of two J-spectral factorization. The sufficiency of this result was extended to the quotient field of H_∞ in [13].

Since the J-spectral factorization plays an essential role in H_∞-control, it is important to know when the matrix-valued function Z possesses such a factorization. Here we show that the existence of a J-spectral factorization is equivalent with the nonexistence of equalizing vectors. This result was obtained in [16] for finite-dimensional systems. There state space techniques are used. However, our proof is completely in frequency-domain and uses factorization results of [4] in an essential way.

For a matrix-valued function in the Wiener algebra we prove that the existence of a J-spectral factorization is equivalent to the invertibility of the associated Toeplitz operator. This generalizes the result obtained in [18], where it was done for exponentially stable Pritchard-Salamon class of systems. The proof there is very long and uses Riccati equations. Our proof is entirely done in frequency-domain. Although the transfer matrix of every system in this class lies in the Wiener algebra the converse is not true.

For the Riesz-spectral systems defined in [14] an equivalent condition for the existence of a J-spectral factorization in terms of the Hamiltonian of the system is given.

Section 2 introduces our notation and quote some general results. In Section 3 the main result is presented. It is shown that an invertible matrix-valued function Z in the Wiener algebra admits a J-spectral factorization if and only if Z has no equalizing vectors, or if and only if the associated Toeplitz operator is invertible. In Section 4 our result is reformulated for the exponentially stable Pritchard-Salamon systems. Also an equivalent condition with the solvability of an algebraic Riccati equation is given. The connection with the Riesz-spectral systems and the Hamiltonian is presented in Section 5.

2 Notation and preliminaries

In this section we quote some general results and introduce our notation. We begin with our class of stable systems. We say that $f \in \mathcal{A}$ if f has the representation

$$f(t) = \begin{cases} f_a(t) + f_0\delta(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where $f_0 \in \mathbb{C}$, $\int_0^\infty |f_a(t)|dt < \infty$ and δ represents the delta distribution at zero. Let $\hat{f}$ denote the Laplace transform of f. Then $\hat{\mathcal{A}}$ is defined as

$$\hat{\mathcal{A}} := \left\{ \hat{f} \mid f \in \mathcal{A} \right\}.$$

By the definition of $\mathcal{A}$ it is easy to see that for every $f \in \mathcal{A}$, $\hat{f}$ is well-defined on $\overline{\mathbb{C}}_+ := \{s \in \mathbb{C} \mid Re(s) \geq 0\}$, it is holomorphic and bounded on $\mathbb{C}_+ := \{s \in \mathbb{C} \mid Re(s) > 0\}$, and continuous on $\mathbb{C}_0 := \{s \in \mathbb{C} \mid Re(s) = 0\}$. Furthermore, $\hat{\mathcal{A}}$ is a commutative Banach algebra with identity under pointwise addition and multiplication (see [7], Corollary A.7.48).
For any complex number s we make the following notation

$$f^{\sim}(s) = \overline{f(-\overline{s})}. \tag{1}$$

We consider the next class of transfer functions, known as the *Wiener algebra*

$$\hat{\mathcal{W}} = \left\{ \hat{f} \in L_\infty \mid \hat{f} = \hat{f}_1 + \hat{f}_2, \text{ with } \hat{f}_1, \hat{f}_2^{\sim} \in \hat{\mathcal{A}} \right\},$$

where

$$L_\infty = \{f : \mathbb{C}_0 \to \mathbb{C} \mid \|f\|_{L_\infty} = \operatorname{ess\,sup}_{s \in \mathbb{C}_0} |f(s)| < \infty\}.$$

$\hat{\mathcal{W}}$ is a Banach algebra under pointwise addition, multiplication, and scalar multiplication. The elements of $\hat{\mathcal{W}}$ are bounded and continuous on the imaginary axis, and their limit at infinity is well-defined. For more properties of the elements of Wiener algebra, see [9].
We denote by $L_\infty^{n\times m}$, $\hat{\mathcal{A}}^{n\times m}$, $\hat{\mathcal{W}}^{n\times m}$, the classes of $n \times m$ matrices with entries in L_∞, $\hat{\mathcal{A}}$, $\hat{\mathcal{W}}$, respectively. We omit the size of the matrix when it is no danger of confusion. For properties of these classes of transfer functions see [1]-[3] and [9]. In [6] the following result is given.

