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Computing the Maximum Turnable State of Simple Parts

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Abstract: Machining planning is an important task which directly affects the cost of a product. Computer aided methods are increasingly being developed and used to assist in the planning task. We describe a new concept, the Maximum Turnable State of a part. The concept is intuitive and relates to an intermediate state of every turned part. By using this idea in machining planning, efficiency can be increased. We describe the concept, study the mathematical, geometrical and computational aspects of the problem and give examples.

Keywords: machining planning, maximum turned state, mill-turns, computer-aided process planning, computational methods, optimization techniques.

AMS Classification: 90C26, 65D17

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1 Introduction

In this paper, we consider the Maximum Turnable State (MTS) - a novel concept in product design and manufacturing process planning. This concept was recently introduced by Yip-Hoi and Dutta [Yip-Hoi et al., 1998] and shown to be of practical use in the process planning tasks involving machining on mill-turn machines. Figure 1 illustrates a common mill/turn configuration.

The concept of MTS is simple. Consider a generic final part (FP) that is composed of planar and non-planar surfaces. The MTS is an intermediate state of the part beyond which no more turning operations can be done without gouging the surfaces. A three dimensional object is said to be a spherical cylinder, if it has a central axis which is a line segment, and every cross-section perpendicular to this axis is a circle with its center on the central axis. The diameters of the cross-sections may vary. Given a final part which is a three dimensional object, its Maximum Turnable state is defined to be the smallest volume spherical cylinder containing the final part.

Knowing the MTS of any part can be of use in many instances. For example, by a boolean subtraction of the MTS from the initial workpiece (bar stock) one can obtain the total volume to be turned, and can directly generate the cutter path. Turning is more efficient than milling (w.r.t material removal rates) and therefore this strategy leads to efficient fabrication. On the design side, the MTS can be used effectively to determine near "net-shaped" workpieces that correspond to several different parts. That is, for a part family consisting of say M different parts, the MTS can be used to determine N ($N \ll M$) intermediate turned parts that can then be machined to yield the full part family (of M parts). This enables the effective utilization of turning resources. As needed, the appropriate intermediate state can be further machined (possibly some additional turning and milling) to realize the final part. These N intermediate states/shapes can be produced either by turning or even by casting, depending upon the best utilization of the available resources. Manufacturing process planning under such an environment (of intermediate stages of a part) is quite different. Feature extraction for machining is now driven by the MTS and not by machinable volumes.

The efficient determination of the MTS for a general part is quite complex. If one assumes a part axis (e.g. chosen by the designer), as in [1], the problem simplifies. However, for a complex part, it is not easy, or visually possible, to determine the best axis for a part. In what follows we will describe how the MTS can be computed.

The paper is organized as follows. In Section 2 we formulate the problem explicitly and describe a conceptual algorithm for determining the MTS of a general FP. Section 3 is concerned with the mathematical and geometrical properties of the MTS-problem. In Section 4 we describe the numerical computation of the minimum volume of revolution for fixed axis. We discuss the convex and non-convex case and we will consider final parts with holes. In Section 5 we examine algorithms for solving the minimization problem. Section 6 concludes with some remarks on generalizations and comments on future developments.

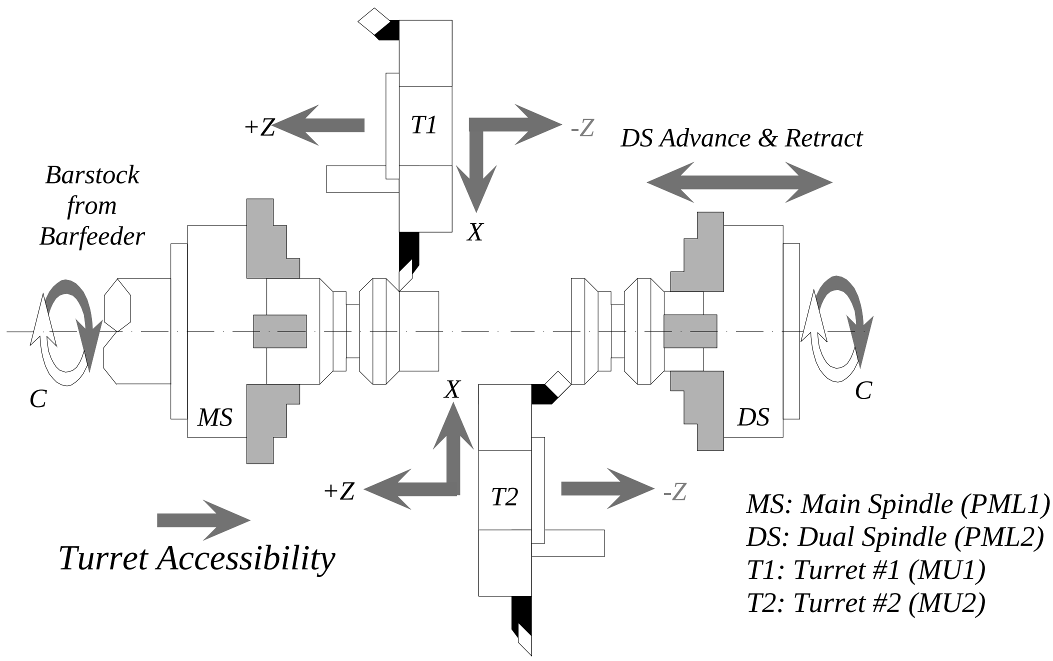

Figure 1: A common Mill/Turn configuration

2 A general description of the MTS-problem

We give an explicit description of the problem and outline a conceptual method for solving it.

MTS-problem: Let FP be a 3D final part. Consider the machine axis of a Mill/Turn and a profile function above that axis in a plane containing the axis such that the spherical cylinder obtained by revolving the profile function around the axis contains the final part. We want to determine such a spherical cylinder with minimum volume. A spherical cylinder with minimum volume is called the MTS of FP.

This formulation leads to the following conceptual method for solving the problem. Here we describe the case that FP does not have holes (or holes are neglected).

First, we choose a representation of the machine axis ax: Given a support vector $c = (c_1, c_2, c_3) \in I\!R^3$ and a direction $d(\phi, \varphi) = (\cos\phi\cos\varphi, \sin\phi\cos\varphi, \sin\varphi)$, $\phi, \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, the axis is defined by the line trough c along the direction $d(\phi, \varphi)$,

$$\text{ax:} \qquad ax(\gamma) = c + \gamma(\cos\phi\cos\varphi, \sin\phi\cos\varphi, \sin\varphi), \quad \gamma \in I\!R \; . \tag{1}$$

So, the axis ax is given by the parameter $p = (c, \phi, \varphi)$.

