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Optimal strategies for a replacement model

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Abstract

We examine a replacement system with discrete-time Markovian deterioration and finite state space $\{0, ..., N\}$. State 0 stands for a new system, and the higher the state the worse the system; a system in state N is considered to be in a *bad state*. We impose the condition that the fraction of replacements in state N should not be larger than some fixed number. We prove that a generalized control limit policy maximizes the expected time between two successive replacements and we explain explicitly how to derive this optimal policy. Some numerical examples are given.

Keywords and phrases: replacement system, inspection, maintenance, average cost, generalized control limit policy.

1991 Mathematics Subject Classification: 93E20, 90B25

1 Introduction

We consider a model for a system which deteriorates stochastically in time but may be replaced by a new system. The state of the system is an element of the set $I = \{0, 1, ..., N\}$, 0 being the best and N being the worst state. We assume that the state of the system is detected by inspection at times $n = 0, 1, ...$ and that a decision to replace the system can be taken immediately after inspection.

The most famous replacement models of this type are those of Derman [3] and Ross [10]. They assume operating cost/rewards which are higher/lower as the system gets worse, constant replacement cost and Markovian deterioration. Optimal replacement strategies for these models are *control limit policies*, which are policies prescribing replacement if the system state exceeds a particular level. Also in many related models optimal policies are of the control-limit type, e.g. Stadje and Zuckerman [11], Parlar and Perry [8], Perry and Posner [9] and Jensen [5]. A comprehensive review of replacement models is given in Valdes-Florez [12].

Replacement models may also be interpreted as systems with *minimal* and *maximal repair*. A repair without improving the system is called minimal repair. Different interpretations of this kind of repair (black box minimal repair, physical minimal repair) are described in detail in the recent book of Aven and Jensen [1], p.82f. Maximal repair means to repair a system such that it is as good as new after repair, so the state after repair is 0.

The model we are interested in has Markovian deterioration (also known as *Derman's condition*) but there are no cost involved. Rather we want the fraction of replacements when the systems state is N (the *bad state*) to be not larger than a fixed $\epsilon_0 \in [0, 1]$. A system which is in this bad state has to be replaced. We shall show that the optimal policy is not a control limit policy but a randomized bang-bang strategy. Since this is a control limit replacement policy with a randomized threshold, it may be interpreted as a generalized control limit policy.

An example of a system whose status is an element of $\{0, ..., N\}$ is a parallel N-component system: the status reveals the number of failed components; the machine functions if at least one component is working. Therefore only status N identifies a failed system. Such a system with parallel components has been dealt with by Nakagawa [7], for example. With his model, which will be mentioned again later, the deterioration is caused by shocks. Since the components are identical and independent, every component fails with a constant probability p. Nakagawa used constant cost c_2 for a replacement and constant cost c_1 ($> c_2$) for a replacement of a failed system and discovered that a control limit policy minimizes the long-term average cost.

Another example of a system whose status is an element of $\{0, ..., N\}$ is a stand-by system studied, e.g. by Kistner, Subramanian and Venkatakrishnan [6]. In this system only one component is used and the other $N-1$ units are stand-by components. Again the status reveals the number of failed components. State N stands for system-failure in these models. An overview of $N-$component models was published by Cho and Parlar [2].

A third example of a system with one bad state is a system identifying a chemical fluid which is needed for production. The fluid becomes worse during production and if it is replaced too late (meaning in state N), there will be a problem of recycling it.

The remainder of this paper is organized as follows. In Section 2 we introduce some basic notation and terminology. In Section 3 we change the model to a cost model by introducing a cost c for every replacement. In this new model we do not consider the restriction regarding the fraction of replacements taking place in state N. The structure of the strategy minimizing the average cost of this new system will be obtained. Using this result we find two different kinds of strategies optimizing our original model in Section 4. In Section 5 we present formulas to compute the optimal strategies and in Section 6 we present some numerical examples.

2 Preliminaries and notations

Our model deals with a system starting at time 0. X_0 stands for the initial state of the system. For all $n \in I\!N$ we call the time period $[n-1, n)$ the nth interval.

We let p_{ij} be the probability of deterioration from state i to state $j \geq i$ in one interval. Furthermore we assume, as previously mentioned, Markovian Deterioration (MD), that is, the probability $\sum_{j=k}^{N} p_{ij}$ is non-decreasing in i for all $k \in I$.

Since we do not wish to deal with the trivial case in which deterioration is impossible, we furthermore impose the condition $p_{ii} < 1$ for every state $i \in I$. Finally, we assume $p_{0N} > 0$, so MD implies $p_{iN} > 0$ for every i so that the system may be in the bad state at the end of any time-interval with probability greater than zero.

At the end of each interval an inspection takes place after which the manager of the system can choose between two actions: to replace or not to replace. If the state of the system is N it has to be replaced. An admissible (randomized) strategy δ can be represented as a family of random variables $\{\delta(i),\ i \in I\}$ with $P(\delta(i) \in \{0,1\}) = 1$ for all $i \in I$ and $\delta(N) = 1$. Decision $\delta(i) = 1$ stands for replacing a system being in state i and $\delta(i) = 0$ stands for not replacing it. The space of all admissible strategies is denoted by Π.

We let X_n^- be the state before and X_n be the state after the $n-$th action. Obviously, under any strategy δ the processes $(X_n^-)_{n\in \mathbb{N}}$ and $(X_n)_{n\in \mathbb{Z}^+}$ are Markov chains. We let $\left(q_{ij}^{\delta}\right)_{i,j\in I}$ and $\left(\tilde{q}_{ij}^{\delta}\right)_{i,j\in I}$ be the transition probabilities, and $(\pi_\delta(i))_{i\in I}$ and $(\tilde{\pi}_\delta(i))_{i\in I}$ be the stationary distributions of the stochastic processes $(X_n)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$ under strategy δ, respectively. We observe that in a stationary setting

$$\begin{aligned}
\pi_\delta(0) = P_\delta(X_n = 0) &= P_\delta(X_n = 0,\ X_n^- = 0) + P_\delta(X_n = 0,\ X_n^- \neq 0) \\
&= P_\delta(X_n^- = 0,\ X_{n-1} = 0) + P_\delta(X_n = 0,\ X_n^- \neq 0) \\
&= \pi_\delta(0) = \pi_\delta(0)p_{00} + P_\delta(X_n = 0,\ X_n^- \neq 0),
\end{aligned}$$

so that

$$P_\delta(X_n = 0,\ X_n^- \neq 0) = \pi_\delta(0)(1 - p_{00}). \tag{1}$$

We will show later that both processes are ergodic. Hence, the expected cycle length, that is the time between two replacements, under δ is $\frac{1}{\pi_\delta(0)(1-p_{00})}$, since $\pi_\delta(0)(1-p_{00})$ is the expected relative frequency of state 0 occurring under strategy δ, excluding direct visits from state 0 (in this case there was no replacement).

¿From $\{X_n^- = N\} \subset \{X_n = 0,\ X_n^- \neq 0\}$ and (1) we get

$$P_\delta(X_n^- = N | X_n = 0,\ X_n^- \neq 0) = \frac{P_\delta(X_n^- = N)}{P_\delta(X_n = 0,\ X_n^- \neq 0)} = \frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)(1 - p_{00})}. \tag{2}$$

As a consequence, we can phrase our problem in the following way. We look for a strategy $\delta \in \Pi$ which minimizes $\pi_\delta(0)(1 - p_{00})$ and observes the subsidiary condition

$$\frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)(1 - p_{00})} \leq \epsilon_0, \tag{3}$$

since from (2) we know that the last fraction equals the probability that a replacement at time n (that is, the event $\{X_n = 0,\ X_n^- \neq 0\}$) was caused by a failure (that is, the event $\{X_n^- = N\}$).
We let

$$\epsilon_1 := \epsilon_0(1 - p_{00}).$$

In the next Theorem we give a condition for a strategy satisfying (3) to exist.