Remark 2.1 *$\hat{f} \in \hat{\mathcal{W}}$ is invertible over $\hat{\mathcal{W}}$ if and only if* $\det \hat{f}(j\omega) \neq 0$ *for all* $\omega \in \mathbb{C}_0 \cup \{\infty\}$.

For matrix valued functions we define

$$F^{\sim}(s) = [F(-\overline{s})]^*,$$

where * denotes the transpose complex conjugate. For the scalar functions this corresponds to (1).
In order to define factorizations we have to split our class of transfer functions into a stable and an antistable part. By the definition of $\hat{\mathcal{W}}$ this is easily done.

$$\hat{\mathcal{W}}^+ = \hat{\mathcal{A}}, \;\; \hat{\mathcal{W}}^- = \left\{ F | F^{\sim} \in \hat{\mathcal{A}} \right\}.$$

From this we see that the intersection of $\hat{\mathcal{W}}^+$ and $\hat{\mathcal{W}}^-$ consists only of the constants. $\hat{\mathcal{W}}^+$ could be seen as the functions in $\hat{\mathcal{W}}$ which have a bounded analytic extension to the right-half plane. Let $G\hat{\mathcal{W}}^+$ and $G\hat{\mathcal{W}}^-$ be the set of invertible elements of $\hat{\mathcal{W}}^+$ and $\hat{\mathcal{W}}^-$, respectively.

Definition 2.2 *The matrix-valued function $Z \in G\hat{\mathcal{W}}$ is said to admit a* (right-) standard factorization relative to the imaginary axis *if Z can be decomposed as*

$$Z = Z_- D Z_+, \tag{2}$$

with $Z_- \in G\hat{\mathcal{W}}^-$,$Z_+ \in G\hat{\mathcal{W}}^+$ and D a diagonal matrix-valued function of the form

$$D(s) = diag\left[\left(\frac{s - s_{+,1}}{s - s_{-,1}} \right)^{k_1}, ..., \left(\frac{s - s_{+,n}}{s - s_{-,n}} \right)^{k_n} \right], \;\; s \in \mathbb{C}_0, \tag{3}$$

with $s_{+,i} \in \mathbb{C}_-$, $s_{-,i} \in \mathbb{C}_+$, $k_i \in \mathbb{Z}$ and $k_1 \geq ... \geq k_n$. The integers k_i are called (the right-) partial indices *of the factorization. In the case $k_1 = ... = k_n = 0$, so that,*

$$Z = Z_- Z_+, \tag{4}$$

then Z is said to admit a (right-) canonical factorization *relative to the imaginary axis.*

The question about the existence of a standard factorization for the Wiener class of matrix functions was answered in [4], Chapter II.

Theorem 2.3 *Every element $Z \in \hat{\mathcal{W}}$ satisfying*

$$\det Z\,(s) \neq 0, \text{ for all } s \in \mathbb{C}_0 \cup \{\infty\}$$

admits a factorization relative to the imaginary axis.

Hence every invertible element of $\hat{\mathcal{W}}$ admits a factorization. As mentioned in the introduction we are interested in J-spectral factorization. For its definition we need to consider the matrix

$$J_{n,m} = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix},$$

where $n, m \in \mathbb{N}$. Sometimes we simply use J without indices.

Definition 2.4 *Let $Z = Z^{\sim} \in \hat{\mathcal{W}}$. Z has a J-*spectral factorization *if there exists a matrix-function $V \in G\hat{\mathcal{W}}^{+}$ such that*

$$Z(s) = V^{\sim}(s) J V(s) \text{ for all } s \in \mathbb{C}_0.$$

*Such a matrix V is called J-*spectral factor*.*

As consequence of the definition of J-spectral factorization we have the following properties. The proofs are left to the reader.

Lemma 2.5 *If $Z \in \hat{\mathcal{W}}$ has a $J_{n,m}$-spectral factorization, then it satisfies the following:*

1. *$Z = Z^{\sim}$;*
2. *Z has no poles on $\mathbb{C}_0 \cup \{\infty\}$;*
3. *$\det(Z)$ has no zeros on $\mathbb{C}_0 \cup \{\infty\}$;*
4. *$Z(s)$, for $s \in \mathbb{C}_0$, has n positive and m negative eigenvalues.*

