

---

Faculty of Mathematical Sciences

University of Twente

University for Technical and Social Sciences

---

---

P.O. Box 217  
7500 AE Enschede  
The Netherlands

Phone: +31-53-4893400

Fax: +31-53-4893114

Email: [memo@math.utwente.nl](mailto:memo@math.utwente.nl)

---

MEMORANDUM No. 1487

Best data-dependent triangulations

L. ALBOUL, G. KLOOSTERMAN,  
C.R. TRAAS AND R.M.J. VAN DAMME

JUNE 1999

ISSN 0169-2690

# Best data-dependent triangulations

Lyuba Alboul      Gertjan Kloosterman      Cornelis Traas  
Ruud van Damme

## Abstract

When reconstructing a surface from irregularly spaced data we need to decide how to identify a good triangulation. As a measure of quality we consider various differential geometrical properties, namely integral absolute Gaussian curvature, integral absolute mean curvature and area. A comparison is made with data-dependent triangulation methods that exist in the literature.

**Key words:** Polyhedral metrics, data-dependent triangulations, surface reconstruction, curvature.

**1991 Mathematics Subject Classification:** 65Y25, 53C42.

## 1 Introduction

Triangulating a data set is an essential technique in solving problems in surface reconstruction, *i.e.*, in scattered data interpolation and approximation, or in applications where one needs to recover 3D objects. The surface reconstruction problem can be formulated as follows:

Given a discrete data set  $\{(x^i, y^i, z^i) \in \mathbb{R}^3\}_{i=1}^N$ , fit a surface to it.

The data sites are supposed to be irregularly-located.

In general it is not possible to interpolate or approximate the data without preprocessing. One first needs to organise the data or, in other words, to put a structure on the data. Therefore the first step in evaluating a surface is to obtain a triangulation of the data. Determining a triangulation, the simplest  $C^0$ -approximation, is the quickest and cheapest way to take an initial look at the data before turning to higher order interpolation/approximation methods. A 'good' triangulation can help to solve many problems. These problems are not limited to finding a smooth interpolant, but also concern the definition of the shape of the object [Zah71, SH79, Boi84, Car87, DLR90b], or, the other way around, the reduction of the number of points without much damaging the actual shape of the object [FHMB84, DvD98], the control of the automatic processing of surfaces [DM83], or the estimates of some geometric properties, such as area, volume, axes of inertia, or the extraction of elementary shapes [LP82, FS91].

Using pure geometric criteria, Alboul and van Damme have introduced new triangulations for irregularly-located 3D data [vDA95, AvD95, AvD96].

The first triangulation is called the Tight triangulation and is based on optimising a discrete analogon of integral absolute Gaussian curvature. Tight triangulations are evidently better than Delaunay triangulations, as for example, the Tight triangulation automatically preserves convexity.

The second optimality criterion that was proposed in [AvD96] concerns triangulations that minimise integral absolute mean curvature.

Up to now it was not clear which of those two criteria was the 'best' (if any), and how they compare with known methods in the literature such as

- minimising the area of the resulting object [O'R81],
- heuristic criteria such as can be found in [DLR90b, DLR90a],
- methods based on minimising a certain functional, like the energy of a bending plate, constrained by the interpolation conditions. Such a method can *e.g.*, be found in [QS89, QS90b, QS90a] (with the obvious disadvantage of this method is that it only works for *functional* data).

This article is organised as follows. We first briefly review the the triangulation methods proposed in [vDA95, AvD95, AvD96], based on minimising absolute Gaussian and mean curvature. Next we perform numerical experiments for closed objects with these criteria from which it turns out that the latter in general performs better, although this method may not be shape preserving: it is not known whether the method conserves convexity in general. Both perform much better than minimising the area of the object as it comes to visual pleasingness.

In the second part of the paper we investigate the quality of the methods for the special case of functional data: both tight as well as minimal absolute mean curvature methods can be compared with methods known in the literature. Also in these cases the mean criterion seems to be superior when it comes to precision.

## 2 Geometric criteria for best triangulations

In this section we review the geometric quantities integral Gaussian and mean curvature for triangulations, as well as *absolute* Gaussian and mean curvature. For smooth objects Gaussian curvature  $K$  and mean curvature  $H$  are defined in terms of the principal curvatures  $\lambda_1, \lambda_2$ , being the eigenvalues of the differential of the Gauss map [dC76]:

$$K = \lambda_1 \cdot \lambda_2, \quad H = (\lambda_1 + \lambda_2)/2.$$

The integral version of these quantities can also be defined for triangulations, and they lead to different criteria to define the best triangulation. We consider two of these criteria and compare them with the aid of experiments.

