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On the structure of transitively differential algebras

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ON THE STRUCTURE OF TRANSITIVELY DIFFERENTIAL ALGEBRAS

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ABSTRACT. We study finite-dimensional Lie algebras of polynomial vector fields in n variables that contain the vector fields $\frac{\partial}{\partial x_i}$ $(i = 1, \dots, n)$ and $x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$. We derive some general results on the structure of such Lie algebras, and provide the complete classification in the cases $n = 2$ and $n = 3$. Finally we describe a certain construction in high dimensions.

Keywords: Lie algebras, vector fields, graded Lie algebras.
MSC 1991: 17B66, 17B70, 17B05.

0. INTRODUCTION

The purpose of this paper is to study finite-dimensional Lie algebras $\mathfrak{L}$ of polynomial vector fields in n variables. We always assume that $\mathfrak{L}$ contains the translations $\frac{\partial}{\partial x_i}$ $(i = 1, \dots, n)$ and usually assume the same for the Euler vector field $E = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$. Consequently we can (and always will) assume that the polynomial coefficients of $X \in \mathfrak{L}$ are all homogeneous of the same degree. Note that in any finite-dimensional Lie algebra $\mathfrak{g}$ of C^∞-vector fields we have a natural filtering by the minimal degree of the Taylor series (around 0) of the coefficients. The associated graded Lie algebra will be of the type we consider in case $\mathfrak{g}$ (or rather G) acts transitively (around 0). This explains a part of the interest in these algebras. For further motivations, we refer to [1].

Known at present is a complete classification in two special cases. The first case is that $\mathfrak{L}$ is (semi-)simple. Now $\mathfrak{L}$ is related to simple Jordan triples, see [2, 3], that can be described completely. Hence we will usually assume that $\mathfrak{L}$ is not semi-simple.

The second case in which a good description of $\mathfrak{L}$ is available, is when $\mathfrak{L}$ is multi-graded, see [4]. Such $\mathfrak{L}$ is described (almost completely) by a diagram made from the integers a_{ij} with $a_{ij} = \max\{\alpha | x_i^\alpha \partial_{x_j} \in \mathfrak{L}\}$.

It turns out that there are many cases in which $\mathfrak{L}$ is not multi-graded. However by putting maximality requirements on $\mathfrak{L}$ (so that $\mathfrak{L}$ becomes a so called transitively differential algebra of certain order), most examples drop out; it is a non-trivial task to distinguish the remaining ones.

We describe the contents of this paper. In section 1 we give some general results on the structure of the Lie algebras under consideration. Our main result is that the radical is contained in a multi-graded Lie algebra. Next we apply these structural results to the cases $n = 2$ and $n = 3$ in section 2; we obtain a complete classification of transitively differential algebras here. In section 3 we describe an elegant construction in higher dimensions.

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1. General structure of graded algebras

1.1. **Definitions.** Throughout $\mathfrak{L}$ will denote a finite-dimensional Lie algebra of polynomial vector fields on $\mathbb{C}^n$, containing the elements of $\partial_i = \dfrac{\partial}{\partial x_i}$ ($i = 1, \dots, n$) and $E = \sum_{i=1}^{n} x_i \partial_i$. The maximal degree ν of the polynomial coefficients of $X \in \mathfrak{L}$ is called the *order* ν of $\mathfrak{L}$, $\mathrm{ord}(\mathfrak{L})$; in this case $\mathfrak{L}$ is said to belong to the *class* $\mathcal{D}^\nu$.

In fact, due to the presence of E in $\mathfrak{L}$, we may assume that the coefficients P_i of $X = \sum P_i(x)\partial_i \in \mathfrak{L}$ are all homogeneous polynomials of the same degree k; this degree is then called the order k of X, $\mathrm{ord}(X)$. It follows that $\mathfrak{L}$ is $\mathbb{Z}$-graded:

$$\mathfrak{L}_d = \{X \in \mathfrak{L} \mid \mathrm{ord}(X) = d + 1\}$$

with

$$\mathfrak{L} = \bigoplus_{d=-1}^{\nu-1} \mathfrak{L}_d \quad \text{and} \quad [\mathfrak{L}_{d_1}, \mathfrak{L}_{d_2}] \subset \mathfrak{L}_{d_1+d_2}.$$

In particular we have $\mathfrak{L}_{-1} = \langle \partial_1, \dots, \partial_n \rangle$.

A transitively differential algebra is a maximal algebra in $\mathcal{D}^\nu$. To be more explicit

Definition 1.1. *Suppose $\mathfrak{L}$ belongs to $\mathcal{D}^\nu$ and $\mathfrak{L}$ is maximal in $\mathcal{D}^\nu$, i.e. $\mathfrak{L}'$ belongs to $\mathcal{D}^\nu$ and $\mathfrak{L}' \supset \mathfrak{L}$ implies $\mathfrak{L}' = \mathfrak{L}$, then $\mathfrak{L}$ is called a transitively differential algebra (TDA) of order ν.*

While all $\mathfrak{L}$ in the class $\mathcal{D}^\nu$ are $\mathbb{Z}$-graded, we will also encounter $\mathfrak{L}$ that are $\mathbb{Z}^n$-graded. Remember that $\mathbb{C}[x_1, \dots, x_n]$ has a $\mathbb{Z}^n$-graded, mdeg, by

$$\mathrm{mdeg}(x^\alpha) = \alpha$$

Correspondingly, the polynomial vector fields, being realized as special elements of $\mathrm{End}(\mathbb{C}[x_1, \dots, x_n])$ attain a $\mathbb{Z}^n$-grading by

$$\mathrm{mdeg}(x^\alpha \partial_j) = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \dots, \alpha_n). \tag{1.1}$$

We say that $\mathfrak{L}$ is *multi-graded* if it has a basis of homogeneous elements w.r.t. the $\mathbb{Z}^n$-grading (1.1). Equivalently $\mathfrak{L}$ has a basis of simultaneous eigenvectors for the operators ad $x_1\partial_1$, ad $x_2\partial_2, \dots,$ ad $x_n\partial_n$. Hence we have a decomposition

$$\mathfrak{L} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathfrak{L}_\alpha \quad ; \quad [\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \subset \mathfrak{L}_{\alpha+\beta}$$

where

$$\{\mathfrak{L}_\alpha = \{X \in \mathfrak{L} \mid [x_i\partial_i, X] = \alpha_i X \quad \text{for} \quad i = 1 \dots n\}$$

Note that any multi-graded TDA contains the elements $x_1\partial_1, \dots x_n\partial_n$.