For the definition and the properties of Toeplitz operators we need the following notations

$$\begin{aligned}
L_2^n &= \{f : \mathbb{C}_0 \to \mathbb{C}^n \mid \|f\|_{L_2}^2 = \int_{-\infty}^{+\infty} \mid f(j\omega) \mid^2 d\omega < \infty\}, \\
H_2^n &= \{f : \mathbb{C}_+ \to \mathbb{C}^n \mid f \text{ is analytic in } \mathbb{C}_+ \\
&\qquad \text{and } \|f\|_{H_2}^2 = \sup_{r>0} \int_{-\infty}^{+\infty} \|(f(r + j\omega))\|^2 d\omega < \infty\}, \\
H_2^{n,\perp} &= \{f : \mathbb{C}_- \to \mathbb{C}^n \mid f \text{ is analytic in } \mathbb{C}_- \\
&\qquad \text{and } \|f\|_{H_2^\perp}^2 = \sup_{r<0} \int_{-\infty}^{+\infty} \|(f(r + j\omega))\|^2 d\omega < \infty\}. \\
H_\infty^{n\times m} &= \{F : \mathbb{C}_+ \to \mathbb{C}^{n\times m} \mid F \text{ is analytic in } \mathbb{C}_+ \\
&\qquad \text{and } \|F\|_{H_\infty} = \sup_{s\in\mathbb{C}_+} \|F(s)\| < \infty\}
\end{aligned}$$

It is well known that L_2^n is the direct sum of H_2^n and $H_2^{n,\perp}$. Now we define the Toeplitz operator.

Definition 2.6 *Let $G \in L_\infty^{n\times m}$. The* Toeplitz operator *with symbol G is defined by*

$$T_G : H_2^m \to H_2^n,\ T_G x = P_+ G x, \tag{5}$$

where $P_+ : L_2^n \to H_2^n$ is the orthogonal projection.

The following property of Toeplitz operators can be found in [18].

Lemma 2.7 *If $G^\sim \in H_\infty$ or $K \in H_\infty$, then*

$$T_{GK} = T_G T_K. \tag{6}$$

3 Equivalent conditions for the J-spectral factorization

In this section we prove that an invertible function in the Wiener algebra admits a J-spectral factorization if and only if the associated Toeplitz operator is invertible. The proof of this result uses Theorem 2.3 and the concept of equalizing vectors.

Definition 3.1 *A vector u is an* equalizing vector *of $Z \in \hat{\mathcal{W}}$ if u is a nonzero element of H_2 and Zu is in $H_2^\perp$.*

The following lemma shows that the nonexistence of equalizing vectors is a necessary condition for the existence of a J-spectral factorization (see also [16]).

Lemma 3.2 *Let $Z \in \hat{\mathcal{W}}$ such that Z admits a J-spectral factorization in $\hat{\mathcal{W}}$. Then Z has no equalizing vectors.*

Proof: Suppose that Z admits a J-spectral factorization in $\hat{\mathcal{W}}$

$$Z(s) = V^\sim(s) J V(s) \tag{7}$$

with $V \in G\hat{\mathcal{W}}^+$, and consider u an equalizing vector for Z. Applying (7) to u and multiplying to the left with the inverse of $V^\sim$, we obtain

$$V^{-\sim} Z u = J V u. \tag{8}$$

Since $V \in G\hat{\mathcal{W}}^+$ and $u \in H_2$, the right-hand side of the equality (8) is in H_2. From the definition of an equalizing vector we have that $Zu \in H_2^\perp$. Since $V \in G\hat{\mathcal{W}}^+$ gives $V^{-\sim} \in G\hat{\mathcal{W}}^-$, which implies that the left-hand side of the equality (8) is in $H_2^\perp$. The equality (8) is verified only when $u = 0$. But, by definition, the equalizing vectors are nonzero. This means that Z has no equalizing vectors. ■

Our main result is the following.

Theorem 3.3 *Let $Z \in \hat{\mathcal{W}}$ be such that $\det Z(s) \neq 0$, for all $s \in \mathbb{C}_0 \cup \{\infty\}$. The following statements are equivalent*

