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Abstract

Various infinite families of Z-related pairs in microtonal systems are presented. Soderberg's dual inversion is compared to a more special transformation, the one-pitch shift. The material is illustrated by several examples.

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1 Introduction

We consider microtonal systems in which the number of tones or pitches in one octave is n. The letter n will be used throughout this paper in that sense. A *pitch class* is a set of pitches where octave equivalent and enharmonically equivalent pitches are identified. The elements of a pitch class are all represented by an integer between 0 and $n-1$ inclusive; $c = 0$, c sharp=d flat=1, etc. So a *pitch class set*, abbreviated *PC set* can be seen as a set of integers reduced modulo n. By the *order* of a PC set we mean the number of pitch classes in the PC set.
Two PC sets A and B are said to be *equivalent* if there is a number t such that for each $p \in A$ there is a $q \in B$ such that either $p + t = q \pmod{n}$ or $p + q = t \pmod{n}$. In the first case B is a *transposition* of A, denoted by $B = T_t A$, in the second case it is an *inversion*, denoted by $B = I_t A$.
The *interval* between two pitch classes p and q with $p < q$ is the minimum of $q - p$ and $n + p - q$. The *interval vector* of a PC set is a vector the i-th entry of which is the number of intervals of length i, where i runs from 1 to $\lfloor n/2 \rfloor$. Two PC sets are said to be *Z-related* if they have identical interval vectors whereas they are not equivalent.

In Section 2, 3, 4 we consider Z-related pairs of order 4, 5, $n/2$, respectively. In Section 5 we present Z-related pairs for scales of odd orders. Finally, in Section 6 we investigate for which pairs it is possible to obtain one member of a Z-related pair from the other by a shift of just one pitch class.

Before starting off, we make three simple observations.

1. If the PC sets A and B form a Z-related pair of order k in an n-tone scale, then the complements of A and B form a Z-related pair of order $n - k$.

2. No Z-related pairs of order 3 exist.

3. If A and B are Z-related in an n-tone scale then mA and mB are Z-related in an mn-tone scale, where mA is the PC set that is obtained by multiplying all integers of A by m, and similarly for mB.

2 PC sets of order 4

Our first property gives an infinite collection of Z-related pairs of PC sets of order 4.

Property 1 *When n is a multiple of 4, say $n = 4m$ with $m \geq 2$, the PC sets $[0, a, a + m, 2m]$ and $[0, a, m, 2m + a]$ are Z-related provided $a < m$.*

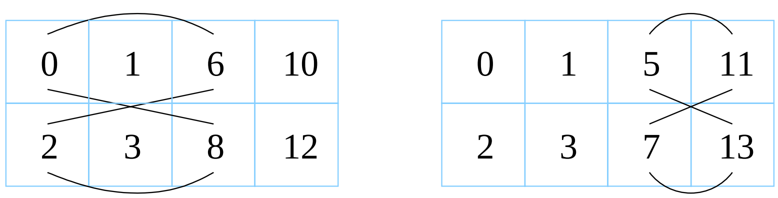
Proof In each of the above PC sets, the six intervals are: $a, m-a, m$, $m+a, 2m-a, 2m$. □

Since the number of values the parameter a can assume is $m-1$, it seems that the above result gives $m-1$ Z-related pairs of order 4. However, the value $a=x$ gives the same pair as the value $a=m-x$. Otherwise, no pairs are counted twice, so the number of pairs given by Property 1 is roughly $n/8$.

Example 1 Let $n=20$, so $m=5$. Meaningful values for a are 1 and 2. For $a=1$ we obtain the pair $[0,1,6,10]$, $[0,1,5,11]$ with interval vector [1001110011], and for $a=2$ the pair $[0,2,7,10]$, $[0,2,5,12]$ with interval vector [0110101101].

From the pairs given by Property 1 it is possible to derive pairs for other orders. We illustrate the procedure by an example.

Example 2 In the first pair from Example 1 we add four pitch classes to each PC set by simply increasing the existing values by 2. The result is the pair $[0,1,2,3,6,8,10,12]$, $[0,1,2,3,5,7,11,13]$, a Z-related pair with $n=20$ of order 8 with interval vector [3523232332].
For, in $[0,1,6,10]$ there are 6 intervals, and each of these leads to 4 intervals in the augmented PC set. In $[0,1,5,11]$ the same 6 intervals occur, and these lead to the same 6×4 intervals. See the figure below where the situation for the interval 6 is indicated.



