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How to split the eigenvalues of a one-parameter family of matrices

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Abstract

We are concerned with families F of $n \times n$-matrices $F(t)$ depending smoothly on the parameter $t \in I\!R$. We survey results on the behavior of eigenvalues of $F(t)$ for certain classes of matrices. We are especially interested in the question whether multiple eigenvalues can be avoided generically. In the set of families of symmetric matrices $F(t)$, for example, generically all eigenvalues of $F(t)$ are simple for all $t \in I\!R$. We consider a class of natural perturbations $\widetilde{F}$ of a given matrix family F such that $\widetilde{F}$ lies in the generic class, i.e. $\widetilde{F}$ avoids double eigenvalues 'as far as possible'.

Keywords: one-parametric eigenvalue problems, eigenvalue perturbation, genericity properties, continuation methods.

AMS Classification: 65F15, 65H17, 57Q65.

1 Introduction

Given $n \in I\!N$, we define the sets of matrices,

$$\begin{array}{l} A^n = \{A \mid A \text{ is a real } n \times n\text{-matrix}\} \equiv I\!R^{n^2} \\ S^n = \{S \in A^n \mid S \text{ is symmetric, } S^T = S\} \equiv I\!R^{(n+1)n/2} , \end{array}$$

and consider one-parameter families of matrices,

$$F : I\!R \rightarrow A^n \ (S^n) , \qquad t \mapsto F(t) .$$

We are interested in the behavior of the eigenvalues $\lambda(t)$ and corresponding eigenvectors $v(t)$ of $F(t),\ t \in I\!R$.
A classical question in perturbation theory is the following. How smoothly depend the eigenvalues and eigenvectors of $F(t)$ on t? It is well-known that for $F \in C(I\!R, A^n)$ also the eigenvalues $\lambda(t)$ are continuous in t. Differentiability is assured if the eigenvalue is simple. In fact, by applying the implicit function theorem to the equation

$$H(t, \lambda, x) = \begin{array}{lcl} F(t)x - \lambda x & = & 0 \\ x^T x - 1 & = & 0 \end{array} \tag{1}$$

the following is immediate for $F \in C^r(I, S^n),\ r \geq 1,\ I = (a, b)$, an open interval:

If $\lambda \in C(I, I\!R)$ and $\lambda(t)$ is a simple eigenvalue of $F(t)$ for all $t \in I$ $\Rightarrow$ $\lambda \in C^r(I, I\!R)$ and $v \in C^r(I, I\!R^n)$.

At points $\bar{t}$, where an eigenvalue $\lambda(\bar{t})$ is not simple the situation is more complicated. This case is analyzed in the landmark book of T. Kato [8]. We briefly outline some results needed later on.
Consider, as a first example, the analytic family of non-symmetric matrices $F(t) = \binom{0\ 1}{t\ 0}$ with eigenvalues $\pm\sqrt{t}$ (not differentiable at $t = 0$). Assume now, $F \in C^r(I, S^n),\ r \in I\!N \cup \{\infty\},\ I = (a, b)$. Then, even in presence of multiple eigenvalues, the eigenvalue functions λ_j of F can be defined in such a way that $\lambda_j \in C^r(I, I\!R),\ j = 1, \ldots, n$. Such a smoothness result is not valid for the eigenvectors as is shown by the following example due to Rellich (cf. [8, p.111]):

$$F(t) = e^{-\frac{1}{t^2}} \begin{pmatrix} \cos\frac{2}{t} & \sin\frac{2}{t} \\ \sin\frac{2}{t} & -\cos\frac{2}{t} \end{pmatrix},\ t \neq 0,\ F(0) = 0 .$$

This family F is a C^∞-function as well as the eigenvalue functions $\pm e^{-\frac{1}{t^2}}$. However, it can be seen that no eigenvector function $v(t)$ can be defined which is continuous in $t = 0$.
Under the stronger assumptions, that $F : I\!R \rightarrow S^n$ is an analytic function of t such a singular behavior of the eigenvectors is excluded (cf. [8, II, Th.6.1]).

Theorem 1 *Suppose, $F : I \rightarrow S^n$ is analytic on a real interval $I = (a, b)$. Then, the (appropriately defined) eigenpaires $(\lambda_j, v_j),\ j = 1, \ldots, n$, are analytic on I and orthonormal, i.e. $v_j^T(t) v_k(t) = 0$ for $k \neq j,\ ||v_j(t)|| = 1$.*

The assumption in this theorem, that all matrices $F(t)$ of the family F are symmetric (or at least normal) is essential.
In this paper we are concerned with the question to what extend a one-parametric matrix family F will generically avoid double eigenvalues. Moreover, we are interested in smooth perturbations of a given matrix family F with intersecting eigenvalue functions into a family $\widetilde{F}$ with simple eigenvalues. To answer this question, in Section 2, we give a survey of genericity results concerning multiplicities of eigenvalues. In particular, we consider the case where $F(t)$ is restricted to the subclass S^n of symmetric matrices. The results are based on stratifications of S^n, A^n and on Thom's transversality theory. A main theorem is as follows: The set P^s of smooth functions $F : I\!R \to S^n$ contains a dense and open subset P_0^s such that for all families F in P_0^s all eigenvalues $\lambda_j(t)$ are simple for all $t \in I\!R$.
Section 3 deals with smooth perturbations of a matrix family F. We discuss perturbations based on Sard's theorem. Then, we consider a concrete perturbation which transforms a given family $F \in P^s \setminus P_0^s$ into a generic family $\widetilde{F} \in P_0^s$. These perturbations only depend on the eigenvectors of the unperturbed family F. Section 4 deals with numerical aspects of the perturbation results. Note that, to avoid double eigenvalues in an eigenvalue problem describing a mechanical system is an important problem. Often double eigenvalues are related to instabilities in the mechanical system.

