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Optimal maintenance strategies for systems with partial repair options and without assuming bounded costs

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Abstract

We study a repairable system with Markovian deterioration and partial repair options, carried out at fixed times $n = 1, 2, ...$ and look for optimal strategies under certain conditions. Two optimality criteria are considered: expected discounted cost and long-run average cost. Douer and Yechiali found conditions under which a policy in the class of generalized control limit policies is optimal. In this paper conditions are found under which an optimal policy is a control-limit policy. We explicitly explain how to derive this optimal policy; numerical examples are given, too.
As in the book of Hernandez and Lerma, we are interested in the case of possibly unbounded costs.

Keywords and phrases: reliability, availability, maintenance, inspection, control-limit policy, partial repair, discounted cost, average cost, unbounded cost.

Subject Classification: 93E20, 90B25

1 Introduction

Many authors have considered stochastically deteriorating repairable systems. We are interested in models whose status can only be detected by inspection. This inspection is carried out at fixed times $n = 1, 2, ...$ In most of the models studied, a decision has to be made either to *replace* the system immediately by a new one or to *do nothing* after inspection. The most famous models of this type are those by Derman [5] and Ross [16]. The state of the system inspected at time $n \in \mathbb{Z}^+$ is an element of $\{0, ..., N\}$ in Derman [5] and $\mathbb{Z}^+$ in Ross [16]. State 0 stands for a new system, and the higher the state, the worse the system. The action space is finite in both models. There is a cost or reward function which is dependent of the system state and level of repair. The deterioration of the system during the time interval $(n, n + 1]$ is Markovian and depends on the state after the nth repair.

Some papers deal with *general repair* in the sense of partial repair options. After undergoing such a repair the state of the system is somewhere between the state of the system before the repair and the state of a new system. A general repair system with Markovian deterioration and a finite number of repair stages was first studied by Klein [11]. In his model, the decision-maker decides on the number of periods in which the system will not be checked, thus enabling money to be saved. Klein presented a linear program giving the strategy which minimizes the average cost. Stadje and Zuckerman [18] also extended the models of Derman and Ross. They used an reward function, a cost function and also a finite number of repair stages. They found conditions under which the optimal strategy is of the form (j, l) with $j < l$, meaning the system goes to state j as soon as a state higher than $l-1$ is observed. Examples show that in some cases it is better to carry out a general repair which is not a replacement. Under the condition that the system may deteriorate one step only or fail during a time period, Lesanovsky computed the strategy (j, l) minimizing the discounted cost (with Kasumu, R. A. [9]) and minimizing the average cost [12].

In another interesting paper by Stadje and Zuckerman [17], general repair is also allowed. Some structural characteristics of the value function are proved for a system with a random lifetime. Additional conditions are found under which a strategy using complete repair is optimal. A system in which replacement and general repair are mixed is the system studied by Kijima at al [10]. Here the system is periodically replaced by a new one at scheduled times kT and probably imperfectly repaired after a failure. Kijima was looking for an optimal value of T and a repair strategy minimizing the expected long run cost per unit time.

The model of Douer and Yechiali [6] has some similarities to our model. They use a finite state space $\{0, 1, \ldots, N\}$, an expected operating cost $r_k \geq 0$ for the next time unit of a system which is in state k after repair and repair costs $c_{ik} \geq 0$ if the state is changed from state i to state k by repair. Hence a general repair is allowed. They found conditions under which there are strategies belonging to the class of *generalized control limit policies* that optimize the discounted and the average costs. A *control-limit policy* used by Derman [5] or Ross [16] is a policy replacing a system if and only if its state is larger than a certain value $k \in \mathbb{N}$. A generalized control-limit policy is a policy which repairs the system in some certain way if its state is larger than a value $k \in \mathbb{N}$. Related papers dealing with these control-limit policies are Cho and Parlar [3], Jensen [8], Parlar and Perry [13], Perry and Posner [14] and Perry [15]. A survey of articles concerning *maintenance* until 1989 is written by Valdes-Florez and Feldman [19].