## 2.1 Gaussian curvature

The notion of Gaussian curvature is one of the central concepts in differential geometry and is strictly related to the concept of angle. On the basis of the notion of angle, we can define the following curvatures for a triangulation  $\Delta$  considered as a polyhedral surface:

- The (integral) curvature  $\kappa$  (an analogue of the integral Gaussian curvature).

The total angle  $\theta(v)$  around the vertex  $v$  is the sum of angles of all the plane polygons incident to  $v$ , called the star of  $v$ ,  $\text{Star}(v)$ . For any point  $x \in \Delta$ :  $\kappa(x) = 2\pi - \theta(x)$ . The quantity  $\kappa$  is also known as the angle deficit.

Only for vertices we have:  $\kappa(x) \neq 0$ , see figure 1.

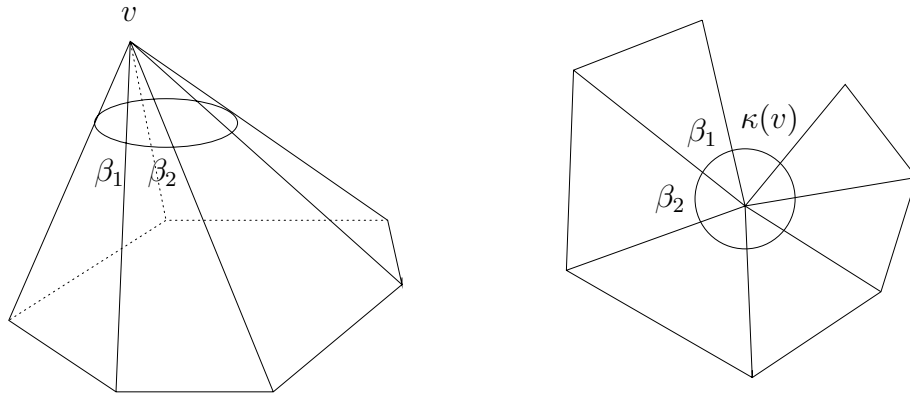


Figure 1: Curvature around a vertex:  $\kappa(v) = 2\pi - \sum \beta_i = 2\pi - \theta(v)$ .

- The positive (extrinsic) curvature  $\kappa^+(v)$ .

Suppose that through the vertex  $v$  there passes some (local) supporting plane of a triangulation  $\Delta$ . Then this vertex lies on the boundary of the convex hull of  $\text{Star}(v)$ . We denote the star of  $v$  in the boundary of this convex hull by  $\text{Star}^+(v)$  and will call it the star of the convex cone of a vertex. The curvature  $\kappa^+(v)$  of  $\text{Star}^+(v)$  is called the positive (extrinsic) curvature of  $v$ . If there is no supporting plane through  $v$  then we put  $\kappa^+(v)$  equal to zero.

- The negative (extrinsic) curvature  $\kappa^-(v)$ :  $\kappa^-(v) = \kappa^+(v) - \kappa(v)$ .
- The absolute (extrinsic) curvature  $\hat{\kappa}(v)$ :  $\hat{\kappa}(v) = \kappa^+(v) + \kappa^-(v)$ .

We can identify the following sets of vertices:

1. Proper convex vertices:  $\kappa(v) = \kappa^+(v) = \hat{\kappa}(v)$  and  $\kappa^-(v) = 0$ .

Geometrically this means that  $\text{Star}(v)$  coincides with  $\text{Star}^+(v)$ .

2. Proper saddle vertices:  $\hat{\kappa}(v) = \kappa^-(v) = -\kappa(v)$  and  $\kappa^+(v) = 0$ .

The Gaussian curvature  $\kappa$  of a proper saddle vertex is less than zero and there exists no supporting plane, *i.e.*, there does not exist a plane which passes through vertex  $v$  such that all the neighbouring vertices lie on the same side of (or in) this plane.

3. Mixed vertices:  $\kappa^-(v) > 0$ ,  $\kappa^+(v) > 0$ .

A vertex is mixed if it is neither convex nor a saddle. Therefore it has a supporting plane, but there exist two successive edges incident on the vertex which span a plane that divides the set of adjacent vertices.

Examples of the three described vertices are given in figure 2.