2 Genericity results

In this section we answer the question, whether in the generic case (general situation) double eigenvalues of a family $F(t)$ can be excluded for all parameter values $t \in I\!R$.
We begin with the non-parametric case which is easy to answer: Given any matrix $A \in A^n$ with multiple eigenvalues, there exist arbitrarily small perturbations E such that $\widetilde{A} = A + E$ has simple eigenvalues. To prove this, consider the transformation of A into Jordan canonical form,

$$Q^{-1}AQ = J \ , \quad Q \text{ a regular (complex) } n \times n\text{-matrix}, \ J = \begin{pmatrix} \lambda_1 & * & 0 \\ & \ddots & * \\ 0 & & \lambda_n \end{pmatrix} ,$$

with $* = 0$ or 1, $\lambda_1, \dots, \lambda_n$, the eigenvalues of A. We can choose real $\varepsilon_1, \dots, \varepsilon_n$, arbitrarily small such that the numbers $\widetilde{\lambda}_j = \lambda_j + \varepsilon_j$ are distinct. Then obviously, the perturbation of A,

$$\widetilde{A} = A + QE_0Q^{-1} \ , \quad \text{with} \quad E_0 = \text{diag}(\varepsilon_1, \dots, \varepsilon_n) \ ,$$

has the distinct eigenvalues $\widetilde{\lambda}_j$.
A more precise analysis of how densely matrices with double eigenvalues are lying in S^n or A^n is provided by stratification theory. For definitions and details on stratification theory we refer to [5].

We begin with the case $n = 2$:

$$A^2 = \{A = \begin{pmatrix} a_1 & a_3 \\ a_4 & a_2 \end{pmatrix} \mid a_1, \dots, a_4 \in I\!R\} \equiv I\!R^4,\ S^2 = \{S = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix} \mid a_1, a_2, a_3 \in I\!R\} \equiv I\!R^3$$

The eigenvalues of $\begin{pmatrix} a_1 & a_3 \\ a_4 & a_2 \end{pmatrix}$ are , $\lambda_{1,2} = \frac{a_1+a_2}{2} \pm \frac{1}{2}\sqrt{(a_1-a_2)^2 + 4a_3a_4}$. Consequently, a matrix $A \in A^2$ has double eigenvalues iff $(a_1 - a_2)^2 + 4a_3a_4 = 0$ is valid (codimension 1). A matrix $S \in S^2$ has a double eigenvalue iff $a_1 = a_2,\ a_3 = 0$ (codimension 2). Thus, in $A^2 \equiv I\!R^4$ the set of matrices with double eigenvalues has dimension 3, whereas in $S^2 \equiv I\!R^3$ this set is of dimension 1.

We now consider the general case S^n for $n \geq 2$. Let the multiplicities of the eigenvalues of a matrix $S \in S^n$ be denoted by $m_j(S),\ j = 1, \dots l$, where l is the number of distinct eigenvalues of S. So, we have $m_1(S) + \dots + m_l(S) = n$. We introduce the symbol $\sigma(S)$,

$$\sigma(S) = \{m_1(S), \dots, m_l(S)\}.$$

Let σ be any partition of n into strictly positive integers, say $m_j,\ j = 1, \dots, k$. So, $\sigma = \{m_1, \dots, m_k\}$, where $m_1 + \dots + m_k = n$. The set of all such partitions σ is denoted by $\mathcal{S}$. For any $\sigma \in \mathcal{S}$, the subset S^n_σ of S^n is defined by

$$S^n_\sigma = \{S \in S^n \mid \sigma(S) = \sigma\}.$$

Apparently, the collection $\{S^n_\sigma\}_{\sigma \in \mathcal{S}}$ constitutes a finite partition for S^n into mutually distinct subsets. For the proof of the following result and further details cf. [6].

Theorem 2 *The partition $\Sigma = \{S^n_\sigma,\ \sigma \in \mathcal{S}\}$ of S^n is a Whitney regular stratification. For $\sigma = \{m_1, \dots, m_k\}$ we have*

$$\text{codim } S^n_\sigma = \sum_{j=1}^{k} \left(\frac{(m_j+1)m_j}{2} - 1 \right).$$

In particular for $\sigma_0 = \{1, \dots, 1\}$ the set $S^n_{\sigma_0}$ (all eigenvalues simple) is open (codimension 0). All other strata $S^n_\sigma,\ \sigma \in \mathcal{S},\ \sigma \neq \sigma_0$ have codimension ≥ 2.

For A^n we have to choose a partition into subsets $A^n_\rho,\ \rho = 1, \dots, L$, such that any set A^n_ρ is characterized by a specific structure of the Jordan canonical form of its elements given by the so-called Segre symbol.