Let $L : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. This transformation induces a transformation on the polynomial vector fields, preserving the order. In particular a transitively differential algebra is mapped to another one; these two will be called *equivalent*. Note that the multi-grading of an element is *not* preserved.

We will call $\mathfrak{L}$ *essentially multi-graded* if $\mathfrak{L}$ is equivalent to a multi-graded Lie algebra.

There is a convenient way to construct multi-graded Lie algebras. Suppose we give all variables $x_1, x_2, \dots, x_n$ a degree:

$$\deg(x_1) = d_1;\ \deg(x_2) = d_2;\ \dots \deg(x_n) = d_n$$

with $d_i > 0$ for all i. This induces a degree on the vector field terms by

$$\deg(x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \partial_j) = \alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n - d_j. \tag{1.2}$$

Now suppose that $\mathfrak{L}$ is the Lie algebra with basis consisting of all vector field terms of degree 0 or less. By our assumption that $d_i > 0$ we have that $\mathfrak{L}$ in indeed finite-dimensional. Moreover it is clear that ∂_i and the Euler vector field are contained in $\mathfrak{L}$. Hence $\mathfrak{L}$ belong to $\mathcal{D}^\nu$ for some big ν.

A very useful description of vector fields is in term of multi-linear mappings, see e.g. [2]. Let $X \in \mathfrak{L}$, $\operatorname{ord}(X) = k$, and denote $U_{-1} = \langle \partial_1, \dots, \partial_n \rangle$. We associate to X the k-linear map $A : U_{-1} \times U_{-1} \times \cdots \times U_{-1} \to U_{-1}$ by

$$A(v_1, v_2, \dots, v_k) = \frac{1}{k!}[v_1, [v_2, \dots, [v_k, X] \dots]]$$

for $v_1, v_2, \dots, v_k \in U_{-1}$. Using the Jacobi identity one easily checks that A is symmetric. Therefore A is completely determined by its diagonal $A(x, \dots, x)$, $x \in U_{-1}$. If A is associated to X of order $k \geq 1$ and B is associated to Y of order $\ell \geq 1$, then to $[X, Y]$ is associated the $(k + \ell - 1)$-linear map C, defined by

$$C(x, \dots, x) = \ell B(A(x, \dots, x), x, \dots, x) - kA(B(x, \dots, x), x, \dots, x).$$

If $\operatorname{ord}(X) = -1$, then we put $v = X \in U_{-1}$ and we have

$$C(x, \dots, x) = kB(v, x, \dots, x)$$

Definition 1.2. *Let V be a linear subspace of U_{-1}, and $\mathfrak{L}$ in $\mathcal{D}^\nu$.*

(a) *We call V a reducing subspace of $\mathfrak{L}$ if for all A associated to $X \in \mathfrak{L}$ holds*

$$A(v, x, \dots, x) \in V$$

for all $v \in V$ and $x \in U_{-1}$.

(b) *We call $\mathfrak{L}$ reducible if there exists a non-zero reducing subspace V for $\mathfrak{L}$, with $V \neq U_{-1}$. If such V does not exist, $\mathfrak{L}$ is called irreducible.*

Usually we simply identify X and A above.

Lemma 1.3. *Let V be a reducing subspace of $\mathfrak{L}$ and define*

$$I_V = \{A \in \mathfrak{L} \mid A(x, \dots, x) \in V \text{ for all } x \in U_{-1}\} \tag{1.3}$$

Then : I_V is an ideal in $\mathfrak{L}$.

Proof. Let us give a proof of this useful lemma [1]. Remember $A(v, x, \dots, x)$ is (up to a factor k) simply the commutator of A of order k and $v \in U_{-1}$. Fix coordinates $\{\partial_1, \dots, \partial_n\}$ of U_{-1} such that $\langle \partial_{r+1}, \dots, \partial_n \rangle = V$. Since V is a reducing subspace, $X \in \mathfrak{L}$ is of the form

$$X = \sum_{i=1}^{r} P_i(x_1, \dots, x_r)\partial_i + \sum_{i=r+1}^{n} P_i(x_1, \dots, x_n)\partial_i.$$

Define $\varphi : \mathfrak{L} \to \bar{\mathfrak{L}}$ by

$$\varphi(X) = \sum_{i=1}^{r} P_i(x_1, \dots, x_r)\partial_i.$$

One easily checks that φ is a Lie algebra morphism. Hence $I_V = \ker(\varphi)$ is an ideal. □

[1] $\mathfrak{L}/I_V$ ("die verkürzte Guppe") appears already in [5] as an important technique.

Since $V \subset I_V$ we conclude that any simple $\mathfrak{L}$ is irreducible. The converse is also true for $ord(\mathfrak{L}) \neq 1$ (see prop. 1.7). A counter example for $\mathrm{ord}(\mathfrak{L}) = 1$ is the Lie algebra $\mathfrak{L} = U_{-1} \oplus U_0$ of all polynomial vector fields of order 0 and 1.

1.2. **The radical and the nilradical of $\mathfrak{L}$.** Let $\mathfrak{L}$ be a Lie algebra in $\mathcal{D}^\nu$. We will study the radical $\mathfrak{R}$ of $\mathfrak{L}$ (the maximal solvable ideal) and the nilradical $\mathfrak{N}$ of $\mathfrak{L}$ (the maximal nilpotent ideal). Of course, $\mathfrak{N} \subset \mathfrak{R}$; according to [6], § 5, N° 3, we have that

$$\mathfrak{N} = \{X \in \mathfrak{R} | \mathrm{ad}\ X \text{ is nilpotent}\}.$$

Consequently if $X \in \mathfrak{R}$ and $\mathrm{ord}(X) \neq 1$ then $X \in \mathfrak{N}$.

First we state

Proposition 1.4. *Let $\mathfrak{L}$ be in $\mathcal{D}^\nu$ with $\nu \neq 2$ and $\mathfrak{R}$ the nilradical of $\mathfrak{L}$. Then $\mathfrak{R} \neq 0$.*

Proof. If $\nu < 2$ then $U_{-1} \subset \mathfrak{R}$. Hence assume $\nu > 2$. Let $K(X,Y)$ be the Killing form of $X, Y \in \mathfrak{L}$. We show that if $\mathrm{ord}(X) = k > 2$ then $K(X,Y) = 0$ for all $Y \in \mathfrak{L}$, hence $X \in \mathfrak{R}$.