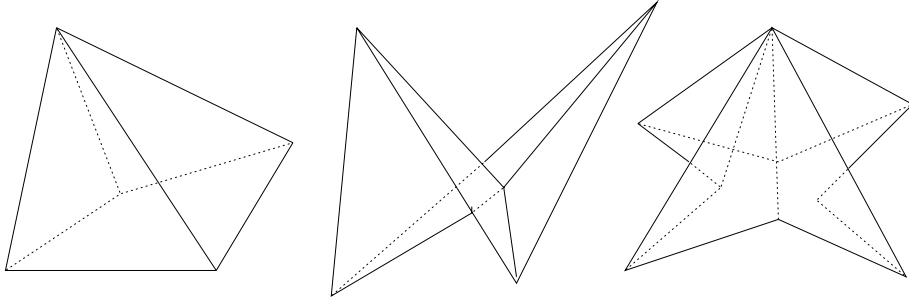


Figure 2: The three types of vertices: proper convex, proper saddle and mixed

The total absolute (extrinsic) curvature  $K_{abs}(\Delta)$  of a triangulation  $\Delta$  is given by the following expression:

$$K_{abs}(\Delta) = \sum_{v_\alpha \text{ convex}} \kappa^+(v_\alpha) + \sum_{v_\beta \text{ saddle}} \kappa^-(v_\beta) + \sum_{v_\gamma \text{ mixed}} (\kappa^+(v_\gamma) + \kappa^-(v_\gamma)).$$

For any triangulation proper convex, proper saddle and mixed vertices form a partition of all vertices.

**Definition 2.1** Given a data set  $\{x^i\}, i = 1, \dots, N$ . A triangulation  $\bar{\Delta}$  of the data set is said to be the Tight triangulation, if it is proper, and if  $K_{abs}(\bar{\Delta})$  is minimal, *i.e.*,

$$K_{abs}(\Delta) \geq K_{abs}(\bar{\Delta})$$

for all possible other proper triangulations  $\Delta$ .

Some properties of the Tight triangulation were given in [AvD97], the most important being that it preserves convexity. Moreover it can be proved that with the local swapping

algorithm, first suggested by Lawson [Law77], this global optimum can actually be obtained if the data are convex [AvD97].

Unfortunately this optimisation criterion has some undesirable results, as it seems to create unwanted long thin triangles. This already happens on a relatively well sampled object as shown in figure 3. The initial triangulation (left) is very good, but it was obtained in a laborious fashion, namely by hand.

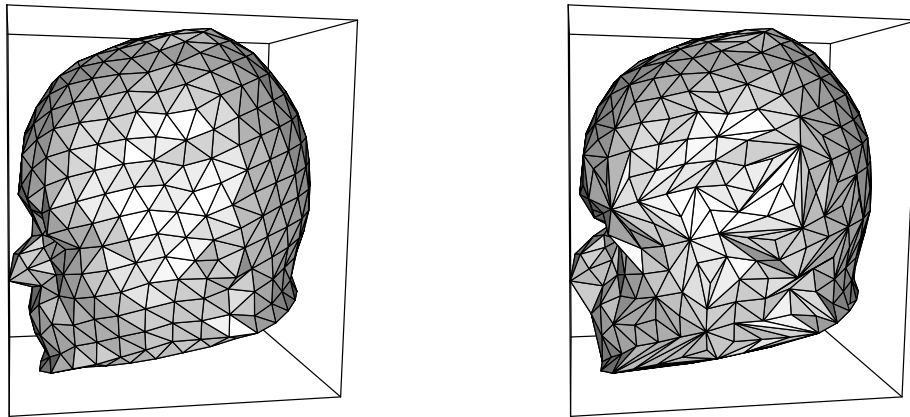


Figure 3: The scalp before (left) and after (right) applying the tight criterion

## 2.2 Mean curvature

Another curvature that can be defined for polyhedra and consequently for a triangulation, is mean curvature. One easily shows that a proper definition for integral mean curvature on a strip along an edge  $e$  is given by:

$$H(e) = \alpha \cdot \|x_i - x_j\|.$$

Here  $x_i, x_j$  are the coordinates of the vertices of  $e$ , and  $\alpha$  is the angle between the normals of the two triangles which have  $e$  in common. (This can be derived by considering a  $C^1$  approximation of the triangulation by replacing edges with cylinder of small radius.)

The criterion of minimising absolute mean curvature is given by

$$H_{abs} = \sum_e |H(e)|,$$

and consequently a triangulation of a data set is said to be the triangulation of minimal absolute mean curvature if it is proper and minimises  $H_{abs}$ .

This definition implies that we look for triangulations that minimise

$$\int_K (|\kappa_1 + \kappa_2|) dS, \quad (2.1)$$

as well as

$$\int_K (|\kappa_1| + |\kappa_2|) dS, \quad (2.2)$$

because

- vertices have no contribution in the integral, since they have zero measure,
- points inside triangles have no contribution, since both principal curvatures are zero,
- edges do have contribution, but only of one principal curvature, the other being zero.