Theorem 3 *The partition $\Sigma = \{A^n_\rho,\ \rho = 1, \dots, L\}$ of A^n according to the Segre symbol is a Whitney regular stratification. In particular, we have*

$A^n_0 = \{A \in A^n \mid$ *all eigenvalues of A are simple*$\}$ *is open in A^n (codimension 0)*

$A^n_1 = \{A \in A^n \mid$ *A has canonical Jordan form J_1 in* $(2)\}$ *is a manifold of codimension 1*

$$J_1 = \begin{pmatrix} \boxed{T} & & & 0 \\ & \lambda_3 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \quad \text{with } T = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix},\ \lambda_j \neq \lambda_l,\ j \neq l\,. \tag{2}$$

All other strata $A^n_\rho,\ \rho \geq 2$, have codimension ≥ 2.

Proof. For a proof of this statement for the case of complex matrices, i.e for $B^n = \{B \mid B \text{ is a complex } n \times n\text{-matrix}\} \equiv \mathbb{C}^{n^2}$, see [2], [3]. The proof for A^n can be done by showing that the (analytic) functions $f_j : B^n \to \mathbb{C},\ j = 1, \ldots, s$, defining locally a manifold B^n_ρ of codimension s are real-valued, i.e. $f_j(A) \in \mathbb{R}$ for $A \in A^n$.
Here, we only give an elementary proof of the fact that A^n_1 is a manifold of codimension 1. Let be given $\overline{A} \in A^n_1$. After applying a transformation to Jordan canonical form we can assume that $\overline{A} = J_1$ (cf. (2)). Let U be a sufficiently small neighborhood of J_1. Then, by continuity, for $A \in A^n \cap U$ with $f_1(A, \tau) := \det(A - \tau I),\ f_2(A, \tau) := \frac{d}{d\tau} \det(A - \tau I)$ we have

$$A \in A^n_1 \iff \text{ with some } \lambda \approx \lambda_1 \quad f_1(A, \lambda) = f_2(A, \lambda) = 0\ .$$

To prove our statement, we have to construct a C^∞-function $f : U \to \mathbb{R}$ such that $Df(J_1) \neq 0$ and for $A \in U \cap A^n$ it follows: $A \in A^n_1$ iff $f(A) = 0$. Here, $Df(A)$ denotes the gradient of f and $D_A f_1(A, \tau)$ stands for the partial derivatives with respect to (the elements of) A. Now, since $\frac{d}{d\tau} f_2(J_1, \lambda_1) \neq 0$ (double eigenvalue) the implicit function theorem can be applied to the equation $f_2(A, \tau) = 0$. This yields a C^∞-function $\lambda : U \to \mathbb{R}$ such that $\lambda(J_1) = \lambda_1$ and for A near J_1 we have $f_2(A, \lambda(A)) = 0$. By construction, for $A \in U \cap A^n$ it follows

$$A \in A^n_1 \iff f(A) := f_1(A, \lambda(A)) = 0\ .$$

Using $\frac{d}{d\tau} f_1(J_1, \lambda_1) = f_2(J_1, \lambda_1) = 0$ we find $Df(J_1) = D_A f_1(J_1, \lambda_1) + \frac{d}{d\tau} f_1(J_1, \lambda_1) D\lambda(J_1) = D_A f_1(J_1, \lambda_1)$. A short calculation shows $D_{a_{21}} f_1(J_1, \lambda_1) = -\prod_{j=3}^n (\lambda_1 - \lambda_j) \neq 0$, i.e. $DF(J_1) \neq 0$. Here, $D_{a_{21}}$ denotes the partial derivative with respect to the element a_{21} of A. ◇

To analyze the generic behavior of parametric eigenvalue problems we have to apply Thom's transversality theory to the above stratification results. Roughly speaking, a one-parametric family $F : \mathbb{R} \to S^n\ (A^n)$ generically avoids manifolds of codimension ≥ 2 in $S^n\ (A^n)$.
To be more precise, let $C^\infty(\mathbb{R}, \mathbb{R}^K)$, $(K \in \mathbb{N})$ be endowed with the so-called strong C^∞-topology, denoted by C^∞_s (cf. [4]). The following stability and density result has been given in [6].

Theorem 4 *The subset P^s_0 of $C^\infty(\mathbb{R}, S^n)$ is C^∞_s-open and dense, where*

$$P^s_0 = \{F \in C^\infty(\mathbb{R}, S^n) \mid F(t) \text{ has } n \text{ simple eigenvalues for all } t \in \mathbb{R}\}.$$

By Theorem 4, the eigenvalue functions of a generic family $F \in P^s_0$ never intersect, i.e. behave as indicated in Figure 1a. The application of Thom's theory to Theorem 3 leads to the following result.

Theorem 5 *The subset P^a_0 of $C^\infty(\mathbb{R}, A^n)$ is C^∞_s-open and dense, where*

$$P^a_0 = \{F \in C^\infty(\mathbb{R}, A^n) \mid \text{there is a discrete set } D \subset \mathbb{R} \text{ such that } F(t) \in A^n_0, t \in \mathbb{R} \setminus D \text{ (simple eigenvalues) and } F(t) \in A^n_1, t \in D, \text{ ‘transversal intersection’ } \}.$$

By Theorem 5, the (real) eigenvalue functions of a generic family $F \in P_0^a$ never intersect, i.e. behave as indicated in Figure 1b. At possible turning points $\bar{t}$, marked by $\bullet$, an eigenvalue of algebraic multiplicity 1 (geometric multiplicity 2) occurs, $F(\bar{t}) \in A_1^n$.