Obviously, in both cases, 4 extra intervals of length 2 are introduced.

The procedure can be applied to a much more general case, as formulated in the next property.

Property 2 *Let A, B be a Z-related pair of order k, for which the t-th entry in the common interval vector is* 0. *Then $A \cup (A+t)$, $B \cup (B+t)$ is a Z-related pair of order $2k$.*
Here $A+t$ is the PC set obtained from A by increasing all pitch class numbers by t, and similarly for $B+t$. The symbol $\cup$ has its usual meaning: union of sets.

A formal proof of this property is not hard, and will be omitted.

So far, all our Z-related pairs of order 4 belong to tone systems in which the number of pitches is a multiple of 4. In Section 5 we present a Z-related pair of order 4 for the case $n = 13$.

3 PC sets of order 5

Contrary to what one might expect, it seems that for order 5 Z-related pairs are more abundant than for order 4.

Property 3 *When n is at least 10 and even, say $n = 2m$, the PC sets $[0, a, 3a, m-a, m]$ and $[0, a, 2a, m-2a, m+a]$ are Z-related provided:*
$0 < 2a < m$, $a \neq m/4$ *and* $a \neq m/3$.

Proof In each of the above PC sets the intervals are: $a, a, 2a, 3a, m-a, m-a, m-2a, |m-3a|, |m-4a|$ and m. Note that $m-a$ and $m+a$ are to be considered as identical intervals. □

The number of Z-related pairs of order 5, obtained by Property 3, is roughly $n/4$.

Example 3 Let $n = 14$, so $m = 7$. The following Z-related pairs are obtained.
For $a = 1$: $[0, 1, 3, 6, 7]$, $[0, 1, 2, 5, 8]$ with interval vector [2121121].
For $a = 2$: $[0, 2, 5, 6, 7]$, $[0, 2, 3, 4, 9]$ with interval vector [2211211].
For $a = 3$: $[0, 3, 4, 7, 9]$, $[0, 1, 3, 6, 10]$ with interval vector [1122211].

The next property gives a different class of pairs of order 5.

Property 4 *When n is at least 10 and even, say $n = 2m$, the PC sets $[0, a, m-2a, m-a, m+a]$ and $[0, a, 2a, m-a, m+2a]$ are Z-related provided $0 < 2a < m$ and $a \neq m/3$.*

Proof The sequence of intervals is for both PC sets given by $a, a, 2a, 3a, m-a, m-a, m-2a, m-2a, |m-3a|, m$. □

Although the two classes of Z-related pairs given by the above properties are justly claimed to be different, they are not disjoint! For example, take $n = 10$, so $m = 5$. Then the case $a = 1$ of Property 3 yields the same pair as the case $a = 2$ of Property 4. On the other hand, the following example shows that for $n = 14$, the three pairs obtained from Property 4 are all 'new'.

Example 4 Let $n = 14$, $m = 7$. The interval vectors obtained for $a = 1, 2, 3$ are [2111221], [1221211], [2122111], respectively.

The two above properties certainly do not cover all Z-related pairs of order 5. Even for $n = 10$ one pair is missing, viz. $[0,1,2,5,7]$, $[0,1,3,5,6]$ with interval vector [22222].

In Section 5 we present a class of pairs of order 5 for odd n.

4 PC sets of order $n/2$

A well-known property in 12-tone systems is (Forte): a PC set of order 6 is either part of a Z-related pair, or it is self-complementary. This property also holds in n-tone systems with n even for PC sets of order $n/2$.
In n-tone systems with a small even value of n, Z-related pairs of order $n/2$ are not abundant. However, as n grows large, almost all PC sets belong to a Z-related pair. Or, to put it differently, self-complementary PC sets will be extremely rare. A precise formulation is given below.