Since only one of the two principal curvatures is non zero, the following quantities are also minimised:

$$\int_{\Omega} \max(|\kappa_1|, |\kappa_2|) dS, \quad \int_{\Omega} \sqrt{\kappa_1^2 + \kappa_2^2} dS. \quad (2.3)$$

This criterion gives very promising results, see figure 4. This result is obtained using a local swap algorithm with as initial condition the Tight triangulation, obtained in the previous subsection: it almost recovers with the good initial result which was use there (see figure 3).

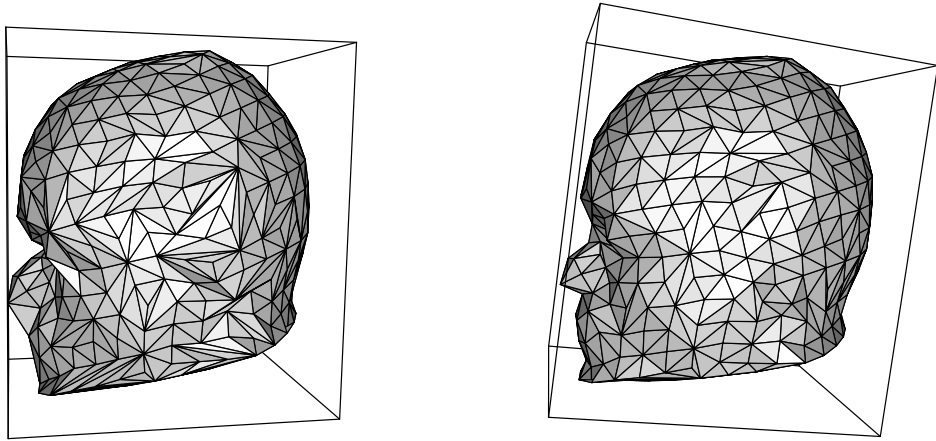


Figure 4: The scalp before (Tight, left) and after (right) applying the mean curvature criterion

### 3 Other data-dependent triangulations

O'Rourke suggested to define polyhedra of minimum area [O'R81] as the best; however, this criterion may yield strange results, even for very simple data, see figure 5. Here the data are drawn from two parallel circles. Both the Tight triangulation as well as the triangulation of minimal absolute mean curvature lead to the good result in this case, namely the convex triangulation.

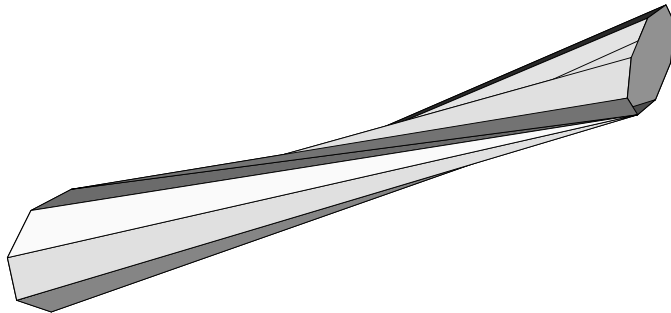


Figure 5: Two slices skewed cylinder

There are more criteria which, in a certain sense, are related to minimisation of mean curvature, such as JND (jump in normal derivatives) or ABN (angle between normals) (see [DLR90b]). The variational criterion, proposed by Quak and Schumaker [QS90b], is connected with the integral absolute mean curvature. They use as energy measure the expression:

$$\mathcal{E}(s, \Delta) = \sum_{\{T_j\}} \int_{T_j} [(\frac{\partial^2}{\partial x^2} s(x, y))^2 + 2(\frac{\partial^2}{\partial x \partial y} s(x, y))^2 + (\frac{\partial^2}{\partial y^2} s(x, y))^2] dx dy, \quad (3.4)$$

where  $\{T_j\}$  denotes the set of triangles of the triangulation  $\Delta$ , and  $s$  is a interpolation cubic  $C^1$  spline using a Clough-Tocher split [Law77].