If $\mathrm{ord}(Y) = \ell$, then

$$(k-1)K(X,Y) = K([E,X],Y) = -K(X,[E,Y]) = -(\ell - 1)K(X,Y).$$

Hence $(k + \ell - 2)K(X,Y) = 0$. Since $k \geq 3$ and $\ell \geq 0$, we have $K(X,Y) = 0$. □

It follows that any semi-simple $\mathfrak{L}$ is $\mathcal{D}^\nu$ is of order 2. The following notion is useful for us.

Definition 1.5. *Let $S \subset \mathfrak{L}$ be a linear subspace. We call S* layered *if $[v,s] \in S$ for all $v \in U_{-1}$ and $s \in S$.*

In particular $\mathfrak{L}$ itself is layered, as $U_{-1} = \mathfrak{L}_{-1}$. We note that also any ideal $I \subset \mathfrak{L}$ is layered. Moreover we have

Lemma 1.6. *Let $Z(I)$ be the center of an ideal I in $\mathfrak{L}$. Then $Z(I)$ is layered.*

Proof. Choose $x \in I, z \in Z(I)$ and $v \in U_{-1}$ arbitrary. We have to prove that $[x,[v,z]] = 0$. This follows directly from the Jacobi identity:

$$[x,[v,z]] = [[x,v],z] + [v,[x,z]] = 0 + 0 = 0 \qquad \square$$

Of special interest is the case $I = \mathfrak{N}$.

Proposition 1.7. *Let $W = Z(\mathfrak{N}) \cap U_{-1}$ be as above. Then W is a reducing subspace.*

Proof. Let $A \in \mathfrak{L}$. We have to prove that for $w \in W$ holds $A(w,x,\ldots,x) \in W$ for all $x \in U_{-1}$. Fix an $\bar{x} \in U_{-1}$ and define $\bar{A} \in \mathfrak{L}_0$ by

$$\bar{A}(v) = A(v,\bar{x},\ldots,\bar{x})$$

Take $B \in \mathfrak{N}$, arbitrary, say $\mathrm{ord}(B) = p$. Then

$$[\bar{A},B](x,\ldots,x) = pB(\bar{A}(x),x,\ldots,x) - \bar{A}(B(x,\ldots,x))$$

and by commutation with w we get

$$\begin{aligned} 0 =& [\bar{A},B](w,x,\ldots,x) = pB(\bar{A}(w),x,\ldots,x) \\ &+ p(p-1)B(\bar{A}(x),w,x,\ldots,x) - \bar{A}(B(w,\ldots,x)) \end{aligned}$$

Now clearly the last two terms of the right-hand side are 0. Hence also the first term:

$$B(\bar{A}(w),x,\ldots,x) = 0$$

for all $x \in U_{-1}$ and $B \in \mathfrak{N}$. Hence $[\bar{A}(w), B] = 0$ for all $B \in \mathfrak{N}$, i.e. $\bar{A}(w) \in W$. So we obtain

$$A(w, \bar{x}, \ldots, \bar{x}) \in W$$

for all $\bar{x} \in U_{-1}$. □

Let us summarize our results. If $\mathfrak{R} \neq 0$, then also $\mathfrak{N} \neq 0$ and $V = \mathfrak{R} \cap U_{-1} = \mathfrak{N} \cap U_{-1}$. This space V is a reducing subspace, as is easily checked. It is possible that $V = U_{-1}$, e.g. for $n = 2$ one can take

$$\mathfrak{L} = \langle x_1^k \partial_2 \ (k \leq \nu), \partial_1 \rangle$$

Moreover if $\mathfrak{R} \neq 0$ then also $W = Z(\mathfrak{N}) \cap U_{-1} \neq 0$. This W is also a reducing subspace. Inductively, using lemma 1.3, we have a flag of subspaces $0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_s = V$ in V such that any $X \in \mathfrak{N}$ satisfies

$$[W_{i+1}, X] \in W_i \qquad \text{for all } i = 0, \ldots, s-1. \tag{1.4}$$

If $W = U_{-1}$ then $\mathfrak{N} = U_{-1}$ and hence $\mathfrak{L} \subset U_{-1} \oplus U_0$, where U_0 denotes the space of *all* vector fields of order 0.

1.3. **The semi-simple part of $\mathfrak{L}$.** Let $\mathfrak{L}$ be of class $\mathcal{D}^\nu$ and $\mathfrak{R}$ its radical. Then $\mathfrak{L}/\mathfrak{R}$ is semi-simple; there exists a subalgebra $\mathfrak{S} \subset \mathfrak{L}$ such that $\mathfrak{S} \simeq \mathfrak{L}/\mathfrak{R}$. Such $\mathfrak{S}$ is called a Levi factor. $\mathfrak{S}$ is the direct sum of (say d) simple Lie algebras $\mathfrak{S}_1, \ldots, \mathfrak{S}_d$. Set, as before $V = \mathfrak{R} \cap U_{-1}$, and $I_V \supset \mathfrak{R}$ the corresponding ideal. It is possible that I_V contains some of the factors $\mathfrak{S}_1, \ldots, \mathfrak{S}_d$. However we have

Proposition 1.8. *Let $\mathfrak{S}_i \subset I_V$, then $\mathfrak{S}_i \subset U_0$.*

Proof. Consider the Killing form K on I_V. For $X \in V$, $K(X, Y) = 0$ for all $Y \in I_V$, as $V \subset \mathfrak{N}$. Using the proof of proposition 1.4, it follows for $X \in I_V$ and $\mathrm{ord}(X) \neq 1$, that $K(X, Y) = 0$ for all $Y \in I_V$. So the semi-simple part of I_V is contained in the part of order 1. □

Using all results till now we are able to formulate a proposition on the structure of $\mathfrak{L}$ in case that $V = U_{-1}$.