Theorem 1 *There exists a strategy δ with $\frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \leq \epsilon_1$ if and only if $p_{0N} \leq \epsilon_1$.*

Proof: We note that

$$\tilde{\pi}_\delta(N) = \sum_{i=0}^{N} \pi_\delta(i) p_{iN} \;=\; \pi_\delta(0) p_{0N} + \sum_{i=1}^{N} \pi_\delta(i) p_{iN}. \tag{4}$$

The last summand is non-negative and vanishes if the strategy chosen is the control limit policy with threshold one, defined as δ_1. Thus $p_{0N} = \frac{\tilde{\pi}_{\delta_1}(N)}{\pi_{\delta_1}(0)} = \min_{\delta \in \Pi} \frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)}$. ■

Next we define the strategies used in this paper.

Definition 1
(i) A control limit policy with threshold i^*, denoted by δ_{i^*}, is a policy which prescribes replacement in state i if and only if $i \geq i^*$.
(ii) A pre-randomized bang-bang strategy with parameter $(i^*, p) \in I \times [0,1]$ is a replacement strategy which is a control limit policy with threshold i^* with probability $1-p$ and a control limit policy with threshold i^*+1 with probability p. This strategy we identify by $(i^*, p)_{\text{pre}}$.
(iii) In contrast to a pre-randomized strategy a post-randomized bang-bang strategy is not deterministic. It is a strategy in which a new decision is made for **every** *machine independent of the past either to choose the bang-bang strategy with threshold i^* or the bang-bang strategy with threshold i^*+1. We reselect threshold i^* with probability $(1-p)$ and threshold i^*+1 with probability p. This strategy we identify by $(i^*, p)_{\text{post}}$.*

Using a pre-randomized strategy means making one decision before starting the process; using a post-randomized strategy means making a new decision for every machine. Obviously with the use of any bang-bang strategy δ (pre-, post-randomized) for the process $(X_n)_{n \in \boldsymbol{N}}$ only one stationary distribution exists because of the Markovian deterioration and the condition $p_{ii} < 1$ for every state i: if $\delta \in \Pi$ meaning the threshold i^* is less than or equal to N state 0 is reachable from every other state. In the opposite case ($i^* > N$) this is valid for state N. Of course for the process $(X_n^-)_{n \in \boldsymbol{N}}$ there is only one stationary distribution, too. In the next Lemma some properties of the stationary probabilities are given. We write $\pi_{i^*}(i)$ instead of $\pi_{\delta_{i^*}}(i)$.

Lemma 1
(i) The stationary probability $\pi_{i^*}(0)$ is non-increasing in i^*.
(ii) The stationary probability $\tilde{\pi}_{i^*}(N)$ is non-decreasing in i^*.
(iii) $\tilde{\pi}_{i^*}(N) = \tilde{\pi}_{i^*+1}(N)$ yields $\pi_{i^*}(i) = \pi_{i^*+1}(i)$ or $p_{i,N} = p_{i+1,N} = ... = p_{i^*,N}$ for all $i \in \{0, ..., i^*-1\}$.
(iv) The stationary probabilities $\pi_{(i^*,p)_{pre}}(0)$ and $\pi_{(i^*,p)_{post}}(0)$ are non-increasing in p on $[0,1]$ for every threshold i^*.
(v) The stationary probabilities $\pi_{(i^*,p)_{pre}}(i)$ and $\pi_{(i^*,p)_{post}}(i)$ are continuous in p on $[0,1]$ for every threshold i^* and for every state i.
(vi) The stationary probabilities $\tilde{\pi}_{(i^*,p)_{pre}}(i)$ and $\tilde{\pi}_{(i^*,p)_{post}}(i)$ are continuous in p on $[0,1]$ for every threshold i^* and for every state i.

Proof:
(i) For any subsets $A, B \subset I$ and a family of random variables $(Z_n)_{n\in\mathbb{N}}$ on I with $P(Z_{n+1} = j|Z_n = i) = p_{ij}$, $i, j \in I$, $n \in \mathbb{N}$, let

$$A \overset{\geq n}{\to} B \;:=\; \{Z_0 \in A,\, Z_1 \notin B, ..., Z_{n-1} \notin B,\, \exists m \geq n:\; Z_m \in B\} \tag{5}$$

and

$$T(i) \;:=\; \inf\{n \in \mathbb{N}:\; Z_n = i \,|\, Z_0 = i\} \tag{6}$$

be a stopping-time for $(Z_n)_{n\in\mathbb{N}}$. We define $q^n_{ij}(i^*)$ as the probability that process $(X_n)_{n\in\mathbb{N}}$, using the bang-bang strategy with threshold i^* and starting in state i, will reach state j after at least n steps. If $i \neq j$ process $(X_n)_{n\in\mathbb{Z}^+}$ has to leave state i before reaching it again. Furthermore, let (p^n_{ij}) be the elements of the matrix P^n, where $P = (p_{ij})_{\{0\leq i,j,\leq N\}}$. The condition $p_{ii} \neq 0$ for every $i \in I$ yields that the process $(Z_n)_{n\in\mathbb{N}}$ will visit a state of $\{i^*+1, ..., N\}$ after a visit in i^* in a finite number of steps. If $i = j$ the process has to leave state i before reaching it again. Thus we have for $n \in \mathbb{N}$:

$$\{\{i\} \overset{\geq n}{\to} \{i^*, i^*+1, ..., N\}\} \quad \subset \quad \{\{i\} \overset{\geq n}{\to} \{i^*+1, i^*+2, ..., N\}\},$$

so

$$P\left(\{i\} \overset{\geq n}{\to} \{i^*, i^*+1, ..., N\}\right) \quad \leq \quad P\left(\{i\} \overset{\geq n}{\to} \{i^*+1, i^*+2, ..., N\}\right),$$

that is $q^n_{i0}(i^*) \leq q^n_{i0}(i^*+1)$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} q^n_{ii}(i^*) \;&=\; \sum_{m=1}^{n-1} q^m_{i0}(i^*)P\left(Z_n = i,\, Z_{n-1} \neq i, ..., Z_{m+1} \neq i|Z_m = 0\right) \\ &\quad + \sum_{m=n}^{\infty} P\left(Z_m = i,\, Z_{m-1} \neq i, ..., Z_1 \neq i|Z_0 = 0\right) \qquad (7) \\ &\leq\; \sum_{m=1}^{n-1} q^m_{i0}(i^*+1)P\left(Z_n = i,\, Z_{n-1} \neq i, ..., Z_{m+1} \neq i|Z_m = 0\right) \\ &\quad + \sum_{m=n}^{\infty} P\left(Z_m = i,\, Z_{m-1} \neq i, ..., Z_1 \neq i|Z_0 = 0\right) = q^n_{ii}(i^*+1). \qquad (8) \end{aligned}$$

Now we explain (7). As mentioned before, $q^n_{ii}(i^*)$ stands for the probability that process $(X_n)^{\infty}_{n=1}$ first visits state i after visiting a state which is different from i after at least n steps, under the condition of starting at state i and using the bang-bang strategy with threshold i^*. Thus during the time between these two visits there is a visit to state 0. Now we define the random variable X as the number of steps until the first visit at state 0 and the random variable Y as the number of steps to the subsequent visit at state i. Then we get

$$\begin{aligned} P(X+Y \geq n) \;&=\; \sum_{m=1}^{n-1} P(X+Y \geq n,\, Y = m) + P(Y \geq n) \\ &=\; \sum_{k=1}^{n-1} P(X \geq n-m,\, Y = m) + P(Y \geq n). \qquad (9) \end{aligned}$$

Hence equation (7) holds.
Furthermore we have for all $i < i^*$

$$\begin{aligned}\frac{1}{\pi_{i^*}(i)} &= E_{\delta_{i^*}}(T(i)) = \sum_{n\in\boldsymbol{N}} P_{\delta_{i^*}}(T(i)\geq n) = \sum_{n\in\boldsymbol{N}} q_{ii}^n(i^*) \\ &\leq \sum_{n\in\boldsymbol{N}} q_{ii}^n(i^*+1) = E_{\delta_{i^*+1}}(T(i)) = \frac{1}{\pi_{i^*+1}(i)}.\end{aligned} \tag{10}$$