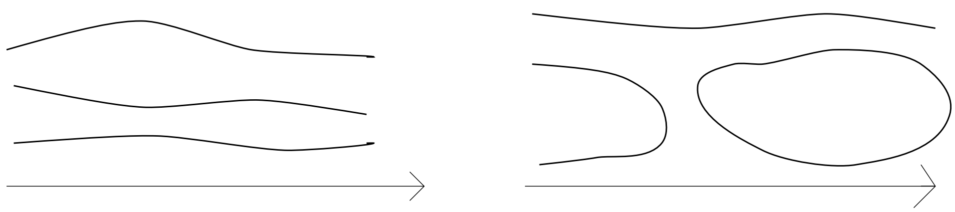

Figure 1 a) eigenvalues of $F \in P_0^s$ b) eigenvalues of $F \in P_0^a$

Genericity results can also be derived for other subclasses of matrices. In the following remark we deal with Hermitian matrices (see [7] for skew-symmetric matrices)

Remark. Let H^n denote the set of Hermitian matrices,

$$H^n = \{H = S + iB \mid S, B \in A^n,\ S^T = S,\ B^T = -B\} \equiv I\!R^{n^2}\ .$$

With the notations as in Theorem 2 the following stratification result holds for H^n: The partition $\Sigma = \{H_\sigma^n,\ \sigma \in \mathcal{S}\}$ of H^n is a Whitney regular stratification. For $\sigma = \{m_1, \ldots, m_k\}$ we have codim $H_\sigma^n = \sum_{j=1}^k (m_j^2 - 1)$. Consequently, the manifolds of H^n containing matrices with multiple eigenvalues are of codimension ≥ 3. Thus, parametric families $F \in C^\infty(I\!R^k, H^n)$ for $k = 1$ and $k = 2$ (two-parametric case) generically avoid multiple eigenvalues for all $t \in I\!R^k$.

3 Generic perturbations

The result of Theorem 4 in particular asserts that for any given matrix family $F \in C^\infty(I\!R, S^n)$ with intersecting eigenvalue functions $\lambda_j(t),\ j = 1, \ldots, n$, by an appropriate, smooth, arbitrary small perturbation (in the C_s^∞-sense) we obtain a family $\widetilde{F} \in C^\infty(I\!R, S^n)$ such that the eigenvalue functions $\widetilde{\lambda}_j(t)$ of $\widetilde{F}(t)$ never intersect on the whole $I\!R$. However, the proofs of the genericity results do not give any concrete idea how such a perturbation $\widetilde{F}$ can be constructed. This section deals with such perturbations.

We firstly discuss perturbations based on Sard's theorem. Consider the characteristic polynomial of a given family $F \in C^\infty(I\!R, A^n)$,

$$h : I\!R^2 \to I\!R, \qquad h \in C^\infty(I\!R^2, I\!R), \quad h(t, \lambda) := \det(F(t) - \lambda I).$$

We are interested in the real eigenvalues $\lambda(t)$ of $F(t)$, which obviously are given by the solution set $h^{-1}(0) = \{(t, \lambda) \mid h(t, \lambda) = 0\}$.

Definition 1 *A point $(\bar{t}, \bar{\lambda}) \in I\!R^2$ is called a regular point for h if $Dh(\bar{t}, \bar{\lambda}) \neq 0$ (maximal rank). Points which are not regular are called critical. A value $\varepsilon \in I\!R$ is called a regular value of h if all points (t, λ) in $h^{-1}(\varepsilon)$ (i.e. $h(t, \lambda) = \varepsilon$) are regular points for h.*

By the implicit function theorem, for any regular value ε the solution set $h^{-1}(\varepsilon)$ consists of non-intersecting C^∞-curves. The famous Sard theorem applied to h yields the following.

Theorem 6 *For almost all $\varepsilon \in I\!R$ (in the sense of Lebesgue measure), ε is a regular value of h, i.e. the solution set $h^{-1}(\varepsilon)$ consist of non-intersecting C^∞-curves.*

Now, let be given a one-parametric matrix family $F \in C^\infty(I\!R, A^n)$ with intersecting 'eigenvalue-functions'. Then, if instead of $h = 0$ (eigenvalues of F) we solve the perturbed equation

$$h(t, \lambda) := \det(F(t) - \lambda I) = \varepsilon \ , \tag{3}$$

in view of Theorem 6, the solution set $h^{-1}(\varepsilon)$ will have the form of the eigenvalue-curves of a family $\widetilde{F}$ in the generic class P_0^a as indicated in Figure 1b.
Unfortunately, if we consider families of symmetric matrices, $F \in C^\infty(I\!R, S^n)$, the perturbation (3) need no more correspond to an eigenvalue problem with symmetric matrices. Consider, as an example, the family $F(t) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$, $t \in I\!R$, with eigenvalue functions $\lambda_1(t) = t$, $\lambda_2(t) = -t$ and $h(t, \lambda) = \lambda^2 - t^2$. The solutions of the perturbed equation $h(t, \lambda) = \varepsilon$ are, $\lambda_{1,2}(t) = \pm\sqrt{t^2 + \varepsilon}$. Hence, depending on the choice $\varepsilon = 0$, $\varepsilon > 0$, $\varepsilon < 0$ the solution set of $h(t, \lambda) = \varepsilon$ looks like indicated in Figure 2. When $\varepsilon < 0$, then the real solutions disappear for $t \in (-\sqrt{-\varepsilon}, \sqrt{-\varepsilon})$, which is not compatible with a family of symmetric matrices.