Proposition 1.9. *Suppose $\mathfrak{L}$ is a Lie algebra in the class $\mathcal{D}^\nu$ with radical $\mathfrak{R}$ such that $\mathfrak{R} \cap U_{-1} = U_{-1}$. Then $\mathfrak{L}$ is a subalgebra of a multi-graded $\bar{\mathfrak{L}}$ of class $\mathcal{D}^{\bar{\nu}}$ with $\bar{\nu} \geq \nu$.*

Proof. By proposition 1.7 we have a flag

$$U_{-1} = W_r \supset W_{r-1} \supset \cdots \supset W_1 \supset W_0 = \{0\}$$

with W_r/W_{r-1} a reducing subspace for $\mathfrak{L}/I_{W_{r-1}}$. In particular we have that *all* elements of $\mathfrak{L}$ have a common "triangular" form: if $x^{(s)} = (x_1^{(s)}, x_2^{(s)}, x_{r_s}^{(s)})$ are coordinates for W_s/W_{s-1} then $X \in \mathfrak{L}$ takes the form

$$X = P_r(x^{(r)})\partial_{x^{(r)}} + P_{r-1}(x^{(r-1)}, x^{(r)})\partial_{x^{(r-1)}} + \cdots + P_n(x^{(1)}, x^{(2)}, \ldots, x^{(r)})\partial_{x^{(1)}}$$

where we used some obvious vector notations. If $X \in \mathfrak{N}$ then X even takes the strictly triangular form ($\mathrm{ord}(X) \geq 1$):

$$X = P_{r-1}(x^{(r)})\partial_{x^{(r-1)}} + \cdots + P_n(x^{(2)}, \ldots, x^{(r)})\partial_{x^{(1)}}$$

We inductively give degrees to the variables in $x^{(r)}, x^{(r-1)}$ down to $x^{(1)}$, cf. (1.2). First we put $\deg(x_i^{(r)}) = 1$. Then look at all terms in $P_{r-1}(x^{(r-1)}, x^{(r)})$ that are independent of $x^{(r-1)}$. Suppose d_{r-1} is the maximal degree (which in this case coincides with the maximal polynomial degree), then we put $\deg(x_i^{(r-1)}) = d_{r-1}$. Now look at all terms in P_{r-2} independent of $x^{(r-2)}$. Let d_{r-2} be the maximal degree. Then we put $\deg(x_i^{(r-2)}) = d_{r-2}$. This way we continue, and obtain that all these terms have degree 0 or less. It remains to consider the "diagonal terms" in P_i. We know from $\mathfrak{R} \cap U_{-1} = U_{-1}$ and proposition 1.8 that the diagonal terms are of order 1, hence of the form $x^{(i)}\partial_{x^{(i)}}$. Consequently these terms have degree 0. Hence $\mathfrak{L}$ is a subalgebra of the multi-graded Lie algebra $\mathfrak{L}'$ consisting of all terms of degree 0 or less in the grading constructed above. □

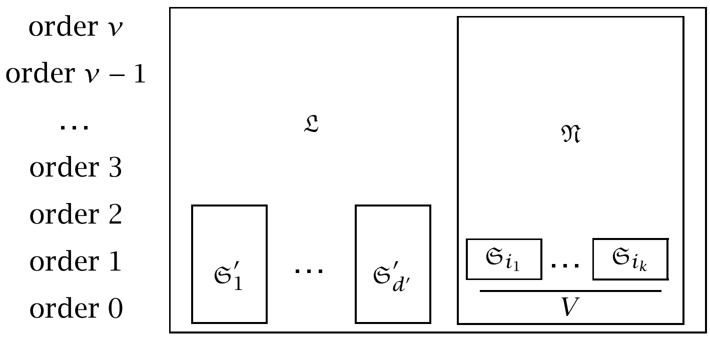
It remains to study the situation in which $V \neq U_{-1}$. In this case we can consider $\mathfrak{L}' = \mathfrak{L}/I_V$. This is a Lie algebra of vector fields on $\mathbb{C}^{n'}$ with $n' = n - \dim(V)$. Note that the Euler field is $E' = E \bmod I_V$. Moreover $\mathfrak{L}'$ belongs to $\mathcal{D}^2$, as $\mathfrak{L}'$ is semi-simple. Let $\mathfrak{S}'_1, \dots, \mathfrak{S}'_{d'}$ be the simple Lie algebras that constitute $\mathfrak{L}'$. Thank to the presence of E' it is immediate that also $\mathfrak{S}'_1, \dots, \mathfrak{S}'_{d'}$ are graded. In particular if we denote $U'_{-1} = U_{-1}/V$ we have

$$U'_{-1} = \bigoplus_{i=1}^{d'} \mathfrak{S}'_i \cap U'_{-1}$$

Clearly for any $\mathfrak{S}'_i$ we have $\mathfrak{S}'_i \cap U'_{-1} \neq \{0\}$. Hence we have

Proposition 1.10. *Let $\mathfrak{L}' = \mathfrak{S}'_1 \oplus \cdots \oplus \mathfrak{S}'_{d'}$ be as above. Then each simple summand $\mathfrak{S}'_i$ is in $\mathcal{D}^2$.*

Consequently, we know the form of the possible factors $\mathfrak{S}'_i$, thanks to [3]. We summarize the results in a table.



2. Transitive Differential algebras in low dimensions

We will discuss the structure of Lie algebras in the class $\mathcal{D}^\nu$ for $n = 2$ and $n = 3$. Apart from being interesting in its own right, we consider this a demonstration of the theorems from section 1.

2.1. We will start with $n = 2$. Remember from before the spaces $V = \mathfrak{N} \cap U_{-1}$ and $W = Z(\mathfrak{N}) \cap U_{-1}$. If $\dim V = 2$, we have that $\mathfrak{L}$ is a subalgebra of a multi-graded one. Hence we see (cf. [4]) that $\mathfrak{L}$ is subalgebra of

$$\langle \partial_x, \partial_y, x\partial_y, \dots, x^\nu\partial_y, x\partial_x, y\partial_y \rangle \qquad (\nu \geq 2)$$

or $\mathfrak{L}$ is contained in $U_{-1} \oplus U_0$.