1. *Z admits a J-spectral factorization;*
2. *Z has no equalizing vectors;*

3. The Toeplitz operator T_Z is boundedly invertible.

Proof: 1. $\Rightarrow$ 2. Suppose that Z admits a J-spectral factorization $Z = V^{\sim}JV$, where $V \in G\hat{\mathcal{W}}^{+}$. We define $Z_- = V^{\sim}J$ and $Z_+ = V$. Obviously, $Z_+ \in G\mathcal{W}^+$ and $Z_- \in G\mathcal{W}^-$. Since $\mathcal{W}^+ \subset H_\infty$, we obtain by Lemma 2.7 that

$$T_{Z_+^{-1}}T_{Z_+} = T_{Z_+^{-1}Z_+} = I = T_{Z_+}T_{Z_+^{-1}}$$

Thus T_{Z_+} is invertible and $T_{Z_+}^{-1} = T_{Z_+^{-1}}$. Similarly, one can show that $T_{Z_-}^{-1} = T_{Z_-^{-1}}$. From this it is easy to see that $T_{Z_+}^{-1}T_{Z_-}^{-1}$ is the bounded inverse of the Toeplitz operator T_Z (see also [18], page 25).

3. $\Rightarrow$ 2. Suppose that u is an equalizing vector for Z. Then $T_Z u = 0$, which means that T_Z is not injective, so T_Z is not invertible.

2. $\Rightarrow$ 1. Since $\det Z(s) \neq 0$ for all $s \in \mathbb{C}_0 \cup \{\infty\}$, from Theorem 2.3 we have that Z admits a factorization relative to the imaginary axis in $\hat{\mathcal{W}}$. Let this factorization be given as

$$Z = Z_- D Z_+, \tag{9}$$

where $Z_+ \in G\hat{\mathcal{W}}^+$, $Z_- \in G\hat{\mathcal{W}}^-$, and D a diagonal matrix function of the form (3) (see Definition 2.2). It remains to prove that this standard factorization leeds to a J-spectral factorization. First we show that this factorization is canonical. Suppose that the factorization (9) is not canonical, then there exists a $k_i \neq 0$. Without loss of generality we may assume that $k_1 \neq 0$. Let $u \in H_2$ be such that

$$Z_+ u = \begin{bmatrix} (\frac{1}{s-s_{+,1}})^{k_1} \\ 0 \\ . \\ . \\ 0 \end{bmatrix},$$

where $s_{+,1} \in \mathbb{C}_-$. This is possible because $Z_+ \in G\hat{\mathcal{W}}^+ \subset H_\infty$, and $Z_+ u \in H_2$. We have that

$$DZ_+u = diag\left[\left(\frac{s-s_{+,1}}{s-s_{-,1}}\right)^{k_1}, ..., \left(\frac{s-s_{+,n}}{s-s_{-,n}}\right)^{k_n}\right] \begin{bmatrix} (\frac{1}{s-s_{+,1}})^{k_1} \\ 0 \\ . \\ . \\ 0 \end{bmatrix} = \begin{bmatrix} (\frac{1}{s-s_{-,1}})^{k_1} \\ 0 \\ . \\ . \\ 0 \end{bmatrix},$$

where $s_{-,1} \in \mathbb{C}_+$, which means that $DZ_+u \in H_2^{\perp}$. Since $Z_-^{\sim} \in G\hat{\mathcal{W}}^+ \subset H_\infty$, we have that $Z_- D Z_+ u \in H_2^{\perp}$. Since $Z = Z_- D Z_+$, this shows that u is an equalizing vector for Z. This is in contradiction with item 2, and thus we conclude that the factorization (9) is canonical. Hence we have

$$Z_- Z_+ = Z = Z^{\sim} = Z_+^{\sim} Z_-^{\sim}. \tag{10}$$

We rewrite (10) as

$$(Z_+^{\sim})^{-1} Z_- = Z_-^{\sim} Z_+^{-1}. \tag{11}$$

The left-hand side is an element of $\hat{\mathcal{W}}^-$ and the right-hand side is an element of $\hat{\mathcal{W}}^+$. Thus $Z_-^\sim Z_+^{-1}$ is a constant matrix, which we denote by C. The equation (11) shows $C = C^*$, and $\det(C) \neq 0$. Thus there exists an unitary matrix U such that

$$C = U^\sim JU. \tag{12}$$

Using once again (11) we get that $A_- = A_+^\sim U^\sim JU$, and thus $A = A_+^\sim U^\sim JUA_+$. Choosing $V = UA_+$ proves the assertion. ∎

The above characterizations are completely in frequency domain. In the following sections we add time-domain characterizations. In Section 4 we prove that the existence of a J-spectral factorization is equivalent with the existence of a stabilizing solution of an algebraic Riccati equation. We show this for exponential stable systems in the Pritchard-Salamon class. Systems in this class may have unbounded input and/or output operators. If the input and output operators are bounded and if the Hamiltonian possesses a Riesz basis of eigenfunctions, then we obtain an extra equivalent condition (see Section 5).