Expression 3.4 represents the energy of a so-called 'thin plate'. Due to the specific nature of a 'thin plate' (it is almost flat), it does not seem convenient to apply the above-mentioned criterion directly to a closed surface (see [AvD96]). However, if we regard it from a geometric point of view, we can see a criterion, that generalises the given one and that can also be applied to a closed surface. Indeed, for a 'thin plate' the functional in (3.4) is approximately equal to  $\int_{\Omega} (\lambda_1^2 + \lambda_2^2) dA$  where  $\lambda_1, \lambda_2$  are the principal curvatures of a surface. The last functional was used in some work on surface approximation as well. For example, in [Hag88] it is minimised for smoothing an approximating surface. The expression  $(\lambda_1^2 + \lambda_2^2)$  can be rewritten as follows:

$$\lambda_1^2 + \lambda_2^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 - 2\lambda_1\lambda_2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = 4H^2 - 2K$$



Thus we have

$$\int_{\Omega} (\lambda_1^2 + \lambda_2^2) dS = 4 \int_{\Omega} H^2 dS - 2 \int_{\Omega} K dS \quad (3.5)$$

Note that this expression also makes sense in the case of closed surfaces.

We can try to minimise the integral in the left-hand side of (3.5). Only the first integral is subject to minimisation, since the Gauss-Bonnet theorem tells us that the second integral is constant. This latter statement holds for closed data, as well as for functional data.

Consequently, minimisation of the expression  $\int_{\Omega} (\lambda_1^2 + \lambda_2^2) dS$  is equivalent to minimisation of the expression  $\int_{\Omega} H^2 dS$  and vice versa.

It is interesting to note that this is also equivalent to minimising  $|H|$  (albeit in a different norm), and not of  $|K|$ .

## 4 Examples for the functional case

In this section we make a comparison of all the methods described in this article for the special case of functional data, as we also want to compare all methods with the method of [QS89, QS90b, QS90a]. Although our final findings and conclusions hold for many more examples we only present here two typical examples.

The test-functions that we use are

$$\begin{aligned} f^{(1)}(x, y) &= \exp(-81/4((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2)) \\ f^{(2)}(x, y) &= \sin \pi(x - y). \end{aligned} \quad (4.6)$$

For  $f^{(1)}$  we use a regular  $20 \times 30$  grid, called  $\Delta^{(1)}$ , and for  $f^{(2)}$  we use  $\Delta^{(2)}$  which is one of the standard grids from [Fra87] with 36 points.

For finding the minimum we use the following strategy:

- We use the swapping procedure of Lawson ([Law77]).
- Secondly, in order to get as close to the real minimum as possible, we also use simulated annealing. This method of course does not guarantee that we obtain the global minimum.

Furthermore we measure the quality of a method in two ways:

1. The error as the maximal distance of the resulting triangulation to the original function, in the tables denoted as  $\epsilon(\Delta)$ .
2. We apply the subdivision scheme from [vD97] to the data supplemented with exact derivatives, and again measure the error compared with the original function. In the tables this error is denoted with  $\epsilon(\text{subdiv})$ .

For the first example, using  $f^{(1)}$  on data  $\Delta^{(1)}$  we obtain the results shown in table 1: in the first row the results from minimising absolute mean curvature ( $|H|$ ) are shown, the second row contains the results when applying the tight criterion ( $|K|$ ), the third and fourth are

the results from applying the method from [QS89, QS90b, QS90a] (QS) resp. the JND criterion from [DLR90b] (the ABN criterion gives almost identical results).

In the first four columns one finds the resulting energy measured according to the same criteria (and as a consequence it is minimal on the diagonal in each column). In figure 6 the four resulting triangulations are shown from which we can draw the following conclusions:

- The criteria minimising  $|H|$  and JND (as well as ABN) yield almost identical results. The methods of Dyn, Rippa and Levin have the tendency to create longer edges than the first one. From table 1 one can conclude that JND and ABN are overdoing it a little, indeed.

Of course we have to be cautious with such a remark as this conclusion might depend on the specific subdivision scheme (or any other approximation method) we use.

- The tight criterion gives a triangulation that has the same characteristics as the triangulations of  $|H|$  and JND, but less pronounced.

From table 1 it is clear that the tight criterion performs considerably worse, which is the same conclusion as we arrived at in the previous section.

- The criterion QS is even more conservative if we consider figure 6. The triangulation has a totally different behaviour than the other three around the peak of the function, near the point  $(1/2, 1/2)$ . It is interesting to note that if we change the energy of (3.4) into

$$\mathcal{E}(s, \Delta) = \sum_{\{T_j\}} \int_{T_j} |(\frac{\partial^2}{\partial x^2} s(x, y) \cdot (\frac{\partial^2}{\partial y^2} s(x, y)) - (\frac{\partial^2}{\partial x \partial y} s(x, y))^2| dx dy \approx \int_{\Omega} |K| dS$$

we obtain a triangulation that is very similar to the Tight triangulation. However, we could not find an energy (like in (2.1), (2.2) or (2.3)) that reproduced the triangulation of minimal absolute mean curvature.