A repairable system with finite state space $\{0, 1, ..., N\}$ and general repair can also be identified as a special $N-$component replacement system: the state of the system reveals the number of failed components. Since a general repair means perhaps not replacing every failed component, such a multi-component system must be a system with *grouping corrective maintenance* in the terminology of Dekker and Wildeman [4]. Such a model with similarities to our own model is considered by Assaf and Shantikumar [1] in their Problem II. There, however, the time of the next inspection is not fixed: it is also a decision variable. The cost of complete repair of n of the failed machines is linear in n and Assam and Shantikumar consider the cost of production loss for failed machines. They showed that a control-limit policy $f(n, t_0, ..., t_n)$, with threshold n and t_i being the waiting time until the next inspection if i machines have failed ($t_{i+1} \leq t_i$) optimizes the expected cost per unit time over an infinite horizon. For further information on multi-component systems we refer to Cho and Parlar's survey [3].

1.1 The Model

We consider a system which is inspected at discrete time instants $n \in I\!N$ and is classified by an element i_n of the state space $I = Z\!\!Z^+$ or $I = \{0, 1, ..., N\}$, $N \in Z\!\!Z^+$. If $I = Z\!\!Z^+$ we set $N = \infty$. X_n^- will denote the state of the system just before time n. At time n a repair action $A_n \in \{0, 1, ..., X_n^-\}$ is chosen which immediately improves the state of the system to $X_n = X_n^- - A_n$. The random variable X_0 is defined as the initial state. The length of time required for inspection and repair is negligible. The change from state X_n to X_{n+1}^- is Markovian. If we do not repair, the stochastic process $(X_n)_{n \in Z\!\!Z^+}$ forms a Markov chain in discrete time with transition probabilities $p_{ij} := P\left(X_{n+1} = j | X_n = i\right),\ i, j \in I$. A repair without improving the system is called *minimal repair*. A minimal repair process, a process where minimal repair is used only, is described in detail at Aven and Jensen [2]. Maximum repair means complete repair or replacement by a new system, so the state after repair is zero. In this model the state is not the system age but describes the degree of deterioration.
As time passes by, the system deteriorates, so the probabilities p_{ij} with $j < i$ are zero. In the nth interval $(n, n+1]$, $n \in I\!N$, there is a manufacturing cost $r(i)$ if the state after the nth repair is i, and the cost of repair will be $d(j, \delta(j))$ if the state is j before repair and δ is the strategy giving the amount of repair (only depending on the present state). The cost of repair depending on the first component might also be interpreted not as real repair cost but as cost of production loss for *bad* machines. The cost of repair (energy, personnel, etc) are paid additionally after each period together with the cost of manufacturing. This is important if cost are discounted.

The situation in the nth period is shown in the following table with $i,\ a,\ j\ \in I$, $\max\{i, a\} \leq j$. Also, we define $q_j^\delta(a)$ as the probability $P\left(\delta(j) = a\right)$ for all $j, a \in I$. The strategy δ may be identified with the sequence $(q_j^\delta)_{j \in I}$ of functions on I. The space of all admissible strategies is $\Pi = \left\{\delta | q_j^\delta \text{ is a probability measure on } \{0, ..., j\}\right\}$.

time	n		$(n+1)-$		$n+1$
random variable	X_n		X_{n+1}^-		X_{n+1}
state	i	$\longrightarrow$	j	$\stackrel{A_{n+1}=a}{\longrightarrow}$	$j-a$
probability		p_{ij}		$q_j^\delta(a)$	
cost			$r(i)$		$d(j,a)$
type of cost		manuf. cost		repair cost	

The random variables X_n and X_n^- depend on the strategy δ used, but we will not use the notations X_n^δ or $X_n^{-,\delta}$ in this paper. Note that for every state i and j the following identity holds:

$$P_\delta\left(X_{n+1}=j|X_0,A_1,X_1,A_2,\cdots,A_{n-1},X_{n-1},A_n,X_n=i\right)=\sum_{k=j}^{N}p_{ik}q_k^\delta(k-j), \qquad (1)$$

because at time $(n+1)$ the machine is in some state k which must be larger than or equal to i because of deterioration and also larger than or equal to j because the machine can't become worse after repair.

