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The standard H_∞-Suboptimal Control Problem for LTI Infinite Dimensional Systems

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Abstract

In this paper we provide sufficient conditions for the solvability of the standard H_∞-suboptimal control problem for linear, time invariant, infinite-dimensional systems with finite-dimensional input and output spaces. The sufficient conditions are formulated in terms of the existence of two J-lossless factorizations. For the Wiener algebra class of the transfer functions an algorithm for solving J-spectral factorization is given.

Keywords: H_∞-control, coprime factorization, J-spectral factorization, linear, infinite-dimensional systems, J-lossless.

Mathematics Subject Classification: 93B36, 93C20, 93C80, 47A68.

Contents

1 Introduction

The standard H_∞-control problem was introduced in 1984 by J.C. Doyle [13]. In a few words the H_∞-suboptimal control problem is to find a controller which stabilizes a given plant and which makes the H_∞-norm of the associated transfer function less than a given positive real number.

Nowadays there are different techniques for solving the H_∞-control problem. We will use coprime factorizations for solving this problem. The idea of factorizing the transfer function of a (not necessarily stable) system as a ratio of two stable transfer functions was first introduced in 1972 by Vidyasagar [29]. Using this (coprime) factorization Green [20, 21] showed that for rational transfer functions the H_∞-control problem can be solved if and only if two J-spectral factorizations are solvable. This result has been extended to the infinite-dimensional case for systems in state space form $\Sigma(A, B, C, D)$ with B and C bounded by Curtain and Rodriguez [11] and for the Pritchard-Salamon class of state space systems in Weiss [31].

A generalization of the results presented by Green [21] was found by Curtain and Green [10] for the Wiener algebra, using a result of Ball and Helton [1, 2]. The same result for the rational case was proved by Meinsma [24], using different techniques and extended to dead-time systems by Meinsma and Zwart [25]. In this paper we will show that this approach extends (partially) to a large class of infinite-dimensional systems, namely to systems with their transfer functions in the quotient field of H_∞. We prove that if two J-spectral factorizations have a solution, then the standard H_∞-suboptimal control problem has a solution. Furthermore, we parametrize the set of solutions. For the Wiener algebra class of transfer functions, we present an algorithm for solving the J-spectral factorization. This algorithm is based on the work of Clancey and Gohberg [9].

For systems with a state space realization, the existence of a J-spectral factorization is equivalent to the existence of a stabilizing solution to an algebraic Riccati equation. This is shown by Green [21] for the finite-dimensional situation and by Oostveen [27] for infinite-dimensional systems. The solution of the algebraic Riccati equation is used to present an explicite state space formula for all solutions to the H_∞ control problem.

This paper basically generalizes the proofs of Meinsma as presented in [24]. We do not make use of a state-space representation of our system, neither we use Riccati equations. Since there are many systems for which the given state space realization is not well-posed, and thus the state space approach using Riccati equations is not guaranteed to work, our approach can be applied to a larger class of systems. Furthermore, we expect that our approach will have some numerical advantages. We remark that our class of systems does include dead-time systems. The procedure for solving the J-spectral factorization as presented in Section 8 can be applied to systems that can be approximated by rational functions. However, this procedure does not apply to a pure delay.

Section 3 introduces basic definitions and notations. Furthermore, some preliminary results are also presented here. Section 4 presents the standard H_∞-suboptimal control problem in two equivalent forms. The main results are presented in Section 5, it is shown that the standard H_∞-suboptimal control problem can be solved, provided that two J-spectral factorizations have a solution. We give also a formula of all stabilizing controllers for the standard H_∞-suboptimal control problem. In Section 6 an example illustrates the method of solving J-spectral factorization, for a system with time delay. A nice class of transfer functions, the Wiener algebra, is introduced in Section 7. For this class, an algorithm for solving the J-spectral factorization is presented in Section 8 .

2 Notation

$\oplus$ direct sum
$\langle \cdot, \cdot \rangle$ inner product in a Hilbert space
$\mathbb{R}$ the set of real numbers
$\mathbb{R}_- = (-\infty, 0]$
$\mathbb{R}_+ = [0, +\infty)$
$\mathbb{C}$ the set of complex numbers
$\mathbb{C}_+ = \{s \in \mathbb{C} \mid Re(s) > 0\}$
$\mathbb{C}_- = \{s \in \mathbb{C} \mid Re(s) < 0\}$
$\overline{\mathbb{C}_+} = \{s \in \mathbb{C} \mid Re(s) \geq 0\}$
$\overline{\mathbb{C}_-} = \{s \in \mathbb{C} \mid Re(s) \leq 0\}$
$j\mathbb{R} = \{s \in \mathbb{C} \mid Re(s) = 0\}$
$\overline{\sigma}(A)$ the maximum singular value of the matrix A
$L_\infty^{n\times m} = \{F : j\mathbb{R} \to \mathbb{C}^{n\times m} \mid \|F\|_{L_\infty} = \operatorname{ess\,sup}_{\omega\in\mathbb{R}} \overline{\sigma}(F(j\omega)) < \infty\}$
$H_\infty^{n\times m} = \{F : \mathbb{C}_+ \to \mathbb{C}^{n\times m} \mid F \text{ is analytic in } \mathbb{C}_+ \text{ and } \|F\|_{H_\infty} = \sup_{s\in\mathbb{C}_+} \overline{\sigma}(F(s)) < \infty\}$
$F_\infty^{n\times m} = \{H^{-1}G \mid G \in H_\infty^{n\times m},\ H \in H_\infty^{n\times n},\ \det H \neq 0\}$
$RH_\infty{}^{n\times m} = \{A \in H_\infty^{n\times m} \mid A \text{ has rational entries}\}$
$GH_\infty{}^{n\times n}$ the units of $H_\infty{}^{n\times n}$
$L_2^n = \{f : j\mathbb{R} \to \mathbb{C}^n \mid \|f\|_{L_2}^2 = \int_{-\infty}^{+\infty} \mid f(j\omega) \mid^2 d\omega < \infty\}$
$H_2^n = \{f : \mathbb{C}_+ \to \mathbb{C}^n \mid f \text{ is analytic in } \mathbb{C}_+ \text{ and } \|f\|_{H_2}^2 = \sup_{r>0} \int_{-\infty}^{+\infty} \|(f(r+j\omega))\|^2 d\omega < \infty\}$
$H_2^{n,\perp} = \{f : \mathbb{C}_- \to \mathbb{C}^n \mid f \text{ is analytic in } \mathbb{C}_-,\ \|f\|_{H_2^\perp}^2 = \sup_{r<0} \int_{-\infty}^{+\infty} \|(f(r+j\omega))\|^2 d\omega < \infty\}$
δ the delta distribution
$\overline{s}$ the complex conjugate of the complex number s
A^* the transpose conjugate of the matrix $A \in \mathbb{C}^{n\times m}$
$A^\sim(s) = [A(-\overline{s})]^*$
$J_{\gamma,m,n} = \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_n \end{bmatrix}$ for real $\gamma > 0$
$B_A = \{x \in H_2^{m,\perp} \mid Ax \in H_2^n \text{ for given } A \in H_\infty^{n\times m}\}$
Γ the unit circle
F_Γ^+ the open unit disc
F_Γ^- the exterior of the unit disc
A algebra of complex matrix value functions on Γ
GA the group of invertible elements of the algebra A
$R(\Gamma)$ the space of rational complex-valued functions on Γ
$C(\Gamma)$ the space of complex-valued continuous functions on Γ
$R^{n\times n}(\Gamma)$ the algebra of $n\times n$-matrices whose entries are elements in $R(\Gamma)$
$C^{n\times n}(\Gamma)$ the algebra of $n\times n$-matrices whose entries are elements in $C(\Gamma)$
$C^\pm(\Gamma)$ the closed subalgebras of $C(\Gamma)$ consisting of those continuous functions that are restrictions to Γ of functios holomorphic in $F_\Gamma^\pm$ and continuous on $F_\Gamma^\pm \cup \Gamma$
$R^\pm(\Gamma) = C^\pm(\Gamma) \cap R(\Gamma)$
$GR^\pm(\Gamma)$ the units of $R(\Gamma)^{n\times n}$
$GC^\pm(\Gamma)$ the units of $C(\Gamma)^{n\times n}$

3 Preliminaries

Before we can formulate and solve our problem, we need to introduce some definitions and notations. This we will do in the first three subsections of this section. In the last subsection we present some preliminary results.

3.1 Transfer matrices

We begin by recalling the definition of Hardy spaces. The Hardy spaces $H_2^{n,\perp}$ and H_2^n, are defined as

$$H_2^n = \{f : \mathbb{C}_+ \to \mathbb{C}^n \mid f \text{ is analytic in } \mathbb{C}_+ \text{ and } \|f\|_{H_2}^2 = \sup_{r>0} \int_{-\infty}^{+\infty} \|(f(r+j\omega))\|^2 d\omega < \infty\},$$

$$H_2^{n,\perp} = \{f : \mathbb{C}_- \to \mathbb{C}^n \mid f \text{ is analytic in } \mathbb{C}_-,\ \|f\|_{H_2^\perp}^2 = \sup_{r<0} \int_{-\infty}^{+\infty} \|(f(r+j\omega))\|^2 d\omega < \infty\}.$$

The elements of $H_2^{n,\perp}$ and H_2^n may be identified with those elements of $L_2(j\mathbb{R}, \mathbb{C}^n)$ that have an analytic extension in $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively. Moreover, under this identification, $H_2^{n,\perp}$ and H_2^n are Hilbert spaces in the usual inner product on $L_2(j\mathbb{R}, \mathbb{C}^n)$;

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(j\omega) g(j\omega) d\omega.$$

$H_2^{n,\perp}$ and H_2^n are orthogonal subspaces of $L_2(j\mathbb{R}, \mathbb{C}^n)$, and

$$L_2(j\mathbb{R}, \mathbb{C}^n) = H_2^{n,\perp} \oplus H_2^n. \tag{1}$$

Under the two-sided Laplace transform, $H_2^{n,\perp}$ and H_2^n are isomorphic to $L_2(\mathbb{R}_-, \mathbb{C}^n)$ and $L_2(\mathbb{R}_+, \mathbb{C}^n)$, respectively

$$H_2^{n,\perp} = Lap(L_2(\mathbb{R}_-, \mathbb{C}^n)), \quad H_2^n = Lap(L_2(\mathbb{R}_+, \mathbb{C}^n)). \tag{2}$$

This transformation is bijective and preserves the inner product.
From (1) and (2) we see that we can uniquely decompose $F = Lap(f)$, $f \in L_2(\mathbb{R}, \mathbb{C}^n)$, as $F = F_- + F_+$, where $F_- \in H_2^{n,\perp}$ and $F_+ \in H_2^n$. Obviously, $F_- = Lap(\pi_- f)$ and $F_+ = Lap(\pi_+ f)$, where π_- and π_+ are the orthogonal projections from $L_2(\mathbb{R}, \mathbb{C}^n)$ to $L_2(\mathbb{R}_-, \mathbb{C}^n)$ and $L_2(\mathbb{R}_+, \mathbb{C}^n)$, respectively.
The Hardy spaces H_2^n and $H_2^{n,\perp}$ will denote our signals and the Hardy space H_∞ will denote our class of stable matrices.

Definition 3.1 (Stable matrices) *Let $\overline{\sigma}(A)$ denotes the maximum singular value of the matrix A. We consider the Hardy space*

$$H_\infty^{n\times m} = \{F : \mathbb{C}_+ \to \mathbb{C}^{n\times m} \mid F \text{ is analytic in } \mathbb{C}_+ \text{ and } \|F\|_{H_\infty} = \sup_{s\in\mathbb{C}_+} \overline{\sigma}(F(s)) < \infty\},$$

to be the set of our stable *matrices. So a matrix F is stable if it is analytic and bounded in the open right half plane. We say that a square matrix valued function is* bistable *if it is stable and the inverse exists and it is also stable. We denote by GH_∞ (the units of H_∞) the set of bistable matrices.*

Definition 3.2 (Transfer matrices) *The quotient field of* $H_\infty^{n\times m}$ *is denoted by* $F_\infty^{n\times m}$*,i.e.,*

$$F_\infty^{n\times m} = \{H^{-1}G \mid G \in H_\infty^{n\times m},\ H \in H_\infty^{n\times n},\ \det H \neq 0\}$$

and this will be our class of transfer matrices.

We will usually write H_∞ or F_∞ when we mean $H_\infty^{n\times m}$ or $F_\infty^{n\times m}$ and when there is no danger for confusion. In the following five lemmas we summarise some standard results on stable matrices.
We introduce the following notation

$$A^\sim(s) = [A(-\overline{s})]^*,$$

where A^* is the transpose conjugate of the matrix $A \in \mathbb{C}^{n\times m}$. It is easy to see that

$$G^\sim(j\omega) = G(-\overline{j\omega})^* = G(j\omega)^*,$$

for all $\omega \in \mathbb{R}$.