	$ H $	$ K $	QS	JND	$\epsilon(\Delta)$	$\epsilon(\text{subdiv})$
$ H $	0.403	0.890	30.9	89.0	$4.91 \cdot 10^{-3}$	$1.39 \cdot 10^{-4}$
$ K $	0.457	0.883	30.9	90.0	$4.91 \cdot 10^{-3}$	$3.56 \cdot 10^{-4}$
QS	0.506	0.907	28.7	92.1	$7.67 \cdot 10^{-3}$	$1.66 \cdot 10^{-4}$
JND	0.506	0.907	28.7	88.2	$4.91 \cdot 10^{-3}$	$2.03 \cdot 10^{-4}$

Table 1: Results for  $f^{(1)}$  on data  $\Delta^{(1)}$

Table 2 and figure 7 show the same results, but for data given by  $\Delta^{(2)}$  drawn from test-function  $f^{(2)}$ . The conclusions are the same as in the previous example.

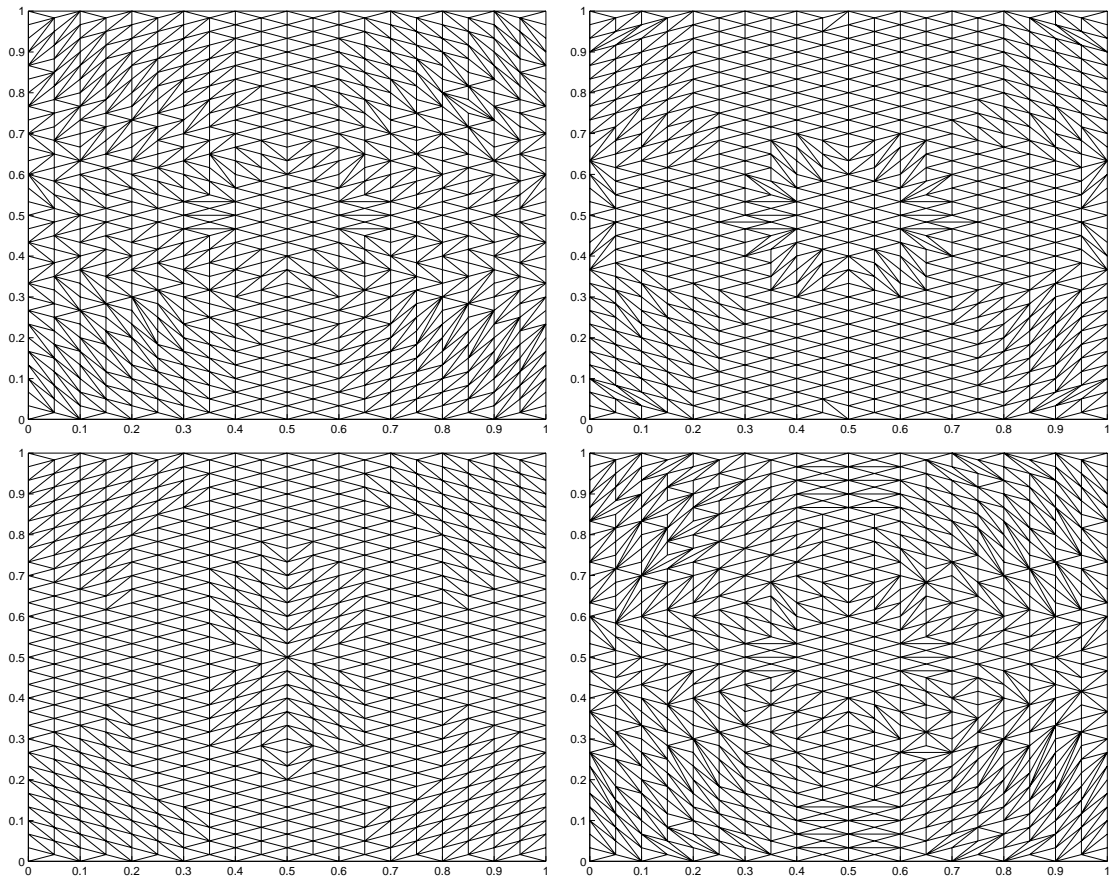


Figure 6: Best triangulations according to the mean and tight (up) and QS and JND criterion (down) for  $f^{(1)}$  in (4.6)

## 5 Conclusions

We can conclude that minimising the absolute Gaussian curvature, *i.e.*, the tight criterion, is not very applicable to general datasets. The area criterion has been shown to lead to poor results even in the globally optimal case. The most promising criterion seems to be minimising absolute mean curvature, or (at least almost as good) the two methods ABN and JND from [DLR90b]. The method of Quak and Schumaker gives too conservative triangulations, in general. Apart from that, their method is more expensive as all other criteria can be computed at the  $C^0$ -level.