2 The Discounted Case

The cost considered in this section are discounted by some factor $\alpha \in (0,1)$. We assume throughout that the cost of the nth interval are polynomially bounded in n, that is, we assume that for all $i \in I$ there exist constants $B_i \in I\!R$ and $\kappa_i \in I\!N$, such that

$$\sum_{j=i}^{N}p_{ij}^{(n)}\left(\max_{\{0\le k_1\le j\}}\{|r(k_1)|\}+\max_{\{0\le k_3\le k_2\le j\}}\{|d(k_2,k_3)|\}\right)<B_i n^{\kappa_i}, \quad n\in Z\!\!Z^+, \qquad (2)$$

where $(p_{ij}^{(n)})$ are the elements of the matrice product $P^{(n)}$. We define this condition as **condition A**. We also assume the validity of **Markovian deterioration** (MD), that is:

the term $\sum_{j=k}^{N}p_{ij}$ is non-decreasing in $i\in I$ for all $k\in I$. (3)

MD means the probability $P\left(X_{n+1}^-\ge k\,\middle|\,X_n=i\right)$ is non-decreasing in $i\in I$ for fixed $k\in I$. The expected discounted cost function using strategy $\delta\in\Pi$ and discount factor $\alpha\in(0,1)$ is given by

$$V_{\delta,\alpha}:\ I\to I\!R \quad \text{with} \quad V_{\delta,\alpha}(i)=E_\delta\left(\sum_{n=0}^{\infty}\alpha^n c_\delta(X_n)\,\middle|\,X_0=i\right), \qquad (4)$$

where $c_\delta(X_n)=r(X_n)+d(X_n,\,\delta(X_n))$ are the cost of the $(n+1)^{th}$ interval if the state after the nth repair is X_n and the policy chosen is δ. The expected cost of the nth period is given by

$$E_\delta(c_\delta(X_n))=r(X_n)+\sum_{j=0}^{N}P\left(X_{n+1}^-=j|X_n\right)\sum_{a=0}^{j}d(j,a)q_j^\delta(a). \qquad (5)$$

Thus we can write

$$V_{\delta,\alpha}(i)=E_\delta\left(\sum_{n=0}^{\infty}\alpha^n\left(r(X_n)+\sum_{j=0}^{N}P\left(X_{n+1}^-=j|X_n\right)\sum_{a=0}^{j}d(j,a)q_j^\delta(a)\right)\,\middle|\,X_0=i\right). \qquad (6)$$

The next theorem guarantees the existence of $V_{\delta,\alpha}$. The following lemma is required for its proof:

Lemma 1 *The process* $(X_n^{\delta_\infty})$, *resp.* $(X_n^{\delta_\infty,-})$, *is stochastically larger than the process* (X_n^{δ}), *resp.* $(X_n^{\delta,-})$, *for every strategy* $\delta \in \Pi$*:*

The identity $X_0^{\delta_\infty} = X_0^{\delta}$ *yields to* $X_n^{\delta_\infty} \overset{st}{\geq} X_n^{\delta}\ \forall n \in I\!N$ *and* $X_n^{\delta_\infty,-} \overset{st}{\geq} X_n^{\delta,-}\ \forall n \in I\!N$.

We **prove** this lemma by induction using MD, (3):

$n = 1:$ $\quad X_0^{\delta_\infty} = X_0^{\delta} \Rightarrow X_1^{\delta_\infty,-} \overset{D}{=} X_1^{\delta,-}$. $X_1^{\delta_\infty} = X_1^{\delta_\infty,-}$ and $X_1^{\delta} \leq X_1^{\delta,-}$ yield $X_1^{\delta_\infty} \overset{st}{\geq} X_1^{\delta}$ and so $X_1^{\delta_\infty,-} \overset{st}{\geq} X_1^{\delta,-}$.

$n \to n+1:$ Defining $f_k(i) := \sum_{j=k}^{N} p_{ij},\ i, k \in I,$ the subsequent inequality holds:

$$\begin{aligned}
P(X_{n+1}^{\delta_\infty,-} \geq k) &= \sum_{i=0}^{N} P\left(X_{n+1}^{\delta_\infty,-} \geq k \,\middle|\, X_n^{\delta_\infty} = i\right) P\left(X_n^{\delta_\infty} = i\right) \\
&= \sum_{i=0}^{N} \left(\sum_{j=k}^{N} p_{ij}\right) P\left(X_n^{\delta_\infty} = i\right) = E\left(f_k\left(X_n^{\delta_\infty}\right)\right) \\
&\overset{\text{MD}}{\geq} E\left(f_k\left(X_n^{\delta}\right)\right) = P\left(X_{n+1}^{\delta,-} \geq k\right).
\end{aligned}$$

As a consequence $X_{n+1}^{\delta_\infty,-} \overset{st}{\geq} X_{n+1}^{\delta,-}$ and thus $X_{n+1}^{\delta_\infty} \overset{st}{\geq} X_{n+1}^{\delta}$.