At the other extreme, we have that $V = \{0\}$, which means that $\mathfrak{L}$ is semi-simple. This gives two (multi-graded) possibilities, namely

$$\mathfrak{L} = \langle y^2\partial_y, x^2\partial_x, y\partial_y, x\partial_x, \partial_y, \partial_x \rangle \cong sl_2 \oplus sl_2$$

or

$$\mathfrak{L} = \langle y^2\partial_y + xy\partial_x, x^2\partial_x + xy\partial_y, y\partial_y, x\partial_y, y\partial_x, x\partial_x, \partial_x, \partial_y \rangle \cong sl_3$$

Finally we have the case that $\dim V = \dim W = 1$. Hence $X \in \mathfrak{N}$ is of the form

$$X = x^k \partial_y \quad (k \le \nu)$$

while $\mathfrak{L}$ contains an element Y of the form

$$Y = x^2\partial_x + (\alpha x^2 + \beta xy + \gamma y^2)\partial_y$$

Now $[\partial_y, Y] \in \mathfrak{N}$ implies that $\gamma = 0$. Moreover $[Y, x^\nu \partial_y] = 0$ implies $\beta = \nu$. For $\nu \le 1$ it follows that one can take $\alpha = 0$, while for $\nu \ge 2$ we have $x^2\partial_y \in \mathfrak{L}$, hence we can assume that $\alpha = 0$. All together we find that $\mathfrak{L}$ is a subalgebra of

$$\langle x^\nu\partial_y, x^{\nu-1}\partial_y, \dots, \partial_y, x^2\partial_x + \nu xy\partial_y, x\partial_x, y\partial_y, \partial_x\rangle$$

2.2. For $n = 2$ we gave a description of all Lie algebras in $\mathcal{D}^\nu$. In particular, we derived that for $\nu \ge 2$ all these algebras are (essentially) multi-graded. For $n \ge 3$ this is no longer the case. A simple counterexample is the smallest Lie algebra in $\mathcal{D}^\nu$ containing the element $P(x_1, x_2)\partial_{x_3}$ for a homogeneous polynomial P of order ν. The construction of a TDA that is not essentially multi-graded is not so simple. We discuss this construction of the TDA $\mathfrak{L}$ by considering different cases for the dimensions of $W = Z(\mathfrak{N}) \cap U_{-1}$ and $V = \mathfrak{N} \cap U_{-1}$. We will not discuss $\dim W = 0$, as in this case $\mathfrak{L}$ is a direct sum of simple Lie algebras in $\mathcal{D}^2$, see proposition 1.10. The case $\dim W = 1$ and $\dim V \ge 2$ is discussed in subsection 2.3.

If $\dim W = 3$, we have that $\mathfrak{L} \subset U_{-1} \oplus U_0$, hence by maximality, $\mathfrak{L} = U_{-1} \oplus U_0$. If $\dim W = 2$, we can assume that $W = \langle \partial_y, \partial_z\rangle$ and $\mathfrak{N}$ contains $\langle x^k\partial_z, x^\ell\partial_y\rangle$ for $k \le \kappa$, $\ell \le \lambda$ (and possibly $\partial_x \in \mathfrak{N}$). We can assume that $\kappa \ge \lambda$ and $\kappa \ge 1$. The only possible element of order 2 can be put in the form

$$X = x^2\partial_x + \lambda xy\partial_y + (\kappa xz + \gamma xy)\partial_z.$$

Further considerations yield that $\mathfrak{L}$ is not maximal in case that $\dim W = 2$.

We end up at the most difficult case, $\dim W = 1$. This case we split in two subcases, namely $V = W$, and $\dim V \ge 2$. In the first case, the nilradical has only elements of the form $X = P(x, y)\partial_z$. Let κ be the maximal k such that $x^k\partial_x \in \mathfrak{N}$ and λ the maximal ℓ such that $y^\ell\partial_z \in \mathfrak{N}$. We know that $\bar{\mathfrak{L}} = \mathfrak{L}/I_V$ is semi-simple, and according to subsection 2.1 we have only two possibilities: $\bar{\mathfrak{L}} = sl_2 \oplus sl_2$ or $\bar{\mathfrak{L}} = sl_3$.

If $\bar{\mathfrak{L}} = sl_2 \oplus sl_2$, then $\mathfrak{L}$ contains

$$Y_1 = x^2\partial_x + Q_1(x, y, z)\partial_z \text{ and } Y_2 = y^2\partial_y + Q_2(x, y, z)\partial_z$$

for some quadratic polynomials Q_1 and Q_2.
As $[Y_1, x^\kappa\partial_z] = 0$ and $[Y_2, y^\lambda\partial_z] = 0$ and $[Y_1, Y_2] = 0$, we find

$$Q_1(x, y, z) = \kappa xz + \alpha_1 x^2 + \alpha_2 xy \text{ and } Q_2(x, y, z) = \lambda yz + \beta_2 xy + \beta_3 y^2.$$

for some α_i and β_i. We may assume that these α_i and β_i are 0 in case $\kappa \ge 2$ and $\lambda \ge 2$. By direct check one can show that this holds in the remaining cases $\kappa \le 1$ or $\lambda \le 1$ as well. Hence $\mathfrak{L}$ has order $\nu = \kappa + \lambda$ and basis

$$\{x^k y^\ell \partial_z \ (k \le \kappa, \ell \le \lambda), z\partial_z, x^2\partial_x + \kappa xz\partial_z, x\partial_x, \partial_x, y^2\partial_y + \lambda yz\partial_z, y\partial_y, \partial_y\}$$

Next we consider the case that $\bar{\mathfrak{L}} = sl_3$. Now $\mathfrak{L}$ contains elements of the form

$$Y_1 = x^2\partial_x + xy\partial_y + Q_1(x, y, z)\partial_z \text{ and } Y_2 = xy\partial_x + y^2\partial_y + Q_2(x, y, z)\partial_z.$$

By the same method as in the previous case, we obtain $\kappa = \lambda$ and that we can put $\alpha_i = 0$ and $\beta_i = 0$ in all cases. Now $\mathfrak{L}$ has order $\nu = \kappa$ and a basis for $\mathfrak{L}$ is

$$\{x^k y^\ell \partial_z \ (k+\ell \le \kappa), z\partial_z, x^2\partial_x + xy\partial_y + \kappa xz\partial_z,$$
$$x\partial_x, y\partial_x, \partial_x, xy\partial_x + y^2\partial_y + \kappa yz\partial_z, x\partial_y, y\partial_y, \partial_y\}$$