Lemma 3.3 *For a matrix valued function* $F \in H_\infty^{n\times m}$*, we have that* $\|F\|_{H_\infty} < \gamma$ *if and only if* $F^\sim F - \gamma^2 I < 0$ *on the imaginary axis.*

Proof: For the matrix $F \in H_\infty^{n\times m} \subset L_\infty^{n\times m}$ we have the relation

$$\|F\|_{H_\infty} = \|F\|_{L_\infty} = \sup_{u\in L_2^m} \frac{\|Fu\|_2}{\|u\|_2},$$

see [12], Theorem A.6.26. Hence $\|F\|_{H_\infty} < \gamma$ is equivalent to

$$\|Fu\|_2^2 - \gamma^2 \|u\|_2^2 < 0,$$

for all $u \in L_2^m$. Using the definition of $\|\cdot\|_2$ and $A^\sim$, we see that this is equivalent with

$$\int_{-\infty}^{\infty} \left[u^\sim(j\omega) F^\sim(j\omega) F(j\omega) u(j\omega) - \gamma^2 u^\sim(j\omega) u(j\omega)\right] d\omega < 0$$

for all $u \in L_2^m$. Rewriting this gives

$$\int_{-\infty}^{\infty} u^\sim(j\omega) \left[F^\sim(j\omega) F(j\omega) - \gamma^2 I\right] u(j\omega)\, d\omega < 0,$$

for all $u \in L_2^m$. This expression is equivalent with $F^\sim F - \gamma^2 I < 0$ on the imaginary axis. ■

Lemma 3.4 *Consider a matrix valued function* $F \in L_\infty^{n\times n}$*. The following conditions are equivalent*

1. *F is invertible in $L_\infty^{n\times n}$;*
2. *$F^\sim F > \epsilon I$ for some strictly positive ϵ;*
3. *$\|Fu\|_{L_2^n} > \sqrt{\epsilon}\|u\|_{L_2^n}$.*

Proof: 1.$\Rightarrow$ 3. Suppose that $F \in L_\infty^{n\times n}$ is invertible in $L_\infty^{n\times n}$ and let $L \in L_\infty^{n\times n}$ be its inverse. Then $LFu = u$ for all $u \in L_2^n$ and

$$\|u\|_{L_2^n} = \|LFu\|_{L_2^n} \leq \|L\|_{L_\infty} \|Fu\|_{L_2^n} \Rightarrow \|Fu\|_{L_2^n} \geq \frac{1}{\|L\|_{L_\infty}} \|u\|_{L_2^n}.$$

Choosing $\epsilon > 0$ such that

$$\frac{1}{\|L\|_{L_\infty}} > \sqrt{\epsilon},$$

gives the desired result.
3.$\Rightarrow$1. If $\|Fu\|_{L_2^n} > \sqrt{\epsilon}\,\|u\|_{L_2^n}$, then $\ker F = \{0\} \subset L_2^n$, and so the correspondence between the domain and the range is one-to-one and thus F^{-1} exists. Suppose that F^{-1} is not bounded almost everywhere. Then, for eny $M > 0$ there exists a $v \in L_2$ such that

$$\|F^{-1}v\|_2 \geq M\|v\|_2. \tag{3}$$

If we replace now u with $F^{-1}v$ in the inequality

$$\|Fu\|_{L_2^n} > \sqrt{\epsilon}\,\|u\|_{L_2^n}$$

and we use also (3), we obtain that

$$\|v\|_2 = \|FF^{-1}v\|_2 > \sqrt{\epsilon}\|F^{-1}v\|_2 \geq \sqrt{\epsilon}M\|v\|_2.$$

This is not true if we choose M big enough. As a conclusion we have that $F^{-1} \in L_\infty$.
2.$\Leftrightarrow$3. The condition $F^\sim F > \epsilon I$ on the imaginary axis is equivalent to

$$\int_{-\infty}^{\infty} u^\sim(j\omega)\left[F^\sim(j\omega)\,F(j\omega) - \epsilon I\right] u(j\omega)\, d\omega > 0, \tag{4}$$

for all $u \in L_2^m$. Expanding the expression in (4) gives

$$\int_{-\infty}^{\infty} u^\sim(j\omega)\, F^\sim(j\omega)\, F(j\omega)\, u(j\omega) - \epsilon u^\sim(j\omega)\, u(j\omega)\, d\omega > 0$$

which is equivalent with

$$\|Fu\|_{L_2}^2 \geq \epsilon\|u\|_{L_2}^2$$

for all $u \in L_2^n$. ■

Using the previous lemma and the fact that F^T is invertible if and only if F is, the following result easily follows.

Lemma 3.5 *A matrix $F \in L_\infty^{n\times n}$ is invertible in $L_\infty^{n\times n}$ if and only if $FF^\sim > \epsilon I$ for some strictly positive ϵ.*

For stable transfer matrices there exists a similar result.

Lemma 3.6 *A stable transfer matrix F is bistable if and only if*

$$F(s)^* F(s) \geq \epsilon I \tag{5}$$

for all $s \in \mathbb{C}_+$.

Proof: From (5) it follows that for every $s \in \mathbb{C}_+$ the matrix $F(s)$ has an inverse. Standard complex analysis gives that this inverse is an anaalytic function of $s \in \mathbb{C}_+$. Multiplying (5) from the right with this inverse and from the left by the transposed of the inverse gives

$$I \geq \epsilon F(s)^{-*} F(s)^{-1}.$$

Or equivalently

$$\|F^{-1}(s)u\|^2 \leq \frac{1}{\epsilon}\|u\|^2$$

for all $s \in \mathbb{C}_+$ and $u \in \mathbb{C}^n$. This shows that $F^{-1} \in H_\infty$. The other implication follows the opposite direction. ■

Lemma 3.7 *Suppose that $F \in L_\infty^{n\times n}$ with $\|F\|_{L_\infty} < 1$. Then F is stable if and only if $(I+F)^{-1}$ is stable.*

Proof: The fact that $(I+F)^{-1}$ is stable implies that F is stable is proved in Lemma 8.3.5 in [12]. Now suppose that F is stable. It has to be shown that $(I+F)^{-1}$ is stable. We prove that $(I+F)^{-1} = \sum_{k=0}^{\infty}(-F)^k$. F is stable implies that the sequence of partial sums

$$f_n = \sum_{k=0}^{n}(-F)^k$$

is also stable. Since F is stable we have that $\|F\|_{L_\infty} = \|F\|_{H_\infty}$. Furthermore, we see that

$$\|f_n - f_m\|_{H_\infty} \leq \sum_{k=m+1}^{n} \|F\|_{H_\infty}^k\,,\ m < n.$$

Since $\|F\|_{H_\infty} < 1$, this implies that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. H_∞ is a Banach space, so it is complete, and thus the sequence f_n has a limit in H_∞. Let us denote the limit with f. We have that

$$f\,(I+F) = \lim_{n\to\infty} f_n\,(I+F) = \lim_{n\to\infty}\sum_{k=0}^{n}(-F)^k\,(I+F) = \lim_{n\to\infty}\left(I - (-F)^{n+1}\right) = I.$$

The equality

$$f_n(I+F) = (I+F)f_n$$

implies that $(I+F)f = I$. Therefore $f = (I+F)^{-1}$, and since $f \in H_\infty$ we have that $(I+F)^{-1}$ is stable. ■

The proof of the next lemma will be omitted, since it is straightforward.

Lemma 3.8 *If the matrix F is an element of $H_\infty^{n\times m}$ then $F^T \in H_\infty^{m\times n}$ and $\left\|F^T\right\|_{H_\infty} = \|F\|_{H_\infty}$.*

3.2 Coprime Factorization

The concept of coprime factorization was introduced for transfer matrices by M. Vidyasagar in [30], and it will play an important role in the sequel.

Definition 3.9 (Left-coprime factorisation) *An element $G \in F_\infty$ has a left-coprime factorisation over H_∞ if there exist stable matrices D, N, X, $Y \in H_\infty$ such that $G = D^{-1}N$ and $DX + NY = I$.*

Definition 3.10 (Right-coprime factorisation) *An element $G \in F_\infty$ has a right-coprime factorisation over H_∞ if there exist stable matrices D, N, X, $Y \in H_\infty$ such that $G = ND^{-1}$ and $XD + YN = I$.*

The right(left)-coprime factorization is unique except for the possibility of multiplying the "numerator" and "denominator" matrices on the right (left) by a bistable matrix (see [30]). Not every transfer matrix has a coprime factorization. However, the set of stabilizable elements have such a factorization, as was proved by Smith [28].

Theorem 3.11 *Every stabilizable plant $G \in F_\infty$ has a left(right)-coprime factorization over H_∞.*

3.3 Inner and J-lossless matrices

The concept of inner and J-lossless matrices are closely related as will be shown in Lemma 3.17. We begin with the definition of inner matrices.

Definition 3.12 (Inner matrices) *A matrix valued function $G \in H_\infty^{r\times c}$ is said to be* inner *if*

$$G^{\sim}(j\omega)G(j\omega) = I_c \tag{6}$$

for almost all $\omega \in \mathbb{R}$.

Observe that if F is inner, then $r \geq c$. In other words, the matrix is tall.

Lemma 3.13 *A matrix valued function $G \in H_\infty^{r\times c}$ is inner if and only if*

$$G^*(s)G(s) \leq I_c \tag{7}$$

for almost all $s \in \overline{\mathbb{C}_+}$, with equality on the imaginary axis.

Proof: If (7) is an equality almost everywhere on the imaginary axis then obviously the condition from the definition of inner matrices is satisfied.
For the other implication suppose that the matrix valued function $G \in H_\infty^{r\times c}$ is inner, which means that the equality (6) is satisfied almost everywhere on the imaginary axis, so

$$\operatorname{ess\,sup}_{s\in j\mathbb{R}} \|G(s)\| = 1.$$

Since $G \in H_\infty^{r\times c}$, we have that

$$\sup_{s\in\mathbb{C}_+} \|G(s)\| = \operatorname{ess\,sup}_{s\in j\mathbb{R}} \|G(s)\|$$

which implies that

$$\|G(s)\|^2 \leq 1 \text{ for almost all } s \in \overline{\mathbb{C}_+}.$$

This means that all the singular values of the matrix $G(s)^*G(s)$ are less or equal to one for every fixed value of s. It follows that

$$G(s)^*G(s) - I \leq 0, \text{ for almost all } s \in \overline{\mathbb{C}_+}.$$

■

For the definition of J-lossless we need to consider the matrix

$$J_{\gamma,n,m} = \begin{bmatrix} I_n & 0 \\ 0 & -\gamma I_m \end{bmatrix},$$

where γ is a given nonnegative real number and $n, m \in \mathbb{N}$. When $\gamma = 1$ we will write $J_{n,m}$ instead of $J_{1,n,m}$. Sometimes we will use J without indices.

Definition 3.14 (J-lossless) *A matrix $G \in H_\infty^{(r+p)\times(q+p)}$ is $J_{\gamma,q,p}$-lossless (or $J_{\gamma,q,p}$-inner) if*

$$G(s)^* J_{\gamma,r,p} G(s) \leq J_{q,p} \text{ for all } s \in \overline{\mathbb{C}_+}$$

with equality on the imaginary axis. This means that

$$G^\sim(j\omega) J_{\gamma,r,p} G(j\omega) = J_{q,p}, \quad \text{for almost all } \omega \in \mathbb{R}, \tag{8}$$

$$G(s)^* J_{\gamma,r,p} G(s) \leq J_{q,p}, \text{ for all } s \in \mathbb{C}_+. \tag{9}$$

Notice that a J-lossless matrix need not to be square; it is tall ($r \geq q$), and it is assume to be partitioned so that the lower-right corner is square.

Definition 3.15 (co-J-lossless) *A partitioned matrix $G \in H_\infty^{(q+p)\times(r+p)}$ is co-J-lossless if it satisfies*

$$G(j\omega) J_{\gamma,q,p} G^\sim(j\omega) = J_{r,p}, \text{ for almost all } \omega \in \mathbb{R}, \tag{10}$$

$$G(s) J_{\gamma,q,p} G(s)^* \leq J_{r,p}, \text{ for all } s \in \mathbb{C}_+. \tag{11}$$

Remark 3.16 *It is easy to see that a matrix $G \in L_\infty$ is co-J-lossless if and only if $G(s)^T$ is J-lossless.*

The following result descibes the relation between inner and J-lossless transfer matrices.

Lemma 3.17 *We have the following relation between inner and J-lossless.*

1. *If $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ belongs to $H_\infty^{(n_z+n_y)\times(n_u+n_y)}$, and it is is J_{n_u,n_y}-lossless, then*

$$\left\{(z, w, u, y) \mid \begin{bmatrix} z \\ w \end{bmatrix} = M \begin{bmatrix} u \\ y \end{bmatrix}, \begin{bmatrix} u \\ y \end{bmatrix} \in H_2^{n_u+n_y}\right\} \tag{12}$$

   equals

$$\left\{(z, w, u, y) \mid \begin{bmatrix} z \\ y \end{bmatrix} = G \begin{bmatrix} w \\ u \end{bmatrix}, \begin{bmatrix} w \\ u \end{bmatrix} \in H_2^{n_y+n_u}\right\}, \tag{13}$$

   where G is the matrix defined as

$$G = \begin{bmatrix} M_{12}M_{22}^{-1} & M_{11} - M_{12}M_{22}^{-1}M_{21} \\ M_{22}^{-1} & -M_{22}^{-1}M_{21} \end{bmatrix}. \tag{14}$$

   Moreover, G is inner.

2. *If $G \in H_\infty^{(n_z+n_y)\times(n_y+n_u)}$ is an inner matrix whose lower left $n_y \times n_y$ block element is in $GH_\infty^{n_y\times n_y}$, then there exists a unique stable J_{n_u,n_y}-lossless M for which the sets given in (12) and (13) coincide. Moreover, if G is given as*

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

then M is given as

$$M = \begin{bmatrix} G_{12} - G_{11}G_{21}^{-1}G_{22} & G_{11}G_{21}^{-1} \\ -G_{21}^{-1}G_{22} & G_{21}^{-1} \end{bmatrix}. \tag{15}$$

Proof: We begin by proving the first assertion. It is easy to see that for $s \in \overline{\mathbb{C}_+}$

$$M(s)^* J_{n_u,n_y} M(s) = \begin{bmatrix} M_{11}(s)^*M_{11}(s) - M_{21}(s)^*M_{21}(s) & M_{11}(s)^*M_{12}(s) - M_{21}(s)^*M_{22}(s) \\ M_{12}(s)^*M_{11}(s) - M_{22}(s)^*M_{21}(s) & M_{12}(s)^*M_{12}(s) - M_{22}(s)^*M_{22}(s) \end{bmatrix} \tag{16}$$

Since M is J_{n_u,n_y}-lossless, we have that this, in particular, implies that

$$-M_{22}(s)^*M_{22}(s) \le M_{12}(s)^*M_{12}(s) - M_{22}(s)^*M_{22}(s) \le -I_{n_y} < 0 \tag{17}$$

for all $s \in \overline{\mathbb{C}_+}$. With Lemma 3.6 this implies that M_{22} is bistable, and thus G in (14) is well-defined.
If (z, w, u, y) is such that $\begin{bmatrix} z \\ w \end{bmatrix} = M \begin{bmatrix} u \\ y \end{bmatrix}$, with $\begin{bmatrix} u \\ y \end{bmatrix} \in H_2$ then, by the stability of M, we have that $\begin{bmatrix} z \\ w \end{bmatrix} \in H_2$. Furthermore,

$$\begin{aligned} z &= M_{11}u + M_{12}y; \\ w &= M_{21}u + M_{22}y, \end{aligned}$$

with M_{22} invertible, implies that

$$y = M_{22}^{-1}w - M_{22}^{-1}M_{21}u \tag{18}$$

and

$$z = \left(M_{11} - M_{12}M_{22}^{-1}M_{21}\right)u + M_{12}M_{22}^{-1}w.$$

Thus (z, w, u, y) lies in the set defined by (13). The other inclusion can be proved similarly. It remains to show that G is inner. We show that

$$G^\sim(j\omega)G(j\omega) = I \Longleftrightarrow M^\sim(j\omega)J_{n_z,n_w}M(j\omega) = J_{n_u,n_y}, \text{ for } \omega \in \mathbb{R}. \tag{19}$$

Let $\begin{bmatrix} w \\ u \end{bmatrix} \in \mathbb{C}^{n_y+n_u}$ and define $\begin{bmatrix} z \\ y \end{bmatrix} \in \mathbb{C}^{n_z+n_y}$ as

$$\begin{bmatrix} z \\ y \end{bmatrix} = G(j\omega) \begin{bmatrix} w \\ u \end{bmatrix}.$$

Then $G^\sim(j\omega)G(j\omega) = I$ implies that

$$\|z\|^2 + \|y\|^2 = \|w\|^2 + \|u\|^2.$$

This is equivalent with

$$\|z\|^2 - \|w\|^2 = \|u\|^2 - \|y\|^2 .$$

Using the definition of G, we see that this is equivalent to

$$\begin{bmatrix} u^* & y^* \end{bmatrix} M(j\omega)^* J_{n_z,n_y} M(j\omega) \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} u^* & y^* \end{bmatrix} J_{n_u,n_y} \begin{bmatrix} u \\ y \end{bmatrix}$$

This holds for all $\begin{bmatrix} u \\ y \end{bmatrix}$ for which there exist a w such that $\begin{bmatrix} z \\ y \end{bmatrix} = G(j\omega) \begin{bmatrix} w \\ u \end{bmatrix}$. From (18), we see that this set equals $\mathbb{C}^{n_u+n_y}$. In other words (19) is shown. In the same way it can be proved the equivalence

$$G^*(s)G(s) \leq I \Longleftrightarrow M^*(s)J_{n_z,n_w}M(s) \leq J_{n_u,n_y}, \text{ for } s \in \mathbb{C}_+. \tag{20}$$

Using now Lemma 3.12 we have proved the first assertion. For the second assertion we define the matrix M as

$$M = \begin{bmatrix} G_{12} - G_{11}G_{21}^{-1}G_{22} & G_{11}G_{21}^{-1} \\ -G_{21}^{-1}G_{22} & G_{21}^{-1} \end{bmatrix}. \tag{21}$$

The proof follows now the oposite direction. ■

In order to check whether a transfer matrix is J-lossless one can replace condition (9) by an equivalent condition.