So the induction is completed. ■

In what follows we let $\tilde{B}_i^{\alpha} := B_i \sum_{n=1}^{\infty} \alpha^{n-1} n^{\kappa_i}, i \in I, \alpha \in (0,1).$ (7)

Theorem 1 *For fixed* $\alpha \in (0,1)$ *and* $i \in I$ *the set* $\{V_{\delta,\alpha}(i),\ \delta \in \Pi\}$ *is bounded, namely,*

$$|V_{\delta,\alpha}(i)| \leq |r(i)| + \tilde{B}_i^{\alpha}. \tag{8}$$

Proof: $|V_{\delta,\alpha}(i)| = \left| E_\delta \left(\sum_{n=0}^{\infty} \alpha^n (r(X_n) + d(X_{n+1}^-, \delta(X_{n+1}^-)) \,\middle|\, X_0 = i \right) \right|$

$$\begin{aligned}
&\leq |r(i)| + E_\delta \left(\sum_{n=1}^{\infty} \alpha^{n-1} \left(\max_{\{0 \leq k_3 \leq k_2 \leq X_n^-\}} \{|d(k_2,k_3)|\} + \alpha \max_{\{0 \leq k_1 \leq X_n\}} \{|r(k_1)|\} \,\middle|\, X_0 = i \right) \right) \\
&\overset{\text{lemma 1}}{\leq} |r(i)| + E_{\delta_\infty} \left(\sum_{n=1}^{\infty} \alpha^{n-1} \left(\max_{\{0 \leq k_3 \leq k_2 \leq X_n^-\}} \{|d(k_2,k_3)|\} + \alpha \max_{\{0 \leq k_1 \leq X_n\}} \{|r(k_1)|\} \,\middle|\, X_0 = i \right) \right) \\
&\leq |r(i)| + \sum_{n=1}^{\infty} \alpha^{n-1} E_{\delta_\infty} \left(\max_{\{0 \leq k_3 \leq k_2 \leq X_n^-\}} \{|d(k_2,k_3)|\} + \alpha \max_{\{0 \leq k_1 \leq X_n\}} \{|r(k_1)|\} \,\middle|\, X_0 = i \right) \\
&\leq |r(i)| + \sum_{n=1}^{\infty} \alpha^{n-1} \sum_{j=i}^{N} p_{ij}^{(n)} \left(\max_{\{0 \leq k_3 \leq k_2 \leq j\}} \{|d(k_2,k_3)|\} + \max_{\{0 \leq k_1 \leq j\}} \{|r(k_1)|\} \right) \\
&\leq |r(i)| + B_i \sum_{n=1}^{\infty} \alpha^{n-1} (n+1)^{\kappa_i} = |r(i)| + \tilde{B}_i^{\alpha}. \quad ■
\end{aligned}$$

Furthermore the cost functions are assumed to be bounded from below, so without loss of generality we only consider non-negative cost functions. Sometimes we use **condition (B)**:

$$\lim_{n\to\infty} \alpha^n \sum_{j=i}^{N} p_{ij}^{(n)} \left(\max_{k\in\{0,\dots,j\}} \{r(k)\} + \max_{k\in\{0,\dots,j\}} \{\tilde{B}_k^\alpha\} \right) = 0. \tag{9}$$

While condition A - which was defined in (2) - is assumed to hold throughout the paper we always state explicitly when condition B is required.