2.3. Finally we arrive in the case that $\dim W = 1$ and $\dim V \ge 2$. By choosing good coordinates in U_{-1} we can assume that $\mathfrak{N}$ contains elements of the form

$$X = P(x,y)\partial_z \text{ and } Y = x^\ell \partial_y + Q(x,y)\partial_z$$

Let λ be the maximal ℓ such that $\mathfrak{N}$ contains an element $Y = x^\ell\partial_y + Q(x,y)\partial_z$ for some polynomial Q. For $\lambda = 0$ or $\lambda = 1$ and $\kappa \ne 0$, there is no TDA; one always can add terms such that we end up in the case with $\dim V = 1$ and the semi-simple part $\bar{\mathfrak{L}}$ being $sl_2 \oplus sl_2$ or sl_3, respectively. The case $\lambda = 1$ and $\kappa = 0$ is a special case in the series below. So from now on we assume $\lambda \ge 2$ and try to construct an $\mathfrak{L}$ that is not (essentially) multi-graded. Let κ be the maximal k such that $x^k\partial_z \in \mathfrak{L}$. Now consider all terms that $x^a y^b \partial_z$ occurring in some $P(x,y)\partial_z \in \mathfrak{L}$, and take the P for which $a + \lambda b$ is maximal for some a, b. As P is homogeneous, b is maximal among the terms $x^a y^b$ occurring in P. By applying ad Y exactly b times see that $x^{a+\lambda b}\partial_z \in \mathfrak{L}$. By definition of κ, we have:

$$\boxed{a + \lambda b \le \kappa} \tag{2.1}$$

Now we can give the variable x, y, z degrees, according the scheme in the proof of proposition 1.9. We put

$$\deg(x) = 1, \quad \deg(y) = \lambda \text{ and } \deg(z) = \kappa.$$

Thanks to (2.1) all terms in $\mathfrak{N}$ have degree 0 or less except, possibly, terms occurring in Q. We can assume that this happens in case that $\ell = \lambda$. If all terms in Q have degree 0 or less, one can prove to end up with $\mathfrak{L}$ being

$$\langle x^a y^b \partial_z \ (a+\lambda b \le \kappa), x^\ell\partial_y \ (\ell \le \lambda), x^2\partial_x + \lambda xy\partial_y + \kappa xz\partial_z, x\partial_x, \partial_x, y\partial_y, z\partial_z\rangle$$

So assume $Y = x^\lambda\partial_y + Q(x,y)\partial_z$ and assume that $x^c y^d$ occurs in Q with $c + d\lambda > \kappa$. We take d maximal. If $d = 0$ then Q does not depend on y, and we can eliminate $Q(x) = \beta x^\lambda$ by the change of variables $y' = y - \beta z$; so the case $d = 0$ belongs to the multi-graded case above. Now Q depends on y, we consider

$$[Y, [\partial_x, Y]] = x^{\lambda-1}(\deg_x Q - \lambda Q)\frac{\partial Q}{\partial y}\partial_z$$

Hence $[Y, [\partial_x, Y]]$ contains the term $x^{\lambda-1+c}y^{d-1}\partial_z$. Using (2.1) for $a = \lambda - 1 + c$ and $b = d - 1$ gives that $\lambda - 1 + c + \lambda(d-1) \le \kappa$, or $c + \lambda b \le \kappa + 1$. Together with $c + \lambda d > \kappa$ and $c + d = \lambda$ we obtain that $d = \frac{\kappa}{\lambda-1} - 1$. Hence necessarily $\kappa = \alpha(\lambda - 1)$ for an integer α. Consequently $d = \alpha - 1$ (so $\alpha \ge 2$) and $c = \lambda + 1 - \alpha$. Hence also $\alpha \le \lambda + 1$. By rescaling z we can assume that the coefficient of $x^c y^d$ in Q is 1.

We now derived more or less the structure of $\mathfrak{N}$. However, it could be that $\dim V = 2$, so that $\bar{\mathfrak{L}} = \mathfrak{L}/I_V = sl_2$. Direct calculations show that this is only possible if $\alpha = 2$. Hence for $\alpha \ge 3$ we have

$$\begin{aligned}\mathfrak{L} = \langle x^a y^b \partial_z \ (a+\lambda b \le \kappa), (\text{ad } \partial_x)^k \left(x^\lambda\partial_y + x^{\lambda-\alpha+1}y^{\alpha-1}\partial_z\right) \ (k \le \lambda),\\ x\partial_x + y\partial_y + z\partial_z, \partial_x, y\partial_y + \alpha z\partial_z\rangle,\end{aligned} \tag{2.2}$$

In the case $\alpha \ge 3$ we see that $\mathfrak{L}$ is a subalgebra of

$$\langle x^a y^b \partial_z \ (a+\lambda b \le \kappa+1), x^\ell\partial_y \ (\ell \le \lambda), x\partial_x, y\partial_y, z\partial_z, \partial_x\rangle,$$

which is multi-graded, but of order $\kappa + 1$.

For $\alpha = 2$ we can add to $\mathfrak{L}$ in (2.2)

$$Z = x^2\partial_x + \lambda xy\partial_y + (\kappa xz + \frac{1}{2}y^2)\partial_z. \tag{2.3}$$

In this case $\mathfrak{L}$ is maximal in the sense that adding any vector field will generate an infinite-dimensional Lie algebra.

To summarize this section we formulate the following proposition.

Proposition 2.1. *Let $\mathfrak{L}$ be a TDA in 3 dimensions that is not semi-simple. Then either $\mathfrak{L}$ is (essentially) multi-graded or $\mathfrak{L}$ belongs to the series of TDAs given in (2.2) and (2.3) with integers $\lambda \geq 1$, $2 \leq \alpha \leq \lambda+1$ and $\kappa = \alpha(\lambda-1)$. Moreover* $\operatorname{ord}\mathfrak{L} = \kappa$ *for* $\lambda \geq 2$ *and* $\operatorname{ord}\mathfrak{L} = 2$ *if* $\lambda = 1$.

3. A construction in high dimensions

We describe a construction for TDAs of order 3, which generalizes the construction in [7]. Let $\mathfrak{L}$ be one of the constructed algebras. The global structure of $\mathfrak{L}$ is as follows. As $\nu = 3$, the nilradical of $\mathfrak{L}$ is non-zero; hence we have an invariant subspace $W = \langle z_1, z_2, \dots, z_m\rangle$; $W = U_{-1} \cap Z(\mathfrak{N})$. Next we consider $\bar{\mathfrak{L}} = \mathfrak{L}/I_W$. Now $\bar{\mathfrak{L}}$ has also order 3, and we obtain by a similar procedure $\bar{W} = \langle y_1, y_2, \dots, y_\ell\rangle$. The remaining algebra $\bar{\bar{\mathfrak{L}}} = \bar{\mathfrak{L}}/I_{\bar{W}}$ is the affine algebra $U_{-1} \oplus U_0$ in the variables $\bar{\bar{W}} = \langle x_1, x_2, \dots, x_k\rangle$. It will turn out that ℓ is related to k by $\ell = \binom{k+1}{2}$. So $\mathfrak{L}$ only depends on the two parameters k and m, and $n = k + \ell + m$.