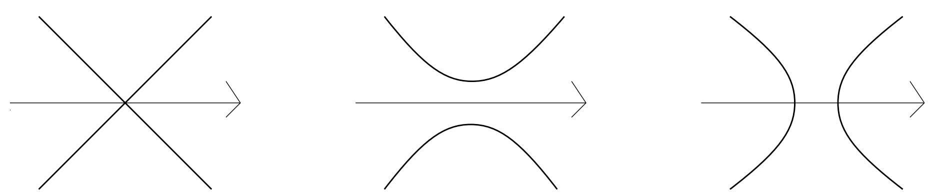

Figure 2 a) $\varepsilon = 0$ b) $\varepsilon > 0$ c) $\varepsilon < 0$

We are interested in perturbations such that for small $\varepsilon \in I\!R$ a solution set as in Figure 2a is perturbed into two disjoint real eigenvalue curves $\lambda_1^\varepsilon, \lambda_2^\varepsilon$ as in Figure 2b. For an eigenvalue of multiplicity two, such a perturbation is possible by choosing the sign of ε appropriately.

Lemma 1 *Let be given $F \in C^\infty(I\!R, A^n)$ such that at $\bar{t} \in I\!R$ the matrix $F(\bar{t})$ has p real eigenvalues satisfying with some $k \in \{1, \ldots, p-1\}$*

$$\lambda_1(\bar{t}) < \ldots < \lambda_k(\bar{t}) = \lambda_{k+1}(\bar{t}) < \ldots < \lambda_p(\bar{t})$$

and complex eigenvalues $\lambda_{p+l}(\bar{t}), \lambda_{p+l}^*(\bar{t})$, $l = 1, \ldots, q$ $(q = (n-p)/2)$. (Here λ^* denotes the complex conjugate of λ.) Then, for a sufficiently small $\varepsilon \in I\!R$ there is a neighborhood $U(\bar{t})$ of $\bar{t}$ such that the equation $h(t, \lambda) = \varepsilon$ has p real and pairwise different solutions $\lambda_j^\varepsilon(t)$, $j = 1, \ldots, p$, $t \in U(\bar{t})$ iff

$$(-1)^{p-1} \cdot \varepsilon > 0. \tag{4}$$

Proof. With the eigenvalues $\lambda_j(t)$ of $F(t)$ the values $\lambda_j^\varepsilon(t)$ are solutions of $h(t,\lambda)-\varepsilon = \prod_{j=1}^n(\lambda_j(t)-\lambda)-\varepsilon=0$. Since the functions $\lambda_j^\varepsilon(t)$ depend continuously on ε and t we only have to analyze for t near $\bar{t}$ the behavior of the solutions $\lambda_k^\varepsilon(t), \lambda_{k+1}^\varepsilon(t)$ near $\lambda_k(\bar{t})$. The relation $h(t,\lambda)-\varepsilon=0$ can be written as

$$(\lambda_k(t)-\lambda)(\lambda_{k+1}(t)-\lambda)=\varepsilon\cdot\prod_{\substack{j=1\\ j\neq k,k+1}}^{p}\frac{1}{\lambda_j(t)-\lambda}\prod_{l=1}^{q}\frac{1}{(\lambda_{p+l}(t)-\lambda)(\lambda^*_{p+l}(t)-\lambda)}$$

After solving the left-hand side for λ we find

$$\lambda=\frac{\lambda_k(t)+\lambda_{k+1}(t)}{2}\pm\frac{1}{2}\sqrt{(\lambda_k(t)-\lambda_{k+1}(t))^2+\frac{4\varepsilon}{\alpha(t,\lambda)}} \tag{5}$$

with $\alpha(t,\lambda)=\prod_{j=1,\,j\neq k,k+1}^{p}(\lambda_j(t)-\lambda)\prod_{l=1}^{q}|\lambda_{p+l}(t)-\lambda|^2$. The expression under the square-root shows, that for small $\varepsilon\in\mathbb{R}$ in a neighborhood $U(\bar{t})$ of $\bar{t}$ there will be two different real solutions $\lambda_k^\varepsilon(t), \lambda_{k+1}^\varepsilon(t)$ of (5) iff for $\bar{\lambda}=\lambda_k(\bar{t})$ the relation $\frac{4\varepsilon}{\alpha(\bar{t},\bar{\lambda})}>0$ is valid, which is equivalent with (4). $\diamond$

Remark. Unfortunately, for a family F of symmetric matrices the perturbation (3) can't be used to split eigenvalue functions at a point $\bar{t}$ where an eigenvalue has multiplicity $m>2$. It is not difficult to show that in this case a (small) perturbation (3) will always produce non-real solutions (cf. [9]). A perturbation of the equation (1),

$$H(t,\lambda,x)=\varepsilon,\qquad \varepsilon\in\mathbb{R}^{n+1}\ (\varepsilon\text{ small})$$

behaves even worse. For fixed $\bar{t}$ this equation may have up to $2n$ real or complex solutions (cf. [9] for details).