Assume the axis, i.e. the vector p, is fixed. The profile function defining the body of revolution containing FP is calculated as follows. For fixed γ we consider the plane $pl(\gamma)$ through the axis point $ax(\gamma)$, perpendicular to the direction $d(\phi, \varphi)$,

$$pl(\gamma) = \{x \in I\!R^3 \mid (x - c)^T d(\phi, \varphi) = \gamma\} \; . \tag{2}$$

Here, $x^T y$ denotes the dotproduct, $x^T y = x_1 y_1 + x_2 y_2 + x_3 y_3$ and we define the distance between x and y by $||x - y|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$, $x, y \in I\!R^3$.

The slice $sl(\gamma)$ is the intersection of the plane $pl(\gamma)$ with FP (see Figure 2). The value $b(\gamma)$ of the profile function is the maximum distance between the axis point $ax(\gamma)$ and the points in the slice $sl(\gamma)$,

$$b(\gamma) = \max\{||x - ax(\gamma)|| \mid x \in sl(\gamma)\} \; . \tag{3}$$

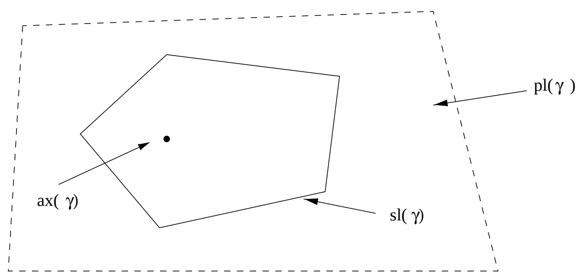

Figure 2: The slice sl(γ)

In the sequel, let γ_b and γ_e denote the minimum and maximum values of γ, respectively, such that the plane $pl(\gamma)$ intersects the final part. The value γ_b can be calculated by solving the optimization problem

$$(\text{P}_b): \qquad \min \; x^T d(\phi, \varphi) \qquad \text{subject to} \quad x \in \text{FP} \; . \tag{4}$$

With a solution $\overline{x}$ of (P$_b$) the value γ_b is given by $\gamma_b = (\overline{x} - c)^T d(\phi, \varphi)$. Accordingly, γ_e is found by solving the problem (P$_e$), obtained by replacing minimize by maximize in (P$_b$).

The final part is contained in the body obtained by revolving the profile function $b(\gamma)$, $\gamma \in [\gamma_b, \gamma_e]$ around the axis ax (see Figure 3). Using the formula $r^2\pi$ for the area of a circle with radius r, the volume $\text{vol}(p)$ of this body (for fixed p) is given by

$$\text{vol}(p) = \pi \int_{\gamma_b}^{\gamma_e} (b(\gamma))^2 \; d\gamma \; . \tag{5}$$

Solving the MTS-problem is equivalent with solving the following optimization problem.

$$(\mathrm{P_{MTS}}): \quad \text{find } p = (c_1, c_2, c_3, \phi, \varphi), \quad \phi, \varphi \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \text{ such that } \mathrm{vol}(p) \text{ is minimal} . \qquad (6)$$

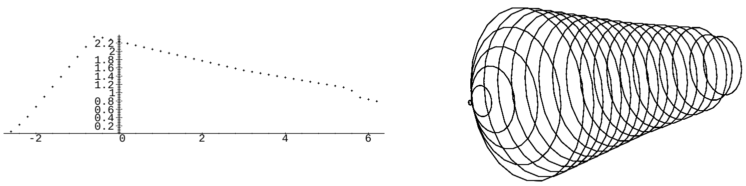

Figure 3: a. The profile $b(\gamma)$ b. The MTS for a fixed axis

3 Properties of the MTS-problem

In this section we study the mathematical and geometrical properties of the function $\mathrm{vol}(p)$.

In (1) the axis ax is described by the 5 parameters $p = (c_1, c_2, c_3, \phi, \varphi)$. This parameterization however contains some redundancy. In fact, given an axis $ax(\gamma) = c + \gamma d$, any point $\overline{c}$ on ax leads to an axis $ax(\gamma) = \overline{c} + \gamma d$ with the same profile function and the same volume of revolution. We now show that $(\mathrm{P_{MTS}})$ is a 4-dimensional minimization problem.

Suppose we are given $a \in I\!R^3,\ a \neq 0,\ \alpha \in I\!R$. The set $H = \{x \in I\!R^3 \mid a^T x = \alpha\}$ is a plane with normal vector a. We consider the three special coordinate-planes (e_i unit vectors)

$$H_i = \{x \in I\!R^3 \mid x_i = 0\} \quad \text{with normal vectors } a_i = e_i,\ i = 1, 2, 3 .$$

Lemma 1 *Any line ax (cf. (1)) intersects at least one of the hyperplanes* H_1, H_2, H_3.

Proof. Consider a plane $H = \{x \in I\!R^3 \mid a^T x = \alpha\}$, $a \neq 0$, and the axis ax. Assume that $a^T d \neq 0$. Then the equation

$$\alpha = a^T ax(\gamma) = a^T c + \gamma a^T d$$

has a unique solution $\gamma = (\alpha - a^T c)/a^T d$, i.e. ax intersects H. Now, suppose that ax does not intersect the hyperplanes H_1, H_2, H_3. Then, necessarily $e_i^T d = 0,\ i = 1, 2, 3$, implying $d = 0$ in contradiction to $||d|| = 1$. □

In view of Lemma 1, any line in $I\!R^3$ can be written in the form (1) with support point c from the union $H_1 \cup H_2 \cup H_3$. Consequently, with the four-dimensional parameter sets $P_i := \left\{p = (c_1, c_2, c_3, \phi, \varphi) \mid c_i = 0,\ \phi, \varphi \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\right\},\ i = 1, 2, 3$, the MTS-problem can be written as

$$(\mathrm{P_{MTS}}): \qquad \min\ \mathrm{vol}(p) \quad \text{subject to } p \in P_1 \cup P_2 \cup P_3 , \qquad (7)$$

This proves that $(\mathrm{P_{MTS}})$ is a 4-dimensional problem.

It is clear that we can restrict the minimum search in (P_{MTS}) to all support vectors c in a bounded subset of the planes H_1, H_2, H_3. One could think that the vector c can be restricted to the final part itself. Equivalently this would mean that the optimal MTS-axis intersects FP. The following example shows that this is not true in general.

Example 1 Consider the final part given by the triangular block as indicated in Figure 4a. The optimal MTS-axis ax does not intersect FP.