## References

- [AvD95] L. Alboul and R. van Damme. Polyhedral metrics in surface reconstruction: tight triangulations. Memorandum no. 1275, University of Twente, Faculty of

	$ H $	$ K $	QS	JND	$\epsilon(\Delta)$	$\epsilon(\text{subdiv})$
$ H $	0.372	1.37	1800	10.6	$7.03 \cdot 10^{-2}$	$9.06 \cdot 10^{-4}$
$ K $	0.422	1.30	1830	10.8	$1.58 \cdot 10^{-1}$	$6.50 \cdot 10^{-3}$
QS	0.993	1.55	1660	10.7	$1.55 \cdot 10^{-1}$	$5.12 \cdot 10^{-3}$
JND	0.373	1.37	1800	10.6	$7.03 \cdot 10^{-2}$	$9.10 \cdot 10^{-4}$
Delaunay	0.661	1.49	1730	10.7	$1.55 \cdot 10^{-1}$	$5.12 \cdot 10^{-3}$

Table 2: Results for  $f^{(2)}$  on data  $\Delta^{(2)}$

Mathematical Sciences, 1995.

- [AvD96] L. Alboul and R. van Damme. Polyhedral metrics in surface reconstruction. In G. Mullineux, editor, *The Mathematics of Surfaces VI*, pages 171–200, Oxford, 1996. Clarendon Press.
- [AvD97] L. Alboul and R. van Damme. Polyhedral metrics in surface reconstruction: tight triangulations. In T. N. T. Goodman, editor, *The Mathematics of Surfaces VII*, pages 309–336, Oxford, 1997. Clarendon Press.
- [Boi84] J.D. Boissonnat. Geometric structure for 3-dimensional shape representation. *ACM Trans. on Graphics*, 3:266–286, 1984.
- [Car87] R.E. Carlson. Shape preserving interpolation. In J. C. Mason and M. G. Cox, editors, *Algorithms for Approximation*, pages 143–167, Oxford, 1987. Clarendon Press.
- [dC76] M. P. do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1976.
- [DLR90a] N. Dyn, D. Levin, and S. Rippa. Algorithms for the construction of data dependent triangulations. In Mason and Cox [MC90], pages 185–192.
- [DLR90b] N. Dyn, D. Levin, and S. Rippa. Data dependent triangulations for piecewise linear interpolation. *IMA J. Numer. Anal.*, 10:137–154, 1990.
- [DM83] J.P. Duncan and S.G. Mair. *Sculptured Surfaces in Engineering and Medicine*. Cambridge University Press, Cambridge, 1983.
- [DvD98] D. Djokovic and R. van Damme. Algorithmic data reduction for voxel-based approximations. Technical report, University of Twente, Faculty of Mathematical Sciences, 1998. Submitted to CAGD.
- [FHMB84] O.D. Faugeras, M. Hubert, P. Mussi, and J.D. Boissonnat. Polyhedral approximation for 3D objects without holes. *Computer Graphics and Image Processing*, 25:169–183, 1984.