Therefore $\pi_{i^*}(i)$ is non-increasing in $i^* \in \{i+1, ..., N\}$ and the first part of the Lemma is proven.
(ii) Part (i) yields to the existence of a constant $a_{i^*}(i) \geq 0$ for every state $i < i^*$ such that the following identity holds.

$$\pi_{i^*}(i) = \pi_{i^*+1}(i) + a_{i^*}(i). \tag{11}$$

We have

$$\sum_{i=0}^{i^*-1} a_{i^*}(i) = \sum_{i=0}^{i^*-1} \pi_{i^*}(i) - \sum_{i=0}^{i^*-1} \pi_{i^*+1}(i) = \pi_{i^*+1}(i^*)$$

and

$$\begin{aligned}\tilde{\pi}_{i^*}(N) &= \sum_{i=0}^{i^*-1} \pi_{i^*}(i)p_{iN} = \sum_{i=0}^{i^*-1} \pi_{i^*+1}(i)p_{iN} + \sum_{i=0}^{i^*-1} a_{i^*}(i)p_{iN} &(12)\\ &\leq \sum_{i=0}^{i^*-1} \pi_{i^*+1}(i)p_{iN} + p_{i^*N}\sum_{i=0}^{i^*-1} a_{i^*}(i) \text{ since } p_{iN} = \sum_{k=N}^{N} p_{ik} \uparrow (i) &(13)\\ &\overset{(2)}{=} \sum_{i=0}^{i^*-1} \pi_{i^*+1}(i)p_{iN} + \pi_{i^*+1}(i^*)p_{i^*N} = \sum_{i=0}^{i^*} \pi_{i^*+1}(i)p_{iN} = \tilde{\pi}_{i^*+1}(N). &(14)\end{aligned}$$

which proves the second part of the Lemma.
(iii) $\tilde{\pi}_{i^*}(N) = \tilde{\pi}_{i^*+1}(N)$ yields $\sum_{i=0}^{i^*-1} a_{i^*}(i)p_{i^*N} = \sum_{i=0}^{i^*-1} a_{i^*}(i)p_{iN}$ (see (12), (13)) and because of $p_{iN} \leq p_{i^*,N}$ for every $i \in \{0, ..., i^*-1\}$ the subsequent statement holds.
If $a_{i^*}(i) = 0$ then $\pi_{i^*}(i) = \pi_{i^*+1}(i)$ or $p_{iN} = p_{i+1,N} = ... = p_{i^*N}$.
Hence part (iii) is also proven.

(iv) For all $j \in \{1, ..., i^*-1\}$, $i \in \{0, ..., i^*\}$ the following identities hold. $q_{ij}^{(i^*,p)_{post}} = p_{ij}$,

$$q_{i,i^*}^{(i^*,p)_{post}} = p \cdot p_{i,i^*} \quad \text{and} \quad q_{i0}^{(i^*,p)_{post}} = p_{i0} + (1-p)p_{ii^*} + \sum_{k=i^*+1}^{N} p_{ik} \text{ so } q_{i0}^{(i^*,p)_{post}}.$$

Thus the probability $q_{i,i^*}^{(i^*,p)_{post}}$ is non-decreasing and the probability $q_{i0}^{(i^*,p)_{post}}$ is non-increasing in p. This means the probability $\pi_{(i^*,p)_{post}}(0)$ is non-increasing in p. The reason is that this probability is equal to the reciprocal of the mean number of steps process $(X_n)_{n=1}^{\infty}$ uses from starting at state 0, leaving it and returning the first time: if we increase the value p, probability $q_{i,0}^{(i^*,p)}$ will decrease or not change and the probability $q_{i,i^*}^{(i^*,p)}$ will increase or not change for every $i \in \{0, ..., i^*-1\}$; hence this mean number either increases or stays the same. Part (iv) of the Lemma is therefore also proven.
(v) The continuity of $\pi_{(i^*,p)_{pre}}$ in p follows from the identity

$$\pi_{(i^*,p)_{pre}} = (1-p)\pi_{i^*} + p\pi_{i^*+1}.$$

Now we look at $\pi_{(i^*,p)_{post}}$. Define $Q^{(i^*,p)_{post}}$ as the square matrix of size i^*+1 with elements $(q_{ji}^{(i^*,p)_{post}})_{\{(i,j)\in I^2\}}$. $(x_0, ..., x_{i^*}) = \left(\pi_{(i^*,p)_{post}}(0), ..., \pi_{(i^*,p)_{post}}(N)\right)$ is the unique solution of the system of equations

$$\begin{pmatrix} Q^{(i^*,p)_{post}} & - & I_{i^*+1} \\ & & \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_{i^*} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Thus the rank of the $(i^*+2) \times (i^*+1)$ matrix

$$\begin{pmatrix} Q^{(i^*,p)_{post}} & - & E_{i^*+1} \\ & & \\ 1 & \cdots & 1 \end{pmatrix}$$

is i^*+1. After elimination of one row being linear dependent on the others, this non-singular quadratic matrix of size i^*+1 will be defined as $A(p) = (a_{ij}(p))_{\{0 \leq i,j \leq i^*\}}$, where $(A(p))^{-1} = \left(a_{ij}^{-1}(p)\right)_{\{0 \leq i,j \leq i^*\}}$. Then

$$\pi_{(i^*,p)_{post}} = \left(a_{ij}^{-1}(p)\right) e_{i^*+1} = \frac{\left(a_{ji}^*(p)\right)}{|\left(a_{ij}\right)(p)|} e_{i^*+1}. \tag{15}$$

Since the elements $\left(a_{ji}^*(p)\right)$ are also determinants and thus polynomials in p, $\pi_{(i^*,p)_{post}}(i)$ is continuous in p. Thus part (iv) is proven completely and it remains to prove the last part.

(vi) To prove the continuity of the probabilities $\tilde{\pi}_{(i^*,p)_{\text{post}}}(i)$, we are faced with the problem that the process (X_n^-) using strategy $\delta_{(i^*,p)_{\text{post}}}$ does not form a Markov process, since the probability $P(X_n^- = j | X_{n-1} = i^*, X_{n-2} = i^*)$ is in general not equal to the probability $P(X_n^- = j | X_{n-1}^- = i^*, X_{n-2} < i^*)$. We avoid this problem by splitting the state $X_n^- = i^*$ into states $X_n^- = (i^*, i^*)$ if $X_{n-1} = X_n^- = i^*$ and $X_n^- = (i^*, < i^*)$, if $X_{n-1} < X_n^- = i^*$. Then process $(X_n^-)_{n \in \mathbb{N}}$ forms a Markov chain with the following transition probabilities: for $i \in \{0, ..., i^*-1, i^*+1, ..., N)$ we have

$$\tilde{q}_{ij}^{(i^*,p)_{\text{post}}} = p_{ij} 1_{\{i<i^*\}} + p_{0j} 1_{\{i>i^*\}}, \quad \tilde{q}_{(i^*,i^*),(i^*,i^*)}^{(i^*,p)_{\text{post}}} = p_{i^*i^*}, \quad \tilde{q}_{(i^*,<i^*),(i^*,i^*)}^{(i^*,p)_{\text{post}}} = (1-p) p_{i^*i^*},$$

$$\tilde{q}_{(i^*,i^*),(i^*,<i^*)}^{(i^*,p)_{\text{post}}} = 0, \quad \tilde{q}_{(i^*,<i^*),(i^*,<i^*)}^{(i^*,p)_{\text{post}}} = p \cdot p_{0i^*}, \quad \tilde{q}_{(i^*,i^*),j}^{(i^*,p)_{\text{post}}} = p_{i^*,j},$$

$$\tilde{q}_{(i^*,<i^*),j}^{(i^*,p)_{\text{post}}} = (1-p) \cdot p_{0j} + p p_{i^*j}, \quad \tilde{q}_{j,(i^*,i^*)}^{(i^*,p)_{\text{post}}} = 0, \quad \tilde{q}_{j,(i^*,<i^*)}^{(i^*,p)_{\text{post}}} = p_{ji^*} 1_{\{j<i^*\}} + p_{0i^*} 1_{\{j>i^*\}}.$$