Lemma 2

1. *If the cost functions are bounded there exists a sequence* $(\tilde{B}_k^\alpha)$ *for which condition B is fulfilled.*
2. *Condition B yields* $\lim_{n\to\infty} \alpha^n E_\delta \left(V_{\delta,\alpha}(X_n|X_0 = i)\right) = 0$ *for every strategy* $\delta \in \Pi$ *chosen.*

Proof: The first part is obvious and the second part is valid since

$$\begin{aligned}
&\lim_{n\to\infty} \alpha^n E_\delta(V_{\delta,\alpha}(X_n|X_0 = i)) \overset{\text{theorem 1}}{\leq} \lim_{n\to\infty} \alpha^n E_\delta \left(r(X_n) + \tilde{B}_{X_n}^\alpha | X_0 = i \right) \\
\leq\ & \lim_{n\to\infty} \alpha^n E_{\delta_\infty} \left(\max_{\{0\leq k\leq X_n\}} \{r(k)\} + \max_{0\leq k\leq X_n} \left\{ \tilde{B}_k^\alpha \right\} \middle| X_0 = i \right) \\
\leq\ & \lim_{n\to\infty} \alpha^n \sum_{j=i}^{N} p_{ij}^{(n)} \left(\max_{k\in\{0,\dots,j\}} \{r(k)\} + \max_{k\in\{0,\dots,j\}} \{\tilde{B}_k^\alpha\} \right) = 0. \quad \blacksquare
\end{aligned}$$

Using the definitions

$$\begin{aligned}
\mathcal{U} &:= \left\{ u : I \to I\!R \middle| \lim_{n\to\infty} \alpha^n \sum_{j=i}^{N} p_{ij}^{(n)} \max_{\{0\leq k\leq j\}} \{|u(k)|\} = 0\ \forall i \in I \right\}, \\
\mathcal{V} &:= \left\{ \lim_{n\to\infty} V_{\delta_n,\alpha} \middle| \ (\delta_n) \subset \Pi,\ \lim_{n\to\infty} V_{\delta_n,\alpha} \text{ exists},\ \alpha \in (0,1) \right\},
\end{aligned}$$

the second part of lemma 2 yields the subsequent corrollary:

Corollary 1 *If condition B holds* $\mathcal{V}$ *is a subset of* $\mathcal{U}$.

Proof: $u \in \mathcal{U}$ means that $\lim_{n\to\infty} \alpha^n E(|u(X_n)|) = 0$. Hence every function $V_{\delta,\alpha} \in \mathcal{U}$ and also every function $\lim_{n\to\infty} V_{\delta_n,\alpha}$, if it exists, is a member of $\mathcal{U}$. $\blacksquare$

The operator $T_\alpha : \quad \{g : I \to I\!R^+\} \to \{g : I \to I\!R^+\}$ is defined as:

$$(T_\alpha(u))(i) := r(i) + \sum_{j=i}^{N} p_{ij} \min_{a\in\{0,\dots,j\}} \{d(j,a) + \alpha u(j-a)\}. \tag{10}$$

The following theorem gives standard results:

Theorem 2 *Apart from the first part of this theorem condition B has to be fulfilled:*
(i) The optimal value function $V_\alpha(i) := \inf_{\delta\in\Pi} V_{\delta,\alpha}(i)$ *satisfies the*
optimality equation:

$$V_\alpha(i) = r(i) + \sum_{j=i}^{N} p_{ij} \min_{a\in\{0,\dots,j\}} \{d(j,a) + \alpha V_\alpha(j-a)\}. \tag{11}$$

(ii) **Existence** *of an optimal strategy:*
let δ_α be the stationary and deterministic strategy satisfying

$$d(j,\delta_\alpha(j)) + \alpha V_\alpha(j-\delta_\alpha(j)) = \min_{a\in\{0,\dots,j\}} \{d(j,a)+\alpha V_\alpha(j-a)\} \quad \text{for all } \delta \in \Pi. \tag{12}$$

Then δ_α is optimal, i.e. $V_{\delta_\alpha,\alpha} = V_\alpha$. Thus an optimal strategy always exists.
(iii) **Uniqueness** *of V_α:*
If condition B holds, V_α is the unique solution of the optimality equation in $\mathcal{V}$.
(iv) **Policy iteration***:*
If for some initial strategy $\delta_0 \in \Pi$ and the strategies $(\delta_n)_{n\in\mathbb{N}} \subset \Pi$, recursively defined by

$$d(j,\delta^n(j)) + \alpha V_{\delta^{n-1},\alpha}(j-\delta^n(j)) = \min_{a\in\{0,\dots,j\}} \{d(j,a)+\alpha V_{\delta^{n-1},\alpha}(j-a)\}, \tag{13}$$

the equality $\delta_{n_0} = \delta_{n_0+1}$ holds for a $n_0 \in \mathbb{N}$, then δ_n is optimal. Moreover, under condition B the following identity holds: $\lim_{n\to\infty} V_{\delta_n,\alpha} = V_\alpha$.