Property 5 *Let n be even, let $P(n)$ be the number of PC sets of order $n/2$, and $S(n)$ the number of self-complementary PC sets of order $n/2$. Then $S(n)/P(n)$ tends to 0 as n tends to infinity.*

Proof Let $n = 2m$. Each self-complementary PC set coincides with its complement after a suitable rotation or after a suitable reflection.
Case 1 - Suppose the transformation is a rotation through k steps. Then $2k$ is a divisor of n. The pitch classes in a section of k consecutive places can be chosen in 2^k ways. Summing over the possible values of k, we obtain a number that is certainly less than 2^{m+1}.
Case 2 - Suppose the transformation is a reflection. For the position of the axis there are m possibilities. For each position there are 2^m possibilities to choose the PC set. Hence the total number of possibilities is at most $m \times 2^m$.
We conclude that $S(n)$ is at most $(m+2) \times 2^m$.
On the other hand, $P(n)$ is equal to the binomial coefficient $\binom{n}{m}$. This is asymptotically equivalent to $\frac{2^n}{\sqrt{\pi \times n/2}}$, according to Stirling's formula. From these estimates, the result easily follows. In fact, the ratio $S(n)/P(n)$ is so small that we can conclude that the claim is true not only for PC sets, but also for certain standardized PC sets, the so called prime forms. □

5 Odd scales

The Z-related pairs that we encountered in Sections 2, 3, 4 all come from scales with an even number of pitches. In this section we consider n-pitch scales for odd values of n.

Property 6 *Let $n = 5m$, $0 < a < m$. The PC sets $[0, a, m, 2m, 2m + a]$ and $[0, a, m, m + a, 3m]$ are Z-related.*

Proof Both PC sets contain the intervals $a, a, m-a, m, m, m+a, 2m-a, 2m, 2m, 2m + a$. Note that $2m$ and $3m$ are identical intervals when $n = 5m$. □

The number of Z-related pairs obtained by Property 6 is $m - 1$.

When m is odd in Property 6, then so is n. It follows that the corresponding Z-related pairs are new. But also when m and hence n are even, some new pairs occur.

Example 5 When $m = 4$, $n = 20$, the case $a = 1$ yields the pair $[0, 1, 4, 8, 9]$, $[0, 1, 4, 5, 12]$ with interval vector [2012101210], and $a = 3$ yields $[0, 3, 4, 8, 11]$, $[0, 3, 4, 7, 12]$ with interval vector [1022101210]. From the interval vectors it is clear that these pairs are indeed new.

Property 7 *Let $n = 2m+1$ with $m \geq 6$. Then the PC sets (of order $m-2$) $[0, 1, 4, 6(1)m]$ and $[0, 1, 3(1)m - 3, m + 3]$ are Z-related.*
Here $6(1)m$ is an abbreviation for the PC's 6 up to and including m, and similarly for $3(1)m - 3$.

Proof In the first PC set the intervals are 1, 4, $6(1)m$, 3, $5(1)m$-1, $2(1)m$-4 and the intervals between the PC's in the set $6(1)m$. In the second PC set the intervals are 1, $3(1)m$-3, m+3, $2(1)m$-4, m+2, $6(1)m$ and the intervals between the PC's in the set $3(1)m$-3. In both cases we see immediately the common intervals 1, $2(1)m$-4 and $6(1)m$. The 'internal' intervals within the sets $3(1)m$-3 and $6(1)m$ are of course the same in both cases since the first is just a transposition of the second. In the first PC set we are left with 3, 4, $5(1)m$-1, i.e. $3(1)m$-1. Since the intervals $m + 2$ and $m + 3$ are equivalent to $m - 1$ and $m - 2$, respectively, we see that in the second PC set, too, the remaining intervals are $3(1)m$-1. □

In the table below we present the pairs generated by Property 7 for $n = 13(2)21$.

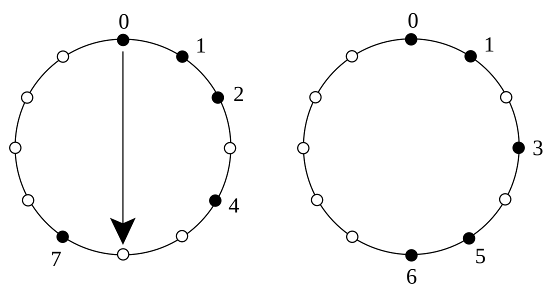
n	Order	PC set	PC set	Interval vector
13	4	[0,1,4,6]	[0,1,3,9]	[111111]
15	5	[0,1,4,6,7]	[0,1,3,4,10]	[2121121]
17	6	[0,1,4,6,7,8]	[0,1,3,4,5,11]	[32221221]
19	7	[0,1,4,6,7,8,9]	[0,1,3,4,5,6,12]	[433222221]
21	8	[0,1,4,6,7,8,9,10]	[0,1,3,4,5,6,7,13]	[5443232221]

The PC sets are not presented in their prime forms. This has been done to make the structure in this class clearer. Note that in passing we have found Z-related pairs of all orders from the fourth order on.