Defining the corresponding matrix $\tilde{Q}^{(i^*,p)_{\text{post}}}$ for these probabilities (dim $\tilde{Q}^{(i^*,p)_{\text{post}}}$ $=$ dim $Q^{(i^*,p)_{\text{post}}} + 1$), the continuity of the probabilities $\tilde{\pi}_{(i^*,p)_{\text{post}}}$ can be proven similiar to the continuity of the probabilities $\pi_{(i^*,p)_{\text{post}}}$. ■

3 A cost model

In this section we look at the model of the previous section without the restriction on the percentage of replacements in the bad state N. We introduce the following cost function $d^{(c)}$ for $c \in \mathbb{R}^+$.

$$d^{(c)}(i,0) = 0, \ i \in I, \quad d^{(c)}(i,1) = 1, \ i \in \{1, ..., N-1\} \text{ and } d^{(c)}(N,1) = 1 + c. \tag{16}$$

The first component is the status before repair and the second one represents the action that is chosen at this replacement model. Hence we take the cost function Nakagawa [7] used with $c_2 = 1$ and $c_1 = 1 + c$. His model was already described in the introduction. The value c may be identified as a penalty cost for being in the bad state N. Derman [3], p. 125, has shown that the strategy optimizing this average cost replacement model is a control limit policy. Next we look at the average cost of this new cost model.

With $\phi_\delta(c)$ being the average cost function of this cost replacement problem under strategy δ and cost function $d^{(c)}$, we have

$$\phi_{i^*}(c) \quad = \quad \pi_{i^*}(0)(1 - p_{00}) + c\tilde{\pi}_{i^*}(N) = \pi_{i^*}(0)\left(1 - p_{00} + c\frac{\tilde{\pi}_{i^*}(N)}{\pi_{i^*}(0)}\right). \tag{17}$$

Hence, ϕ_{i^*} is obviously continuous on $I\!R^+$. (18)

The goal of this section is to find out how the strategy optimizing this cost model depends on the constant c. This result is given in Theorem 2 and helps us finding a strategy optimizing the original model in the next section. For the proof of Theorem 2 we need the next three lemmas.

Lemma 2 *Let $\alpha := p_{0N} > 0$, then*

$$\phi_\delta(c) = \alpha\left(d^{(c)}(N,1) + \tilde{V}^{(c)}_{\delta,1-\alpha}(0))\right), \tag{19}$$

where $\tilde{V}^{(c)}_{\delta,1-\alpha}$ is the mean $(1-\alpha)$ discounted cost function of the cost model with functions

$$\tilde{d}^{(c)} = d^{(c)} \cdot (1-\alpha), \quad \tilde{p}_{ij} = \frac{p_{ij} - \alpha 1_{\{N\}}(j)}{1-\alpha} \in [0,1).$$

Proof: A simple computation (e.g. Hernandez-Lerma, Lasserre [4], formula (4.2.15)) leads to

$$\begin{aligned}\tilde{V}^{(c)}_{\delta,1-\alpha}(i) &= \sum_{j=i}^{N} \tilde{p}_{ij}\left(\tilde{d}^{(c)}(j,\delta(j)) + (1-\alpha)\tilde{V}^{(c)}_{\delta,1-\alpha}(j(1-\delta(j)))\right) \qquad (20)\\ &= \sum_{j=i}^{N} \frac{p_{ij}}{1-\alpha}\left((1-\alpha)d^{(c)}(j,\delta(j)) + (1-\alpha)\tilde{V}^{(c)}_{\delta,1-\alpha}(j(1-\delta(j)))\right)\\ &\quad -\frac{\alpha}{1-\alpha}\left((1-\alpha)d^{(c)}(N,1) + (1-\alpha)\tilde{V}_{\delta,1-\alpha}(0)\right).\end{aligned}$$

Thus

$$\tilde{V}^{(c)}_{\delta,1-\alpha}(i) + \alpha\left(d^{(c)}(N,N) + \tilde{V}^{(c)}_{\delta,1-\alpha}(0)\right) = \sum_{j=i}^{N} p_{ij}\left(d^{(c)}(j,\delta(j)) + \tilde{V}^{(c)}_{\delta,1-\alpha}(j(1-\delta(j)))\right\}. \tag{21}$$

Using $g^{(c)} := \alpha\left(d^{(c)}(N,N) + \tilde{V}^{(c)}_{\delta,1-\alpha}(0)\right)$ and $h^{(c)} := \tilde{V}^{(c)}_{\delta,1-\alpha}$ (bounded because I is finite) we get $h^{(c)}(i) + g^{(c)} = \sum_{j=i}^{N} p_{ij}\left(d(j,\delta(j)) + h^{(c)}(j(1-\delta(j))\right)$. Hence $g^{(c)}$ equals $\phi_\delta(c)$ (e.g. Ross [10], p. 93). ■

Lemma 3 *Let $0 \le i^* < j^*$ and $\phi_{i^*}(c_0) = \phi_{j^*}(c_0)$ for a constant $c_0 \in I\!R^+$. Then we have for $c \in I\!R^+$*

$$\phi_{i^*}(c) \le \phi_{j^*}(c) \quad \Leftrightarrow \quad c \ge c_0. \tag{22}$$

Proof: We have

$$\sum_{i=i^*}^{N} \tilde{\pi}_{i^*}(i) + c_0\tilde{\pi}_{i^*}(N) = \phi_{i^*}(c_0) = \phi_{j^*}(c_0) = \sum_{i=j^*}^{N} \tilde{\pi}_{j^*}(i) + c_0\tilde{\pi}_{j^*}(N).$$

Lemma 2 yields $\tilde{\pi}_{j^*}(N) \geq \tilde{\pi}_{i^*}(N)$. So if c_0 becomes larger (smaller), $\phi_{j^*}(c_0)$ will become smaller (larger) than or equal to $\phi_{i^*}(c_0)$. Hence this Lemma is also proven. ∎

Lemma 4 *Let cost c be fixed. If the control limit policies with thresholds i^* and $j^*(> i^*)$ optimize the average cost, every control limit policy with threshold $i \in \mathbb{Z}^+ :\ i^* < i < j^*$ is optimal, too.*

Proof: From Lemma 2 we recall (19).

$$\phi_\delta(c) = (1-\alpha)\left(d^{(c)}(N,1) + \tilde{V}^{(c)}_{\delta,\alpha}(0)\right).$$

We have $\phi_{\delta_{i^*}} \leq \phi_\delta$ and $\phi_{\delta_{j^*}} \leq \phi_\delta$ for every strategy $\delta \in \Pi$.

Hence $\tilde{V}^{(c)}_{\delta_{i^*},\alpha}(0) \leq \tilde{V}^{(c)}_{\delta,\alpha}(0)$ and $\tilde{V}^{(c)}_{\delta_{j^*},\alpha}(0) \leq \tilde{V}^{(c)}_{\delta,\alpha}(0)$ for every $\delta \in \Pi$.