4 Pritchard-Salamon systems

In this section we reformulate the main result for a particular class of systems, the Pritchard-Salamon class of systems. The definition of this class is done via the state space representation. First we recall the definitions of two standard concept in operator theory. Let us denote by $L(X)$ the set of bounded linear operators on the Hilbert space X.

Definition 4.1 *A* strongly continuous semigroup *is an operator-valued function $T(t)$ from $\mathbb{R}^+$ to $L(X)$ that satisfies the following properties:*

1. *$T(t+s) = T(t)T(s)$ for $t, s \geq 0$;*
2. *$T(0) = I$;*
3. *$\|T(t)z_0 - z_0\| \to 0$ as $t \to 0^+$ for all $z_0 \in X$.*

Definition 4.2 *A C_0-semigroup, $T(t)$, on a Hilbert space X is* exponentially stable *if there exist positive constants M and α such that*

$$\|T(t)\| \leq Me^{-\alpha t} \text{ for } t \geq 0. \tag{13}$$

The α is called the decay rate, and the supremum over all possible values of α is the *stability margin* of $T(t)$; this is minus its growth bound.

For two Hilbert spaces X and Y, $X \hookrightarrow Y$ means that $X \subset Y$, X is dense in Y and the canonical injection is continuous. In particular, there exists some constant c such that for all $x \in X$ there holds $\|x\|_Y \leq c\|x\|_X$.
If $T(\cdot)$ is a C_0-semigroup on two Hilbert spaces X and Y, then its infinitesimal generator will be denoted by using the corresponding space as a superscript, e.g. A^X and A^Y.
Now we give the definition of a Pritchard-Salamon system as in [18]. This class of systems was first introduced in [17].

Definition 4.3 *Let V, W, U and Y be complex separable Hilbert spaces with $W \hookrightarrow V$ and let $T(\cdot)$ be a C_0-semigroup on V which restricts to a C_0-semigroup on W.*

1. *An operator* $B \in L(U, V)$ *is called an* admissible input operator for $T(\cdot)$, *with respect to* (W, V), *if there exist a constant* $\beta > 0$ *and* $t > 0$ *such that*

$$\int_0^t T(t-s)Bu(s)ds \in W \text{ and}$$
$$\| \int_0^t T(t-s)Bu(s)ds\|_W \leq \beta \|u\|_{L_2(0,t;U)} \tag{14}$$

   for all $u(\cdot) \in L_2(0, t; U)$.

2. *An operator* $C \in L(W, Y)$ *is called an* admissible output operator for $T(\cdot)$ *with respect to* (W, V), *if there exists a constant* $\gamma > 0$ *and* $t > 0$ *such that*

$$\|CT(\cdot)x\|_{L_2(0,t;Y)} \leq \gamma \|x\|_V \text{ for all } x \in W. \tag{15}$$

Suppose that $D \in L(U, Y)$. *Under the above assumptions the system given by*

$$\begin{cases} x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds, \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{16}$$

where $x_0 \in V$, $t \geq 0$ *and* $u(\cdot) \in L_2(0, t; U)$ *is called a* Pritchard-Salamon system *and will be denoted by the quadruple* $(T(\cdot), B, C, D)$. *If, in addition,*

$$D(A^V) \hookrightarrow W, \tag{17}$$

the system is called a smooth Pritchard-Salamon system. *Moreover, if* $T(\cdot)$ *is an exponentially stable semigroup on* V *and* W *the system is called an* exponentially stable Pritchard Salamon system.

Before we define the Popov function we need to introduce the concepts of admissible weighting operator and Popov triple.

Definition 4.4 *Let* V *and* W *be complex separable Hilbert spaces with* $W \hookrightarrow V$ *and let* $T(\cdot)$ *be a* C_0*-semigroup on* V *which restricts to a* C_0*-semigroup on* W. *An operator* $Q = Q^* \in L(W)$ *is said to be* an admissible weighting operator *for* $T(\cdot)$ *with respect to* (W, V), *if for some* $t > 0$, *there exists an* $M > 0$ *such that for every* $x, y \in W$

$$\int_0^t |(QT(t)x, T(t)y)_W| dt \leq M\|x\|_V \|y\|_V \tag{18}$$

and Q *is an admissible output operator for* $T(\cdot)$ *with respect to* (W, V).