6 One-pitch shifts

Several Z-related pairs have the property that a change of just one pitch class, possibly followed by a transposition or an inversion, is sufficient to transform one of the PC sets of the pair into the other.

Example 6 Consider the pair $[0, 1, 2, 4, 7]$, $[0, 1, 3, 5, 6]$. If we replace the element 0 in the first PC set by 6, we obtain $[1, 2, 4, 6, 7]$, which is equivalent to $[0, 1, 3, 5, 6]$.



This property holds for all Z-related pairs of order 4 as given by Property 1. Moreover, in all cases there is a choice of two possibilities. Below we give a formal statement.

Property 8 *Let $n = 4m$ with $m \geq 2$ and $a < m$. Each PC set of the form $[0, a, m+a, 2m]$ can be transformed to the Z-related PC set $[0, a, m, 2m+a]$ by a shift of just one PC, possibly followed by a transposition or an inversion.*

Proof In $[0, a, m + a, 2m]$, replace a by $2m + a$. This gives $[0, m + a, 2m, 2m + a]$. Next form an inversion by subtracting all PC's from $2m + a$. This gives $[2m + a, m, a, 0]$. Alternatively, in $[0, a, m + a, 2m]$, replace $m + a$ by $3m + a$, and subtract all PC's from a: $[a, 0, -2m + a, -3m]$, which is equivalent to $[0, a, m, 2m + a]$. □

The one-pitch shift is reminiscent of Soderberg's dual inversion (Soderberg). For, let the PC set P be the disjoint union of the sets A and B with $A = \{a\}$. Then the shift from a to b can be interpreted as an inversion w.r.t. x where $x = a + b$. Then $I_y(I_xA \cup B) = I_yI_xA \cup I_yB$. In order that $I_yI_x = I_x$, we have to choose $y - b = b$ or $y = 2b$, and $I_y(I_xA \cup B) = I_xA \cup I_yB$, which has the appearance of a dual inversion.

Example 7 In the situation of Property 8 and the shift $a \to 2m + a$, $x = 2m + 2a$ and the value of y is found as follows: $y - (2m + a) = 2m + a$ hence $y = 4m + 2a = 2a$. Then $I_y(I_x A \cup B) = I_x A \cup I_y B = [2a,$ 2m+a, 2m+2a, $3m + a]$, which is equivalent to $[0, a, m, 2m + a]$.

For the Z-related pairs of order 5 given by Property 3 and 4, the one-pitch shift property holds as well.

Property 9 *Let $n = 2m$, $n \geq 10$, $0 < 2a < m$, $a \neq m/4$ and $a \neq m/3$. Each PC set of the form $[0, a, 3a, m - a, m]$ can be transformed to the Z-related PC set $[0, a, 2a, m - 2a, m + a]$ by a shift of just one PC, possibly followed by a transposition or an inversion.*

Proof In $[0, a, 3a, m - a, m]$, replace a by $m + a$, which gives $[0, 3a,$ m-a, m, m+a]. An inversion is formed by subtracting all PC's from $m + a$, resulting in $[m + a, m - 2a, 2a, a, 0]$. □

Property 10 *Let $n = 2m$, $n \geq 10$, $0 < 2a < m$ and $a \neq m/3$. Each PC set of the form $[0, a, m - 2a, m - a, m + a]$ can be transformed to the Z-related PC set $[0, a, 2a, m - a, m + 2a]$ by a shift of just one PC, possibly followed by a transposition or an inversion.*

Proof In $[0, a, m - 2a, m - a, m + a]$, replace $m - a$ by $2m - a$. This gives $[0, a, m - 2a, m + a, 2m - a]$. Form a transposition by adding a to all PC's: $[a, 2a, m - a, m + 2a, 0]$. □

In summary, in all these cases, a very simple transformation exists between the members of a Z-related pair. However, properties 8, 9 and 10 do not hold for all Z-related pairs of order 4 and 5.

Example 8 For $n = 13$, the Z-related pair $[0, 1, 4, 6]$, $[0, 1, 3, 9]$ does not have the one-pitch shift property. Nor does the pair $[0, 1, 3, 6, 7]$, $[0, 1, 3, 4, 9]$ for $n = 15$.

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