A control limit strategy with threshold i^* optimizes $V^{(c)}_{\delta,\alpha}(0)$ in δ, if and only if

$$\left(\alpha\sum_{j=0}^{N}\tilde{p}_{ij}\tilde{V}^{(c)}_{\alpha}(j) \geq (1-\alpha) + \alpha\sum_{j=0}^{N}\tilde{p}_{0j}\tilde{V}^{(c)}_{\alpha}(j) \text{ or } \sum_{n=0}^{\infty}\tilde{p}^{(n)}_{0i} = 0\right) \text{ for } i \in \{i^*, i^*{+}1, ..., N{-}1\}$$

and

$$\left(\alpha\sum_{j=0}^{N}\tilde{p}_{ij}\tilde{V}^{(c)}_{\alpha}(j) \leq (1-\alpha) + \alpha\sum_{j=0}^{N}\tilde{p}_{0j}\tilde{V}^{(c)}_{\alpha}(j) \text{ or } \sum_{n=0}^{\infty}\tilde{p}^{(n)}_{0i} = 0\right) \text{ for } i \in \{1, 2, ..., i^*-1\}.$$

This we prove indirectly. First we look at a state i with $\sum_{n=0}^{\infty} p^{(n)}_{0i} > 0$. If the inequality regarding this state i is not fulfilled, we will look at strategy δ being equal to δ_{i^*}, apart from state i, where it will behave contrarily: we have $\tilde{V}^{(c)}_{\delta,\alpha} < \tilde{V}^{(c)}_{\delta_{i^*},\alpha}$, i.e. δ_{i^*} is not optimal. Now we look at a state i with $\sum_{n=0}^{\infty}\tilde{p}^{(n)}_{0i} = 0$: starting at state 0 it is almost sure that state i will never be visited. Thus for every strategy δ the value $\tilde{V}^{(c)}_{\delta,\alpha}(0)$ is independent of δ's behaviour in i. It makes no sense to look what to do in state N since the condition $\delta(N) = 1$ holds for every stategy $\delta \in \Pi$. Using the function $\tilde{V}^{(c,n)}_{\alpha}$ defining the discounted cost up to the nth interval it is standard to prove that $\tilde{V}^{(c)}_{\alpha}$ is non-decreasing, since $\tilde{d}$ is non-decreasing and the Markovian deterioration of the (p_{ij}) yields the Markovian deterioration of the $(\tilde{p}_{ij})$ (e.g. Ross, [10], p. 37f). If the control limit policies with thresholds i^* and j^* are both optimal, we get the following.

$$f(i) := \sum_{j=0}^{N}\tilde{p}_{ij}\tilde{V}^{(c)}_{\alpha}(j) \text{ constant in } \{i^*, ..., j^*\}\backslash\left\{i \,\middle|\, \sum_{n=0}^{\infty} p^{(n)}_{0i} = 0\right\}.$$

If the state is an element of $\left\{i \,\middle|\, \sum_{n=0}^{\infty} p^{(n)}_{0i} = 0\right\}$ both actions are optimal, so all control limit policies with thresholds $\{i^*, ..., j^*\}$ minimize $\tilde{V}_{\delta,\alpha}(0)$ and hence also ϕ_δ in δ. ∎

Theorem 2 *There is a j_0 in I and there are positive real numbers $c_{j_0} \geq c_{j_0+1} \geq ... \geq c_N$ such that the control limit policy δ_{i^*} with threshold*

$$i^* = \begin{cases} N & \text{if} \quad 0 \le c < c_N, \\ j & \text{if} \quad c_{j+1} \le c < c_j, \\ j_0 & \text{if} \quad c \ge c_{j_0}, \end{cases}$$

is optimal. The condition $p_{0N} \neq p_{1N}$ *is sufficient and if* $p_{01} \neq 0$ *also necessary such that* j_0 *can be taken equal to 1.*

We remark that we expect $p_{01} > 0$ since $p_{01} = 0$ yields $\tilde{\pi}_\delta(1) = \pi_\delta(1) = 0$ for every replacement strategy δ so state 1 will never be visited.

Proof: First we look at the trivial case $c = 0$: Using the control limit policy with threshold i^* the average cost are $\pi_{i^*}(0)(1 - p_{00})$. These average cost decrease if i^* increases, so the control limit policy with threshold N is optimal. Now there are two possibilities:
Case a: The control limit policy with threshold N, δ_N, is optimal for every $c \in I\!R^+$.
Case b: Suppose δ_N is not optimal for every $c > 0$, that is, for some $c > 0$ and some state $i \neq N$ we have $\phi_N(c) > \phi_i(c)$. Then, let

$$\tilde{c} = \inf\{c > 0 \,|\phi_N(c) > \phi_i(c) \text{ for some } i \in I\}$$

and

$$i_0 = \limsup_{\epsilon \downarrow 0} \{\min\{i \in I \,|\phi_N(\tilde{c} + \epsilon) > \phi_i(\tilde{c} + \epsilon)\}\}.$$

$\phi_{i_0}(\tilde{c}) = \phi_N(\tilde{c})$ holds because of the continuity of the function $\phi_{i_0} - \phi_N$ (proven at (18)) and $\phi_{i_0}(c) \ge \phi_N(c)$ for $c > \tilde{c}$. Obviously we have $i_0 < N$. For all i with $i_0 < i \le N$ let $c_i := \tilde{c}$. We remark that if $c = \tilde{c}$, every control limit policy with threshold $i \in \{i_0, ..., N\}$ is optimal because of Lemma 4. For every c less than $\tilde{c}$ we found the optimal strategy. To do the same for a value of c being bigger than $\tilde{c}$ we look at the following recursive procedure.
Case b(i): The control limit policy with threshold i_0, δ_{i_0}, is optimal for every $c \in [\tilde{c}, \infty)$. We choose $j_0 := \tilde{i}$ and $c_{\tilde{i}} = \tilde{c}$.
Case b(ii): Suppose δ_{i_0} is not optimal for every $c \in [\tilde{c}, \infty)$, that is, for some $c \in [\tilde{c}, \infty)$ and for some state $i \in \{0, ..., i_0 - 1\}$, $\phi_i(c) < \phi_{i_0}(c)$. Then, let

$$\tilde{\tilde{c}} = \inf\{c \in [\tilde{c}, \infty) \,|\phi_{i_0}(c) > \phi_i(c) \text{ for some } i \in I\}$$

and

$$i_1 = \limsup_{\epsilon \downarrow 0} \left\{\min\left\{i \in I \,\middle|\phi_{i_0}(\tilde{\tilde{c}} + \epsilon) > \phi_i(\tilde{\tilde{c}} + \epsilon)\right\}\right\}.$$

We let $c_i = \tilde{\tilde{c}}$ for all i with $i_1 < i \le i_0$. For every c less than $\tilde{\tilde{c}}$ we found the optimal strategy. With $i_0 := i_1$ and $\tilde{c} := \tilde{\tilde{c}}$ we repeat this procedure recursively until case b(i) is valid. This will be the case at $j_0 = 1$ at the latest.

Next we prove indirectly that the condition $p_{0N} < p_{1N}$ is sufficient that j_0 can be taken equal to 1. We have $p_{0N} < p_{1N}$ and see what happens if case $j_0 = 1$ is not suitable. ¿From (18) we recall the following identity for every threshold i^*.

$$\phi_{i^*}(c) = \pi_{i^*}(0)(1 - p_{00}) + \tilde{\pi}_{i^*}(N) \cdot c.$$

Case I: The probability $\tilde{\pi}_{\delta_1}(N)$ is less than the probability $\tilde{\pi}_{\delta_{j_0}}(N)$. We get

$$\phi_{j_0}(c) - \phi_1(c) = (1 - p_{00})(\pi_{\delta_{j_0}}(0) - \pi_{\delta_1}(0)) + c(\tilde{\pi}_{\delta_{j_0}}(N) - \tilde{\pi}_{\delta_1}(N)). \tag{23}$$

The first summand is indeed less than or equal to zero, but since the factor of c is positive, there is a $c_0 \in I\!R$ so that the above term is positive for all values $c > c_0$. For these values c the control limit strategy δ_1 is better than δ_{j_0}. This contradiction yields $j_0 = 1$.
Case II: The probability $\tilde{\pi}_{\delta_1}(N)$ is equal to the probability $\tilde{\pi}_{\delta_{j_0}}(N)$. Lemma 1$(ii)$ yields to $\tilde{\pi}_{\delta_1}(N) = ... = \tilde{\pi}_{\delta_{j_0}}(N)$ and Lemma 1(iii) to $\pi_{\delta_1}(0) = ... = \pi_{\delta_{j_0}}(0)$. Thus we have $\phi_{j_0}(c) = \phi_1(c)$ for every $c \in I\!R^+$. So apart from the control limit policy with threshold j_0 the control limit policy with threshold 1 is optimal, too. Again we obtained a contradiction.