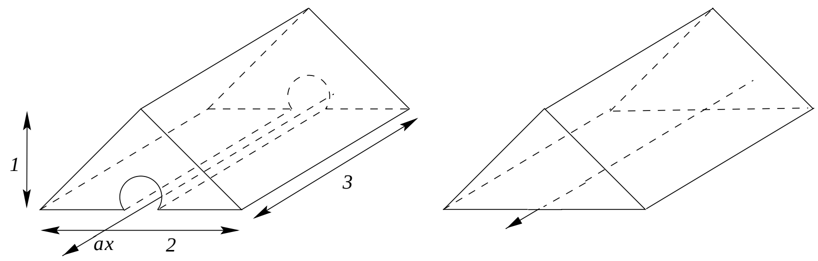

Figure 4: a. non-convex triangular block b. convex triangular block

The final part in Figure 4a is not convex. A set $S \subset I\!R^n$ is said to be convex if with any $x, y \in S$ the whole line segment $\overline{xy}$ is contained in S, where

$$\overline{xy} := \{ (1-t)x + ty \mid t \in [0, 1] \} .$$

Let S^c denote the convex hull of S, i.e. the smallest convex set containing S. The following theorem (in particular) implies that for any convex final part the optimal MTS-axis must intersect the final part.

Theorem 1 *Suppose ax is the optimal MTS-axis of a final part FP. Then ax must intersect the convex hull FP^c.*

Proof. Suppose that ax ($ax(\gamma) = c + \gamma d$) does not intersect FP^c. Then, the distance δ between ax and FP^c is positive,

$$\delta := \min_{y \in ax,\ x \in \mathrm{FP}^c} ||y - x|| > 0 .$$

Suppose the minimum distance is attained at $\overline{c} \in ax$, $\overline{x} \in \mathrm{FP}^c$. Given this point $\overline{c}$, the vector $\overline{x}$ is the solution of the minimization problem $\min_{x \in \mathrm{FP}^c} ||\overline{c} - x||$. By a theorem in convex analysis (cf. [Luenberger, 1974]) we have

$$(\overline{c} - \overline{x})^T (x - \overline{x}) \leq 0 \qquad \text{for all } x \in \mathrm{FP}^c . \tag{8}$$

Moreover, $\overline{c}$ is the orthogonal projection of $\overline{x}$ onto ax, i.e. $(\overline{c} - \overline{x})^T d = 0$. Thus, for any $x \in \mathrm{FP}$ it follows $||\overline{c} - x||^2 = ||\overline{c} - \overline{x} + \overline{x} - x||^2 = ||\overline{c} - \overline{x}||^2 + ||\overline{x} - x||^2 + 2(\overline{c} - \overline{x})^T(\overline{x} - x) \geq ||\overline{c} - \overline{x}||^2 + ||\overline{x} - x||^2$ or

$$||\overline{c} - x||^2 - ||\overline{x} - x||^2 \geq |\overline{c} - \overline{x}||^2 = \delta^2 > 0 . \tag{9}$$

Consider the optimal axis $ax(\gamma) = \overline{c} + \gamma d$ and the shifted axis $\overline{ax}(\gamma) = \overline{x} + \gamma d$. We will show that the volume of revolution around $\overline{ax}$ is smaller than that for ax. To do so, consider the slice $sl(\gamma)$ through the point $ax(\gamma)$. Note, that in view of $(\overline{c} - \overline{x})^T d = 0$ we have $(\overline{ax}(\gamma) - ax(\gamma))^T d = (\overline{x} - \overline{c})^T d = 0$, i.e. $\overline{ax}(\gamma) \in pl(\gamma)$. For any $x \in sl(\gamma)$ in view of $(\overline{ax}(\gamma) - x) \perp d$ and $(\overline{ax}(\gamma) - \overline{x}) = \gamma d$ we find by the theorem of Pythagoras

$$\begin{aligned} ||\overline{x} - x||^2 &= ||\overline{ax}(\gamma) - x + \overline{x} - \overline{ax}(\gamma)||^2 = ||\overline{ax}(\gamma) - x||^2 + ||\overline{x} - \overline{ax}(\gamma)||^2 \\ ||\overline{c} - x||^2 &= ||ax(\gamma) - x + \overline{c} - ax(\gamma)||^2 = ||ax(\gamma) - x||^2 + ||\overline{c} - ax(\gamma)||^2 \ . \end{aligned}$$

Subtraction yields using (9) and $\overline{x} - \overline{ax}(\gamma) = \overline{c} - ax(\gamma)$

$$||ax(\gamma) - x||^2 - ||\overline{ax}(\gamma) - x||^2 = ||\overline{c} - x||^2 - ||\overline{x} - x||^2 \geq \delta^2$$

and $||ax(\gamma) - x||^2 \geq ||\overline{ax}(\gamma) - x||^2 + \delta^2$. Consequently, by taking the maximum over all $x \in sl(\gamma)$ on the right-hand side, we find

$$b^2(\gamma) \geq \overline{b}(\gamma) + \delta^2$$

for the profile functions b of ax, $\overline{b}$ of $\overline{ax}$, respectively. Using (5) this proves that the volume of revolution for $\overline{ax}$ is smaller than that for ax. □

For convex final parts the optimal axis needs not to go through the interior of FP. The convex FP in Figure 4b for example is the convex hull of the final part in Figure 4a. The optimal axis remains on the boundary of FP. Using similar arguments as those used in the proof of Theorem 1 the following sharper results can be proven:

> Suppose that the optimal MTS-axis $ax(\gamma)$ of a final part does not pass through the interior of the convex hull FP^c. Then the boundary $\partial\mathrm{FP}^c$ of FP^c contains a whole line segment $\{ax(\gamma) \mid \gamma \in [\gamma_1, \gamma_2]\},\ \gamma_1 < \gamma_2$, of the axis.

We now show that the volume function $\mathrm{vol}(c, \phi, \varphi)$ is not convex in general. A function $f : K \to I\!R,\ K \subset I\!R^n$, K convex, is said to be convex if for any $x, y \in K,\ t \in [0, 1]$ the inequality

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

holds. A point $\overline{x} \in K$ is called a local minimizer of f if for some $\varepsilon > 0$ we have

$$f(x) \geq f(\overline{x}) \qquad \text{for all } x \in K,\ ||x - \overline{x}|| < \varepsilon \ .$$

If $f(x) \geq f(\overline{x})$ is valid for all $x \in K$, then $\overline{x}$ is said to be a global minimizer. It is a standard result in convex analysis that any local minimizer of a convex function f must be a global minimizer (cf. [Luenberger, 1974]). In view of this result the volume function $\mathrm{vol}(c, \phi, \varphi)$ in the following example is not convex.