The **proof** of (i) is standard, to prove (ii) we get similar to the proof of theorem II.2.2 of Ross [16] - using (4) - the following identity:

$$V_\alpha(i) = \lim_{n\to\infty} E_\delta\left(\sum_{m=0}^{n} \alpha^m c_\delta(X_m)\middle| X_0 = i\right) + \alpha^n E_{\delta_\alpha}\left(V_\alpha(X_n)|\, X_0=i\right). \tag{14}$$

Condition B yields to the identity $V_\alpha(i) = V_{\delta_\alpha,\alpha}(i)$. Now we first present the standard proof of (iii) if the cost functions are bounded:

$$T_\alpha V_\alpha(i) = r(i) + \sum_{j=i}^{N} p_{ij} \min_{a\in\{0,\dots,j\}} \{d(j,a)+\alpha V_\alpha(j-a)\}. \tag{15}$$

Using the optimality equation we have $T_\alpha V_\alpha(i) = V_\alpha(i)$. So V_α is a fixed point of T_α. It is standard to prove that T_α is contracting. By Banach's fixed point theorem, V_α is the only fixed point of T_α and thus the only solution of the optimality equation.
If the cost functions are probably unbounded, but of course the conditions A and B are valid, we take the proof of Lemma 4.2.7 of Hernandez-Lerma, Lasserre [7] using our operator T_α instead of T. For every function $u \in \mathcal{U}$ the identity $u = T_\alpha u$ yields to the identity $u = V_\alpha$. Thus corollary 1 yields that V_α is the only fixed point of the operator T_α in $\mathcal{V}$.
For the proof of (iv) see theorem 4.4.1 (b) of Hernandez-Lerma, Lasserre [7], since corollary 1 yields to their equation (4.4.7). ■

¿From the third part of this theorem we know that, if condition B is valid, a strategy $\delta \in \Pi$ will minimize the $\alpha-$discounted cost if the function $V_{\delta,\alpha}$ fulfills the optimality equation.

To find an optimal strategy that can be calculated, we make the following reasonable monotonicity conditions:

Condition (1) *$r(i)$ is non-decreasing in $i \in I$.*
Condition (2) *MD: $\sum_{j=k}^{N} p_{ij}$ is non-decreasing in $i \in I$ for fixed $k \in I$.*
Condition (3) *$d(i,j)$ is non-decreasing in $i \in I$ and in $j \in \{0,\dots,i\}$.*

Under conditions (1), (2) and (3) conditions A and B reduce to:

condition A: $\forall i \in I\ \exists B_i \in I\!R,\ \kappa_i \in I\!N: \quad \sum_{j=i}^{N} p_{ij}^{(n)} \left(r(j) + p_{jk} d(k,k) \right) < B_i n^{k_i}\ \forall n \in I\!N.$

Without loss of generality the sequence (B_i) as well as the sequence (κ_i) and thus also the sequence $(\tilde{B}_i^{\alpha})$ is non-decreasing under the conditions (1), ..., (3). Hence

condition B: $\lim_{n\to\infty} \alpha^n \sum_{j=i}^{n} p_{ij}^{(n)} \left(r(j) + \tilde{B}_j^{\alpha} \right) = 0.$

Definition 1 *A stationary strategy δ with $\delta(i) \in \{0, i\}\ \forall i \in I$ is called a bang-bang strategy. Π^b is the set of all bang-bang-strategies and $V_\alpha^b(i) := \inf_{\delta\in\Pi^b}\{V_{\delta,\alpha}(i)\}$ $\forall i \in I$.*

After replacing the expressions $\min_{\{0,\dots,k\}}$ by $\min_{\{0,k\}}$ and V_α by V_α^b in (11) and (12) we obtain the following result:

Lemma 3 *(i) The following identity holds:*

$$V_\alpha^b(i) = r(i) + \sum_{j=i}^{N} p_{ij} \min\left\{ d(j,0) + \alpha V_\alpha^b(j),\ d(j,j) + \alpha V_\alpha^b(0) \right\}. \tag{16}$$

(ii) If condition B holds and if the subsequent equality is valid:

$$d(j, \delta_\alpha(j)) + \alpha V_\alpha^b(j - \delta_\alpha(j)) = \min\left\{ d(j,0) + \alpha V_\alpha^b(j),\ d(j,j) + \alpha V_\alpha^b(0) \right\} \tag{17}$$

then $V_{\delta_\alpha,\alpha}$ is equal to V_α^b. ■

Theorem 3 *$V_\alpha(i)$ and $V_\alpha^b(i)$ are non-decreasing in $i \in I$ for all $\alpha \in (0,1)$ if the conditions (1), (2), (3) are valid.*

Proof: Let $\psi(i,\alpha,N)$ be the discounted cost up to time N using the optimal strategy,

$$\text{that is} \qquad \psi(i,\alpha,N) = \inf_{\delta\in\Pi} E_\delta \left(\sum_{n=0}^{N} \alpha^n c_\delta(X_n) \middle| X_0 = i \right). \tag{18}$$

We prove by induction on N that $\psi(i,\alpha,N)$ is non-decreasing in $i \in I$ for all $\alpha \in (0,1)$:

$N = 0:$ $\quad \psi(i,\alpha,0) = r(i)$ and thus, by condition 1, non-decreasing in $i \in I$.

$N \to N+1:$ We have $\psi(i,\alpha,N{+}1) = r(i) + \sum_{j=i}^{N} p_{ij} \min_{a\in\{0,\dots,j\}} \{d(j,a) + \alpha\psi(j-a,\alpha,N)\}$,

$$\begin{aligned}
&\min_{a\in\{0,\dots,j\}} \{d(j,a) + \alpha\psi(j-a,\alpha,N)\} \\
&= \min\{d(j,0) + \alpha\psi(j,\alpha,N), \dots, d(j,j) + \alpha\psi(0,\alpha,N)\} \\
&\le \min\{d(j{+}1,0) + \alpha\psi(j{+}1,\alpha,N), \dots, d(j{+}1,j) + \alpha\psi(1,\alpha,N), d(j{+}1,j{+}1) + \alpha\psi(0,\alpha,N)\} \\
&= \min_{a\in\{0,\dots,j+1\}} \{d(j{+}1,a) + \alpha\psi(j{+}1-a,\alpha,N)\}.
\end{aligned}$$

The inequality is valid because the i^{th} element in the left-hand set is not smaller than the i^{th} element in the right-hand set and the last element in the set above (the

$(j+2)^{th}$) is not smaller than the last element in the set below (the $(j+1)^{th}$) either. The induction is completed using the monotonicity of $r(i)$ and the fact that $\sum_{j=k}^{N} p_{ij} f(j)$ is non-decreasing for all non-decreasing f (Ross, [16], p. 37). Take $f(j) = \min_{a\in\{0,...,j\}}\{d(j,a) + \alpha\psi(j-a,\alpha,N)\}$.
Clearly $\lim_{N\to\infty}\psi(i,\alpha,N)$ solves the optimality equation and thus is equal to $V_\alpha(i,\alpha)$ by theorem 2(3). Hence, $i \to V_\alpha(i)$ is non-decreasing for all $\alpha \in (0,1)$. The proof for V_α^b is identical after replacing the expression $\min_{a\in\{0,...,k\}}$ again by $\min_{a\in\{0,k\}}$. ■

The following observation is important:

Theorem 4 (i) *Under the conditions* $(1),(2),(3)$ *and* B *an optimal strategy* δ_α^* *in the subclass of the bang-bang strategies is given by:*

$$\delta_\alpha^*(j) = \begin{cases} 0 & if \quad \alpha(V_\alpha^b(i) - V_\alpha^b(0)) \le d(i,i) - d(i,0), \\ j & if \quad \alpha(V_\alpha^b(i) - V_\alpha^b(0)) > d(i,i) - d(i,0). \end{cases}$$