3.1. Elements of order 3.

We now give the detailed construction of $\mathfrak{L}$, starting at order 3. First, $\mathfrak{L}$ will contain the elements

$$\boxed{x_a x_b x_c \partial_{z_d}} \qquad (a \leq b \leq c \leq k; \quad d \leq m). \tag{3.1}$$

Secondly we construct the vector

$$v = (x_1^2, x_1x_2, \dots, x_1x_k, x_2^2, \dots, x_k^2)$$

of all quadratic monomials in $x_1, \dots, x_k$. Hence v is vector with $\ell = \binom{k+1}{2}$ components. Let $\partial_y = (\partial_{y_1}, \dots, \partial_{y_\ell})$ and $v \cdot \partial_y = x_1^2\partial_{y_1} + \cdots + x_k^2\partial_{y_\ell}$. Then $\mathfrak{L}$ will contain

$$\boxed{x_a(v \cdot \partial_y)} \qquad (a \leq k). \tag{3.2}$$

Now we fixed v, we look for all vector w, quadratic in $x_1, \dots, x_k$ satisfying

$$\frac{\partial v}{\partial x_a} \cdot w = 0 \qquad (\text{for all } a \leq k)$$

Note that this automatically implies that also $v \cdot w = 0$, by using

$$\frac{\partial v}{\partial x_a} \cdot w = 0 \Rightarrow \sum x_a \frac{\partial v}{\partial x_a} \cdot w = 0 \Rightarrow 2v \cdot w = 0. \tag{3.3}$$

The space of quadratic vectors has dimension $\binom{k+1}{2} \cdot \binom{k+1}{2}$. The linear constraints $\frac{\partial v}{\partial x_a} \cdot w = 0$ are independent and yield each $\binom{k+2}{3}$ equations on the coefficients. So the space of solutions w has dimension

$$r = \frac{1}{4}k^2(k+1)^2 - k\binom{k+2}{3} = \frac{1}{2}k\binom{k+1}{3}.$$

In subsection 3.5 we will describe a construction for the space of solutions. For now let $w_1, \dots, w_r$ be a basis for this space and $y = (y_1, \dots, y_\ell)$. Then $\mathfrak{L}$ contains the elements

$$\boxed{(w_a \cdot y)\partial_{z_b}} \qquad (a \le r; \quad b \le m). \tag{3.4}$$

3.2. **Elements of order 2.** Now we consider the quadratic elements. These are simply the derivatives of the cubic ones. So the elements in (3.1) give

$$\boxed{x_a x_b \partial_{z_c}} \qquad (a \le b \le k; \quad c \le m). \tag{3.5}$$

The elements in (3.2) yield

$$\boxed{x_a \frac{\partial v}{\partial x_b} \cdot \partial_y} \qquad (a, b \le k) \tag{3.6}$$

By (3.3) this is exactly the linear span of the elements $[\partial_{x_b}, x_a v \cdot \partial_y]$.

Next we have the elements

$$\boxed{(\frac{\partial w_a}{\partial x_b} \cdot y)\partial_{z_c}} \qquad (a, b \le k; c \le m). \tag{3.7}$$

These elements are not linearly independent; we will discuss this in 3.5.

3.3. **Elements of order 1.** First we have the elements

$$\boxed{x_a \partial_{z_c}}\ \boxed{x_a \partial_{y_b}}\ \boxed{y_b \partial_{z_c}} \text{ and } \boxed{z_d \partial_{z_c}} \qquad (a \le k; \quad b \le \ell; \quad c, d \le m). \tag{3.8}$$

Secondly we have the element

$$\boxed{y_1 \partial_{y_1} + y_2 \partial_{y_2} + \cdots + y_\ell \partial_{y_\ell}} \tag{3.9}$$

Finally we have elements of the form $x_a \partial_{x_b} + \dots$. The vector fields $x_a \partial_{x_b}$ act on $v \cdot \partial_y$. This action can be view as a linear action on the space $\bar{W} = \langle \partial_{y_1}, \dots \partial_{y_\ell} \rangle$. As such, modulo the vector field (3.9), it can be uniquely represented by a vector field of the form

$$\sum_{c,d} \alpha_{c,d}^{(a,b)} y_c \partial_{y_d}.$$

If we write $v = (v_1, \dots, v_\ell)$ we have explicitly

$$x_a \partial_{x_b}(v_1 \partial_{y_1} + v_2 \partial_{y_2} + \cdots + v_\ell \partial_{y_\ell}) =$$
$$-v_1(\sum_d \alpha_{1,d}^{(a,b)} \partial_{y_d}) - v_2(\sum_d \alpha_{2,d}^{(a,b)} \partial_{y_d}) - \cdots - v_\ell(\sum_d \alpha_{\ell,d}^{(a,b)} \partial_{y_d})$$

By this construction we automatically have that

$$[x_a \partial_{x_b} - \sum_{c,d} \alpha_{c,d}^{(a,b)} y_c \partial_{y_d}, v \cdot \partial_y] = 0.$$

To $\mathfrak{L}$ we add the vector fields

$$\boxed{x_a \partial_{x_b} - \textstyle\sum_{c,d} \alpha_{c,d}^{(a,b)} y_c \partial_{y_d}} \qquad (a, b \le k). \tag{3.10}$$

This completes the description of $\mathfrak{L}$.