Example 2 Given $l_1, l_2, l_3,\ l_1 \gg l_2, l_3 > 0$, we define the rectangular final part, $\mathrm{FP} = \{x \in I\!R^3 \mid -l_i \leq x_i \leq l_i,\ i = 1, 2, 3\}$ and the axes $ax_i(\gamma) = c + \gamma d(\phi_i, \varphi_i)$ with

$$(\phi_1, \varphi_1) = (0, 0), \quad (\phi_2, \varphi_2) = (0, \frac{\pi}{2}) \quad (\phi_3, \varphi_3) = (\frac{\pi}{2}, 0)$$

and $c = 0$ (see Figure 5). One can show that these three axis correspond to local minimizers of $\mathrm{vol}(c, \phi, \varphi)$. However, in view of $l_1 \gg l_2, l_3$ only the axis ax_1 is a global minimizer.

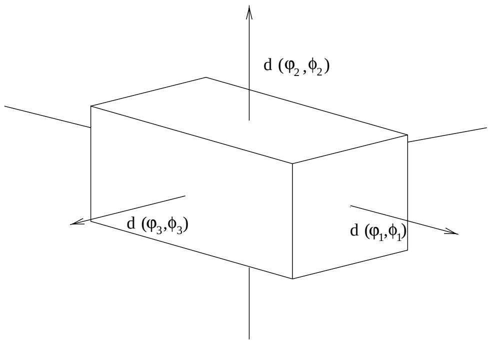

Figure 5: rectangular block final part

Next we will show that the function $\mathrm{vol}(c, \phi, \varphi)$ needs not to be differentiable everywhere. Consider in the x_1x_2-plane the triangle with vertices (cf. Figure 6a)

$$p^0 = (0,0), \quad p^1 = (1, -\tan\beta), \quad p^2 = (1, \tan\alpha)\ ,$$

$\alpha, \beta,\ 0 < \alpha \le \beta < \frac{\pi}{2}$ and the axis $ax_0(\gamma) = \gamma T(\varphi)(1,0)$ with the matrix $T(\varphi) = \binom{\cos\varphi\ \ -sin\varphi}{\sin\varphi\ \ \cos\varphi}$ depending on the parameter $\varphi \approx 0$. Revolving the triangle around this axis sweeps a 3D-volume. It is clear that one can extend the triangle to a 3D final part which sweeps the same volume. We now will prove that this volume function does not depend differentiably on φ. First we apply the transformation (cf. Figure 6b)

$$q^0(\varphi) = T^{-1}(\varphi)p^0, \quad q^1(\varphi) = T^{-1}(\varphi)p^1, \quad q^2(\varphi) = T^{-1}(\varphi)p^2\ .$$

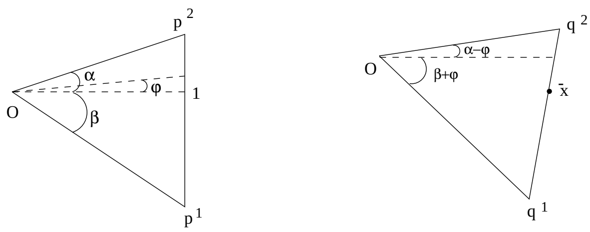

Figure 6: a. triangular part b. rotated triangular part

Obviously, for $\varphi \le 0$ the profile function is given by the segment $\overline{q^0q^1}$,

$$b_1(\gamma) = \frac{-q_2^1(\varphi)}{q_1^1(\varphi)}\gamma = \frac{\sin\varphi + \cos\varphi\ \tan\beta}{\cos\varphi - \sin\varphi\ \tan\beta}\ \gamma\ .$$

For $\varphi > 0$ the profile function is given by pieces of the segments $\overline{q^0q^1}$, $\overline{q^1q^2}$ and $\overline{q^0q^2}$. One finds

$$b_2(\gamma) = \begin{cases} b_1(\gamma) & \text{for } \gamma \in [0, q_1^1(\varphi)] \\ \frac{q_2^1(\varphi)-q_2^2(\varphi)}{q_1^1(\varphi)-q_1^2(\varphi)}\gamma - \frac{1}{\sin\varphi} = \frac{\gamma}{\tan\varphi} - \frac{1}{\sin\varphi} & \text{for } \gamma \in (q_1^1(\varphi), \overline{\gamma}(\varphi)) \\ \frac{-q_2^2(\varphi)}{q_1^2(\varphi)}\gamma = \frac{-\sin\varphi+\cos\varphi\ \tan\alpha}{\cos\varphi+\sin\varphi\ \tan\alpha}\ \gamma & \text{for } \gamma \in [\overline{\gamma}(\varphi), q_1^2(\varphi)] \end{cases}$$

where $\overline{\gamma}(\varphi) = \frac{\cos\varphi\cos\alpha+\sin\varphi\sin\alpha}{\cos\varphi\cos\alpha+\sin\varphi\tan\alpha}$. The volume function is given by

$$\mathrm{vol}(\varphi) = \begin{cases} \mathrm{vol}_1(\varphi), \varphi \le 0 \\ \mathrm{vol}_2(\varphi), \varphi > 0 \end{cases} \text{ with } \mathrm{vol}_i(\varphi) = \pi \int_0^{q_1^i(\varphi)} b_i^2(\gamma) d\gamma,\ i = 1, 2.$$

Since vol_1 is continuously differentiable around $\varphi = 0$, we obtain after some manipulations

$$\begin{aligned} \lim_{\varphi\downarrow 0} \frac{\mathrm{vol}(\varphi) - \mathrm{vol}(0)}{\varphi} &= \lim_{\varphi\downarrow 0}\left(\frac{\mathrm{vol}_1(\varphi) - \mathrm{vol}_1(0)}{\varphi} + \frac{\pi}{\varphi} \int_{q_1^1(\varphi)}^{q_1^2(\varphi)} b_2^2(\gamma) d\gamma \right) \\ &= \lim_{\varphi\uparrow 0} \frac{\mathrm{vol}(\varphi) - \mathrm{vol}(0)}{\varphi} + \frac{1}{3}(\tan^3\alpha + \tan^3\beta) . \end{aligned}$$

This proves that the volume function is not differentiable at $\varphi = 0$.

4 Calculation of the volume of revolution for fixed axis

In this section we discuss the numerical computation of the volume function $\mathrm{vol}(p)$ for a fixed axis (1), i.e. for fixed $p = (c, \phi, \varphi)$. We confine ourselves to the special case of parts bounded by planes. In this case most of the calculations can be done explicitly.