Lemma 3.18 (Characterization of J-lossless) *A partitioned matrix $M \in H_\infty$ is $J_{\gamma,q,p}$-lossless if and only if the following two conditions are satisfied*

1. *$M^\sim(j\omega)J_{\gamma,q,p}M(j\omega) = J_{r,p}$, for $\omega \in \mathbb{R}$;*
2. *the lower right $p \times p$ block element of the matrix M is in $GH_\infty^{p\times p}$.*

Proof: From Lemma 3.17 it follows that M is $J_{\gamma,q,p}$-lossless if and only if the matrix G defined as

$$G = \begin{bmatrix} M_{12}M_{22}^{-1} & M_{11} - M_{12}M_{22}^{-1}M_{21} \\ M_{22}^{-1} & -M_{22}^{-1}M_{21} \end{bmatrix}.$$

where M is partitioned

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

is inner and its lower-left block, i.e. M_{22}^{-1} is in GH_∞. Using now the definition of an inner matrix and the equivalence (19) we have that G is inner is equivalent with

$$M^\sim(j\omega)J_{\gamma,q,p}M(j\omega) = J_{r,p}, \tag{22}$$

so the equivalence is proved. ■

For co-J-lossless we have a similar result. The proof is easy by using Remark 3.16.

Lemma 3.19 (Characterization of co-J-lossless) *A partitioned matrix valued function $M \in H_\infty$ is co-$J_{\gamma,q,p}$-lossless if and only if the next two condition are satisfied*

1. *$M(j\omega)J_{\gamma,q,p}M^\sim(j\omega) = J_{r,p}$, for $\omega \in \mathbb{R}$;*
2. *the lower right $p \times p$ block element of M is in $GH_\infty^{p\times p}$.*

3.4 Preliminary results

In this last subsection we will present some results that will be useful in the sequel.

Definition 3.20 (Positivity) *Consider the Hardy space* $H_2^{m+n,\perp}$ *with the inner product* $\langle \cdot, \cdot \rangle$*. A subspace* B *of* $H_2^{m+n,\perp}$ *is* positive *with respect to the* $J_{\gamma,m,n}$*-inner product*

$$[f, g] := \langle f, J_{\gamma,m,n} g \rangle, \; f, g \in H_2^{m+n,\perp},$$

if for every $x \in B$

$$\langle x, J_{\gamma,m,n} x \rangle \geq 0.$$

It is strictly positive *with respect to the* $J_{\gamma,m,n}$*-inner product if there exists an* $\epsilon > 0$ *such that for all non-zero* $x \in B$ *holds*

$$\langle x, J_{\gamma,m,n} x \rangle \geq \epsilon \langle x, x \rangle.$$

Definition 3.21 (The set generated by a stable matrix) *Given a stable matrix* $G \in H_\infty^{n \times m}$ *define*

$$B_G = \{x \in H_2^{\perp,m} \mid Gx \in H_2^n\}. \tag{23}$$

We say that B_G *is the set generated by* G*.*

Note that the set generated by G is a subset of $H_2^\perp$. The next lemma shows that the set generated by G is invariant under premultiplication by a bistable matrix.

Lemma 3.22 *Consider two stable transfer matrices* G*,* $\overline{G} \in H_\infty^{n \times m}$*. If there exists a* $W \in GH_\infty^{n \times n}$ *such that* $\overline{G} = WG$*, then* G *and* $\overline{G}$ *generate the same set, i.e.,* $B_G = B_{\overline{G}}$*.*

Proof: Suppose that there exists a $W \in GH_\infty^{n \times n}$ such that $\overline{G} = WG$. We have to prove that $B_G = B_{\overline{G}}$. Since $W \in GH_\infty^{n \times n}$ we know that $WH_2^n = H_2^n$ (see [15]). From the definition of B_G, we have that $GB_G \subset H_2^n$ and so

$$WGB_G \subset WH_2^n = H_2^n,$$

which implies that

$$W(GB_G) \subset H_2^n.$$

However,

$$B_{WG} = \{x \in H_2^{\perp,m} \mid WGx \in H_2^n\},$$

which implies that

$$B_G \subset B_{WG}.$$

Using that $\overline{G} = WG$ we get $B_G \subset B_{\overline{G}}$. The other inclusion is very easy to prove by using the equality $G = W^{-1}\overline{G}$. ■

Lemma 3.23 (Stability implies positivity) *Let* H_1 *and* H_2 *be two stable transfer matrices such that* $H_2 \in GH_\infty^{p \times p}$ *and* $\|H_2^{-1} H_1\|_{H_\infty} \leq \gamma$*. Then the set generated by* $[H_1 \; H_2]$*,* $B_{[H_1 \quad H_2]} \subset H_2^{q+p,\perp}$*, is positive with respect to the* $J_{\frac{1}{\gamma^2},q,p}$ *inner product.*

If we have strictly inequality, i.e., $\|H_2^{-1} H_1\|_{H_\infty} < \gamma$*, then* $B_{[H_1 \quad H_2]} \subset H_2^{q+p,\perp}$ *is strictly positive with respect to the* $J_{\frac{1}{\gamma^2},q,p}$*-inner product.*

Proof: Since $H_2 \in GH_\infty$, we have that

$$\begin{bmatrix} H & I_p \end{bmatrix} = H_2^{-1} \begin{bmatrix} H_1 & H_2 \end{bmatrix},$$

where $H := H_2^{-1} H_1$. Applying Lemma 3.22, the following equality holds

$$B_{\begin{bmatrix} H_1 & H_2 \end{bmatrix}} = B_{\begin{bmatrix} H & I_p \end{bmatrix}}.$$

Let $w \in B_{\begin{bmatrix} H & I_p \end{bmatrix}}$ be partitioned as $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, compatibly with the partition of $\begin{bmatrix} H_1 & I_p \end{bmatrix}$. From the definition of $B_{\begin{bmatrix} H & I_p \end{bmatrix}}$ it follows that

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in H_2^{q+p,\perp} \text{ and } \begin{bmatrix} H & I_p \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in H_2^p$$

which is equivalent to

$$w_1 \in H_2^{q,\perp} \text{ and } w_2 = -Q(Hw_1),$$

where Q is the projection from $L_2(j\mathbb{R};\mathbb{C}^p)$ to $H_2^{p,\perp}$.
We have

$$\begin{aligned}
\langle w, J_{\frac{1}{\gamma^2},q,p} w \rangle &= \langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, J_{\frac{1}{\gamma^2},q,p} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \rangle = \|w_1\|_{L_2}^2 - \frac{1}{\gamma^2} \|w_2\|_{L_2}^2 \\
&= \|w_1\|_{L_2}^2 - \frac{1}{\gamma^2} \|Q(Hw_1)\|_{L_2}^2 \\
&\geq \|w_1\|_{L_2}^2 (1 - \frac{1}{\gamma^2} \|H\|_{H_\infty}^2) \\
&= \frac{1}{\gamma^2} \|w_1\|_{L_2}^2 (\gamma^2 - \|H_2^{-1} H_1\|_{H_\infty}) \geq 0. \qquad (24)
\end{aligned}$$

Hence, we proved that for every $w \in B_{\begin{bmatrix} H & I_p \end{bmatrix}}$ there holds $\langle w, J_{\frac{1}{\gamma^2},q,p} w \rangle \geq 0$ which means that $B_{\begin{bmatrix} H & I_p \end{bmatrix}}$ is positive in the J-inner product.
The implication for the strictly positive case follows directly from (24) and using that $\|H_2^{-1} H_1\| < \gamma$. ■

We end this section with three technical results that will be usefull later.

Lemma 3.24 *Consider the stable transfer matrices V_1,V_2,U_1 and U_2, with $V_1 \in H_\infty^{n_z \times n_y}$, $V_2 \in H_\infty^{n_z \times n_z}$, $U_1 \in H_\infty^{n_y \times n_z}$, $U_2 \in H_\infty^{n_z \times n_z}$. Assume that V_2 and U_2 are invertible in $L_\infty^{n_z \times n_z}$, and that the following inequalities are satisfied*

$$\left\| V_2^{-1} V_1 \right\|_{L_\infty} \leq 1 \quad \text{and} \quad \left\| U_1 U_2^{-1} \right\|_{L_\infty} < 1.$$

Then $V_1 U_1 + V_2 U_2$ is bistable if and only if V_2 and U_2 are bistable.

Proof Since $V_1 \in H_\infty^{n_z \times n_y}$, $V_2 \in H_\infty^{n_z \times n_z}$, $U_1 \in H_\infty^{n_y \times n_z}$, $U_2 \in H_\infty^{n_z \times n_z}$ it follows that $V_1 U_1 + V_2 U_2$ is stable. Therefore it is enough to prove that $(V_1 U_1 + V_2 U_2)^{-1}$ is stable if and only if V_2 and U_2 are bistable.

Suppose that V_2 and U_2 are bistable and define $F = V_2^{-1}V_1U_1U_2^{-1}$. Since by assumption V_1, V_2^{-1}, U_1, U_2^{-1} are stable, F is also stable. Furthermore,

$$\|F\|_{L_\infty} = \left\|V_2^{-1}V_1U_1U_2^{-1}\right\|_{L_\infty} \le \left\|V_2^{-1}V_1\right\|_{L_\infty} \left\|U_1U_2^{-1}\right\|_{L_\infty} < 1.$$

Applying Lemma 3.7 it follows that $(I+F)^{-1}$ is stable. Therefore

$$(V_1U_1 + V_2U_2)^{-1} = U_2^{-1}\left(V_2^{-1}V_1U_1U_2^{-1} + I\right)^{-1} V_2^{-1} = U_2^{-1}(F+I)^{-1} V_2^{-1}$$

is stable.

Conversely, if $(V_1U_1 + V_2U_2)^{-1}$ is stable, then

$$\left(I + V_2^{-1}V_1U_1U_2^{-1}\right)^{-1} = U_2 (V_1U_1 + V_2U_2)^{-1} V_2$$

is also stable. Since $\left\|V_2^{-1}V_1U_1U_2^{-1}\right\|_{L_\infty} < 1$ we can apply Lemma 3.7 and conclude that $V_2^{-1}V_1U_1U_2^{-1}$ is stable as well. This trivially implies that $I + V_2^{-1}V_1U_1U_2^{-1}$ is stable. Now we see that

$$\begin{aligned} U_2^{-1} &= U_2^{-1}\left(I + V_2^{-1}V_1U_1U_2^{-1}\right)^{-1}\left(I + V_2^{-1}V_1U_1U_2^{-1}\right) = \\ &= (V_1U_1 + V_2U_2)^{-1} V_2 \left(I + V_2^{-1}V_1U_1U_2^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} V_2^{-1} &= \left(I + V_2^{-1}V_1U_1U_2^{-1}\right)\left(I + V_2^{-1}V_1U_1U_2^{-1}\right)^{-1} V_2^{-1} \\ &= \left(I + V_2^{-1}V_1U_1U_2^{-1}\right) U_2 (V_1U_1 + V_2U_2)^{-1}. \end{aligned}$$

So U_2^{-1} and V_2^{-1} are stable. ∎

Lemma 3.25 *Let* $P \in H_\infty^{(n_w+n_z)\times(n_y+n_z)}$ *, and suppose that*

$$P^{\sim}(j\omega)\, J_{\gamma,n_w,n_z} P(j\omega) = J_{n_y,n_z}, \text{ for almost all } \omega \in \mathbb{R}. \tag{25}$$

Consider the equality

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{26}$$

with $X_1 \in H_\infty^{n_w\times n_z}$, $X_2 \in H_\infty^{n_z\times n_z}$, $Q_1 \in H_\infty^{n_y\times n_z}$, $Q_2 \in H_\infty^{n_z\times n_z}$, $P_{11} \in H_\infty^{n_w\times n_y}$, $P_{12} \in H_\infty^{n_w\times n_z}$, $P_{21} \in H_\infty^{n_z\times n_y}$, $P_{22} \in H_\infty^{n_z\times n_z}$. *Then the following two conditions are equivalent*

1. X_2 *is bistable and* $\left\|X_1X_2^{-1}\right\|_{H_\infty} < \gamma$
2. P_{22} *and* Q_2 *are bistable and* $\left\|Q_1Q_2^{-1}\right\|_{H_\infty} < 1$

Proof: See Theorem 6.2 from [26]. ∎

Lemma 3.26 *Let* $M \in H_\infty^{(n_y+n_z)\times(n_w+n_z)}$ *with* $n_y = n_w$*, and suppose that*

$$M(j\omega)\, J_{\gamma,n_y,n_z} M^{\sim}(j\omega) = J_{n_w,n_z}, \text{ for almost all } \omega \in \mathbb{R}. \tag{27}$$

Consider the equality

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{28}$$

with $H_1 \in H_\infty^{n_z\times n_w}$, $H_2 \in H_\infty^{n_z\times n_z}$, $U_1 \in H_\infty^{n_z\times n_y}$, $U_2 \in H_\infty^{n_z\times n_z}$, $M_{11} \in H_\infty^{n_y\times n_w}$, $M_{12} \in H_\infty^{n_y\times n_z}$, $M_{21} \in H_\infty^{n_z\times n_w}$ *and* $M_{22} \in H_\infty^{n_z\times n_z}$. *Then the following two conditions are equivalent*

1. *H_2 is bistable and* $\left\|H_2^{-1}H_1\right\|_{H_\infty} < \gamma$.

2. *M_{22} and U_2 are bistable and* $\left\|U_2^{-1}U_1\right\|_{H_\infty} < 1$.