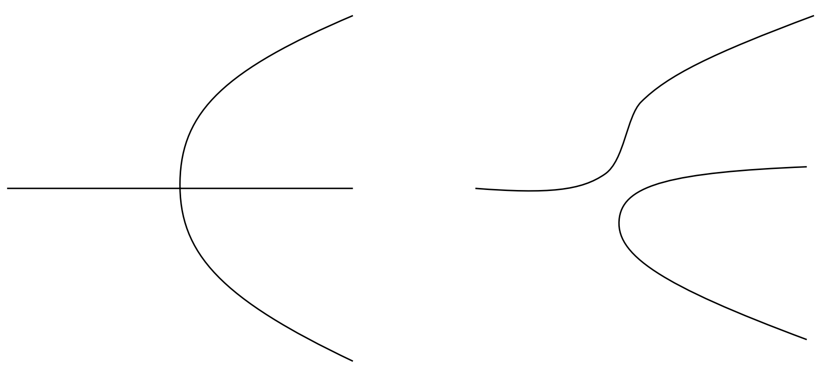

Figure 3 a) eigenvalues of $F(t)$ b) eigenvalues of $\widetilde{F}_\varepsilon(t)$, $\varepsilon>0$)

Our perturbation problem is closely related to singularity theory. Consider for example the ‘pitchfork’ given as solution set of the equation $h(t,\lambda) := -\lambda^3 + \lambda t = 0$ (cf. Figure 3a). The perturbation $h(t,\lambda) = \varepsilon$, $\varepsilon \neq 0$, leads to a solution set as given in Figure 3b, without any intersection points. The equation $h = 0$, $h = \varepsilon$, $(\varepsilon > 0)$ resp., coincide with the characteristic polynomials of the families

$$F(t) = \begin{pmatrix} 0 & 1 & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{F}_\varepsilon(t) = \begin{pmatrix} 0 & 1 & 0 \\ t & 0 & -\sqrt{\varepsilon} \\ \sqrt{\varepsilon} & 0 & 0 \end{pmatrix} \text{ resp.}$$

Besides the problem that the perturbation (3) does not preserve symmetry, this perturbation has the drawback that it cannot be directly written as a perturbation $\widetilde{F}_\varepsilon$ of the given matrix function F. In the following, we will discuss a perturbation which does not have these disadvantages.

We demonstrate the idea with an instructive example. Consider again the family $F \in C^\infty(I\!R, S^2)$, $F(t) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$, $t \in I\!R$, with eigenvalues $\lambda_1(t) = t$, $\lambda_2(t) = -t$ and corresponding eigenvectors $v_1(t) = e_1$, $v_2(t) = e_2$ (e_j the unit vectors). The eigenvalues of $\widetilde{F}_\varepsilon(t) = \begin{pmatrix} t & \varepsilon \\ \varepsilon & -t \end{pmatrix}$, $\varepsilon \neq 0$ are, $\lambda^\varepsilon_{1,2} = \pm\sqrt{t^2+\varepsilon^2}$, and never intersect on $I\!R$. This perturbation can be written with the help of the eigenvectors e_1, e_2 of $F(t)$ in the form

$$\widetilde{F}_\varepsilon(t) = F(t) + \varepsilon(e_1 e_2^T + e_2 e_1^T) .$$

In the following, we generalize this construction. Let again $(\lambda_j(t), v_j(t))$ be the appropriately numbered eigenpairs of $F : I\!R \to S^n$, with $v_j(t)$ orthonormalized.

Theorem 7 *Let be given a family $F \in C^\infty(I, S^n)$, $I = [a,b]$. Suppose that the only multiple eigenvalue of F on I occurs at a point $\bar{t} \in I$ where $F(\bar{t})$ possesses an eigenvalue of multiplicity m $(2 \le m \le n)$, i.e. (choosing an appropriate numbering of the λ_j's)*

$$\lambda_1(\bar{t}) = \ldots = \lambda_m(\bar{t}), \quad \lambda_l(\bar{t}) \neq \lambda_j(\bar{t}) \text{ for } l \neq j;\ l,j = m,\ldots,n.$$

Suppose, the eigenvectors $v_1(t),\ldots,v_m(t)$ are in $C^\infty(I, I\!R^n)$. Then, the perturbation

$$\widetilde{F}_\varepsilon(t) = F(t) + \varepsilon \sum_{j=1}^{m-1} v_j(t) v_{j+1}^T(t) + v_{j+1}(t) v_j^T(t), \quad t \in [-a,a] , \tag{6}$$

is in $C^\infty(I, S^n)$ and for any $\varepsilon \neq 0$, small enough, the n eigenvalue curves of $\widetilde{F}_\varepsilon$ do not have any intersection point on I.

Proof. With the orthogonal matrix $Q(t) = [v_1(t)\ \ldots\ v_n(t)]$ we find

$$Q^T(t)\widetilde{F}_\varepsilon(t)Q(t) = \begin{pmatrix} \boxed{T_\varepsilon(t)} & & & 0 \\ & \lambda_{m+1}(t) & & \\ & & \ddots & \\ 0 & & & \lambda_n(t) \end{pmatrix}, \quad T_\varepsilon(t) = \begin{pmatrix} \lambda_1(t) & \varepsilon & & 0 \\ \varepsilon & \lambda_2(t) & \ddots & \\ & \ddots & \ddots & \varepsilon \\ 0 & & \varepsilon & \lambda_m(t) \end{pmatrix}$$

Obviously, for any λ, the matrix $(T_\varepsilon(t) - \lambda I)$ with $\varepsilon \neq 0$ has rank $\geq m-1$ (it contains a regular $(m-1) \times (m-1)$-matrix). Therefore, the eigenvalues $\lambda_j^\varepsilon(t), j = 1, \ldots, m$, of $T_\varepsilon(t)$ are simple on I. By continuity, for ε small enough, they are different from the other eigenvalues $\lambda_j(t),\ j = m+1, \cdots, n$, of $\widetilde{F}_\varepsilon(t),\ t \in I$. $\diamond$

