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Abstract

A well known Theorem of Vizing states that one can colour the edges of a graph by $\Delta + \alpha$ colours, such that edges of the same colour form a matching. Here, Δ denotes the maximum degree of a vertex, and α the maximum multiplicity of an edge in the graph. An analogue of this Theorem for directed graphs was proved by Frank. It states that one can colour the arcs of a digraph by $\Delta + \alpha$ colours, such that arcs of the same colour form a branching. For a digraph, Δ denotes the maximum indegree of a vertex, and α the maximum multiplicity of an arc.

We prove a common generalization of the above two theorems concerning the colouring of mixed graphs (these are graphs having both directed and undirected edges) in such a way that edges of the same colour form a matching forest.

Keywords: edge colouring, mixed graph, matching, branching
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1 Introduction

The concept of a matching forest was introduced by Giles in [3], [4], [5]. Matching forests in mixed graphs (graphs with both undirected and directed edges) generalize matchings in undirected graphs and branchings in directed graphs. Several important properties that matchings and branchings have in common, have also been proved for matching forests. Giles gave a polynomial-time algorithm for finding a maximum-weight matching forest, and a description of the matching forest polytope (the convex hull of the incidence vectors of matching forests). Schrijver [7] proved total dual integrality for these matching forest constraints.

In the present paper, we give a common generalization, in terms of matching forests, of the following two theorems concerning matchings and branchings.

Theorem 1.1 (Vizing) *Let $G = (V, E)$ be a graph with maximum degree Δ. Let α denote the maximum multiplicity of an edge. Then E can be covered by $\Delta + \alpha$ matchings.*

Theorem 1.2 (Frank) *Let $D = (V, A)$ be a directed graph, with maximum indegree Δ. Let α denote the maximum multiplicity of an arc. Then E can be covered by $\Delta + \alpha$ branchings.*

We start with some definitions and notation. All graphs in this paper are allowed to have multiple edges, but no loops.

A *mixed graph* $G = (V, E \cup A)$ is the union of an undirected graph $H = (V, E)$ and a directed graph $D = (V, A)$, defined on the same vertex set V. (Thus, if one of E or A is empty, G is simply a directed or undirected graph).

For a directed edge a entering a vertex v, we say that v is the (unique) *head* of a. Both endpoints of an undirected edge $e \in E$ are said to be a *head* of e. For any subset F of $E \cup A$, the set of vertices 'covered' by F is by definition $V(F) := \{v \in V \mid v \text{ is head of some } e \in F\}$. Moreover, $d_F^{head}(v)$ denotes the number of edges in F having v as a head. In the case of an undirected graph, $d_F^{head}(v)$ will also be denoted simply by $d_F(v)$, and in the case of a directed graph by $d_F^-(v)$. If F is the whole edge-set, the subscript is usually suppressed.

A *matching* in G is a set of undirected edges $M \subseteq E$ satisfying $d_F^{head}(v) \leq 1$ for every $v \in V$.

A *branching* in G is a set of directed edges $B \subseteq A$ containing no directed circuits, and satisfying $d_F^{head}(v) \leq 1$ for every $v \in V$. The *rootset* $R(B)$ of a branching B is by definition the set $V \setminus V(B)$.

A *matching forest* in G is a set of edges $F \subseteq E \cup A$ containing no directed circuits, and satisfying $d_F^{head}(v) \leq 1$ for every $v \in V$. Equivalently, $F \subseteq E \cup A$ is a matching forest in G if $F \cap A$ is a branching and $F \cap E$ is a matching on the rootset of $F \cap A$ (that is, $V(F \cap E) \cap V(F \cap A) = \emptyset$).

2 Covering with branchings

We start with investigating directed graphs $D=(V,A)$. The weakly connected components of a branching are called *arborescences*. Any arborescence has a unique root $r \in V$, and a unique directed r–v path, for every $v \in V \setminus \{r\}$. If r is the root of the arborescence F, F is said to be an *r-arborescence*. A *spanning r-arborescence* is an r-arborescence B with $V(B)=V \setminus \{r\}$.