We give the definition for the concept of Popov triples associated to the class of Pritchard-Salamon systems as in [18] (see Definition 4.1 page 65).

Definition 4.5 *Let* V, W *and* U *be complex separable Hilbert spaces. The PS(Pritchard-Salamon)-Popov triple on* $(W \hookrightarrow V, U)$ *is defined to be a triple of the form*

$$\Sigma = (T(\cdot), B, M = \begin{bmatrix} Q & N^* \\ N & R \end{bmatrix}) \tag{19}$$

satisfying the assumptions:

1. $T(\cdot)$ *is a* C_0*-semigroup on* V *and* $T(\cdot)$ *restricts to a* C_0*-semigroup on* W;

2. $B \in L(U,V)$ *is an admissible input operator for* $T(\cdot)$ *with respect to* (W,V);

3. $R = R^* \in L(U)$;

4. $N \in L(W,U)$ *is an admissible output operator for* $T(\cdot)$ *with respect to* (W,V);

5. $Q = Q^* \in L(W)$ *is an admissible weighting operator for* $T(\cdot)$ *with respect to* (W,V).

A PS-Popov triple will be called smooth *if condition (17) is satisfied. A PS-Popov triple will be called* regular *if the operator* R *is boundedly invertible.*

Definition 4.6 *Let* $\Sigma = (T(\cdot), B, M)$ *be a PS-Popov triple. The* Popov function *associated with* Σ *is a function associating to every* $s \in \mathbb{C}_0 \cap \rho(A^V)$ *the following operator*

$$\begin{aligned}\Pi_\Sigma(s) = R + N(s-A^V)^{-1}B + (N(s-A^V)^{-1}B)^* \\ + \overline{(s-A^V)^{-1}B}^{*W} Q(s-A^V)^{-1}B,\end{aligned} \tag{20}$$

where $\overline{(s-A^V)^{-1}B}^{*W}$ *denotes the adjoint of* $(s-A^V)^{-1}B$ *not taken as an operator in* $L(U,V)$, *but in* $L(U,W)$.

A simple property of the Popov function is that, if $T(\cdot)$ is an exponentially stable semigroup on V and W, then the relation (20) holds for every real ω and defines a function in $L_\infty(L(U))$. Indeed, in this case $(j\omega - A^V)^{-1}B \in L_\infty(L(U,W))$ (see Lemma 2.12 in [8] or Remark 4.7.d in [18]).

It is easy to verify that

$$G^\sim(s)JG(s) = \Pi_\Sigma(s), \text{ for all } s \in \mathbb{C}_0, \tag{21}$$

where Σ is the Popov triple and G is the transfer function associated to a Pritchard-Salamon system. The transfer function of an exponentially stable Pritchard-Salamon system is in the class $\hat{\mathcal{A}}$ by Proposition 3.5 (ii) in [8] (see also Curtain [5]).
For the Pritchard-Salamon class an equivalent condition for the existence of the J-spectral factorization in terms of Riccati equation is added to our three conditions. We give first the definitions of the Riccati equations and their stabilizing solutions.

Definition 4.7 *Let* $\Sigma = (T(\cdot), B, M)$ *be a smooth regular PS-Popov triple. The Riccati equation associated with* Σ *is the following equation in the unknown* $X = X^* \in L(V)$

$$\begin{aligned}((A^V - BR^{-1}N)x, Xy)_V + (Xx, (A^V - BR^{-1}Ny))_V \\ -(XBR^{-1}B^*Xx, y)_V + ((Q - N^*R^{-1}N)x, y)_W = 0\end{aligned} \tag{22}$$

for $x, y \in D(A^V)$.

Definition 4.8 *If there exists a self-adjoint* $X \in L(V)$ *satisfying the Riccati equation (22) such that* $T_{BF^c}(\cdot)$ *is an exponentially stable semigroup on* V *which restricts to an exponentially stable semigroup on* W, *where*

$$F^c = -R^{-1}(B^*X + N),$$

then X *is called a* stabilizing solution *of the Riccati equation associated with* Σ. *The admissible feedback* F^c *will be called a* stabilizing Riccati feedback.

We have the following theorem giving equivalent conditions for the existence of a J-spectral factorization for an exponentially stable Pritchard-Salamon system.