Proof: Taking the transpose in the relations (27) and (28), we obtain (25) and (26) where $P = M^T$, $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \end{bmatrix}^T$, $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}^T$. Using Lemma 3.8, we have the following equivalent statements

$$\begin{aligned} &H_2 \text{ is bistable and } \left\|H_2^{-1}H_1\right\|_{H_\infty} < \gamma \Leftrightarrow \\ &H_2^T \text{ is bistable and} \left\|\left(H_2^{-1}H_1\right)^T\right\|_{H_\infty} < \gamma \Leftrightarrow \\ &H_2^T \text{ is bistable and} \left\|H_1^T\left(H_2^T\right)^{-1}\right\|_{H_\infty} < \gamma \ \Leftrightarrow \\ &X_2 \text{ is bistable and } \left\|X_1X_2^{-1}\right\|_{H_\infty} < \gamma. \end{aligned} \tag{29}$$

Similarly, it follows that

$$\begin{aligned} &M_{22} \text{ and } U_2 \text{ are bistable and } \left\|U_2^{-1}U_1\right\|_{H_\infty} < 1 \Leftrightarrow \\ &\Leftrightarrow P_{22} \text{ and } Q_2 \text{ are bistable and } \left\|Q_1Q_2^{-1}\right\|_{H_\infty} < 1. \end{aligned} \tag{30}$$

Now using Lemma 3.25 toghether with (29) and (30) we have proved the equivalence between 1. and 2.

■

4 The standard H_∞-suboptimal control problem

The standard H_∞-control problem is shown in Figure 1, where w, u, z and y are vector valued signals:

- w is the exogenous input
- u is the control signal
- z is the output to be controlled
- y is the measured output

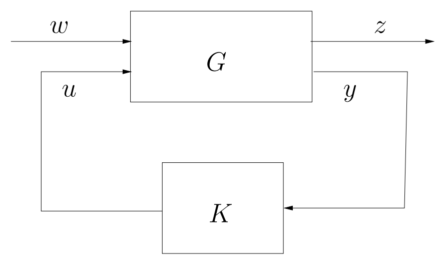

Figure 1: The standard H_∞-control problem

The transfer matrices G and K are assumed to be in F_∞. Let G be a stabilizable plant of the form

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \tag{31}$$

In the closed loop we consider z, y and u as outputs.

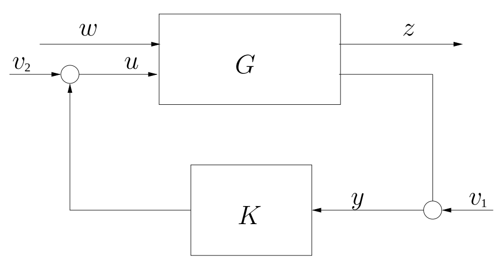

Figure 2: Stability diagram

In order to define stability we consider the diagram given in Figure 2.

Definition 4.1 (Stability of the closed loop system) *The feedback system (G, K) of Figure 2, where $G, K \in F_\infty$, is said to be input output stable if the following holds:*

1. $\det(I - G_{22}(s)K(s)) \neq 0$ *for all s in the open right half plane;*
2. *The transfer functions $S = (I - G_{22}K)^{-1}$, KSG_{22}, $G_{11} + G_{12}KSG_{21}$, KSG_{21}, $G_{12}KS$, KS, SG_{22}, SG_{21} and $G_{12}(I + KSG_{22})$ are in H_∞.*

Next we define the optimal and the suboptimal H_∞-control problem.

Definition 4.2 (The standard H_∞-optimal control problem) *Given a stabilizable plant $G \in F_\infty$ minimize the H_∞ norm of the transfer function T_{zw}, from w to z, over all stabilizable controllers $K \in F_\infty$. Moreover, find*

$$\gamma_{opt} = \inf_K \|T_{zw}\|_{H_\infty}. \tag{32}$$

Definition 4.3 (The standard H_∞-suboptimal control problem) *Given a stabilizable plant $G \in F_\infty$ and the positive bound γ, find a compensator $K \in F_\infty$ such that the transfer function T_{zw}, from w to z, satisfies*

$$\|T_{zw}\|_{H_\infty} < \gamma.$$

If it is possible, describe the general form of all stabilizable controllers which satisfies the above inequality.

We will reformulate the standard H_∞-control problem using coprime factorizations. From Theorem 3.11 we know that every stabilizable G possesses a left-coprime factorization. Let G be a stabilizable plant and let

$$G = D^{-1}N \tag{33}$$

be a left-coprime factorization of G over H_∞, where

$$D \in H_\infty^{(n_z+n_y)\times(n_z+n_y)},\ N \in H_\infty^{(n_z+n_y)\times(n_w+n_u)}.$$

$D = [D_1\ D_2]$ and $N = [N_1\ N_2]$ corresponds to the partitioning of the output and the input of G, where

$$D_1 \in H_\infty^{(n_z+n_y)\times n_z},\ D_2 \in H_\infty^{(n_z+n_y)\times n_y},\ N_1 \in H_\infty^{(n_z+n_y)\times n_w},\ N_2 \in H_\infty^{(n_z+n_y)\times n_u}.$$

Since K stabilizes G if and only if G stabilizes K, we have from Theorem 3.11 that K possesses a coprime factorization. Let $K = K_d^{-1}K_n$ be a left-coprime factorization over H_∞ of the controller, so

$$K_d \in H_\infty^{n_u\times n_u},\ K_n \in H_\infty^{n_u\times n_y}.$$

From Figure 1 we see that we can describe the closed-loop signal equation of the system as

$$\begin{bmatrix} -N_1 & D_1 & -N_2 & D_2 \\ 0 & 0 & K_d & -K_n \end{bmatrix} \begin{bmatrix} w \\ z \\ u \\ y \end{bmatrix} = 0. \tag{34}$$

The upper row block

$$-N_1 w + D_1 z - N_2 u + D_2 y = 0 \tag{35}$$

defines the plant, and the lower row block

$$K_d u - K_n y = 0 \tag{36}$$

defines the controller.

In this case the extended closed-loop of Figure 2 has the equations:

$$\begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} N_1 & D_2 & 0 \\ 0 & 0 & K_d \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}. \tag{37}$$

Let us denote

$$A = \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix}, \; B = \begin{bmatrix} N_1 & D_2 & 0 \\ 0 & 0 & K_d \end{bmatrix}. \tag{38}$$

Lemma 4.4 *Suppose that $G \in F_\infty$ is given in the form (31) and that G has the left-coprime factorization over H_∞ as described before (see (33)). Then the following statements are equivalent:*

1. *The feedback system (G, K) of Figure 2 is input-output stable for the controller $K \in F_\infty$.*
2. *The controller $K \in F_\infty$ has a left-coprime factorization $K = K_d^{-1} K_n$ over H_∞ and the matrix*

$$A = \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix} \tag{39}$$

is invertible over H_∞.

Proof: 1⇒2: Suppose that the feedback system (G, K) of Figure 2 is input-output stable for the controller $K \in F_\infty$ and that the left-coprime factorization over H_∞ of K is given as $K = K_d^{-1} K_n$. Denote

$$T = \begin{bmatrix} I & 0 & -G_{12} \\ 0 & I & -G_{22} \\ 0 & -K & I \end{bmatrix}, \quad \text{and } S = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{40}$$

The inverse of T is

$$T^{-1} = \begin{bmatrix} I & G_{12}K(I - G_{22}K)^{-1} & G_{12} - G_{12}K(I - G_{22}K)^{-1} \\ 0 & (I - G_{22}K)^{-1} & (I - G_{22}K)^{-1} G_{22} \\ 0 & K(I - G_{22}K)^{-1} & I + K(I - G_{22}K)^{-1} G_{22} \end{bmatrix}.$$

Since the feedback system (G, K) is input output stable, it follows that $T^{-1} \in H_\infty$. The loop equations of Figure 2 can be written as

$$T \begin{bmatrix} z \\ y \\ u \end{bmatrix} = S \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix} \tag{41}$$

and multiplying this by the following matrix in H_∞

$$Z = \begin{bmatrix} D & 0 \\ 0 & K_d \end{bmatrix}$$

gives

$$A \begin{bmatrix} z \\ y \\ u \end{bmatrix} = B \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}, \tag{42}$$

where we have used that

$$\begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \end{bmatrix}.$$

For $[A\ B]$ the following equality holds

$$\begin{bmatrix} A & B \end{bmatrix} = LR, \tag{43}$$

where

$$R = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & I \end{bmatrix} \tag{44}$$

and

$$L = \begin{bmatrix} D_1 & D_2 & N_1 & N_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_n & K_d \end{bmatrix}. \tag{45}$$

Since D, N and K_d, K_n are respectively left-coprime there exist the matrices E_1, F_1, E_2 and F_2 such that

$$DE_1 + NF_1 = I$$

and

$$K_n E_2 + K_d F_2 = I.$$

Denote

$$\begin{bmatrix} X & Y \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} R^{-1} = L. \tag{46}$$

We have

$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ F_1 & 0 \\ 0 & E_2 \\ 0 & F_2 \end{bmatrix} = I \tag{47}$$

and using (46) it becomes

$$\begin{bmatrix} A & B \end{bmatrix} R^{-1} \begin{bmatrix} E_1 & 0 \\ F_1 & 0 \\ 0 & E_2 \\ 0 & F_2 \end{bmatrix} = I. \tag{48}$$

So A and B are left coprime; there exists a Q in H_∞, namely

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = R^{-1} \begin{bmatrix} E_1 & 0 \\ F_1 & 0 \\ 0 & E_2 \\ 0 & F_2 \end{bmatrix} \tag{49}$$

such that

$$AQ_1 + BQ_2 = I. \tag{50}$$

From this we see that the inverse of A is given by

$$A^{-1} = Q_1 + A^{-1}BQ_2. \tag{51}$$

It remains to prove that $A^{-1} \in H_\infty$. We know that

$$A^{-1}B = (ZT)^{-1}(ZS) = T^{-1}S$$

with $T^{-1} \in H_\infty$ and $S \in H_\infty$, hence $A^{-1}B \in H_\infty$. Recalling the equality (51) and the fact that $Q_1, Q_2 \in H_\infty$, we obtain that $A^{-1} \in H_\infty$. So A is invertible over H_∞.

2⇐1: We have that the controller $K \in F_\infty$ has a left-coprime factorization $K = K_d^{-1}K_n$ over H_∞ and the matrix A is invertible over H_∞. The closed loop equations (37) can be written as

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = A^{-1}B \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}$$

with $A^{-1}B \in H_\infty$, so K is a stabilizing controller for G. ■

Now we can reformulate the standard H_∞-suboptimal control problem into an equivalent problem involving the coprime factorization of G and K.

Definition 4.5 (The standard H_∞-suboptimal control problem) *Given a transfer matrix* $[\,-N_1 \quad D_1 \quad -N_2 \quad D_2\,] \in H_\infty^{(n_z+n_y)\times(n_w+n_z+n_u+n_y)}$ *find a compensator* $K \in F_\infty^{n_u \times n_y}$ *with a left-coprime factorization* $K_d^{-1}K_n$*, such that the transfer matrix* T_{zw} *(see Figure 1) from* w *to* z *induced by the frequency domain equation*

$$A \begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} N_1 \\ 0 \end{bmatrix} w \tag{52}$$

satisfies $\|T_{zw}\|_{H_\infty} < \gamma$*, where* A *is given by (39) and* $A \in GH_\infty$*.*

The compensators K which solve the H_∞ suboptimal problem will be called admissible compensators. A compensator will be called optimal if it is admissible and minimize the infinity norm of T_{zw} over all admissible compensators.

5 Main result

In this section we show that the standard H_∞-suboptimal control problem, as reformulated in Definition 4.5, can be solved, provided that two J-spectral factorizations have a solution. We begin with a simple observation. Consider Figure 1 and a signal $w \in L_2(\mathbb{R})$ for which $y(t)$ is zero for all $t < 0$. For this signal the feedback input u will be zero for negative times for every causal controller K. This implies a necessary condition for the existence of a causal, stabilizing γ-suboptimal controller K as is given in the following lemma.

Lemma 5.1 (Necessary condition) *A necessary condition for the existence of a solution for the standard H_∞-suboptimal control problem is that the space*

$$B_{[-N_1 \quad D_1]} = \left\{ \begin{bmatrix} w \\ z_- \end{bmatrix} \in H_2^\perp \mid [-N_1 \quad D_1] \begin{bmatrix} w \\ z_- \end{bmatrix} \in H_2 \right\} \tag{53}$$

is strictly positive in the $J_{\frac{1}{\gamma^2},n_w,n_z}$-inner product.

Proof: Suppose that K is a stabilizing suboptimal compensator with bound γ, where γ is a given nonnegative real number, and let $K = K_d^{-1}K_n$ be a left-coprime factorization of K over H_∞. Let $\begin{bmatrix} w \\ z_- \end{bmatrix}$ be an arbitrary element in $B_{[-N_1 \quad D_1]}$. Since the standard H_∞-suboptimal control problem has a solution there exists an input signal u such that the equation (52) is satisfied.
From the equation (52) and from the equality

$$A \begin{bmatrix} z_- \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} D_1 \\ 0 \end{bmatrix} z_-, \tag{54}$$

we obtain

$$A \begin{bmatrix} z - z_- \\ y \\ u \end{bmatrix} = \begin{bmatrix} N_1 & -D_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ z_- \end{bmatrix}. \tag{55}$$

Since, by (53), the right-hand side of this equality is in $H_2^{n_z+n_y+n_u}$, and since $A \in GH_\infty^{(n_z+n_y+n_u)\times(n_z+n_y+n_u)}$, we obtain that $z - z_-$, u and y are in H_2. Obviously, we can write z uniquely as

$$z = z - z_- + z_-,$$

where $z - z_- \in H_2$ and $z_- \in H_2^\perp$.
We have that the transfer matrix T_{zw} from w to z induced by the frequency domain equation (52) satisfies $\|T_{zw}\|_{H_\infty} < \gamma$, provided that A has the structure given in (38) and $A \in GH_\infty$. We have that

$$\gamma^2 \|w\|_2^2 \geq \|z\|_2^2 \geq \|z_-\|_2^2,$$

so

$$\gamma^2 \|w\|_2^2 - \|z_-\|_2^2 \geq \gamma^2 \|w\|_2^2 - \|z\|_2^2 \geq \gamma^2 \|w\|_2^2 - \|T_{zw}\|_\infty^2 \|w\|_2^2 = (\gamma^2 - \|T_{zw}\|_\infty^2) \|w\|_2^2.$$

Using again the inequality $\gamma^2 \|w\|_2^2 \geq \|z_-\|_2^2$, we obtain

$$\gamma^2 \|w\|_2^2 - \|z_-\|_2^2 \geq \frac{1}{2}(\gamma^2 - \|T_{zw}\|_{H_\infty}^2)(\|w\|_2^2 + \frac{1}{\gamma^2} \|z_-\|_2^2).$$

Dividing by γ^2 we see that this inequality is equivalent with

$$\|w\|_2^2 - \frac{1}{\gamma^2}\|z_-\|_2^2 \geq \frac{1}{2}\min\{1, \frac{1}{\gamma^2}\}(1 - \frac{1}{\gamma^2}\|T_{zw}\|_\infty^2)(\|w\|_2^2 + \|z_-\|_2^2).$$

Recalling that $\|T_{zw}\|_{H_\infty} < \gamma$, the last inequality means that $B_{[-N_1 \quad D_1]}$ is strictly positive in $J_{\frac{1}{\gamma^2}, n_w, n_z}$-inner product. ■

The necessary condition (53) holds provided a J-spectral factorization exists.