For a subset of vertices X, the number of arcs (contained in $B \subseteq A$) having both endpoints in X will be denoted by $\gamma(X)$ ($\gamma_B(X)$).

As A. Frank observed in [8], the following result is easily seen to be equivalent to Edmonds' disjoint branchings Theorem [1].

Lemma 2.1 *Let $D=(V+s,A)$ be a directed graph, and let $F_1,\dots,F_k$ be arc-disjoint s-arborescences of D. Then the F_i can be completed to k arc-disjoint spanning s-arborescences if and only if*

$$d^-_{A\setminus\cup_i F_i}(X) \geq |\{i \mid X \cap V(F_i)=\emptyset\}| \quad \forall X \subseteq V \tag{1}$$

□

We can derive the following result on covering the arcs of a directed graph by branchings, where the rootsets of the branchings should contain given sets.

Theorem 2.2 *Let $D=(V,A)$ be a directed graph, and let $U_1,\dots,U_k$ be subsets of V. Then A can be covered by branchings $B_1,\dots,B_k$ of D, with $U_i \subseteq R(B_i)$ for all i, if and only if the indegrees of D satisfy*

$$d^-(v) \leq k - |\{i \mid v \in U_i\}| = |\{i \mid v \notin U_i\}| \quad \forall v \in V, \tag{2}$$

and moreover

$$\gamma(X) \leq \sum_{i:\, U_i \cap X \neq \emptyset} |X \setminus U_i| + \sum_{i:\, U_i \cap X = \emptyset} (|X|-1) \quad \forall X \subseteq V \tag{3}$$

Proof. Clearly, both conditions are necessary. To see sufficiency, we apply Lemma 2.1 to an auxiliary digraph $D'=(V+s,A')$. Here, s is a new vertex, and A' is obtained from A by adding the arcs of the s-arborences $F_i := \{sv \mid v \in U_i\}$, $i=1,\dots,k$, and, in addition, so many parallel arcs sv, $v \in V$, that we obtain an arc set A' for which $d^-_{A'}(v)=k$ for every $v \in V$. (Note that because of the condition (2), after adding only the arcs in $\cup_i F_i$, no indegree exceeds k.)

It is sufficient to prove that in D' we can complete the s-arborescences F_i to arc-disjoint spanning s-arborescences A_i. Indeed, because $d^-_{A'}(v)=k$ for all $v \in V$, the arc-disjoint spanning s-arborescences $A_1,\dots,A_k$ must cover A', and this implies that the arc-disjoint branchings $B_1,\dots,B_k$ defined by $B_i := A_i \cap A$ cover A (note that these B_i satisfy $U_i \subseteq R(B_i)$, since $F_i \subseteq A_i \setminus A$).

To prove that the F_i can be completed, we show that condition (1) holds. In D', we have for every $X \subseteq V$

$$d^-_{A'\setminus\cup_i F_i}(X) = \sum_{v\in X}(k - |\{i \mid v \in U_i\}|) - \gamma_A(X) = \sum_{i=1}^{k} |X \setminus U_i| - \gamma_A(X),$$

so condition (3) implies

$$d^-_{A'\setminus\cup_i F_i}(X) \geq \sum_{i\colon U_i\cap X=\emptyset} (|X\setminus U_i| - (|X|-1)) = |\{i \mid X\cap U_i = \emptyset\}|,$$

which proves (1), since $U_i = V(F_i)$. □

As a corollary, we can derive another theorem of Frank [2] on covering with branchings, which is stronger than Theorem 1.2.

Corollary 2.3 (Frank) *Let $D = (V, A)$ be a directed graph. Then A can be covered by k branchings in D if and only if*

$$d^-(v) \leq k \quad \forall v \in V$$

$$\gamma(X) \leq k(|X|-1) \quad \forall X \subseteq V$$

□

Theorem 2.2 also allows us to prove the following extension of Frank's Theorem 1.2.