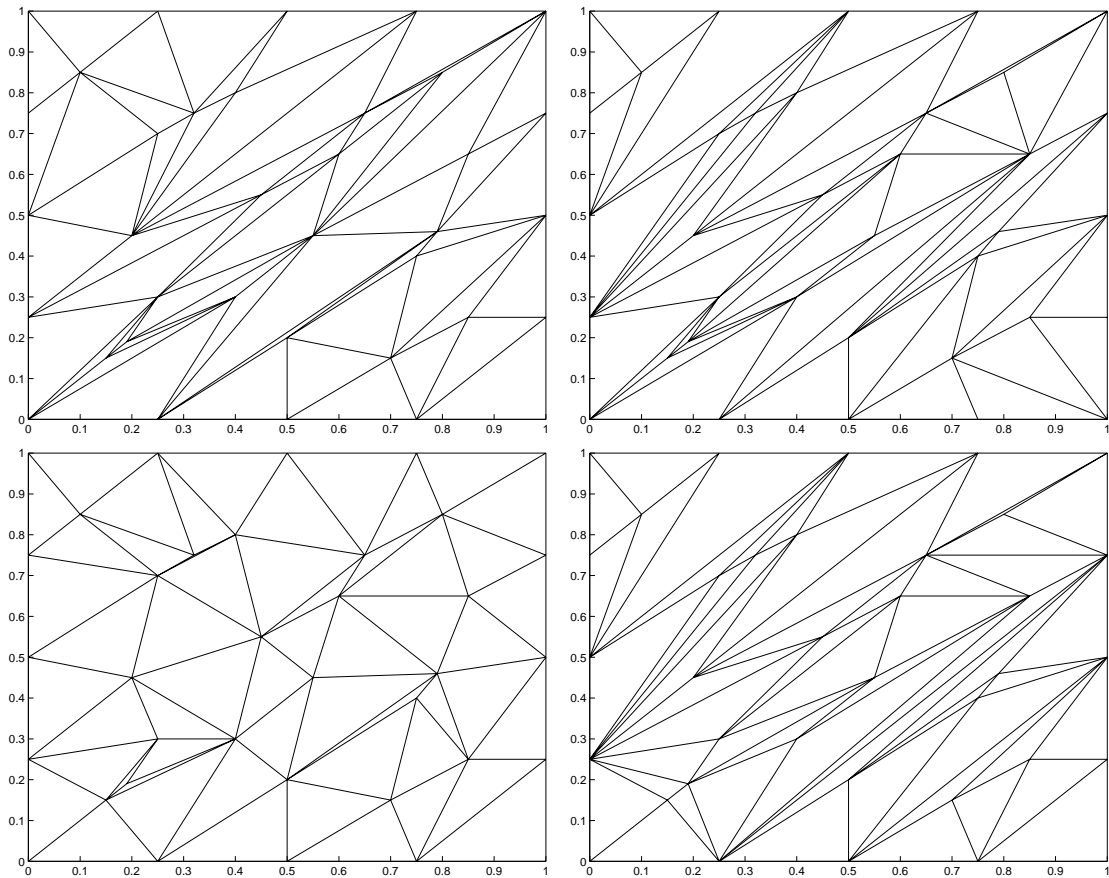


Figure 7: Best triangulations according to the mean and tight (up) and QS and JND criterion (down) for  $f^{(2)}$  in (4.6)

- [Fra87] R. Franke. Recent advances in the approximation of surfaces from scattered data. In C. K. Chui, L. L. Schumaker, and F. I. Utreras, editors, *Topics in Multivariate Approximation*, pages 79–98, New York, 1987. Academic Press.
- [FS91] B. Falcidieno and M. Spagnuolo. A new method for the characterization of topographic surfaces. *Int.J. Geographical information systems*, 5(4):397–412, 1991.
- [Hag88] H. Hagen. Geometric modelling of smooth surfaces. In *Computational geometry and its application. CG88*, 1988.
- [Law77] C.L. Lawson. Software for  $C^1$  surface interpolation. In J. R. Rice, editor, *Mathematical Software III*, New York, 1977. Academic Press.
- [LP82] C. Lin and M.J. Perry. Shape description using surface triangulation. In *Workshop on Computer Vision: Representation and Control*, pages 38–43, 1982.

- [MC90] J. C. Mason and M. G. Cox, editors. *Algorithms for Approximation II*, London, 1990. Chapman and Hall.
- [O'R81] J. O'Rourke. Triangulation of minimal area as 3D object models. In *Proc. of the Intern. Joint Conf. on AI 81*, pages 664–666, Vancouver, 1981.
- [QS89] E. Quak and L.L. Schumaker.  $C^1$  surface fitting using data dependent triangulations. In C. K. Chui, L. L. Schumaker, and J. D. Ward, editors, *Approximation Theory VI, Vol. 2*, pages 545–548, Boston, 1989. Academic Press.
- [QS90a] E. Quak and L.L. Schumaker. Calculation of the energy of a piecewise polynomial surface. In Mason and Cox [MC90], pages 134–143.
- [QS90b] E. Quak and L.L. Schumaker. Cubic spline fitting using data dependent triangulations. *Comput. Aided Geom. Design*, 7:293–301, 1990.
- [SH79] L.G. Shapiro and R.M. Haralick. Decomposition of 2-dimensional shapes by graph-theoretical clustering. In *IEEE Trans. Pattern Anal. Mach. Intell. PAMI-3*, pages 10–20, 1979.
- [vD97] R. van Damme. Bivariate Hermite subdivision. *Comput. Aided Geom. Design*, 14:847–875, 1997.
- [vDA95] R. van Damme and L. Alboul. Tight triangulations. In M. Dæhlen, T. Lyche, and L. L. Schumaker, editors, *Mathematical Methods for Curves and Surfaces*, pages 517–526, Nashville TN, 1995. Vanderbilt University Press.
- [Zah71] C.T. Zahn. Graph-theoretical methods for detecting and describing gestalt cluster. In *IEEE Trans. Comput.*, pages 68–86, 1971.