3.4. **Maximality of $\mathfrak{L}$.** A direct (but long) calculation shows that $\mathfrak{L}$ is indeed a Lie algebra and that it is maximal in $\mathcal{D}^3$. We will discuss some essential steps. Let X be of type (3.2) and Y of type (3.4). They commute by

$$[X,Y] = [x_a(v\cdot\partial_y),(w_c\cdot y)\partial_{z_b}] = x_a(v\cdot w_c)\partial_{z_b} = 0$$

Another commutator to check is type (3.4) and type (3.10). For this we consider

$$0 = [x_a\partial_{x_b} - \sum_{c,d}\alpha_{c,d}^{(a,b)}y_c\partial_{y_d},[(w_c\cdot y)\partial_{z_b}, v\cdot\partial_y]]$$

By the Jacobi identity we find that

$$[[x_a\partial_{x_b} - \sum_{c,d}\alpha_{c,d}^{(a,b)}y_c\partial_{y_d},(w_c\cdot y)\partial_{z_b}], v\cdot\partial_y] = 0$$

Now the inner commutator is of the form $(f\cdot y)\partial_{z_b}$ with f quadratic in x, and moreover $f\cdot v = 0$. Similarly we find $f\cdot\frac{\partial v}{\partial x_i} = 0$ for all i. Hence we conclude that f is a linear combination of $w_1,\dots,w_r$.

Now suppose that $\mathfrak{L}'$ is in $\mathcal{D}^3$ and contains $\mathfrak{L}$. Then $\mathfrak{L}'$ has a reducing subspace W, which must be a subspace of $\langle z_1,\dots,z_m\rangle$. As $\mathfrak{L}$ acts irreducibly on $\langle z_1,\dots,z_m\rangle$, we see that this space coincides with W. So no elements of $\mathfrak{L}'$ contain terms of the form $z_a\partial_{y_b}$ or $z_a\partial_{x_b}$. Next we see that y does not appear quadratically by commuting such elements with type (3.2). After this it is not difficult to check maximality.

Note that $\mathfrak{L}$ is not essentially multi-graded by the fact that the Jacobian of $x_1(v\cdot\partial_y)$ has rank greater than 1 (all essentially multi-graded algebras have a basis in which the basiselements of order 3 and higher have Jacobians with rank 1, see [4]). Moreover by giving the variables degrees according to

$$\deg x_i = 1; \qquad \deg y_i = 2; \qquad \deg z_i = 4$$

we see that $\mathfrak{L}$ is contained in a multi-graded Lie algebra of order 4.

3.5. **The w-space.** Let us now describe the space of w-functions. For this we consider the condition $v\cdot w = 0$. This condition splits in $\binom{k+3}{4}$ constraints on the coefficient of monomials of the form

$$x_a^4; \quad x_a^3x_b; \quad x_a^2x_b^2; \quad x_a^2x_bx_c; \quad x_ax_bx_cx_d.$$

Correspondingly, the solutions w naturally split up. We describe the 5 cases. Without loss of generality, we assume $(a,b,c,d) = (1,2,3,4)$.

- (x_1^4). Then $w = (\alpha_1x_1^2,0,\dots,0)$, and $v\cdot w = 0$ yields $\alpha_1 = 0$.
- $(x_1^3x_2)$. Having $v = (x_1^2, x_1x_2,\dots)$ we get $w = (\alpha_1x_1x_2,\alpha_2x_1^2,0,\dots,0)$. Again $v\cdot w = 0$ and $\frac{\partial v}{\partial x_1}\cdot w = 0$ yield $w = 0$.
- $(x_1^2x_2^2)$. Permute the components of v, so that $v = (x_1^2, x_1x_2, x_2^2,\dots)$. Then $w = (\alpha_1x_2^2,\alpha_2x_1x_2,\alpha_3x_1^2,0,\dots,0)$. It gives one solution
$$w = (x_2^2, -2x_1x_2, x_1^2, 0,\dots,0). \tag{3.11}$$
- $(x_1^2x_2x_3)$. If $v = (x_1^2, x_1x_2, x_1x_3, x_2x_3,\dots)$ then
$$w = (x_2x_3, -x_1x_3, -x_1x_2, x_1^2, 0,\dots,0). \tag{3.12}$$
- $(x_1x_2x_3x_4)$. Finally the most involved case. If
$$v = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4,\dots)$$
then we find two linearly independent solutions:

$$w_1 = (x_3x_4, 0, -x_2x_3, -x_1x_4, 0, x_1x_2, 0, \dots, 0)$$
$$\text{and } w_2 = (0, x_2x_4, -x_2x_3, -x_1x_4, x_1x_3, 0, 0, \dots, 0) \qquad (3.13)$$

Now we can count the dimension of the w-space. For type (3.11) we have $\binom{k}{2}$, for type (3.12) we have $k\binom{k-1}{2}$ and for type (3.13) we have $\binom{k}{4}$ combinations, and the last one doubled. Hence we have totally

$$r = \binom{k}{2} + k\binom{k-1}{2} + 2\binom{k}{4} = \frac{1}{2}k\binom{k+1}{3}$$

independent w-functions.

Our w-space coincides with the space T_n in the paper [7]. However the dimensions do not agree. This is due to the fact that, starting from $n = 4$, the space T_n does not agree with the space of *all* symmetric operators on the set of skew-symmetric matrices.

From the explicit form of the w-functions, it is clear that the derivatives $\frac{\partial w_a}{\partial x_b}$ appearing in (3.7) are not linearly independent. These derivatives span exactly the space of all vectors $f = (f_1, \dots, f_\ell)$, with f_i linear in x, such that $f \cdot v = 0$. Counting the dimension s of this space in the same way as we determined the w-space itself, we obtain[2]

$$s = 2\binom{k+1}{3}.$$

3.6. **Dimension of $\mathfrak{L}$.** Now we can compute the dimension of $\mathfrak{L}$ depending on k and m. We present this in the following table. We divide the elements in certain classes, namely vertically by order and horizontally by appearing variables.

	$x\partial_z$	$x\partial_y$	$x\partial_x$	$(x)y\partial_z$	$y\partial_y$	$z\partial_z$
order 3	$\binom{k+2}{3}m$	k	–	rm	–	–
order 2	$\binom{k+1}{2}m$	k^2	–	sm	–	–
order 1	km	$k\ell$	k^2	ℓm	1	m^2
order 0	m	ℓ	k	–	–	–

Hence for the dimension of $\mathfrak{L}$ we obtain

$$\dim \mathfrak{L} = \frac{1}{12}\left(k^4 + 6k^3 + 17k^2 + 24k + 12m + 12\right)m + \frac{1}{2}\left(k^3 + 6k^2 + 5k + 2\right)$$

In particular we have for $k = 2$ and $m = 1$, that $\dim \mathfrak{L} = 39$, while for $k = 3$ and $m = 6$ (this is the case $n = 3$ in [7]), we have $\dim \mathfrak{L} = 325$.

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[2]This is different in [7] as well.