The same perturbation idea can also be used for families of matrices in A^n. Let be given a family $F \in C^\infty(I, A^n)$, $I = [a,b]$. Suppose that the only multiple eigenvalue of F on I occurs at a point $\bar{t} \in I$, with multiplicity m, $\lambda_1(\bar{t}) = \ldots = \lambda_m(\bar{t})$. Suppose that $F(t)$ is diagonizable on I with regular matrices $Q(t)$ such that $Q \in C^\infty(I, \mathbb{C}^{n^2})$, i.e. $Q^{-1}(t)F(t)Q(t) = \text{diag}(\lambda_1(t), \ldots, \lambda_n(t)),\ t \in I$. Then, by a similar analysis as above, it follows that for any small $\varepsilon \neq 0$ the perturbation

$$\widetilde{F}_\varepsilon(t) = F(t) + \varepsilon \sum_{j=1}^{m-1} Q(t) e_j e_{j+1}^T Q^{-1}(t) \tag{7}$$

does not have multiple eigenvalues on I.

Remark. In general, the perturbation (7) will produce a non-real family $\widetilde{F}_\varepsilon$ if $Q(t)$ is not real, i.e. if some of the eigenvalues of $F(t)$ are not real. Moreover, the perturbation (7) makes use of the knowledge of all eigenvectors of $F(t)$ (columns of $Q(t)$), whereas in the symmetric case, the perturbation (6) only uses the eigenvectors corresponding to the m intersecting eigenvalues.

We emphasize that in general, for families $F \in C^\infty(I, S^n)$, the eigenvectors $v_j(t)$ used in the perturbation (6) need not to be smooth (see Section 1). However, if we assume that F is analytic on I, then by Theorem 1, the perturbation (6) is well-defined and the following holds.

Corollary 1 *Let be given a family $F : I \to S^n,\ I = [a,b]$, such that F is analytic on I. Then, for any small $\varepsilon \neq 0$ the perturbation*

$$\widetilde{F}_\varepsilon(t) = F(t) + \varepsilon \sum_{j=1}^{n-1} v_j(t) v_{j+1}^T(t) + v_{j+1}(t) v_j^T(t) \tag{8}$$

is analytic on I and has simple eigenvalues for all $t \in I$.

Note, that for analytic F, by Theorem 1, the family $F(t)$ can be written as a sum of their 'eigenprojections' in the form $F(t) = \sum_{j=1}^n \lambda_j(t) v_j(t) v_j^T(t),\ t \in \mathbb{R}$.
We give an example of a perturbation (8):

$$F(t) = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & -t \end{pmatrix}, \quad \widetilde{F}_\varepsilon(t) = \begin{pmatrix} t^2 & \varepsilon & 0 \\ \varepsilon & t & \varepsilon \\ 0 & \varepsilon & -t \end{pmatrix}.$$

The eigenvalues of $F(t)$ intersect at $t = \pm 1$ (multiplicity 2), $t = 0$ (multiplicity 3) (see Figure 4a. The eigenvalue curves of the perturbation $\widetilde{F}_\varepsilon(t)$ ($\varepsilon = 0.4$) are given in Figure 4b.

Figure 4 a) eigenvalues of $F(t)$ b) eigenvalues of $\widetilde{F}_\varepsilon(t)$

Remark. A result as in Theorem 1 is also valid for an analytic family $F : I \to H^n$ (cf. [8, p.120-122]). Consequently, a statement as in Corollary 1 is true, if (8) is modified to the form $\widetilde{F}_\varepsilon(t) = F(t) + \varepsilon \sum_{j=1}^{n-1} v_j(t)v_{j+1}^*(t) + v_{j+1}(t)v_j^*(t)$ where v^* denotes the adjoint of the vector v.

4 Numerical aspects

In this section we discuss the numerical aspects of the perturbations in Section 3. Firstly, we comment on the fact that the perturbations (6) are optimal in the following sense. Let be given $S \in S^n$ with eigenpaires (λ_j, v_j), $j = 1, \dots, n$; such that $\lambda_1 = \lambda_2 \neq \lambda_j$, $j = 3, \dots, n$ and let be given $\varepsilon > 0$, (ε small). Then, we ask for a small perturbation $S + E \in S^n$ of S such that for the eigenvalues $\lambda_j(E) \approx \lambda_j$, $j = 1, \dots, n$, of $S + E$ we have

$$|\lambda_1(E) - \lambda_2(E)| = \max_{\|\widetilde{E}\|_2 = \varepsilon,\ \widetilde{E} \in S^n} |\lambda_1(\widetilde{E}) - \lambda_2(\widetilde{E})| \,, \tag{9}$$

i.e. the double eigenvalues $\lambda_1 = \lambda_2$ are separated maximally. ($\|\ \|_2$ denotes the matrix-norm corresponding to the Euclidean norm in $\mathbb{R}^n$.)