Last we prove indirectly that if $p_{01} \neq 0$ holds the condition $p_{0N} \neq p_{1N}$ (so $p_{0N} < p_{1N}$) is necessary to take j equal to 1. Under the condition $p_{0N} = p_{1N}$ and if $p_{01} > 0$ the strategy δ_2 is better than strategy δ_1 for every penalty cost $c \in I\!R^+$, as we now compute.

$$\pi_{\delta_2}(0) = 1 - \pi_{\delta_2}(1) \leq 1 - \pi_{\delta_2}(0)p_{01},$$

so

$$\begin{aligned}
\pi_{\delta_2}(0) &\leq \frac{1}{1+p_{01}} < 1 = \pi_{\delta_1}(0). \\
\tilde{\pi}_{\delta_2}(N) &= \pi_{\delta_2}(0)p_{0N} + \pi_{\delta_2}(1)p_{1N} \\
&= p_{0N}\left(\pi_{\delta_2}(0) + \pi_{\delta_2}(1)\right) = p_{0N} = \pi_{\delta_1}(0)p_{0N} = \tilde{\pi}_{\delta_1}(N),
\end{aligned}$$

since $p_{0N} = p_{1N}$. Hence we have for $c \in I\!R^+$

$$\phi_{\delta_2}(c) = \pi_{\delta_2}(0) + c\tilde{\pi}_{\delta_2}(N) < \pi_{\delta_1}(0) + c\tilde{\pi}_{\delta_2}(N) = \phi_{\delta_1}(c).$$

Hence $j_0 > 1$. ■

4 Optimal strategies for the original model

In this section we look again to the original model. Using Theorem 2 from the last section we first prove that the search for an optimal strategy may be restricted to the class of the pre-randomized strategies or to the class of the post-randomized strategies. The existence and the construction of the optimal strategy will be shown in the next section. Since we need that j_0, defined in Theorem 2, can be taken equal to one, we impose the following

probability-condition: $\qquad 0 < p_{0N} < p_{1N}$

which holds for the subsequent results.

Theorem 3 *For every strategy δ satisfying $\frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \leq \epsilon_1$, there are numbers $j \in I$ and $\lambda \in [0,1]$ with*

$$\frac{\tilde{\pi}_{(j,\lambda)_{\text{pre}}}(N)}{\pi_{(j,\lambda)_{\text{pre}}}(0)} = \frac{\lambda\tilde{\pi}_{\delta_j}(N) + (1-\lambda)\tilde{\pi}_{\delta_{j+1}}(N)}{\lambda\pi_{\delta_j}(0) + (1-\lambda)\pi_{\delta_{j+1}}(0)} \leq \epsilon_1 \tag{24}$$

and

$$\pi_{(j,\lambda)_{\text{pre}}}(0) = \lambda\pi_{\delta_j}(0) + (1-\lambda)\pi_{\delta_{j+1}}(0) = \pi_\delta(0).$$

Proof: Lemma 1(i) yields to

$$\pi_{\delta_N}(0) \leq ... \leq \pi_{\delta_1}(0) = 1. \tag{25}$$

Since a system in state N has to be replaced under every strategy, δ_N is the strategy which replaces most seldom. Thus using δ_N the process $(X_n)_{n\in \boldsymbol{N}}$ visits state N most seldom so this strategy minimizes $\pi_\delta(0)$ in δ. So $\pi_\delta(0) \leq \pi_{\delta_N}(0)$ for every strategy δ. Lemma 1 yields the same result if we look at the subset of control limit policies only.

For every strategy δ we have

$$1 = \pi_{\delta_1}(0) \geq \pi_\delta(0) \geq \pi_{\delta_N}(0). \tag{26}$$

(25) yields that for every δ there is a $j \in I$ with

$$\pi_{\delta_{j+1}}(0) \leq \pi_\delta(0) \leq \pi_{\delta_j}(0). \tag{27}$$

$$\text{Define } \lambda \in [0,1) \text{ with} : \pi_\delta(0) \quad = \quad \lambda\pi_{\delta_j}(0) + (1-\lambda)\pi_{\delta_{j+1}}(0). \tag{28}$$

The cost model with $c := c_{j+1}$ defined in Theorem 2 will be optimized by both control limit policies δ_j and δ_{j+1}. This is why for every strategy δ we have

$$\phi_j(c) = \phi_{j+1}(c) \leq \phi_\delta(c). \tag{29}$$

Hence

$$\pi_j(0)(1-p_{00})+c\tilde{\pi}_j(N) = \pi_{j+1}(0)(1-p_{00})+c\tilde{\pi}_{j+1}(N) \leq \pi_\delta(0)(1-p_{00})+c\tilde{\pi}_\delta(N). \tag{30}$$

¿From the definiton of the pre-randomized bang-bang strategy we obtain

$$\pi_{(j,\lambda)_{pre}}(i) \quad = \quad \lambda\pi_j(i) + (1-\lambda)\pi_{j+1}(i) \text{ for } i \in I, \tag{31}$$
$$\tilde{\pi}_{(j,\lambda)_{pre}}(i) \quad = \quad \lambda\tilde{\pi}_j(i) + (1-\lambda)\tilde{\pi}_{j+1}(i) \text{ for } i \in I. \tag{32}$$

The main idea of this proof stands behind the subsequent inequality.

$$\begin{aligned}
& \pi_{(j,\lambda)_{pre}}(0)(1-p_{00}) + c\tilde{\pi}_{(j,\lambda)_{pre}}(N) \\
= \; & \left(\lambda\pi_{\delta_j}(0) + (1-\lambda)\pi_{\delta_{j+1}}(0)\right)(1-p_{00}) + c\left(\lambda\tilde{\pi}_{\delta_j}(N) + (1-\lambda)\tilde{\pi}_{\delta_{j+1}}(N)\right) \\
= \; & \lambda\left(\pi_{\delta_j}(0)(1-p_{00}) + c\tilde{\pi}_{\delta_j}(N)\right) + (1-\lambda)\left(\pi_{\delta_{j+1}}(0)(1-p_{00}) + c\tilde{\pi}_{\delta_{j+1}}(N)\right) \\
\leq \; & \lambda\left(\pi_\delta(0)(1-p_{00}) + c\tilde{\pi}_\delta(N)\right) + (1-\lambda)\left(\pi_\delta(0)(1-p_{00}) + c\tilde{\pi}_\delta(N)\right) \\
= \; & \pi_\delta(0)(1-p_{00}) + c\tilde{\pi}_\delta(N).
\end{aligned} \tag{33}$$

This, together with

$$\pi_{(j,\lambda)_{pre}}(0) = \lambda\pi_{\delta_j}(0) + (1-\lambda)\pi_{\delta_{j+1}}(0) = \pi_\delta(0)$$

gives us

$$\tilde{\pi}_{(j,\lambda)_{pre}}(N) \leq \tilde{\pi}_\delta(N).$$

Finally, the identity

$$\pi_{(j,\lambda)_{pre}}(0) = \pi_\delta(0)$$

yields

$$\frac{\tilde{\pi}_{(j,\lambda)_{pre}}(N)}{\pi_{(j,\lambda)_{pre}}(0)} \leq \frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \leq \epsilon_1. \quad \blacksquare$$

For the proof of an analogous result regarding to the post-randomized strategies which is formulated in the next Theorem we need the subsidiary Lemma.