First, we introduce a transformation of the axis ax which simplifies the computation of the profile function $b(\gamma)$. Note that the direction vector $d(\phi, \varphi)$ is obtained by applying two rotations to the unit vector $e_1 = (1, 0, 0)$,

$$d(\phi, \varphi) = \hat{Q}(\phi, \varphi)e_1 , \quad \hat{Q}(\phi, \varphi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix} .$$

Thus, the mapping

$$y = Q\ (x - c) \quad \text{with } Q = Q(\phi, \varphi) := \hat{Q}^{-1}(\phi, \varphi) \tag{10}$$

transforms the axis ax in (1) in the x-space into the y_1-axis $ax_y(\gamma) = \gamma(1, 0, 0)$ of the y-space and the final part FP into

$$\mathrm{FP}_y = \{y = Q(x - c) \mid x \in \mathrm{FP}\} .$$

This transformation (translation/rotation) does not alter distances between points. Consequently the following holds: The profile function for FP w.r.t. ax equals the profile function for FP_y w.r.t. y_1-axis.

As an example consider the profile function $b(\gamma)$ obtained by revolving the line segment $\overline{v^1v^2}$ between two points v^1, v^2 around the axis in (1). We apply the transformation $q^1 = Q(v^1 - c)$, $q^2 = Q(v^2 - c)$ and interchange q^1 with q^2 if $q_1^1 > q_1^2$ to keep the orientation. Then, the profile function obtained by revolving $\overline{q^1q^2}$ around the y_1-axis is given by

$$b(\gamma) = \sqrt{\left(q_2^1 + t(\gamma)(q_2^2 - q_2^1)\right)^2 + \left(q_3^1 + t(\gamma)(q_3^2 - q_3^1)\right)^2},\ t(\gamma) = \frac{\gamma - q_1^1}{q_1^2 - q_1^1},\ \gamma \in [q_1^1, q_1^2]. \tag{11}$$

CONVEX FINAL PART: We now describe the calculation of $b(\gamma)$ for polyhedral FP. A polyhedral part is a part bounded by finitely many planes,

$$\text{FP} = \{x \in I\!R^3 \mid x^T a_j \le b_j,\ j \in J\}, \qquad J = \{1, \ldots, m\}. \tag{12}$$

For the calculation of the profile function it is better to describe the polyhedral final part by the vertices and edges of FP. A vertex v of the polyhedron FP is a point on the surface of FP defined as the intersection of three (independent) planes, i.e.

$$v \text{ is a vertex} \Longleftrightarrow v \in \text{ FP and } v^T a_{j_1} = b_{j_1},\ v^T a_{j_2} = b_{j_2},\ v^T a_{j_3} = b_{j_3},$$

with distinct $j_1, j_2, j_3 \in J$ and linearly independent $a_{j_1}, a_{j_2}, a_{j_3}$. Let $V = \{v^1, \ldots, v^k\}$ be the vertices of FP. An edge E_{ij} of FP is a line segment between two vertices v^i, v^j defined by the intersection of two planes, $E_{ij} = \overline{v^i v^j}$. Let $E = \{E_{ij} \mid (i, j) \in L\}$ denote the set of edges of FP. We give an example.

Example 1 FP is given by the inequalities (see Figure 7):

$$x_1 \le 1 \quad -x_2 \le 0 \quad -x_3 \le 0 \quad -x_1 + x_2 + x_3 \le 0$$

The vertices and edges of this object are:

$$v^1 = (0, 0, 0) \quad v^2 = (1, 0, 0) \quad v^3 = (1, 1, 0) \quad v^4 = (1, 0, 1)$$

and $E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}$.

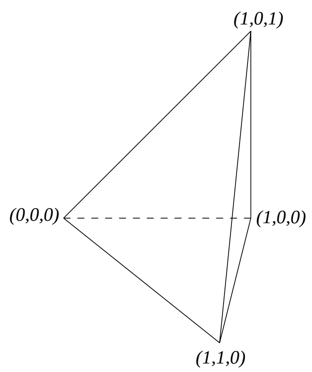

Figure 7: A polyhedral final part.

We now describe the calculation of $b(\gamma)$ and $\text{vol}(p)$ for fixed axis. First apply the transformation (10),

$$q^i = Q(v^i - c), \qquad i = 1, \ldots, k\,.$$

For any edge $\overline{q^i q^j}$, $(i, j) \in L$ the profile function is given by formula (11),

$$b_{ij}(\gamma) = \sqrt{\left(q_2^i + t(\gamma)(q_2^j - q_2^i)\right)^2 + \left(q_3^i + t(\gamma)(q_3^j - q_3^i)\right)^2}, \quad t(\gamma) = \frac{\gamma - q_1^i}{q_1^j - q_1^i}\,,$$

$\gamma \in [q_1^i, q_1^j]$. The maximum distance $b(\gamma)$ in any slice perpendicular to the y_1-axis is attained at the edges. Thus, the profile function is given by

$$b(\gamma) = \max\Big\{\ b_{ij}(\gamma) \mid (i, j) \in L,\ \text{such that}\ \gamma \in [q_1^i, q_1^j]\ \Big\}\ . \tag{13}$$

The values $\gamma_b,\ \gamma_e$ (cf. Section 2) are computed via

$$\gamma_b = \min_{1\leq i\leq k}\ q_1^i, \qquad \gamma_e = \max_{1\leq i\leq k}\ q_1^i\ . \tag{14}$$

Finally the volume function $\mathrm{vol}(p)$ can be calculated approximately as follows: Choose a natural number N, a mesh size $\Delta = \frac{\gamma_e-\gamma_b}{N}$, and approximate $\mathrm{vol}(p)$ in (5) by the trapezium rule,

$$\mathrm{vol}(p) \approx \mathrm{vol}_a(p) = \pi\frac{\Delta}{2}\left(b^2(\gamma_0) + 2\sum_{s=1}^{N-1} b^2(\gamma_s)\ \ + b^2(\gamma_N)\right) \tag{15}$$

on the discretization $\gamma_s = \gamma_b + s\Delta,\ s = 0, \ldots, N$. The algorithm for finding the MTS of a polyhedral FP can now be summarized as follows.

Algorithm 1 (*Calculation of the MTS of a polyhedral FP)*

1. Compute the set V of vertices and the set E of edges of FP.
2. To solve the MTS-problem we have to find a value $p = (c, \phi, \varphi)$ that minimizes the function $\mathrm{vol}_a(p)$. For any given p this function is computed as described above.

NON-CONVEX FINAL PART: Most machined parts are non-convex. In case that the final part is a non-convex body bounded by finitely many planes the boundary of FP can again be described by the sets V and E of vertices and edges of FP. Then, for fixed axis, i.e. for fixed p, the profile function $b(\gamma)$ can be computed by formula (13) and the volume function is approximately given by (15) as in the convex case.