Lemma 2 *Under the assumptions above (for any $\varepsilon > 0$, small enough) the perturbation $S + E$ of S with $E = \varepsilon(v_1 v_2^T + v_2 v_1^T)$ is optimal in the sense of (9).*

Proof. By the Bauer-Fike eigenvalue perturbation result, for given $\widetilde{E} \in S^n$ we have for any $j = 1, \dots, n$, $\min_{1 \le i \le n} |\lambda_j(\widetilde{E}) - \lambda_i| \le \|\widetilde{E}\|_2$. This implies (for small ε)

$$|\lambda_1(\widetilde{E}) - \lambda_2(\widetilde{E})| \le |\lambda_1(\widetilde{E}) - \lambda_1| + |\lambda_2(\widetilde{E}) - \lambda_2| \le 2\|\widetilde{E}\|_2. \tag{10}$$

For $\widetilde{E} = E$ we find $\lambda_{1,2}(E) = \lambda_1 \pm \varepsilon$. Let v_j denote the eigenvectors of S corresponding to λ_j. Since $E(v_1 \pm v_2) = \pm\varepsilon(v_1 \pm v_2)$, $Ev_j = 0$, $j = 3, \dots, n$, the eigenvalues of E are $\varepsilon, -\varepsilon, 0$. Using the formula $\|E\|_2 = \max\{|\mu_j| \mid \mu_j \text{ eigenvalue of } E\}$, valid for symmetric E, it follows $\|E\|_2 = \varepsilon$ and then

$$2\varepsilon = |\lambda_1(E) - \lambda_2(E)| \le 2\|E\|_2 = 2\varepsilon.$$

Consequently, in view of (10) the difference $|\lambda_1(E) - \lambda_2(E)|$ is maximized. $\diamond$

In the sequel, we describe how a perturbation of an analytic family $F : I\!R \to S^n$ can be calculated numerically without knowing the eigenvectors of F explicitly. For brevity, we consider a concrete situation. Suppose, on $I = [a, b]$ we want to split two eigenvalue functions $\lambda_1(t)$, $\lambda_2(t)$ of F which have one intersection at $\bar{t} \in (a, b)$. We could proceed as follows:

- Compute (approximately) the intersection point $\bar{t}$ on (a, b), for example by applying Newton's method to the equation $\lambda_1(t) - \lambda_2(t) = 0$. Calculate eigenvectors $v_1(\bar{t}), v_2(\bar{t})$ corresponding to $\lambda_1(\bar{t}), \lambda_2(\bar{t})$ at $t = \bar{t}$.

- By choosing $\varepsilon, \delta > 0$ (small) appropriately, define the C^1-function

$$\varepsilon(t) := \begin{cases} 0 & \text{for } t \in I \setminus [\bar{t} - \delta, \bar{t} + \delta] \\ \varepsilon\big(\frac{(t-\bar{t})^2}{\delta^2} - 1\big)^2 & \text{for } t \in [\bar{t} - \delta, \bar{t} + \delta] \end{cases}$$

  Note that, $\varepsilon(\bar{t} \pm \delta) = \varepsilon'(\bar{t} \pm \delta) = 0$, $\varepsilon(\bar{t}) = \varepsilon$. Then, a C^1-perturbation $\widetilde{F}$ of F splitting λ_1, λ_2 is given by

$$\widetilde{F}(t) = F(t) + \varepsilon(t)\big(v_1(\bar{t})v_2^T(\bar{t}) + v_2(\bar{t})v_1^T(\bar{t})\big) .$$

Obviously, to separate different intersections, such a construction can be applied locally at any intersection point.

We end with a remark. Under the analyticity assumption on F, for any fixed ε, the perturbation $\widetilde{F}_\varepsilon$ in (8) defines a family which depends analytically on t. By continuity, for small ε the eigenvalues of F will only be slightly perturbed by and $\widetilde{F}_\varepsilon$. This is not the case for the corresponding eigenvectors, which change drastically. To see this, we consider again

$$F(t) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \quad \text{and} \quad F(t, \varepsilon) = \begin{pmatrix} t & \varepsilon \\ \varepsilon & -t \end{pmatrix} = F(t) + \varepsilon(e_1 e_2^T + e_2 e_1^T). \tag{11}$$

The function $F(t, \varepsilon)$ in (11) can be seen as a family in S^n depending on two parameters t, ε. Unfortunately, a result as given in Theorem 1 is no longer true for symmetric matrix-families which depend on more than one real (complex) parameter. The perturbation in (11) is just the counterexample in ([8, p. 116]). The eigenvalues and (non-normalized) eigenvectors of $F(t, \varepsilon)$ read:

$$\begin{aligned} \lambda_1(t,\varepsilon) &= \sqrt{t^2 + \varepsilon^2}, & v_1(t,\varepsilon) &= \begin{pmatrix} 1 \\ \frac{\sqrt{t^2+\varepsilon^2} - t}{\varepsilon} \end{pmatrix}, \quad \text{for } \varepsilon \neq 0, \quad \left(\binom{1}{0}, \ \ \varepsilon = 0\right), \\ \lambda_2(t,\varepsilon) &= -\sqrt{t^2 + \varepsilon^2}, & v_2(t,\varepsilon) &= \begin{pmatrix} \frac{t - \sqrt{t^2+\varepsilon^2}}{\varepsilon} \\ 1 \end{pmatrix}, \quad \text{for } \varepsilon \neq 0, \quad \left(\binom{0}{1}, \ \ \varepsilon = 0\right). \end{aligned}$$

Consequently, the eigenvalues are continuous in $(t, \varepsilon) = (0, 0)$, but not differentiable. The eigenfunctions can't be defined continuously in $(0, 0)$. In particular, for $\varepsilon = 0$ we have $v_1(0, 0) = \binom{1}{0}$, $v_2(0, 0) = \binom{0}{1}$ and for any $\varepsilon \neq 0$ $v_1(0, \varepsilon) = \frac{1}{\sqrt{2}}\binom{1}{1}$, $v_2(0, \varepsilon) = \frac{1}{\sqrt{2}}\binom{-1}{1}$.

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