Lemma 5 *(i) For every strategy $(i,p)_{pre}$, $i \in \{1,...,N-1\}$, $p \in [0,1]$ there is a $q \in [0,1]$ such that for the cost model the subsequent identity is valid.*

$$\phi_{(i,p)_{pre}}(c) \quad = \quad \phi_{(i,q)_{post}}(c) \qquad \text{for all } c \in I\!R^+. \tag{34}$$

(ii) For every strategy $(i,q)_{post}$, $i \in \{1,...,N-1\}$, $q \in [0,1]$ there is a $p \in [0,1]$ such that for the cost model the subsequent identity is valid.

$$\phi_{(i,q)_{post}}(c) \quad = \quad \phi_{(i,p)_{pre}}(c) \text{ for all } c \in I\!R^+ \text{ with } \quad p := \frac{\frac{q}{\pi_{\delta_{i^*}}(0)}}{\frac{q}{\pi_{\delta_{i^*}}(0)} + \frac{1-q}{\pi_{\delta_{i^*+1}}(0)}}. \tag{35}$$

Proof: If $p = 0$ then $\phi_{(i,0)_{pre}}(c) = \phi_{\delta_i}(c) = \phi_{(i,0)_{post}}(c) = \phi_{(i,0)_{post}}(c)$ and if $p = 1$ then $\phi_{(i,1)_{pre}}(c) = \phi_{(i+1,0)_{pre}}(c) = \phi_{(i+1,0)_{post}}(c) = \phi_{(i,1)_{post}}(c) = \phi_{(i,1)_{post}}$, so that (34) holds true. For $p \in (0,1)$ we look at the two statements separately.
(i) $\phi_{(i,p)_{pre}}(c) \quad = \quad p\phi_{i+1}(c) + (1-p)\phi_i(c)$. Thus for every i_0 and c the function $g^c_{i_0}$: $p \rightarrow \phi_{(i_0,p)_{pre}}(c)$ is a line on the unit-interval going through $(0, \phi_{i^*}(c))$ and $(1, \phi_{i^*+1}(c))$. According to Lemma 1, $\phi_{(i_0,p)_{post}}$ is continuous in p. Since $\phi_{(i,0)_{pre}} \equiv \phi_{(i,0)_{post}}$ and $\phi_{(i,1)_{pre}} \equiv \phi_{(i,1)_{post}}$ hold and g^c_i is linear according to the intermediat value Theorem there must be a q of the unit interval with $\phi_{(i_0,p)_{pre}}(c) = \phi_{(i_0,q)_{post}}(c)$.
(ii) For every $n \in I\!N$ we define the random variable τ_n as the time at which the $n-$th visit to state 0 takes place, not counting visits from 0. Let $\tilde{C}^c_\delta(J)$ be the cost during the interval $J \subset Z\!\!\!Z^+$ using any strategy δ such that state 0 is positive recurrent for the process (X_n) and using 'penalty cost' c. Since state 0 is positive recurrent, we have

$$\phi_\delta(c) = \lim_{n\to\infty} \frac{E\left(\tilde{C}^c_\delta([\tau_{n-1},\, \tau_n])\right)}{E(\tau_n - \tau_{n-1})} = \frac{E\left(\tilde{C}^c_\delta([\tau_1,\, \tau_2])\right)}{E(\tau_2 - \tau_1)}. \tag{36}$$

So

$$\begin{aligned}
&\phi_{(i^*,q)_{\text{post}}}(c) \\
= \;& \frac{E(\tilde{C}^c_\delta([\tau_1,\tau_2])|\delta = \delta_{i^*}) \cdot P(\delta = \delta_{i^*}) + E(\tilde{C}^c_\delta([\tau_1,\tau_2])|\delta = \delta_{i^*+1})) \cdot P(\delta = \delta_{i^*+1})}{P(\delta = \delta_{i^*}) \cdot E(\tau_2 - \tau_1|\delta = \delta_{i^*}) + P(\delta = \delta_{i^*+1}) \cdot E(\tau_2 - \tau_1|\delta = \delta_{i^*+1})} \\
= \;& \frac{E(\tilde{C}^c_{\delta_{i^*}}([\tau_1,\tau_2])) \cdot (1-q) + E(\tilde{C}^c_{\delta_{i^*+1}}([\tau_1,\tau_2])) \cdot q}{\frac{1-q}{\pi_{\delta_{i^*}}(0)} + \frac{q}{\pi_{\delta_{i^*+1}}(0)}}
\end{aligned}$$

$$= E\left(\tilde{C}^c_{\delta_{i^*}}([\tau_1,\, \tau_2])\right) \cdot \pi_{\delta_{i^*}}(0) \cdot \frac{\frac{1-q}{\pi_{\delta_{i^*}}(0)}}{\frac{1-q}{\pi_{\delta_{i^*}}(0)} + \frac{q}{\pi_{\delta_{i^*+1}}(0)}} + E\left(\tilde{C}^c_{\delta_{i^*+1}}([\tau_1,\, \tau_2])\right) \cdot \frac{\pi_{\delta_{i^*+1}}(0) \cdot \frac{q}{\pi_{\delta_{i^*+1}}(0)}}{\frac{1-q}{\pi_{\delta_{i^*}}(0)} + \frac{q}{\pi_{\delta_{i^*+1}}(0)}}$$

$= \phi_{i^*}(c) \cdot (1-p) + \phi_{i^*+1}(c) \cdot p = \phi_{(i^*,p)_{\text{pre}}}$.

Theorem 4 *For every strategy δ fulfilling the inequality $\frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \leq \epsilon_1$ there are numbers $j \in Z\!\!\!Z^+$ and $\mu \in [0,1]$ with*

$$\frac{\tilde{\pi}_{(j,\mu)_{\text{post}}}(N)}{\pi_{(j,\mu)_{\text{post}}}(0)} \leq \epsilon_1 \text{ and } \pi_{(j,\mu)_{\text{post}}}(0) = \pi_\delta(0).$$

Proof: Due to (27) there exists a number $j \in Z\!\!\!Z^+$ with $\pi_{\delta_{j+1}}(0) \leq \pi_\delta(0) \leq \pi_{\delta_j}(0)$. Because of $\delta_j = \delta_{(j,0)_{post}}$, $\delta_{j+1} = \delta_{(j,1)_{post}}$ and the fact that $\pi_{(j,\mu)_{post}}(0)$ is continuous and non-increasing on the unit interval, there exists a $\mu \in [0,1]$ with $\pi_\delta(0) = \pi_{(j,\mu)_{post}}(0)$. Because of the last Lemma there exists a $\lambda \in [0,1]$, such that we have with $c := c_{j+1}$

$$\begin{aligned} \pi_{(j,\mu_2)_{post}}(0)(1-p_{00}) + c\tilde{\pi}_{(j,\mu_2)_{post}}(N) &= \phi_{(j,\mu_2)_{post}}(c) \overset{(35)}{=} \phi_{(j,\lambda)_{pre}}(c) \\ &= \pi_{(j,\lambda)_{pre}}(0)(1-p_{00}) + c\tilde{\pi}_{(j,\lambda)_{pre}}(N) \\ &\leq \pi_\delta(0)(1-p_{00}) + c\tilde{\pi}_\delta(N). \end{aligned} \tag{37}$$

We prove (37): for every $\lambda \in [0,1]$ and for all $i \in I$ we have

$$\pi_{(j,\lambda)_{pre}}(i) \overset{(31)}{=} \lambda\pi_{\delta_j}(i) + (1-\lambda)\pi_{\delta_{j+1}}(i) \text{ and } \tilde{\pi}_{(j,\lambda)_{pre}}(i) \overset{(32)}{=} \lambda\tilde{\pi}_{\delta_j}(i) + (1-\lambda)\tilde{\pi}_{\delta_{j+1}}(i).$$

$$\pi_{(j,\mu)_{post}}(0) = \pi_\delta(0) \text{ yields } \tilde{\pi}_{(j,\mu)_{post}}(N) \leq \tilde{\pi}_\delta(N), \text{ so } \frac{\tilde{\pi}_{(j,\mu)_{post}}(N)}{\pi_{(j,\mu)_{post}}(0)} \leq \frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \leq \epsilon_1. \quad \blacksquare$$

Using the cost model we were able to prove the last two Theorems. The content of the first was that for every strategy δ fulfilling the inequality $\frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \leq \epsilon_1$ we find a pre-randomized bang-bang strategy δ_{pre} with $\frac{1}{\pi_{\delta_{pre}}(0)(1-p_{00})} = \frac{1}{\pi_\delta(0)(1-p_{00})}$. The second Theorem states that we find a post-randomized bang-bang strategy δ_{post} with $\frac{1}{\pi_{\delta_{post}}(0)(1-p_{00})} = \frac{1}{\pi_\delta(0)(1-p_{00})}$. Since we want a strategy δ fulfilling the inequality $\frac{\tilde{\pi}_\delta(N)}{\pi_\delta(0)} \leq \epsilon_1$ and minimizing the value $\frac{1}{\pi_\delta(0)(1-p_{00})}$ the search for an optimal strategy can be restricted to the class of the pre-randomized strategies or to the class of the post-randomized strategies.