Theorem 4.9 *We consider an exponentially stable Pritchard-Salamon system with finite-dimensional input and output spaces, and let* Π_Σ *be its associated Popov function. Suppose that* $\det \Pi_\Sigma(s) \neq 0$ *for all* $s \in \mathbb{C}_0 \cup \{\infty\}$*. Then the following statements are equivalent*

1. *The Popov function* Π_Σ *has a* J*-spectral factorization for some unique* J*;*
2. *The Popov function* Π_Σ *has no equalizing vectors;*
3. *The Toeplitz operator* T_{Π_Σ} *is boundedly invertible.*
4. *There exists an invertible matrix* V_∞ *such that*

$$D^*JD = V_\infty^* J V_\infty, \tag{23}$$

and the Riccati equation associated with the PS-Popov triple

$$\Sigma = (T(\cdot), B, \begin{bmatrix} C^*JC & C^*JD \\ D^*JC & D^*JD \end{bmatrix}) \tag{24}$$

has a stabilizing solution P.

In this case, a J*-spectral factor for the Popov function is given by*

$$V(s) = V_\infty + L(sI - A^V)^{-1}B,$$

where $L = -JV_\infty^{-*}(D^*JC + B^*P)$*, and* V_∞^{-*} *is the inverse of* V_∞^**.*

The equivalence between the first three items is a consequence of Theorem 3.3. For the equivalence between 1. and 4. see the proof of Theorem 5.2 in [18]. Note that this equivalence also holds for infinite dimensional input and output spaces.

5 Riesz-spectral systems

In this section we present necessary and sufficient conditions for the existence of a J-spectral factorization in the case when the Hamiltonian is a Riesz-spectral operator. Beside the equivalences established in Theorem 4.9, we add an extra one in terms of the Hamiltonian of the system. First we define Riesz-spectral operator and systems.

Definition 5.1 *Suppose that* A *is a closed linear operator on a Hilbert space* X*, with simple eigenvalues* λ_n*,* $n \in \mathbb{N}$ *and suppose that the corresponding eigenvectors* φ_n*,* $n \in \mathbb{N}$ *form a Riesz basis for* X*. If the closure of* $\{\lambda_n,\ n \in \mathbb{N}\}$ *is totally disconnected, then we call* A *a* Riesz-spectral *operator.*
By totally disconnected *we mean that no two points* $\lambda, \mu \in \overline{\{\lambda_n,\ n \in \mathbb{N}\}}$*, the closure of* $\{\lambda_n,\ n \in \mathbb{N}\}$*, can be joined by a segment lying entirely in* $\overline{\{\lambda_n,\ n \in \mathbb{N}\}}$*.*

In the sequel, we use the notation $\mathbb{Z}_0$ to denote the set of nonzero integers, that is, the set $\mathbb{Z} \setminus \{0\}$. We make the following assumptions throughout this section.

Assumption 5.2 *1. X is a separable Hilbert space.*

2. *$A : D(A)\,(\subset X) \to X$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$.*

3. *$Q_1 = Q_1^* \in L(X)$.*

4. *$Q_2 = Q_2^* \in L(X)$.*

5. *H denotes the Hamiltonian defined by*

$$H = \begin{bmatrix} A & -Q_1 \\ -Q_2 & -A^* \end{bmatrix} \; : \; D(H)\,(\subset X \oplus X) \to X \oplus X, \tag{25}$$

with $D(H) = D(A) \oplus D(A^*)$.

Assumption 5.3 *The Hamiltonian H given by (25) is a Riesz-spectral operator with eigenvalues λ_n, $n \in \mathbb{Z}_0$ and eigenvectors $\Phi_n = \begin{bmatrix} \eta_n \\ \zeta_n \end{bmatrix}$, $n \in \mathbb{Z}_0$.*

Since we assume that the Hamiltonian H given by the equation (25) is a Riesz-spectral operator, we speak about a *Riesz-spectral system*.
We give now the definition of the ARE for the Riesz-spectral systems. In fact this is a particular case of the definition of ARE for Pritchard-Salamon systems, see Weiss [18], page 77.

Definition 5.4 *$P \in L(X)$ is a solution of the* algebraic Riccati equation *(ARE) if*

$$\langle Az_1, Pz_2\rangle + \langle P^* z_1, Az_2\rangle + \langle Q_2 z_1, z_2\rangle - \langle Q_1 P^* z_1, Pz_2\rangle = 0, \tag{26}$$

for all z_1, $z_2 \in D(A)$.