More generally, as a rule, if the final part FP consists of a union of parts $\mathrm{FP}_j,\ \ j = 1, \ldots, s$,

$$\mathrm{FP} = \bigcup_{1\leq j\leq s} \mathrm{FP}_j\ ,$$

then the profile function $b(\gamma)$ of FP can be calculated as follows. Let for fixed axis, i.e. fixed p, b_j be the profile functions of FP_j in the corresponding intervals $[\gamma_b^j, \gamma_e^j]$. We define

$$\gamma_b := \min_{1\leq j\leq s}\ \gamma_b^j, \quad \gamma_e := \max_{1\leq j\leq s}\ \gamma_e^j, \qquad b_j(\gamma) := 0,\ \text{for}\ \gamma \in [\gamma_b, \gamma_e] \setminus [\gamma_b^j, \gamma_e^j]\ .$$

Then the profile function of FP is given by the maximum value,

$$b(\gamma) = \max_{1\leq j\leq s}\ b_j(\gamma)\ , \qquad \gamma \in [\gamma_b, \gamma_e]\ .$$

FINAL PART WITH HOLES: We briefly discuss the calculation of the MTS for holed objects. We are referring to objects with holes that possess at least one approach direction, from outside, for machining. For simplicity we assume that the final part FP^H is the 'set-valued' difference of an outer polyhedron FP and an polyhedron $\hat{\mathrm{FP}}$ representing the hole,

$$\mathrm{FP}^H = \mathrm{FP} \setminus \hat{\mathrm{FP}}\ . \qquad \hat{\mathrm{FP}} \subset \mathrm{FP}$$

Let V and E again denote the set of vertices and edges of FP. Let $\hat{V} = \{\hat{v}^1, \ldots, \hat{v}^{\hat{k}}\}$ be the set of vertices of $\hat{\mathrm{FP}}$ and $\hat{E} = \{\hat{E}_{ij} \mid (i, j) \in \hat{L}\}$ the set of edges. For holed objects we have to calculate two different profile functions. A function $b(\gamma)$ corresponding to FP and the profile function $\hat{b}(\gamma)$ for the hole $\hat{\mathrm{FP}}$. We describe how this can be done.

Again, let the axis ax be fixed, i.e. the vector $p = (c, \phi, \varphi)$ is fixed. Applying the transformation (10) to V and $\hat{V}$ we obtain the vertices $\{q^1, \ldots, q^k\}$ of FP_y and $\{\hat{q}^1, \ldots, \hat{q}^{\hat{k}}\}$ of $\hat{\mathrm{FP}}_y$. We calculate the values γ_b, γ_e for FP as in (14) and the value $b(\gamma)$ is computed via (13).

Let $\gamma \in [\gamma_b, \gamma_e]$ be fixed. Again, we define the slices in the plane $pl(\gamma)$ orthogonal to the y_1-axis ax_y, $sl^H(\gamma) = pl(\gamma) \cap \mathrm{FP}_y^H$, $sl(\gamma) = pl(\gamma) \cap \mathrm{FP}_y$ and $\hat{sl}(\gamma) = pl(\gamma) \cap \hat{FP}_y$. For the calculation of the profile function $\hat{b}(\gamma)$ for the hole instead of the maximum we have to compute the minimium distance,

$$\hat{b}(\gamma) = \min\ \{||x - ax_y(\gamma)|| \mid x \in \hat{sl}(\gamma)\}\ .$$

We have to distinguish between different cases.

Case 1, $ax_y(\gamma) = \gamma e_1 \notin \mathrm{FP}_y^H$ and the slice $sl^H(\gamma)$ is given by two (piecewise linear) closed curves ($sl^H(\gamma)$ is double-connected) such that $ax_y(\gamma) \in \hat{\mathrm{FP}}_y$ (see Figure 8): The intersection points of the transformed edges $\hat{E}_{ij}$ with the plane $pl(\gamma)$ perpendicular to the y_1-axis (for fixed γ) are given by

$$\hat{q}^i + t(\gamma)(\hat{q}^j - \hat{q}^i), \qquad (i, j) \in \hat{L},\ \text{ if }\ t(\gamma) = \frac{\gamma - \hat{q}_1^i}{\hat{q}_1^j - \hat{q}_1^i} \in [0, 1]\ .$$

These points are the vertices of the 2D polygon forming the boundary of the slice $\hat{sl}(\gamma)$. Suppose that these points are represented by $\hat{p}^1, \ldots, \hat{p}^{\hat{r}}$ and numbered in such a way that the boundary of $\hat{sl}(\gamma)$ is defined by the segments (see Figure 8)

$$\overline{\hat{p}^{i+1}\hat{p}^i} = \{\hat{p}^i + t(\hat{p}^{i+1} - \hat{p}^i),\ t \in [0, 1]\}$$

$i = 1, \ldots, \hat{r}$ (put $\hat{p}^{\hat{r}+1} := \hat{p}^1$). Now, we calculate the projections w^i of $ax_y(\gamma)$ onto the segments $\overline{\hat{p}^{i+1}\ \hat{p}^i}$ (points of minimum distance):

$$w^i := \begin{cases} \hat{p}^i + \bar{t}(\hat{p}^{i+1} - \hat{p}^i) & \text{if } \bar{t} \in [0, 1] \\ \min\{\hat{p}^i, \hat{p}^{i+1}\} & \text{otherwise} \end{cases} \qquad \text{with }\ \bar{t} := \frac{(\hat{p}^i - ax(\gamma))^T(\hat{p}^{i+1} - \hat{p}^i)}{(\hat{p}^{i+1} - \hat{p}^i)^T(\hat{p}^{i+1} - \hat{p}^i)}\ .$$

Then, we have

$$\hat{b}(\gamma) = \min_{i=1,\ldots,\hat{r}} \{||w^i - ax_y(\gamma)||\}\ . \tag{16}$$

Case 2, $ax_y(\gamma) \in \mathrm{FP}_y^H$: In this case simply $\hat{b}(\gamma) = 0$.

Case 3, $ax_y(\gamma) \notin \mathrm{FP}_y^H$ and $ax_y(\gamma) \notin \hat{\mathrm{FP}}_y$ and the slice $sl^H(\gamma)$ is given by a (piecewise linear) closed curve ($sl^H(\gamma)$ is simply-connected): In this case $\hat{b}(\gamma)$ is calculated in a similar way as in Case 1.