Lemma 5.2 *If there exists a bistable matrix W such that*

$$N_1(j\omega)N_1^{\sim}(j\omega) - \gamma^2 D_1(j\omega)D_1^{\sim}(j\omega) = W(j\omega)\begin{bmatrix} I_{n_y} & 0 \\ 0 & -I_{n_z} \end{bmatrix} W^{\sim}(j\omega), \text{ for all } \omega \in \mathbb{R}, \quad (56)$$

and the lower-right $n_z \times n_z$ block M_{22} of $M := W^{-1}[-N_1 \quad D_1]$ is bistable, then the set defined in (53) is strictly positive in the $J_{\frac{1}{\gamma^2}}$-inner product and thus the necessary condition for the solvability of the standard H_∞-suboptimal control problem is satisfied.

Proof: Suppose that exists a $W \in GH_\infty^{(n_y+n_z)\times(n_y+n_z)}$ such that (56) is satisfied. Obviously, $M \in H_\infty^{(n_y+n_z)\times(n_w+n_z)}$. Since $W \in GH_\infty$, using Lemma 3.22 we have that

$$B_{[-N_1 \quad D_1]} = B_M$$

Partition M as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where $M_{11} \in H_\infty^{n_y \times n_w}$, $M_{12} \in H_\infty^{n_y \times n_z}$, $M_{21} \in H_\infty^{n_z \times n_w}$ and $M_{22} \in GH_\infty^{n_z \times n_z}$. By (56) M satisfies the conditions 1 and 2 of Lemma 3.19 and thus M is co-J_{γ^2, n_w, n_z}-lossless, i.e.

$$M(s)J_{\gamma^2, n_w, n_z}M(s)^* \leq J_{n_y, n_z}, \text{ for } s \in \overline{\mathbb{C}_+}.$$

The lower-right $n_z \times n_z$ block element of the above inequality is:

$$M_{21}(s)M_{21}(s)^* - \gamma^2 M_{22}(s)M_{22}(s)^* \leq -I_{n_z} < 0, \; for \; s \in \overline{\mathbb{C}_+}.$$

Since $M_{22} \in GH_\infty^{n_z \times n_z}$, it follows that $H = M_{22}^{-1}M_{21}$ has the H_∞-norm stictly less than γ. So by Lemma 3.23 we obtain that $B_{[M_{21}\ M_{22}]}$ is strictly positive with respect to the $J_{\frac{1}{\gamma^2}, q, p}$ inner product. Obviously, $B_M \subset B_{[M_{21}\ M_{22}]}$ and since $B_{[-N_1\ D_1]} = B_M$, the conclusion follows. ■

In general it is difficult to find a bistable W which solves (56). A method for solving the J-spectral factorization problem (56) for the Wiener-class of transfer functions is presented in Section 8.

Definition 5.3 (A two-block problem) *Let $L \in H_\infty^{(n_z+n_w)\times(n_u+n_y)}$, $(n_y = n_w)$ and consider the equality*

$$\begin{bmatrix} H_2 \\ H_1 \end{bmatrix} = L \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} \quad (57)$$

with $H_1 \in H_\infty^{n_z \times n_w}$, $H_2 \in H_\infty^{n_w \times n_w}$, $\tilde{K}_d \in H_\infty^{n_y \times n_y}$, $\tilde{K}_n \in H_\infty^{n_u \times n_y}$. The two-block problem is to find a controller $K \in F_\infty$ with the right-coprime factorization $K = \tilde{K}_n \tilde{K}_d^{-1}$ over H_∞ such that H_2 is bistable and $\left\|H_1 H_2^{-1}\right\|_{H_\infty} < 1$.

In the main theorem of this section we show that if (56) is solvable, then the H_∞-suboptimal control problem is solvable if and only if a related two-block problem is solvable.

Theorem 5.4 (Reduction to a two-block problem) *Consider the standard H_∞-suboptimal control problem of Definition 4.5. If there exists a bistable matrix W such that (56) holds, with the lower right $n_z \times n_z$ block of the matrix $W^{-1}[-N_1\ \ D_1]$ bistable, then there exists a stabilizing controler $K \in F_\infty$ for the system (35) such that the standard H_∞-suboptimal control problem is solved if and only if the two-block problem of Definition 5.3, with*

$$L = W^{-1} \begin{bmatrix} D_2 & -N_2 \end{bmatrix} \tag{58}$$

has a solution.

Proof: The proof will be given in four steps.

In step 1 we obtain the equivalent condition for the stability of the closed-loop system. In step 2 a similar result is obtained but now for a system related to the two-block problem. In the last two steps the necessary and sufficient condition of the theorem is proved.

Step 1: Let $K \in F_\infty$ be any controller for the plant (35), and let $K = K_d^{-1}K_n$ be a left-coprime factorization over H_∞. The closed-loop system is given by

$$\begin{bmatrix} -N_1 & D_1 & D_2 & -N_2 \\ 0 & 0 & -K_n & K_d \end{bmatrix} \begin{bmatrix} w \\ z \\ y \\ u \end{bmatrix} = 0. \tag{59}$$

Denote

$$\Omega = \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix}. \tag{60}$$

From Theorem 3.11 we know that K has a right coprime factorization, i.e. $K = \tilde{K}_n\tilde{K}_d^{-1}$. Furthermore, by [12] Lemma A.7.44, page 661, there exists a $U \in GH_\infty$ of the form

$$U = \begin{bmatrix} \tilde{K}_d & * \\ \tilde{K}_n & * \end{bmatrix}. \tag{61}$$

such that

$$\begin{bmatrix} -K_n & K_d \end{bmatrix} U = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix}. \tag{62}$$

Defining the signals l_1 and l_2 using this U, via

$$\begin{bmatrix} y \\ u \end{bmatrix} = U \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}, \tag{63}$$

we obtain the following equivalent representation for the system

$$\begin{bmatrix} -N_1 & D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n & * \\ 0 & 0 & 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} w \\ z \\ l_1 \\ l_2 \end{bmatrix} = 0 \tag{64}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = U \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}. \tag{65}$$

However $I_{n_u} l_2 = 0$ is the same as $l_2 = 0$, and this representation becomes

$$\begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} \begin{bmatrix} z \\ l_1 \end{bmatrix} = N_1 w \tag{66}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} l_1 \tag{67}$$

$$l_2 = 0. \tag{68}$$

Since

$$\Omega \begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} D_1 & D_2 & -N_2 \\ 0 & -K_n & K_d \end{bmatrix} \begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} = \tag{69}$$

$$= \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n & * \\ 0 & 0 & I_{n_u} \end{bmatrix} \tag{70}$$

and $U \in GH_\infty$, we have the following equivalence

$$\Omega \in GH_\infty \text{ if and only if } \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} \in GH_\infty. \tag{71}$$

Step 2: Using the bistable matrix W which satisfy relation (56) we define

$$\begin{bmatrix} -\tilde{N}_1 & \tilde{D}_1 & \tilde{D}_2 & -\tilde{N}_2 \end{bmatrix} = W^{-1} \begin{bmatrix} -N_1 & D_1 & D_2 & -N_2 \end{bmatrix}. \tag{72}$$

Furthermore, for the plant

$$P = \begin{bmatrix} \begin{bmatrix} 0 \\ I_{n_z} \end{bmatrix} & \tilde{D}_2 \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} I_{n_w} \\ 0 \end{bmatrix} & \tilde{N}_2 \end{bmatrix}, \tag{73}$$

we define the closed-loop system (see (34))

$$\begin{bmatrix} \begin{bmatrix} I_{n_w} \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{n_z} \end{bmatrix} & \tilde{D}_2 & -\tilde{N}_2 \\ 0 & 0 & K_n & K_d \end{bmatrix} \begin{bmatrix} \tilde{w} \\ \tilde{z} \\ y \\ u \end{bmatrix} = 0, \tag{74}$$

where K is a controller of the form $K = K_d^{-1} K_n$, $\tilde{w}$ is the new exogenous input, and the new to be controlled output is $\tilde{z}$.
We define the matrices $\tilde{H}_1$ and $\tilde{H}_2$ via

$$\begin{bmatrix} \tilde{H}_2 \\ \tilde{H}_1 \end{bmatrix} = \begin{bmatrix} \tilde{D}_2 & -\tilde{N}_2 \end{bmatrix} \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} = W^{-1} \begin{bmatrix} D_2 & -N_2 \end{bmatrix} \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix}, \tag{75}$$

where $\tilde{K}_n$ and $\tilde{K}_d$ are given in (61).
Denote

$$\tilde{\Omega} = \begin{bmatrix} \begin{bmatrix} 0 \\ I_{n_z} \end{bmatrix} & \tilde{D}_2 & -\tilde{N}_2 \\ 0 & K_n & K_d \end{bmatrix}. \tag{76}$$

Then

$$\tilde{\Omega}\begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ I_{n_z} \end{bmatrix} & \tilde{D}_2 & -\tilde{N}_2 \\ 0 & K_n & K_d \end{bmatrix}\begin{bmatrix} I_{n_z} & 0 \\ 0 & U \end{bmatrix} = \tag{77}$$

$$= \begin{bmatrix} 0 & \tilde{H}_2 & * \\ I_{n_z} & \tilde{H}_1 & * \\ 0 & 0 & I_{n_u} \end{bmatrix}. \tag{78}$$

Since $U \in GH_\infty$, the following equivalence holds

$$\tilde{\Omega} \in GH_\infty \text{ if and only if } \tilde{H}_2 \in GH_\infty. \tag{79}$$

Using

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = U^{-1}\begin{bmatrix} y \\ u \end{bmatrix} \tag{80}$$

and (77), (78), we obtain the equivalent representation for the new system (74)

$$\begin{bmatrix} 0 & \tilde{H}_2 & * \\ I_{n_z} & \tilde{H}_1 & * \\ 0 & 0 & I_{n_u} \end{bmatrix}\begin{bmatrix} \tilde{z} \\ l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} -I_{n_w} \\ 0 \\ 0 \end{bmatrix}\tilde{w} \tag{81}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = U\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}. \tag{82}$$

Suppose that the controller K stabilizes the closed-loop system (74). Using Lemma 4.4 we have that $\tilde{\Omega} \in GH_\infty$, and thus $\tilde{H}_2 \in GH_\infty$, see (79). Consequently, we can write

$$l_2 = 0,\ l_1 = -\tilde{H}_2^{-1}\tilde{w},\ \tilde{z} = -\tilde{H}_1 l_1 \tag{83}$$

which give us the transfer function from $\tilde{w}$ to $\tilde{z}$, namely

$$T_{\tilde{z}\tilde{w}} = \tilde{H}_1\tilde{H}_2^{-1}. \tag{84}$$

Step 3: In this step we show that if the standard H_∞-suboptimal control problem is solvable, then the two-block problemis solvable. Suppose that the system (35) is stabilized by some controller $K \in F_\infty$ with $K = K_d^{-1}K_n$ a left-coprime factorization over H_∞, such that $\|T_{zw}\|_{H_\infty} < \gamma$, for some positive real number γ. By Lemma 4.4 and the equivalence (71), this implies that $\left[D_1\ D_2\tilde{K}_d - N_2\tilde{K}_n\right] \in GH_\infty$. We will prove that the system (74) is stable, which is equivalent (by (79)) with $\tilde{H}_2 \in GH_\infty$. Furthermore we will prove that $\tilde{H}_1\tilde{H}_2^{-1}\|_{H_\infty} < 1$. Define

$$E = \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix}^{-1}, \tag{85}$$

where $\tilde{K}_d$ and $\tilde{K}_n$ are given in (61), and partition E compatibly as $E = \begin{bmatrix} T \\ V \end{bmatrix}$. Then

$$\begin{bmatrix} T \\ V \end{bmatrix}\begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} = \begin{bmatrix} I_{n_z} & 0 \\ 0 & I_{n_y} \end{bmatrix}. \tag{86}$$

Multiplying both sides of the system (66) from the left by E and using (86) we get

$$\begin{bmatrix} I_{n_z} & 0 \\ 0 & I_{n_y} \end{bmatrix} \begin{bmatrix} z \\ l_1 \end{bmatrix} = \begin{bmatrix} TN_1 \\ VN_1 \end{bmatrix} w.$$

For $[H_1\ H_2]$ defined as

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} = T \begin{bmatrix} -N_1 & D_1 \end{bmatrix}, \tag{87}$$

we have from (86) that $H_2 = I_{n_z} \in GH_\infty$ and from (87) that $\|H_2^{-1}H_1\|_{H_\infty} = \|TN_1\|_{H_\infty} = \|T_{zw}\|_{H_\infty} < \gamma$. Denote

$$\begin{bmatrix} \tilde{T}_1 & \tilde{T}_2 \end{bmatrix} = TW, \tag{88}$$

so

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} = \begin{bmatrix} \tilde{T}_1 & \tilde{T}_2 \end{bmatrix} W^{-1} \begin{bmatrix} -N_1 & D_1 \end{bmatrix}. \tag{89}$$

Since (56) holds, we have that the matrix $X = W^{-1}[-N_1\ D_1]$ satisfies

$$XJX^{\sim} = J$$

almost everywhere on the imaginary axis. Moreover, we made the assumption that the lower-right block of the matrix X is bistable. So, the conditions from Lemma 3.19 are satisfied. Applying this lemma we obtain that $W^{-1}[-N_1\ D_1]$ is co-J-lossless. We apply now Lemma 3.26 and conclude that $\tilde{T}_2 \in GH_\infty$ and $\left\| \tilde{T}_2^{-1}\tilde{T}_1 \right\|_{H_\infty} < 1$.