The following theorem states necessary and sufficient conditions for the existence of a stabilizable solution of an ARE. We give here a short proof. For more details see Theorem 7.1 and Corollary 7.2 in [14].

Theorem 5.5 *Suppose that the Assumptions 5.2 and 5.3 above hold. Then the following are equivalent:*

1. *There exists an index set $\mathbb{J}$ $(\subset \mathbb{Z}_0)$ such that*

$$\left\{ \begin{bmatrix} \eta_n \\ \zeta_n \end{bmatrix}, n \in \mathbb{J} \right\}$$

is contained in the set of eigenvectors of H corresponding to the eigenvalues with negative real parts and $\{\eta_n, n \in \mathbb{J}\}$ is a Riesz basis for Z. Furthermore, H has no spectrum in $\{s \in \mathbb{C} \mid |Re(s)| < \epsilon\}$ for some $\epsilon > 0$.

2. *The Riccati equation (26) has a solution P such that $A - Q_1 P$ is stable.*

Proof. <u>1 ⇒ 2.</u> From Theorem 5.6.2 [14], it follows that the linear operator P defined by

$$P\eta_n = \zeta_n, \quad n \in \mathbb{J}$$

is an element in $\mathcal{L}(U)$ and it is a solution of the ARE. Thus it only remains to show that $A - Q_1P$ is stable. From Lemma 5.2 [14] and the remark following it, we obtain that $A - Q_1P$ is a Riesz-spectral operator and

$$\sigma_p(A - Q_1P) = \{\lambda_n, \; n \in \mathbb{J}\}.$$

Furthermore, H has no spectrum in $\{s \in \mathbb{C} \mid |Re(s)| < \epsilon\}$ for some $\epsilon > 0$ which implies that $A - Q_1P$ has no spectrum in this strip. Consequently, using Theorem 2.3.5 [7], $A - Q_1P$ is exponentially stable.

<u>2 ⇒ 1.</u> The implication follows from Theorem 5.6.3, Theorem 6.1 and Lemma 5.5 in [14]. ■

For the next theorem we suppose that we have a Pritchard-Salamon system , as in (16), but with finite rank and bounded B and C operator. The Hamiltonian of the system has the following form

$$H = \begin{bmatrix} A & 0 \\ -C^*JC & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C^*JD \end{bmatrix} (D^*JD)^{-1} \begin{bmatrix} D^*JC & B^* \end{bmatrix}. \tag{27}$$

Combining the above theorem and Theorem 4.9 we obtain the following result for Riesz-spectral systems.

Theorem 5.6 *Suppose that Π_Σ is the Popov function associated to an exponentially stable Riesz-spectral system with finite-dimensional input and output spaces such that* $\det \Pi_\Sigma(s) \neq 0$ *for all $s \in \mathbb{C}_0 \cup \{\infty\}$ and that the Assumptions 5.2 and 5.3 above hold. Then the following statements are equivalent*

1. *The Popov function Π_Σ has a J-spectral factorization for some unique J;*
2. *The Popov function Π_Σ has no equalizing vectors;*
3. *The Toeplitz operator T_{Π_Σ} is boundedly invertible;*
4. *There exists an index set $\mathbb{J}$ $(\subset \mathbb{Z}_0)$ such that*

   $$\left\{ \begin{bmatrix} \eta_n \\ \zeta_n \end{bmatrix}, n \in \mathbb{J} \right\}$$

   is contained in the set of eigenvectors of H corresponding to the eigenvalues with negative real parts and $\{\eta_n, n \in \mathbb{J}\}$ is a Riesz basis for Z. Furthermore H has no spectrum in $\{s \in \mathbb{C} \mid |Re(s)| < \epsilon\}$ for some $\epsilon > 0$;
5. *There exists an invertible matrix V_∞ such that*

   $$D^*JD = V_\infty^* J V_\infty, \tag{28}$$

   and the Riccati equation associated with the PS-Popov triple

   $$\Sigma = (T(\cdot), B, \begin{bmatrix} C^*JC & C^*JD \\ D^*JC & D^*JD \end{bmatrix}) \tag{29}$$

has a stabilizing solution P.
In this case, a J-spectral factor for the Popov function is given by

$$V(s) = V_\infty + L(sI - A^V)^{-1}B,$$

where $L = -JV_\infty^{-*}(D^*JC + B^*P)$, and V_∞^{-*} is the inverse of V_∞^*.

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