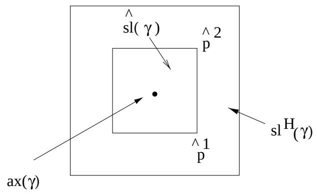

Figure 8: The slice $sl^H(\gamma)$ of a holed object

To calculate the volume of revolution for a given axis $p = (c, \phi, \varphi)$, we distinguish between two different situations. The case that the axis does not have any access to the hole $\hat{\mathrm{FP}}$, i.e. the hole is neglected, or the situation that ax enters or leaves the part FP via $\hat{\mathrm{FP}}$.

To decide which case occurs we determine the γ-values $\tilde{\gamma}_b$ and $\tilde{\gamma}_e$ where the axis enters and leaves FP by solving the (one-dimensional linear) problem

$$\begin{matrix}\tilde{\gamma}_b \\ \tilde{\gamma}_e\end{matrix} := \begin{matrix}\min \\ \max\end{matrix} \left\{\gamma \mid ax(\gamma) \in \mathrm{FP}\right\} = \left\{\gamma \mid (c + \gamma d(\phi, \varphi))^T a_j \leq b_j,\ j = 1, \ldots, m\right\}$$

Finally $\mathrm{vol}_a(p)$ is calculated according to the given situation.

Case that the hole is neclected, $ax_y(\tilde{\gamma}_b) \notin \hat{\mathrm{FP}}_y$ and $ax_y(\tilde{\gamma}_e) \notin \hat{\mathrm{FP}}_y$: In this case the profile function $b(\gamma)$ is calculated for the polyhedron FP_y as in Section 3 and the volume is approximated by formula (15).

Case with hole, $ax_y(\tilde{\gamma}_b) \in \hat{\mathrm{FP}}_y$ or $ax_y(\tilde{\gamma}_e) \in \hat{\mathrm{FP}}_y$ (or both): Then, we determine the profile functions $b(\gamma)$, $\hat{b}(\gamma)$ (according to the cases 1 to 3 above) on a discretization of $[\gamma_b, \gamma_e]$ and we calculate the volumes $\mathrm{vol}_a(p)$, $\hat{\mathrm{vol}}_a(p)$ for these functions using formula (15). The volume of revolution of the holed object FP^H is given approximately by the formula

$$\mathrm{vol}_a^H(p) = \mathrm{vol}_a(p) - \hat{\mathrm{vol}}_a(p)\ .$$

The MTS of holed objects of a more complex structure is a topic of ongoing research.

5 Minimization of the volume

To solve the MTS-problem, for given FP, we have to solve the minimization problem (7). Due to the non-convexity and the non-differentiability of the volume function $\mathrm{vol}(p)$, smooth local minimization methods may have difficulties to solve the problem. Local methods can end up in local minimizers or in points of non-differentiability.

To avoid these problems one could make use of a pure discretization method: Minimize the function $\mathrm{vol}(p)$ on a discretization of the parameter set (see (7))

$$P_M = \{p = (c, \phi, \varphi) \in P_1 \cup P_2 \cup P_3,\ \ -M \leq c_i \leq M,\ i = 1, 2, 3\}$$

where $M > 0$ is choosen appropriately. Such a method is robust but not efficient.

One could also try to use a smooth local method such as the *steepest descent* or the *conjugate gradient method*. We report on some numerical experiments.

As a test example we want to solve the MTS-problem for the final part given in Figure 7. We applied the conjugate gradient method of Fletcher and Reeves for minimizing the function $\text{vol}(p)$ on the set P_1. This method is a local method which under certain assumptions on a local minimizer $\overline{p}$ converges locally super-linearly to the point $\overline{p}$. For details we refer to [Luenberger, 1974]. The method needs values of the gradient $\nabla\text{vol}(p^k)$ at the iteration points p^k of the minimization method. We have computed approximate values of $\frac{\partial}{\partial p_j}\text{vol}(p)$ by the formula

$$\frac{\partial}{\partial p_j}\text{vol}(p) \approx \frac{\text{vol}(p + he_j) - \text{vol}(p)}{h} , \qquad j = 1, \ldots , 5 ,$$

with appropriate stepsize $h > 0$. Here, e_j denote the unit vectors in $I\!R^5$.

Choosing a starting point $p^0 \in P_1$ the conjugate gradient method produces iteration points p^k converging to a limit point $\overline{p}$. Since the function $\text{vol}(p)$ is not differentiable everywhere, the method may have a limit point $\overline{p}$ where the volume function is not differentiable. In this case the condition $\nabla vol(\overline{p}) = 0$ need not be fulfilled and $\overline{p}$ need not be a local minimizer.

The following table gives 7 starting points $p^0 = (0, c_2, c_3, \phi, \varphi) \in P_1$ and the corresponding (approximate) limit points $\overline{p}$ produced by the method and the value of the volume function. The last column indicates whether the iteration process converges to the global minimizer (case GM), to one of the local minimizers (cases LM_2, LM_3, LM_4) or that the method stops at a point of non-differentiability (case ND).

	values p	$\text{vol}(\overline{p})$	case
•	$p^0 = (0, -0.400,\ 0.600,\ 1.300, -0.600)$	6.861	
	$\overline{p}\ = (0,\ 1.000,\ 1.000, -0.579, -0.785)$	0.401	GM
•	$p^0 = (0,\ 0.000,\ 0.600, -0.200,\ 0.500)$	1.331	
	$\overline{p}\ = (0,\ 0.982,\ 0.981, -0615, -0.786)$	0.415	GM
•	$p^0 = (0,\ 0.100,\ 1.400, -0.100,\ 1.300)$	9.435	
	$\overline{p}\ = (0, -1.318, -1.318,\ 0.742,\ 1.160)$	0.751	LM_2
•	$p^0 = (0,\ 2.000, -0.700,\ 0.100,\ 0.500)$	22.82	
	$\overline{p}\ = (0, -0.282,\ 1.000, -0.653,\ 0.699)$	0.755	LM_3
•	$p^0 = (0,\ 0.000,\ 0.500,\ 0.700,\ 0.500)$	3.304	
	$\overline{p}\ = (0, -0.012, -0.012,\ 0.375,\ 0.404)$	0.655	LM_4
•	$p^0 = (0,\ 0.200,\ 0.200,\ 0.100,\ 0.100)$	0.719	
	$\overline{p}\ = (0,\ 0.182,\ 0.182,\ 0.093,\ 0.093)$	0.705	ND
•	$p^0 = (0,\ 0.000,\ 0.400,\ 0.500,\ 0.700)$	2.495	
	$\overline{p}\ = (0, -0.160, -16.520,\ 1.518,\ 0.473)$	0.801	ND

Table 1

Two of the 7 starting points lead to the global minimizer. In 3 cases local minimizers are computed and in 2 cases the minimum search stoped at a point $\overline{p}$ where the volume function is not differentiable.