For $\tilde{H}_1$ and $\tilde{H}_2$ as defined in (75), that is

$$\begin{bmatrix} \tilde{H}_2 \\ \tilde{H}_1 \end{bmatrix} = W^{-1} \begin{bmatrix} D_2 & -N_2 \end{bmatrix} \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix} = L \begin{bmatrix} \tilde{K}_d \\ \tilde{K}_n \end{bmatrix}$$

we have the following sequence of equalities

$$\begin{aligned} \begin{bmatrix} \tilde{T}_1 & \tilde{T}_2 \end{bmatrix} \begin{bmatrix} \tilde{H}_2 \\ \tilde{H}_1 \end{bmatrix} &\overset{(88)}{=} TW \begin{bmatrix} \tilde{H}_2 \\ \tilde{H}_1 \end{bmatrix} \\ &\overset{(75)}{=} TWW^{-1}\left(D_2\tilde{K}_d - N_2\tilde{K}_n\right) \\ &= T\left(D_2\tilde{K}_d - N_2\tilde{K}_n\right). \end{aligned} \tag{90}$$

Now, using (86) and (90), we have that

$$\begin{bmatrix} \tilde{T}_1 & \tilde{T}_2 \end{bmatrix} \begin{bmatrix} \tilde{H}_2 \\ \tilde{H}_1 \end{bmatrix} = 0,$$

which is equivalent to

$$\tilde{T}_1\tilde{H}_2 + \tilde{T}_2\tilde{H}_1 = 0. \tag{91}$$

Using again relation (86) and also (75) gives that

$$\begin{bmatrix} \tilde{T}_1 & \tilde{T}_2 \\ I & 0 \end{bmatrix} W^{-1} \begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} = \begin{bmatrix} I_{n_z} & 0 \\ * & \tilde{H}_2 \end{bmatrix} \tag{92}$$

which, together with $W \in GH_\infty$ and $\begin{bmatrix} D_1 & D_2\tilde{K}_d - N_2\tilde{K}_n \end{bmatrix} \in GH_\infty$, implies the equivalence

$$\tilde{T}_2 \in GH_\infty \text{ if and only if } \tilde{H}_2 \in GH_\infty. \tag{93}$$

Since $\tilde{T}_2 \in GH_\infty$ we get $\tilde{H}_2 \in GH_\infty$. Now, using (91) we obtain

$$\tilde{H}_1 \tilde{H}_2^{-1} = -\tilde{T}_2^{-1} \tilde{T}_1, \tag{94}$$

so

$$\|\tilde{H}_1 \tilde{H}_2^{-1}\|_{H_\infty} < 1.$$

Step 4: In this step we show that if the two-block problem is solvable, then the H_∞-suboptimal control problem is solvable. Using the notation of (75) we know that $\tilde{H}_2 \in GH_\infty$ and $\|\tilde{H}_1 \tilde{H}_2^{-1}\|_{H_\infty} < 1$. Furthermore, let $K = K_d^{-1} K_n$ be a left-coprime factorization of the controller $K = \tilde{K}_n \tilde{K}_d^{-1}$ which solves the two-block problem. Using step 2 and Lemma 4.4 we see that $K = K_d^{-1} K_n$ is a stabilizing controller for the system (74) and that $\|T_{\tilde{z}\tilde{w}}\|_{H_\infty} < 1$. We have to prove that $\begin{bmatrix} D_1 & D_2 \tilde{K}_d - N_2 \tilde{K}_n \end{bmatrix} \in GH_\infty$ and $\|T_{zw}\|_{H_\infty} \leq \gamma$. Define

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} = \begin{bmatrix} -T_{\tilde{z}\tilde{w}} & I \end{bmatrix} W^{-1} \begin{bmatrix} -N_1 & D_1 \end{bmatrix}. \tag{95}$$

Using the fact that the matrix $W^{-1} \begin{bmatrix} -N_1 & D_1 \end{bmatrix}$ is co-J-lossless and the assumption that $\|T_{\tilde{z}\tilde{w}}\| < 1$ we can apply Lemma 3.26 and obtain that

$$\|H_2^{-1} H_1\|_{H_\infty} < \gamma \text{ and } H_2 \in GH_\infty.$$

From (95) we see that $\|T_{zw}\|_{H_\infty} = \|H_2^{-1} H_1\|_{H_\infty}$. Also from (95) we have that

$$\begin{bmatrix} -T_{\tilde{z}\tilde{w}} & I \end{bmatrix} W^{-1} D_1 = H_2 \tag{96}$$

and using (75) it follows that

$$\begin{bmatrix} I & 0 \end{bmatrix} W^{-1} (D_2 \tilde{K}_d - N_2 \tilde{K}_n) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \tilde{H}_2 \\ \tilde{H}_1 \end{bmatrix} = \tilde{H}_2. \tag{97}$$

Combining (96) and (97) the following equality holds

$$\begin{bmatrix} -T_{\tilde{z}\tilde{w}} & I_{n_z} \\ I_{n_w} & 0 \end{bmatrix} W^{-1} \begin{bmatrix} D_1 & D_2 \tilde{K}_d - N_2 \tilde{K}_n \end{bmatrix} = \begin{bmatrix} H_2 & 0 \\ * & \tilde{H}_2 \end{bmatrix}. \tag{98}$$

Since H_2, $\tilde{H}_2$ and W are elements of GH_∞ we have that $\begin{bmatrix} D_1 & D_2 \tilde{K}_d - N_2 \tilde{K}_n \end{bmatrix}$ is in GH_∞. Using the step 1 we conclude that the standard H_∞-suboptimal control problem is solved. ■

Lemma 5.5 *Let L be*

$$L = W^{-1} [-N_2 \; D_2], \tag{99}$$

and

$$R = \begin{bmatrix} 0 & I_{n_w} \\ I_{n_z} & 0 \end{bmatrix} L \begin{bmatrix} 0 & I_{n_y} \\ I_{n_u} & 0 \end{bmatrix}. \tag{100}$$

If there exists a bistable matrix V such that

$$R^\sim(j\omega) J_{n_w, n_z} R(j\omega) = V^\sim(j\omega) J_{n_y, n_u} V(j\omega) \text{ for } \omega \in \mathbb{R}, \tag{101}$$

and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable, then the set of all controllers which solves the two-block problem (see Definition 5.3) is given by $K = K_n K_d^{-1}$ where

$$\begin{bmatrix} K_n \\ K_d \end{bmatrix} = V^{-1} \begin{bmatrix} U \\ I_{n_y} \end{bmatrix}, \tag{102}$$

with $U \in H_\infty$ such that $\|U\|_{H_\infty} < 1$ and $\det K_d \neq 0$.

Proof: Suppose that there exists $V \in GH_\infty$ such that (101) is satisfied, and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable. From (100) and (57) we have that

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = R \begin{bmatrix} K_n \\ K_d \end{bmatrix}. \tag{103}$$

Let K_n and K_d be given by (102). We have that

$$\begin{bmatrix} 0 & I \end{bmatrix} V \begin{bmatrix} K_n \\ K_d \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} VV^{-1} \begin{bmatrix} U \\ I \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} U \\ I \end{bmatrix} = I$$

which means that K_n and K_d are right-coprime. Consider the controller $K = K_n K_d^{-1}$, then

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = RV^{-1} \begin{bmatrix} U \\ I_{n_y} \end{bmatrix}, \tag{104}$$

where $U \in H_\infty$ and $\|U\|_{H_\infty} < 1$. Applying Lemma 3.25 results that H_2 is bistable and $\|H_1 H_2^{-1}\|_{H_\infty} < 1$. So the two-block problem is solved. It remains to prove that for every contoller K which solves the two-block problem there exists a right-coprime factorization $K = K_n K_d^{-1}$ such that K_n and K_d have the form given by (102), with $U \in H_\infty$ and $\|U\|_{H_\infty} < 1$.

Consider a controller K which solve the two-block problem, with the right-coprime factorization $K = X_n X_d^{-1}$. From (103) we have that

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = R \begin{bmatrix} X_n \\ X_d \end{bmatrix} = (RV^{-1})V \begin{bmatrix} X_n \\ X_d \end{bmatrix} = RV^{-1} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \tag{105}$$

where

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = V \begin{bmatrix} X_n \\ X_d \end{bmatrix}. \tag{106}$$

Since K solve the two-block problem, H_2 is bistable and $\|H_1 H_2^{-1}\|_{H_\infty} < 1$. Using also the assumption that the lower-right $n_y \times n_y$ block of the matrix RV^{-1} is bistable, we can apply Lemma 3.25 and obtain that Q_2 is a bistable matrix and $\|Q_1 Q_2^{-1}\|_{H_\infty} < 1$.
We have that

$$\begin{bmatrix} X_n \\ X_d \end{bmatrix} = V^{-1} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = V^{-1} \begin{bmatrix} Q_1 Q_2^{-1} \\ I \end{bmatrix} Q_2$$

and since Q_2 is bistable, $K = K_n K_d^{-1}$, where $K_n = X_n Q_2^{-1}$ and $K_d = X_d Q_2^{-1}$, is another right-coprime factorization of the controller. If we denote now $U = Q_1 Q_2^{-1}$, we see that K_n and K_d satisfy (102), $U \in H_\infty$ and $\|U\|_{H_\infty} < 1$. ■
We summarize the results in the following theorem.

Theorem 5.6 *Consider the standard H_∞-suboptimal control problem in the form (34). If there exist bistable matrices W and V such that (56) and (101) hold, with the lower-right $n_z \times n_z$ block of the matrix $W^{-1} \begin{bmatrix} -N_1 & D_1 \end{bmatrix}$ bistable and the lower-right $n_y \times n_y$ block of the matrix RV^{-1} bistable, then the set of all stabilizing controllers for the standard H_∞-control problem is given by (102).*

Remark 5.7 *The Theorem 5.6 states that if the bistable matrices W and V exists such that (56) and (101) hold, the matrix $W^{-1} \begin{bmatrix} -N_1 & D_1 \end{bmatrix}$ is co-J-lossless and the matrix RV^{-1} is J-lossless, then we have a formula for all stabilizing controllers for the standard H_∞-suboptimal control problem.*

We can write, similar to the rational case (see [24], page 64), the following algorithm.

Remark 5.8 (Algorithm)
Let us consider a stabilizable plant $G \in F_\infty$.

Step 1: *Find a left-coprime factorization of the plant*

$$G = D^{-1}N$$

as in (33), with $D = [D_1\ D_2]$ *and* $N = [N_1\ N_2]$ *corresponds to the partitioning of the output and the input of* G.

Step 2: *Choose a real strictly positive* γ.

Step 3: *Compute, if possible, the matrix* $W \in GH_\infty$ *such that (56) holds.*
If this solution exists and if the lower-right $n_z \times n_z$ *block of the matrix* $W^{-1}\begin{bmatrix} -N_1 & D_1 \end{bmatrix}$ *is bistable then procede with the next step.*

Step 4: *Compute, if possible, the matrix* $V \in GH_\infty$ *such that (101) is satisfied.*
If this solution exists and if the lower-right $n_{n_y} \times n_{n_y}$ *block of the matrix* RV^{-1} *is bistable, then procede with the next step.*

Step 5: *Consider arbitrary* $U \in H_\infty$ *of appropriate size such that* $\|U\|_{H_\infty} < 1$. *A stabilizing controller for the standard* H_∞*-suboptimal control problem is given by* $K = K_n K_d^{-1}$, *where* K_n *and* K_d *are given by (102).*

6 Example

In this section we present an example which ilustrates how we can apply the algorithm as described in the and of the previous section to systems with delay.
Consider the plant G given by

$$G = \begin{bmatrix} 0 & -\frac{s-\sqrt{1+e^{-2\tau}}}{s+1}e^{-\tau s} \\ \frac{s+1}{s-1} & -\frac{1}{s-1}e^{-\tau s} \end{bmatrix}. \tag{107}$$

It is easy to see that $G = D^{-1}N$, where

$$\begin{aligned} D &= \begin{bmatrix} 0 & \frac{s-1}{s+1} \\ 1 & 0 \end{bmatrix}, \\ N &= \begin{bmatrix} 1 & -\frac{1}{s+1}e^{-\tau s} \\ 0 & -\frac{s-\sqrt{1+e^{-2\tau}}}{s+1}e^{-\tau s} \end{bmatrix}, \end{aligned}$$

is a left-coprime factorization of G over H_∞. Let $\gamma > 0$ be a given real number. We will find a matrix $W \in GH_\infty$ such that the equality

$$N_1 N_1^\sim - \gamma^2 D_1 D_1^\sim = W J_{1,1} W^\sim, \tag{108}$$

with

$$\begin{aligned} N_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

holds on the imaginary axis. We compute the left hand side of the equality (108)

$$\begin{aligned} N_1 N_1^\sim - \gamma^2 D_1 D_1^\sim &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \gamma^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix}. \end{aligned}$$

If we take the matrix W to be $W = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$, the equality (108) is always satisfied.
The inverse of the matrix W exists and it is

$$W^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix}.$$

We have that the lower-right element of the matrix

$$W^{-1} \begin{bmatrix} -N_1 & D_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix}.$$

is a constant, so it is bistable.
We will find, using the procedure described in [26], a matrix $V \in GH_\infty$ such that the relation

$$R^\sim J_{1,1} R = V^\sim J_{1,1} V \tag{109}$$

is satisfied on the imaginary axis, where

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad (110)$$

$$L = W^{-1} \begin{bmatrix} D_2 & -N_2 \end{bmatrix}, \qquad (111)$$

$$D_2 = \begin{bmatrix} \frac{s-1}{s+1} \\ 0 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} -\frac{1}{s+1} e^{-\tau s} \\ -\frac{s-\sqrt{1+e^{-2\tau}}}{s+1} e^{-\tau s} \end{bmatrix}.$$

Replacing W^{-1}, D_2 and N_2 in (111), we obtain

$$\begin{aligned} L &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+1} e^{-\tau s} \\ 0 & \frac{s-\sqrt{1+e^{-2\tau}}}{s+1} e^{-\tau s} \end{bmatrix} \\ &= \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+1} e^{-\tau s} \\ 0 & \frac{1}{\gamma} \frac{s-\sqrt{1+e^{-2\tau}}}{s+1} e^{-\tau s} \end{bmatrix}, \end{aligned}$$

and using the definition of the matrix R given in (110) we have that

$$\begin{aligned} R &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+1} e^{-\tau s} \\ 0 & \frac{1}{\gamma} \frac{s-\sqrt{1+e^{-2\tau}}}{s+1} e^{-\tau s} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\gamma} \frac{s-\sqrt{1+e^{-2\tau}}}{s+1} e^{-\tau s} & 0 \\ \frac{1}{s+1} e^{-\tau s} & \frac{s-1}{s+1} \end{bmatrix}. \qquad (112) \end{aligned}$$

We compute now the left hand side of the equality (109)

$$\begin{aligned} R^{\sim} J_{1,1} R &= \begin{bmatrix} \frac{1}{\gamma} \frac{s+\sqrt{1+e^{-2\tau}}}{s-1} e^{\tau s} & \frac{1}{-s+1} e^{\tau s} \\ 0 & \frac{s+1}{s-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma} \frac{s-\sqrt{1+e^{-2\tau}}}{s+1} e^{-\tau s} & 0 \\ \frac{1}{s+1} e^{-\tau s} & \frac{s-1}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\gamma} \frac{s+\sqrt{1+e^{-2\tau}}}{s-1} e^{\tau s} & \frac{1}{s-1} e^{\tau s} \\ 0 & -\frac{s+1}{s-1} \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma} \frac{s-\sqrt{1+e^{-2\tau}}}{s+1} e^{-\tau s} & 0 \\ \frac{1}{s+1} e^{-\tau s} & \frac{s-1}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\gamma^2} \frac{s^2-1-e^{-2\tau}}{s^2-1} + \frac{1}{s^2-1} & \frac{1}{s+1} e^{\tau s} \\ -\frac{1}{s-1} e^{-\tau s} & -1 \end{bmatrix}. \qquad (113) \end{aligned}$$