Theorem 2.4 *Let $D = (V, A)$ be a directed graph, with maximum indegree Δ. Let α denote the maximum multiplicity of an arc of D. Let $k := \Delta + \alpha$. If subsets $U_1, \ldots, U_k$ are given such that the indegrees of D satisfy*

$$d^-(v) \leq \Delta - |\{i \mid v \in U_i\}| \quad \forall v \in V, \tag{4}$$

then A can be covered by branchings $B_1, \ldots, B_k$, with $U_i \subseteq R(B_i)$ for all i.

Proof. By Theorem 2.2, it suffices to check that in this case conditions (2) and (3) are satisfied. Condition (2) is satisfied, because of (4) and the fact that $\Delta \leq k$. To see (3), let $X \subseteq V$.

Suppose first that $|X|\alpha \leq |\{i \mid U_i \cap X = \emptyset\}|$. Then

$$\gamma(X) \leq \alpha|X|(|X|-1) \leq |\{i \mid U_i\cap X=\emptyset\}|(|X|-1) = \sum_{i\colon X\cap U_i=\emptyset}(|X|-1),$$

which certainly implies (3).

Suppose next that $|X|\alpha > |\{i \mid U_i \cap X = \emptyset\}|$. Then

$$\begin{aligned}
\gamma(X) &\leq \sum_{v\in X}(\Delta - |\{i \mid v\in U_i\}|) \\
&= (\sum_{v\in X}|\{i \mid v\notin U_i\}|) - |X|\alpha \\
&< \sum_{v\in X}|\{i \mid v\notin U_i\}| - |\{i \mid U_i\cap X=\emptyset\}| \\
&= \sum_{i\colon X\cap U_i\neq\emptyset}|X\setminus U_i| - \sum_{i\colon X\cap U_i=\emptyset}(|X|-1),
\end{aligned}$$

as required. □

3 Covering with matching forests

By combining Vizing's Theorem and Theorem 2.4, the main result is obtained.

Theorem 3.1 *Let $G = (V, E \cup A)$ be a mixed graph with maximum degree Δ. Let α denote the maximum number of parallel edges of the same type (directed or undirected) in G. Then $E \cup A$ can be covered by $\Delta + \alpha$ matching forests of G.*

Proof. By Vizing's Theorem 1.1, matchings $M_1, \ldots, M_k$ covering E exist. Now, $D = (V, A)$, with $U_i := V(M_i)$ satifies the condition of Theorem 2.4, so branchings $B_1, \ldots B_k$ covering A exist, with $V(M_i) \subseteq R(B_i)$. But this means that the $M_i \cup B_i$ are matching forests covering $E \cup A$. □

Note that the number $\Delta + \alpha$ occuring in the above theorem is best possible: in the mixed graph on three vertices, with three undirected edges (between all possible pairs of vertices) and six directed edges (between all possible ordered pairs of vertices), $\Delta = 4$, and $\alpha = 1$, but the edge set can not be covered by four matching forests, since any matching forest in this graph contains at most two edges.

By a well-known Theorem of Kőnig, the edge set of a bipartite graph G can be covered by $\Delta(G)$ matchings. This gives a strengthening of Vizing's Theorem in the bipartite case. A corresponding strengthening of Theorem 3.1 can be obtained in the case of a (suitably defined) bipartite mixed graph.

A mixed graph $G = (V, E \cup A)$ is said to be *bipartite*, if the vertex set can be partitioned into two sets S, T such that every undirected edge has one end in S and the other in T (i.e. (V, E) is a bipartite graph with bipartition $\{S, T\}$), whereas no directed edge has one end in S and the other in T, (i.e. $A = \gamma_A(S) \cup \gamma_A(T)$). The following theorem was proved in [6]. It is a common generalization of Kőnig's Theorem and Corollary 2.3.

Theorem 3.2 *Let $G = (V, E \cup A)$ be a bipartite mixed graph. Then G can be covered by k matching forests if and only if*

$$d^{head}(v) \leq k \quad \forall v \in V$$

$$\gamma(X) \leq k(|X| - 1) \quad \forall X \subseteq S \text{ or } T$$

□

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