Since one global search with the discretization method is much more time consuming than many local searches the latter method is more efficient. So the local method can in principle be used despite of the drawback of possible failure for certain starting points.

We give another illustrative example. Consider the FP as given in Figure 9.

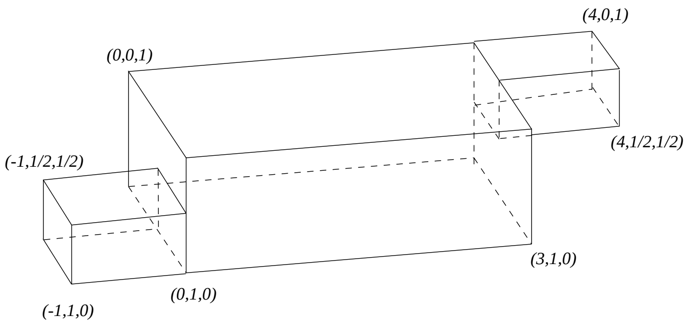

Figure 9: final part

A good adhoc choice for an axis is the axis $p^0 = (1.5, 0.5, 0.5, 0, 0)$ leading to a volume of revolution of $\text{vol}(p^0) = 7.854$. Few steps of the conjugate gradient method lead to the global minimizer $p^1 = (1.5, 0.5, 0.5, 0.098, -0.098)$ with volume $\text{vol}(p^1) = 7.419$, an improvement of more than 5.5%.

6 Final remarks

We make some comments on possible modifications of the MTS-problem. Asking for a production process where we remove as much as possible by turning can also lead to a more complicated problem as follows. Let us reformulate our problem.

Problem: Let be given an initial part IP and a final part FP, FP $\subset$ IP. Find an axis *ax*, a profile function above *ax* in a plane containing this axis, and an intermediate state $\widetilde{\text{FP}}$ produced from IP only by turning such that FP $\subset \widetilde{\text{FP}}$ and that one of the following conditions holds

MTS: The volume of $\widetilde{\text{FP}}$ is minimized subject to the constraint that $\widetilde{\text{FP}}$ is a spherical cylinder obtained by revolving the profile function around *ax*.

MTS$_g$: The volume of $\widetilde{\text{FP}}$ is minimized

These problems are different. By definition, the solution of the MTS-problem is a spherical cylinder not depending on the initial part. We emphasize that we do not assume $\widetilde{\text{FP}} \subset$ IP. The solution of the MTS$_g$-problem does depend on the initial part. We give an illustrative example.

Let FP be the union of two parts, F_1 and F_2. F_1 is the left half part of a cylinder C_1 with radius r_1 and length $l \gg r_1$ and F_2 the part of a cylinder C_2 with radius $r_2 > r_1$ and length l (see Figure 10a). As initial parts we choose $\text{IP}_1 = C_1$ (Figure 10b) and the rectangular

block IP_2 (Figure 10c). Then for the MTS-problem, since $\widetilde{\mathrm{FP}}$ has to be a spherical cylinder, the solution is given by the cylinder C_1, not depending on the initial part.
For MTS_g in case of IP_1 the solution is given by $\widetilde{\mathrm{FP}} = \mathrm{FP}$. This can be seen as follows. Choose for the profile function $b(\gamma) = r_2$ and fix the initial part IP_1 in the mill/turn such that the central axis of C_1 has distance $r_2 - \sqrt{r_2^2 - r_1^2}$ from the axis ax. Consider the cylinder C_2 with central axis coinciding with ax. Then the intersection of C_2 with C_1 equals the final part and FP can be produced by turning.

Suppose now that the initial part is given by IP_2. Then the solution of MTS_g will be as indicated in Figure 10c. The axis ax (indicated by $\bullet$) coincides with the central axis of C_1 and we have to choose $b(\gamma) = r_1$. The intermediate part $\widetilde{\mathrm{FP}}$ is given by the doted line.

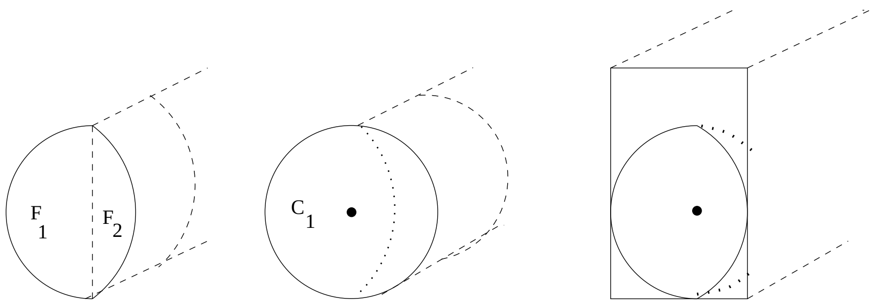

Figure 10: a. final part b. IP_1 and solution of MTS c. IP_2 with FP

We briefly describe a conceptual method for solving the problem MTS_g. Let ax, i.e. $p = (c, \phi, \varphi)$, be fixed. For any $\gamma \in [\gamma_b, \gamma_e]$ we define the sets $\mathrm{IP}(\gamma) = \mathrm{IP} \cap pl(\gamma)$ (cf. (2)), $C(\gamma) = \{x \in \mathrm{IP}(\gamma) \mid ||x - ax(\gamma)|| \leq b(\gamma)\}$ and the surface

$$sf(\gamma) \int_{C(\gamma)} ds \, .$$

Then the volume function for the MTS_g problem is given by

$$vol_g(p) = \int_{\gamma_b}^{\gamma_e} sf(\gamma) \, d\gamma \, .$$

To solve MTS_g we have to minimize this function over all axes ax.

So, compared with MTS the MTS_g-problem is more difficult. For the volume of revolution for fixed ax in the case of MTS we only have to compute the distance function $b(\gamma)$ whereas in the case of MTS_g the whole surfaces $sf(\gamma)$ have to be calculated.

We end up with comments on future developments. Further investigations are necessary in the case of holed objects. In particular, FP's with different holes should be considered.

An efficient implementation of the algorithm for solving the MTS-problem for objects bounded by spherical and ellipsoidal surfaces or for surfaces given by rational Bezier splines has to be developed.

The following generalization leads to a more complex problem, the MTS of *M* different parts: Given *M* different parts, find an axis and a profile function *b* with minimal volume of revolution such that all *M* parts are contained in the spherical cylinder obtained by revolving *b* around the axis.

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