5 Construction of optimal strategies

First we look for an optimal strategy in the class of pre-randomized bang-bang strategies. ¿From the monotonicity of $\pi_{i^*}(0)$ and $\tilde{\pi}_{i^*}(N)$ in i^* (Lemma 1), we get from the identities

$$\pi_{(i^*,p)_{pre}}(i) = (1-p)\pi_{i^*}(i) + p\pi_{i^*+1}(i) \quad \text{and} \quad \tilde{\pi}_{(i^*,p)_{pre}}(i) = (1-p)\tilde{\pi}_{i^*}(i) + p\tilde{\pi}_{i^*+1}(i)$$

the monotonicity of $\pi_{(i^*,p)_{pre}}(0)$ and $\tilde{\pi}_{(i^*,p)_{pre}}(N)$ in i^* and p. Thus $\frac{\tilde{\pi}_{(i^*,p)_{pre}}(N)}{\pi_{(i^*,p)_{pre}}(0)}$ and $\frac{1}{\pi_{(i^*,p)_{pre}}(0)}$ are non-decreasing in i^* and in p. Since the first term may not larger be than ϵ_1 and the second has to be maximized, we look for parameters i^* and p with

$$\frac{\tilde{\pi}_{(i^*,p)_{pre}}(N)}{\pi_{(i^*,p)_{pre}}(0)} = \epsilon_1.$$

Thus we get

$$i^* = \max\left\{ i \,\middle|\, \frac{\tilde{\pi}_i(N)}{\pi_i(0)} \leq \epsilon_1 \right\} \tag{38}$$

and the value p we solve from the equation

$$\frac{(1-p)\tilde{\pi}_{i^*}(N) + p\tilde{\pi}_{i^*+1}(N)}{(1-p)\pi_{i^*}(0) + p\pi_{i^*+1}(0)} = \epsilon_1.$$

That is

$$p = \frac{\epsilon_1\pi_{i^*+1}(0) - \tilde{\pi}_{i^*+1}(N)}{(\tilde{\pi}_{i^*}(N) - \tilde{\pi}_{i^*+1}(N)) - \epsilon_1(\pi_{i^*}(0) - \pi_{i^*+1}(0))}. \tag{39}$$

Hence we have obtained an optimal strategy in the class of pre-randomized bang-bang strategies. Then this strategy optimizes our replacement system in the whole class Π.

Now we are also able to compute an optimal post-randomized bang-bang strategy. We simply have to compute the second parameter, because the first is identical to that of the optimal pre-randomized bang-bang strategy. We obtain the second parameter q using Lemma 5. If $p = 0$ we choose $q = 0$ and if $p \neq 0$ the following identity holds.

$$p = \frac{\frac{q}{\pi_{i^*+1}(0)}}{\frac{1-q}{\pi_{i^*}(0)} + \frac{q}{\pi_{i^*+1}(0)}},$$

hence,

$$q = \frac{\pi_{i^*+1}(0)}{\pi_{i^*+1}(0) - (1 - \frac{1}{p})\pi_{i^*}(0)}. \tag{40}$$

6 Numerical Examples

Here we provide some examples of optimal strategies generated by a computer. We look at the following class of transition probabilities holding Markovian deterioration.

$$\tilde{p}_{ij} = \begin{cases} \left(\frac{i+1}{j+1}\right)^{\beta} - \left(\frac{i+1}{j+2}\right)^{\beta} & N > j \geq i, \\ \left(\frac{i+1}{N+1}\right)^{\beta} & j = N, \\ 0 & \text{otherwise.} \end{cases}$$

In Table 1 the optimal strategies are given as examples.

The transition probabilities $p_{ij} = \frac{1}{N-i-1} \cdot 1_{\{j \geq i\}}$ yield similar results to those in Table 1, but the values of p of $(i^*, p)_{pre}$ are slightly larger than the corresponding values p of $(i^*, p)_{post}$. Looking at the identity

$$p = 0 \Leftrightarrow q = 0 \quad \text{else} \quad p = \frac{\frac{q}{\pi_{i^*+1}(0)}}{\frac{1-q}{\pi_{i^*}(0)} + \frac{q}{\pi_{i^*+1}(0)}},$$

where q is the parameter of the post-randomized strategy and p is the parameter of the pre-randomized strategy, recalling the monotonicity of $\pi_{i^*}(0)$ in i^* we see that parameter q of the optimal post-randomized strategy is always less than the corresponding parameter of the pre-randomized strategy. Comparisons of both optimal strategies would be an interesting topic for further research.

N	β	ϵ_0	p_{0N}	i^*	$p:(i^*,p)_{pre}$	$p:(i^*,p)_{post}$
2	0.75	0.75	0.738	1	0.056	0.047
2	0.75	0.9	0.738	1	0.663	0.619
2	0.75	0.99	0.738	1	0.968	0.962
10	0.5	0.5	0.426	1	0.793	0.768
10	0.5	0.9	0.426	7	0.915	0.912
10	0.5	0.99	0.426	9	0.789	0.785
100	0.75	0.1	0.053	3	0.710	0.694
100	0.75	0.25	0.053	14	0.908	0.907
100	0.75	0.5	0.053	39	0.082	0.082
100	0.75	0.75	0.053	67	0.824	0.824
100	0.75	0.99	0.053	98	0.656	0.656
100	0.9	0.1	0.029	6	0.827	0.821
100	0.9	0.25	0.029	20	0.648	0.646
100	0.9	0.5	0.029	45	0.758	0.758
100	0.9	0.9	0.029	88	0.842	0.842
100	0.9	0.99	0.029	98	0.879	0.878
100	1.0	0.05	0.02	4	0.053	0.050
100	1.0	0.5	0.02	49	0.501	0.500
100	1.0	0.9	0.02	89	0.900	0.900
100	1.0	0.99	0.02	98	0.990	0.990
100	2.0	0.001	0.0004	2	0.192	0.172
100	2.0	0.01	0.0004	9	0.098	0.096
100	2.0	0.1	0.0004	30	0.938	0.938
100	2.0	0.9	0.0004	94	0.816	0.816
100	2.0	0.99	0.0004	99	0.493	0.492

Table 1: Some examples for optimal strategies.

References

[1] Aven, T. and Jensen, U. (1999) *Stochastic Models of Reliability*. Springer, New York, Berlin, Heidelberg.

[2] Cho, D. I. and Parlar, M. (1991) A survey of maintenance models for multi-unit systems. *European Journal of Operational Research* **51**, 1-23.

[3] Derman, C. (1970) *Finite State Markovian Decision Processes*. Academic Press, New York and London.

[4] Hernandez-Lerma, O., Lasserre J. B. (1996) *Discrete-Time Markov Control Processes*. Springer, New York, Berlin, Heidelberg.

[5] Jensen, U. (1996) Stochastic Models of Reliabilty and Maintenance: An overview. In: S. Oezekici (ed.): *Reliability and Maintenance of Complex Systems*. NATO ASI Series F, Springer, Berlin, 3-36.

[6] Kistner, K.P., Subramanian, R. and Venkatakrishnan, K.S.: Reliability of a repairable system with standby failure - *Operations Research* **24**, 169-176.

[7] Nakagawa, T. (1979): Replacement problem of a parallel system in random environment - *Journal of Applied Probability* **16**, 203-205.

[8] Parlar, M. and Perry, D. (1996): Optimal (Q, r, T) Policies in deterministic and random yield inventory models with uncertain future supply. *Naval Research Logistics* **43**, 191-210.

[9] Perry, D. and Posner, M.J.M. (1991): Determining the control limit policy in a replacement model with linear restoration. *Operations Research Letters* **10**, 335-341.

[10] Ross, S. M. (1983) *Introduction to Stochastic Dynamic Programming*. Academic Press, New York.

[11] Stadje, W. and Zuckerman, D. (1991) Optimal maintenance strategies for repairable systems with general degree of repair. *Journal of Applied Probability* **28**, 384-396.

[12] Valdez-Flores, C. and Feldman, R. M. (1989): A survey of preventive maintenance models for stochastically deteriorating single-unit systems. *Naval Research Logistics* **36**, 419-446.