We write $\frac{1}{s-1} e^{-\tau s}$ as a sum of a stable part and a rational part

$$\frac{1}{s-1} e^{-\tau s} = F_{stab}(s) + F_{rat}(s),$$

where

$$F_{stab}(s) = \frac{e^{-\tau s} - e^{-\tau}}{s-1}, \qquad (114)$$

$$F_{rat}(s) = \frac{e^{-\tau}}{s-1}.$$

We multiply the matrix $R^{\sim} J_{1,1} R$ to the left with the matrix $\begin{bmatrix} 1 & -F_{stab}^{\sim} \\ 0 & 1 \end{bmatrix}$ and to the right with the matrix $\begin{bmatrix} 1 & 0 \\ -F_{stab} & 1 \end{bmatrix}$ and choosing $\gamma = 1$ we obtain

$$\begin{bmatrix} 1 & -F_{stab}^{\sim} \\ 0 & 1 \end{bmatrix} R^{\sim} J_{1,1} R \begin{bmatrix} 1 & 0 \\ -F_{stab} & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 & -\frac{e^{\tau s}-e^{-\tau}}{-s-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s^2-1-e^{-2\tau}}{s^2-1} + \frac{1}{s^2-1} & \frac{1}{s+1} e^{\tau s} \\ -\frac{1}{s-1} e^{-\tau s} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{e^{-\tau s}-e^{-\tau}}{s-1} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{s^2-1-e^{-2\tau}}{s^2-1} + \frac{1}{s^2-1} - \frac{e^{\tau s}-e^{-\tau}}{s^2-1} e^{-\tau s} & \frac{1}{s+1} e^{\tau s} - \frac{e^{\tau s}-e^{-\tau}}{s+1} \\ -\frac{1}{s-1} e^{-\tau s} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{e^{-\tau s}-e^{-\tau}}{s-1} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \frac{e^{-\tau}}{s+1} \\ -\frac{e^{-\tau}}{s-1} & -1 \end{bmatrix}.$$

We consider the following matrix function

$$Q = \begin{bmatrix} 1 & \frac{e^{-\tau}}{s+1} \\ 0 & \frac{s+\sqrt{1+e^{-2\tau}}}{s+1} \end{bmatrix}. \tag{115}$$

The equality

$$\begin{bmatrix} 1 & \frac{e^{-\tau}}{s+1} \\ -\frac{e^{-\tau}}{s-1} & -1 \end{bmatrix} = Q^{\sim} J_{1,1} Q \tag{116}$$

holds on the imaginary axis.
Since $\det Q \in GH_{\infty}$, the matrix function Q is bistable. We define the matrix function V to be

$$V = Q \begin{bmatrix} 1 & 0 \\ -F_{stab} & 1 \end{bmatrix}, \tag{117}$$

where F_{stab} and Q are given in (114) and (115). With this V, the equality (109) is satisfied on the imaginary axis.
Explicitely V is given by

$$V = \begin{bmatrix} 1 - \frac{e^{-\tau}(e^{-\tau s}-e^{-\tau})}{s^2-1} & \frac{e^{-\tau}}{s+1} \\ -\frac{(s+\sqrt{a+e^{-2\tau}})(e^{-\tau s}-e^{-\tau})}{s^2-1} & \frac{s+\sqrt{a+e^{-2\tau}}}{s+1} \end{bmatrix}. \tag{118}$$

Using (112) and (118) we obtain that the lower-right element of the matrix valued function RV^{-1} is

$$(RV^{-1})_{22} = \frac{s+1-2e^{-\tau s}e^{-\tau}+e^{-2\tau}}{s+\sqrt{1+e^{-2\tau}}}. \tag{119}$$

We see in the Figure 3 that the Nyquist plot, for $\tau = 0.2$ does not encircle the origin, which means that $(RV^{-1})_{22}$ is bistable.

All the conditions required for applying the algorithm described in Remark 5.8 are satisfied, and thus we can construct a H_{∞}-suboptimal controler

$$K = F_{stab} \frac{s+\sqrt{1+e^{-2\tau}}-e^{-\tau}}{2s+\sqrt{1+e^{-2\tau}}+1-e^{-\tau}}$$

for $\tau = 0.2$

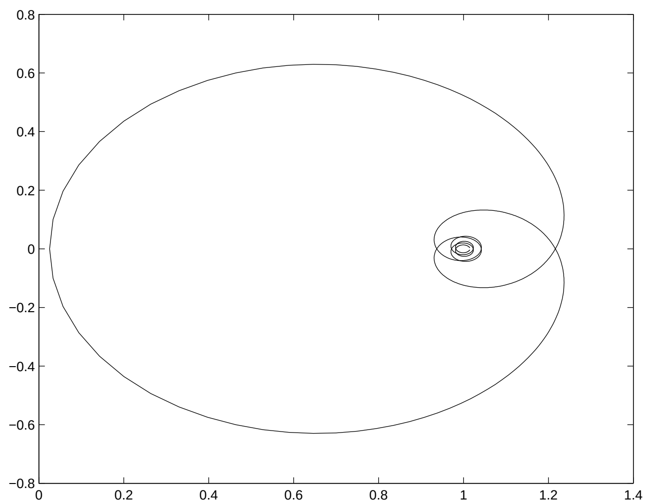

Figure 3: The Nyquist plot of $(RV^{-1})_{22}$

7 Classes of transfer functions

First we define two classes of stable transfer functions via there impulse responses:

$$\mathcal{A} = \left\{ f : \mathbb{R} \to \mathbb{C} \mid f(t) = \begin{cases} f_a(t) + f_0\delta(t), & t \geq 0, \\ 0, & t < 0, \end{cases} \text{ with } f_o \in \mathbb{C} \text{ and } \int_o^\infty |f_a(t)|dt < \infty \right\},$$

$$\hat{\mathcal{A}} = \left\{ \hat{f} \mid f \in \mathcal{A} \right\},$$

where $\hat{}$ denotes the Laplace transform, which is defined on $\overline{\mathbb{C}}_+$, and δ represents the delta distribution. $\hat{f}$ is holomorphic on $\mathbb{C}_+$ and continuous on the imaginary axis (see [12], Lemma A.7.47 page 663). $\hat{\mathcal{A}}$ is a commutative Banach algebra with identity under pointwise addition and multiplication (see [12], Corolarry A.7.48, page 665).
Let us consider the following subalgebra of $\hat{\mathcal{A}}$

$$\begin{aligned} \mathcal{A}_- \;=\; & \left\{ f : \mathbb{R} \to \mathbb{C} \mid f(t) = \begin{cases} f_a(t) + f_0\delta(t), & t \geq 0, \\ 0, & t < 0, \end{cases} \right. \\ & \left. \text{with } f_0 \in \mathbb{C} \text{ and } \int_0^\infty e^{\epsilon t}|f_a(t)|dt < \infty, \text{ for some } \epsilon > 0 \right\} \end{aligned}$$

$$\hat{\mathcal{A}}_- = \left\{ \hat{f} \mid f \in \mathcal{A}_- \right\}.$$

A class of unstable irrational systems is the algebra of fractions

$$\hat{\mathcal{B}} = \hat{\mathcal{A}}_-[\hat{\mathcal{A}}_\infty]^{-1},$$

where $\hat{\mathcal{A}}_\infty$ is the subclass of transfer functions in $\hat{\mathcal{A}}_-$ with the property that they are bounded away from zero at infinity. The limit at infinity of the elements in $\hat{\mathcal{A}}_\infty$ exists and it is non-zero. Furthermore, $\hat{f} \in \hat{\mathcal{B}}$ has only finitely many unstable poles in $\overline{\mathbb{C}_+}$ (see [6]-[8]).
We consider the next class of unstable tranfer functions, known in literature as the *Wiener algebra*

$$\hat{\mathcal{W}} = \left\{ \hat{f} \in L_\infty \mid \hat{f} = \hat{f}_1 + \hat{f}_2, \text{ where } \hat{f}_1 \text{ and } \hat{f}_2^{\sim} \in \hat{\mathcal{A}} \right\}.$$

$\hat{\mathcal{W}}$ is a Banach algebra under pointwise addition, multiplication, and scalar multiplication. The elements of $\hat{\mathcal{W}}$ are bounded and continuous on the imaginary axis.

Remark 7.1 *$\hat{f} \in \hat{\mathcal{W}}$ is invertible over $\hat{\mathcal{W}}$ if and only if $\hat{f}(j\omega) \neq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.*

For more properties of the elements of Wiener algebra, see [16].
For certain applications the following subalgebra in which the functions are defined and holomorphic on a strip surrounding the imaginary axis is important:

$$\hat{\mathcal{W}}_- = \left\{ \hat{f} \in L_\infty \mid \hat{f} = \hat{f}_1 + \hat{f}_2, \text{ where } \hat{f}_1 \text{ and } \hat{f}_2^{\sim} \in \hat{\mathcal{A}}_- \right\}.$$

Obviously, $\hat{\mathcal{W}}_- \subset \hat{\mathcal{W}}$. For more properties of $\hat{\mathcal{W}}$, see [16].
We will denote by $\hat{\mathcal{A}}^{n\times m}$, $\hat{\mathcal{A}}_-^{n\times m}$, $\hat{\mathcal{W}}^{n\times m}$, $\hat{\mathcal{W}}_-^{n\times m}$ the classes of $n \times m$ matrices with entries in $\hat{\mathcal{A}}$, $\hat{\mathcal{A}}_-$, $\hat{\mathcal{W}}$, $\hat{\mathcal{W}}_-$, respectively. We will omit the size of the matrix when it is no danger of confusion. For properties of this classes of transfer functions see [6]-[8] and [18].

8 A method for solving the J-spectral factorization for the Wiener class of transfer functions

In this section we present the algorithm of Clancey and Gohberg for finding a J-spectral factorization. More details can be found in [9], page 47. The notation of this section can be found in Appendix A. Let $\mathcal{C}$ be a decomposing R-algebra of continuous matrix valued functions on the unit circle Γ, and let $A \in G\mathcal{C}$. For any nonsingular element $R \in R(\Gamma)$ we can write

$$A = CR, \tag{120}$$

where

$$C = I - (R - A)\, R^{-1}.$$

Since $\mathcal{C}$ is an R-algebra we have that $R(\Gamma)$ is dense in $\mathcal{C}$, and thus it is possible to choose an invertible $R \in R\,(\Gamma)$ such that

$$\|I - C\| = \left\|(R - A)\, R^{-1}\right\| < \min\left\{\|P\|^{-1}, \|Q\|^{-1}\right\}, \tag{121}$$

where $\|P\|$, and $\|Q\|$ denote the operator norm of P, Q on $\mathcal{C}$, respectively. Since the equation (121) is the same as the equation (145) from the Corrolary A.11, with $b = C$, we can write C as

$$C = C_- C_+,$$

where $C_+ = x_I^{-1}$, with x_I the solution of the equation

$$Q(X) + P\,(CX) = I$$

and $C_- = y_I^{-1}$, with y_I the solution of

$$P(Y) + Q\,(YC) = I.$$

We remark that $C_+ \in G\mathcal{C}^+$ and $C_- \in G\mathcal{C}^-$. At this point we have

$$A = C_- C_+ R. \tag{122}$$

Let q be a polynomial which has as zeroes the roots of all the denominators of R with the real part positive, and the corresponding multiplicities, and p a polynomial of the same degree wich has all its zeroos with the real part negative. We can write

$$R = \frac{p}{q} R_+,$$

where $\frac{p}{q}$ is a scalar and $R_+ \in \mathcal{C}^+$. From Lemma A.16, there exists a $R_1 \in R\,(\Gamma)$ such that

$$C_+ R_+ = R_1 B$$

with $B \in G\mathcal{C}^+$. So with (122) we obtain

$$A = C_- \frac{p}{q} R_1 B, \tag{123}$$

where $C_- \in G\mathcal{C}^-$, $R_1 \in GR(\Gamma)$, and $B \in G\mathcal{C}^+$. At this point we use the Theorem A.3 to factorize $\frac{p}{q}R_1$ as

$$\frac{p}{q}R_1 = R_- D R_+, \tag{124}$$

where

$$D(t) = diag\left[\left(\frac{t-t_+}{t-t_-}\right)^{k_1}, ..., \left(\frac{t-t_+}{t-t_-}\right)^{k_n}\right],\ t \in \Gamma,$$

and $R_\pm \in R^\pm(\Gamma)$. Using (124), we factorize A as follows

$$A = C_- R_- D R_+ B. \tag{125}$$

Denoting by

$$\begin{aligned} A_- &= C_- R_- \in G\mathcal{C}^- \\ A_+ &= R_+ B \in G\mathcal{C}^+ \end{aligned}$$

we obtained the standard factorization relative to the contour Γ

$$A = A_- D A_+. \tag{126}$$

As stated in the appendix, the proof of (124) is constructive, but does not lead to explicit formulae for the partial indices $k_1, ..., k_n$ and thus for D. Using the results from Gohberg, Lerer and Rodman [19] we have formulae for computing the partial indices. These partial indices are unique (see Theorem A.4) and if they are all zero, then we have a canonical factorization relative to the contour Γ. Suppose now that we have a canonical factorization for A

$$A = B_- B_+,$$

with $B_- \in G\mathcal{C}^-$ and $B_+ \in G\mathcal{C}^+$, and that A is self-adjoint, i.e. $A = A^*$, where "$*$" means the transpose conjugate of the matrix valued function A. We make the following notation

$$A^\sim(z) = [A\left(\frac{1}{\bar{z}}\right)]^*.$$

For $z \in \Gamma$, we have that

$$A^\sim(z) = [A\left(\frac{1}{\bar{z}}\right)]^* = [A\left(\frac{z}{\|z\|^2}\right)]^* = [A(z)]^*.$$

Since A is self-adjoint

$$A = B_- B_+ = B_+^\sim B_-^\sim = A^\sim = A^*. \tag{127}$$

From the identity

$$\left(\mathcal{C}^\pm\right)^\sim = \mathcal{C}^\mp$$

it follows that $B_\pm^\sim$ are invertible in $\mathcal{C}^\mp$. Consequently, (127) represents two canonical factorizations of A relative to the contour Γ. By the uniqueness of canonical factorization (see Theorem A.5) we know that

$$\left(B_+^\sim\right)^{-1} B_- = B_-^\sim \left(B_+\right)^{-1} = C,$$

where C is a self-adjoint constant matrix. Replacing B_- with $B_+^{\sim}C$ we get

$$A = B_+^{\sim} C B_+ \tag{128}$$

with $B_+ \in G\mathcal{C}^+$ and $B_+^{\sim} \in G\mathcal{C}^-$. But $B_+^{\sim} = B_+^*$ on the unit circle, so we have that

$$A = B_+^* C B_+,$$

where $B_+ \in G\mathcal{C}^+$ C is an invertible self adjoint matrix with $sgnC = sgnA$. The matrix C may be written in the form

$$C = U^* P \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} PU$$

where $P \geq 0$, U is unitary, $p + q = n$ and $p - q = sgnC$. Denoting now $A_+ = PUB_+$ we obtain

$$A = A_+^* \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} A_+.$$

Up till now we have presented the algorithm for solving the J-spectral factorization for the Wiener algebra on the unit disc. In order to pass from the unit circle to the imaginary axis, we use a Mobius mapping such as

$$\eta(s) = \frac{1-s}{1+s}$$

which is invertible.

Remark 8.1 *This procedure can be applied to find, for the Wiener algebra on the imaginary axis, the bistable W and V (units of the Wiener algebra) which are required in the Step 3 and Step 4 of the Algorithm given in Remark 5.8.*

A Some results from Clancey and Gohberg

In this appendix we collect some results of Clancey and Gohberg on standard and canonical factorization of matrix valued functions.

A.1 The Factorization of Rational Matrix Functions

Let us denote by Γ the unit circle, F_Γ^+ the open unit disc, and F_Γ^- the exterior of the unit disc.

Definition A.1 *A is an* algebra *over the scalar field F if its elements admit the operations of addition, multiplication and scalar multiplication, subject to the following conditions:*

1. *A is a ring;*
2. *A is a linear vector space over the scalar field F;*
3. *Multiplication and scalar multiplication commute i.e. $(\alpha x) * (\beta y) = \alpha\beta\,(x * y)$*

If A is an algebra of complex matrix valued functions on Γ with an identity, then GA will denote the group of invertible elements of A.
Denote by $C(\Gamma)^{n\times n}$ the algebra of $n \times n$-matrices whose entries are elements in the space $C(\Gamma)$ of complex valued continuous functions on Γ. We will omit the dimensions when it is no danger of confusion. Alternately, $C(\Gamma)^{n\times n}$ can be viewed as the space of continuous maps from Γ into the space $\mathbb{C}^{n\times n}$ of complex $n \times n$-matrices. The norm of the element $A \in C(\Gamma)$ is defined by

$$\|A\| = \max_{t\in\Gamma} \|A(t)\|,$$

where the norm of an element in $\mathbb{C}^{n\times n}$ is its norm as an operator on the finite dimensional Euclidian space $\mathbb{C}^n$. The group $GC(\Gamma)$ of invertible elements in $C(\Gamma)$ consists of those elements A satisfying

$$\det A(t) \neq 0, t \in \Gamma.$$

We will use the notation $C^{\pm}(\Gamma)$ for the closed subalgebras of $C(\Gamma)$ consisting of those continuous functions that are restrictions to Γ of functions which are holomorphic in $F_\Gamma^{\pm}$ and continuous on $F_\Gamma^{\pm} \cup \Gamma$. The group $GC^{\pm}(\Gamma)$ of invertible elements in $C^{\pm}(\Gamma)$ consists of the matrix valued functions A whose determinant satisfy

$$\det A(z) \neq 0 \ , \ z \in F_\Gamma^{\pm} \cup \Gamma. \tag{129}$$

Definition A.2 *The matrix valued function $A \in GC(\Gamma)$ is said to admit a* (right-) standard factorization relative to the contour Γ *if A can be decomposed as*

$$A = A_- D A_+, \tag{130}$$

with $A_\pm \in GC^{\pm}(\Gamma)$, and D a diagonal matrix function of the form

$$D(t) = diag\left[\left(\frac{t-t_+}{t-t_-}\right)^{k_1}, ..., \left(\frac{t-t_+}{t-t_-}\right)^{k_n}\right],\ t \in \Gamma, \tag{131}$$

with $t_\pm$ *fixed points in* $F_\Gamma^\pm$, $t_\pm \neq \infty$, $k_i \in \mathbb{Z}$ *and* $k_1 \geq ... \geq k_n$. *The integers are called* (the right-) partial indices *of the factorization. The integer*

$$k = \sum_{i=1}^{n} k_i \tag{132}$$

is called the total index *of the factorization. In the case* $k_1 = ... = k_n = 0$, *so that,*

$$A = A_- A_+, \tag{133}$$

then A *is said to admit a* (right-) canonical factorization *relative to* Γ.

Similar definitions can be formulated for (left-) standard and canonical factorizations relative to Γ. The following theorem is the main result of this subsection.

Theorem A.3 *Let* R *be in* $GR(\Gamma)$. *Then* R *admits a factorization*

$$R = R_- D R_+ \tag{134}$$

relative to Γ, *where* $R_\pm \in GR^\pm(\Gamma)$ *and* D *is a diagonal matrix function of the form (131).*

For a proof of this theorem we refer to [9], pages 14-17. The proof is constructive, however, it does not present the factorization (134) in a manner which, for example leads to explicit formulae for the partial indices.
The following theorem proves the uniqueness of the partial indices for factorization relative to a contour.

Theorem A.4 *Suppose that the non-singular matrix valued function* $A \in C(\Gamma)$ *admits two factorizations* $A = A_- D A_+$, $A = B_- \tilde{D} B_+$ *relative to the contour* Γ, *where* D *and* $\tilde{D}$ *are diagonal matrix functions of the form (131). Then the diagonal factors* D *and* $\tilde{D}$ *are identical.*

The proof can be foun in in [9], Theorem 1.1., page 9. The next result shows the simple relationship which holds between different factorizations of matrix valued functions. For a proof see Theorem 1.2., page 11, of [9].

Theorem A.5 *Suppose that the non-singular matrix valued function* $A \in C(\Gamma)$ *admits a factorization* $A = A_- D A_+$ *relative to the contour* Γ, *then the factors in any other factorization* $A = B_- D B_+$ *are given by*

$$B_+ = C_+ A_+ \tag{135}$$

$$B_- = A_- D C_+^{-1} D^{-1} \tag{136}$$

where $C_+ = \left[c_{ij}^+\right]$ *is a non-singular matrix valued function whose entries satisfy:*

1. $c_{ij}^+ = 0$ *if* $k_j < k_i$;
2. c_{ij}^+ *is a constant if* $k_j = k_i$;
3. c_{ij}^+ *is a polynomial of degree less than or equal to* $k_j - k_i$ *if* $k_j > k_i$.

Conversely, if C_+ *is a non-singular matrix whose entries have the properties 1.-3., then with* $B_\pm$ *given by (136) the matrix valued function* A *admits the factorization* $A = B_- D B_+$.

A.2 General factorizations

In this subsection some general results are stated for elements in standard Banach algebras. Later these results will be applied in the case where the Banach algebra is an algebra of complex valued matrix functions on a contour (unit circle) or on the imaginary axis.

Definition A.6 *An algebra A is a* Banach algebra *if there exists a norm on A such that A is a Banach space with respect to the norm and*

$$\|x * y\| \leq \|x\| \, \|y\| \text{ for all } x, y \in A$$

The Banach algebra is commutative *if the multiplicative operation in the algebra is commutative. If A has an identity e, then $\|e\| = 1$.*

Definition A.7 *A* decomposing Banach algebra *A is a Banach algebra with identity e which is a direct sum*

$$A = A^+ \dot{+} A^-$$

of two closed subalgebras $A^\pm$. The projection of A onto A^+ parallel to A^- will be denoted by P and we set $Q = I - P$.

Definition A.8 *An element $a \in GA$ is said to admit a* (right-) canonical factorization *in case*

$$a = a_- a_+$$

where $a_\pm \in GA$ satisfy $a_\pm - e \in A^\pm$, and $a_\pm^{-1} - e \in A^\pm$.

The following result gives equivalent conditions for an element to admit a canonical factorization.

Theorem A.9 *Let A be a decomposing Banach algebra in which elements that have inverse on one side are invertible. For $a \in A$ the following statements are equivalent:*

1. *The element $b = e - a$ admits a canonical factorization*

$$b = b_- b_+.$$

2. *Each of the equations*

$$\begin{aligned} x - P\,(ax) &= e \\ y - Q\,(ya) &= e \end{aligned}$$

is solvable in A.

3. *For any pair of elements $f, g \in A$, each of the equations*

$$\begin{aligned} x - P\,(ax) &= f \\ y - Q\,(ya) &= g \end{aligned}$$

is uniquely solvable in A.

Proof: The complete proof can be found in [9], page 35-37. From this proof we extracted the proof for the implication 2. to 1. Note that by 3. we have that the equations in 2. are uniquely solvable.
Let the elements x_a and y_a be solutions to the equations

$$\begin{aligned} x - P(ax) &= e && (137) \\ y - Q(ya) &= e. && (138) \end{aligned}$$

Defining

$$u = x_a - e \text{ and } v = y_a - e, \tag{139}$$

we obtain the following relations for u and v

$$\begin{aligned} u + e - P(a(u+e)) &= e \\ v + e - Q((v+e)a) &= e. \end{aligned}$$

Thus $u \in A^+$, $v \in A^-$. If we denote u_- by $u_- = -Q(ax_a) \in A^-$, and v_+ by $v_+ = -P(y_a a) \in A^+$, then we have

$$\begin{aligned} bx_a &= e + u_- \\ y_a b &= e + v_+, \end{aligned}$$

since

$$\begin{aligned} bx_a - e &= (e-a)x_a - e = x_a - ax_a - e = x_a - P(ax_a) - e - Q(ax_a) \overset{(137)}{=} -Q(ax_a) \\ y_a b - e &= y_a(e-a) - e = y_a - y_a a - e = y_a - Q(y_a a) - e - P(y_a a) \overset{(138)}{=} -P(y_a a). \end{aligned}$$

The next equalities hold

$$y_a b x_a = y_a(e + u_-) = (e + v_+)x_a.$$

So, replacing y_a and x_a from (139) in the last two expressions, we get

$$y_a b x_a = (e+v)(e+u_-) = (e+v_+)(e+u). \tag{140}$$

Keeping only the second equality, we see that this is equivalent to

$$v + u_- + vu_- = v_+ + u + v_+ u.$$

Since $u, v_+ \in A^+$, $v, u_- \in A^-$, this implies that $v + u_- + vu_- \in A^-$ and $v_+ + u + v_+ u \in A^+$. Using the definition of the Decomposing Banach algebra (A is a direct sum of the closed subalgebras $A^\pm$) we get

$$\begin{aligned} v + u_- + vu_- &= 0 \\ v_+ + u + v_+ u &= 0, \end{aligned}$$

which are the same as

$$\begin{aligned} (e+v)(e+u_-) &= e && (141) \\ (e+v_+)(e+u) &= e. && (142) \end{aligned}$$

In particular, this shows that $e+v$ and $e+u$ are invertible on one side. By assumption, this implies that they are invertible, and moreover

$$\begin{aligned} (e+v)^{-1} &= e+u_- \quad (143) \\ (e+u)^{-1} &= e+v_+. \quad (144) \end{aligned}$$

Using (140) and (141) we have

$$y_a b x_a = e.$$

Replacing y_a and x_a we have from (139) that

$$(e+v)\,b\,(e+u) = e.$$

Finally, using (143) and (144) we conclude that

$$b = (e+v)^{-1}(e+u)^{-1} = (e+u_-)(e+v_+) = y_a^{-1} x_a^{-1}.$$

where $u, v_+ \in A^+$, $v, u_- \in A^-$. ■

Remark A.10 *We have the relations between the elements defined in the proof of Theorem A.9*

$$\begin{aligned} u - u_- &= a x_a \\ v - v_+ &= y_a a \\ x_a &= e + u = (e+v_+)^{-1} \\ y_a &= e + v = (e+u_-)^{-1} \\ b^+ &= y_a^{-1} \\ b^- &= x_a^{-1}. \end{aligned}$$

From Theorem A.9 we obtain a simple corollary. The proof is given on page 39 of [9].

Corollary A.11 *If an element $b \in GA$ satisfies*

$$\|e-b\| < \min\left\{\|P\|^{-1}, \|Q\|^{-1}\right\}, \quad (145)$$

where $\|P\|$, and $\|Q\|$,respectively denotes the operator norm of the projection P, respectively Q, then b admits a canonical factorization. Moreover whenever the inequality (145) holds, the factors $b_\pm$ in the canonical factorization $b = b_- b_+$ of b may be chosen as $b_+ = x_e^{-1}$, where the x_e is the solution of the equation

$$Q(x) + P(bx) = e \quad (146)$$

and $b_- = y_e^{-1}$, where y_e is the solution of

$$P(y) + Q(yb) = e. \quad (147)$$

A.3 Factorization in decomposing R-algebras of matrix valued functions

Let $\mathcal{C}$ be a decomposing Banach algebra of continuous matrix valued functions on the contour Γ such that $R(\Gamma) \subset \mathcal{C}$. Let $C_0^-(\Gamma)$ denote the subalgebra of $C^-(\Gamma)$ consisting of those matrix valued functions f satisfying $f(\infty) = 0$. The notations $\mathcal{C}^{\pm}$, $\mathcal{C}_0^-$ will stand for the subalgebras of C given by

$$\mathcal{C}^{\pm} = \mathcal{C} \cap C^{\pm}(\Gamma), \, \mathcal{C}_0^- = \mathcal{C} \cap C_0^-(\Gamma).$$

The Banach algebra $\mathcal{C}$ is the direct sum of its subalgebras $\mathcal{C}^+$ and $\mathcal{C}_0^-$, i.e.,

$$\mathcal{C} = \mathcal{C}^+ \dotplus \mathcal{C}_0^-.$$

Remark A.12 *$C(\Gamma)$ itself is not a decomposing. The fact that*

$$C(\Gamma) \neq C^+(\Gamma) \dotplus C_0^-(\Gamma)$$

is demonstrated by Hoffman in [22], page 155, for the scalar case (see also the remark on page 40 in [9]).

The next definition is similar to Definition A.2.

Definition A.13 *The matrix valued function $A \in G\mathcal{C}$ is said to admit a* (right-) standard factorization relative to the contour Γ *if A can be decomposed as*

$$A = A_- D A_+$$

with $A_\pm \in G\mathcal{C}^{\pm}$,and D a diagonal matrix function of the form

$$D(t) = diag\left[\left(\frac{t-t_+}{t-t_-}\right)^{k_1}, ..., \left(\frac{t-t_+}{t-t_-}\right)^{k_n}\right], \ t \in \Gamma, \tag{148}$$

with $t_\pm$ fixed points in $F_\Gamma^{\pm}$, $t_\pm \neq \infty$, $k_i \in \mathbb{Z}$, and $k_1 \geq ... \geq k_n$ integers called (right-) partial indices *of the factorization. The integer*

$$k = \sum_{i=1}^{n} k_i$$

is the total index *of the factorization. In the case that $k_1 = ... = k_n = 0$, so that,*

$$A = A_- A_+,$$

then A is said to admit a (right-) canonical factorization *relative to Γ.*

Definition A.14 *A Banach algebra $\mathcal{C}$ of continuous function on the contour Γ, containing $R(\Gamma)$ as a dense subset, will be called* R-algebra *of continuous functions on the countour Γ.*

The existence of the standard factorization for an alement of a decomposing R-algebra of continuous functions on the countour Γ is stated in the next theorem.

Theorem A.15 *Let $\mathcal{C}$ be a decomposing* R-algebra *of continuous functions on the contour Γ. Then every element $A \in \mathcal{C}$ satisfying*

$$\det A(t) \neq 0, \text{ for all } t \in \Gamma$$

admits a factorization relative to Γ in $\mathcal{C}$.

Lemma A.16 *Let $\mathcal{C}$ be a decomposing R-*algebra *of continuous matrix valued functions on the countour Γ. Then every element $A \in \mathcal{C}^{\pm}$ satisfying*

$$\det A(t) \neq 0, \text{ for all } t \in \Gamma$$

admits factorizations

$$A = BR_1 = R_2C$$

where $R_1, R_2 \in R(\Gamma)$ and B, C are elements in $\mathcal{C}^{\pm}$ satisfying

$$\det B(t) \neq 0, \ \det C(t) \neq 0, \text{ for all } t \in F_{\Gamma}^{\pm} \cup \Gamma.$$

For a proof see [9], page 45-46.

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