(ii) *Especially if* $d(i,i) - d(i,0)$ *is non-increasing in* $i \in I$, *then the following strategies are optimal in the subclass of bang-bang strategies:*

$$\delta_k^*(j) = \begin{cases} 0 & j < k, \\ j & j \ge k, \end{cases} \quad k \in \{j_\alpha^*,\, j_\alpha^*+1, ..., i_\alpha^*\},$$

with $i_\alpha^* = \min A_\alpha,\ A_\alpha := \{j \in I : \alpha(V_\alpha^b(j) - V_\alpha^b(0)) > d(j,j) - d(j,0)\}$
and $j_\alpha^* = \min \tilde{A}_\alpha,\ \tilde{A}_\alpha := \{j \in I : \alpha(V_\alpha^b(j) - V_\alpha^b(0)) \ge d(j,j) - d(j,0)\}$
($i_\alpha^* = \infty$ *if* $A_\alpha = \emptyset$ *and* $j_\alpha^* = \infty$ *if* $\tilde{A}_\alpha = \emptyset$).
We call these control-limit-policies bang-bang-strategy with threshold k.

Proof: According to theorem 2(2) we prove that δ_α^* is optimal in the subclass of bang-bang-strategies iff $d(i,\delta_\alpha^*(i)) + \alpha V_\alpha^b(i-\delta_\alpha^*(i)) = \min\left\{d(i,0) + \alpha V_\alpha^b(i),\ d(i,i) + \alpha V_\alpha^b(0)\right\}$ $\forall i \in I$. So a repair action i [0] in state i is optimal iff:

$$\begin{aligned} d(i,i) + \alpha V_\alpha^b(0) \le d(i,0) + \alpha V_\alpha^b(i) \qquad & \left[d(i,i) + \alpha V_\alpha^b(0) \ge d(i,0) + \alpha V_\alpha^b(i)\right] \\ \Leftrightarrow \alpha(V_\alpha^b(i) - V_\alpha^b(0)) \ge d(i,i) - d(i,0) \qquad & \left[\alpha(V_\alpha^b(i) - V_\alpha^b(0)) \le d(i,i) - d(i,0)\right]. \end{aligned}$$

Since $V_\alpha^b(i)$ is non-decreasing and $d(i,i) - d(i,0)$ is non-increasing the theorem follows immediately. ■

Now we compare our model to the model of Douer and Yechiali [6]. They use the same probability functions, and their cost functions are related to our cost-functions via

$$r_i := r(i), \qquad c_{ik} := d(i,i-k) \quad \forall i \in \{0,...,N\},\ k \in \{0,...,i\}.$$

In their theorem 2.1 they prove, that under the conditions a) r_i is non-decreasing in i, b) c_{ik} is non-decreasing in i and $c_{ii} = 0$, c) MD d) a system being in state N has to be repaired and e) $c_{ik} - r_i$ is non-decreasing in i, a generalized control limit policy optimizes the discounted costs. Since our conditions (1), ..., (3) imply the conditions a),...,c), we get

Theorem 5 *If the state space* I *is finite, a repair-action* $a = 0$ *is not allowed in state* N , $d(i,0)$ *equal zero for all* $i \in I$ *and the function* $d(i,i-k) - r(i)$ *is non-decreasing in* $i \in \{k,...,N\}$ *for all* $k \in I$ *then a generalized control limit policy is optimal.*

Now we will give conditions under which a bang-bang strategy with threshold is optimal:
A function $f: I \rightarrow I\!R$ is called concave if $f(i+1) - f(i)$ is non-increasing in i, $i \in I$. Conditions (1), (2) and (3) (monotonicity of r und d and MD) guarantee the existence of an optimal bang-bang strategy with threshold, being a special kind of generalized control limit policy, under the following **conditions**:

Condition (4): *$d(i,i) - d(i,0)$ is non-increasing in $i \in I$.*
Condition (5): *$r(i)$ is concave in $i \in I$.*
Condition (6): *$\sum_{j=k}^{N}(p_{i+1,j} - p_{ij})$ is non-increasing in $i \in I$ $\forall k \in I$.*
Condition (7): *$d(i,a)$ is concave in $a \in \{0,...,i\}$ for all i in I.*

We already used condition (4) in the last theorem. In addition to the assumptions $(1),...,(3)$ which are usually valid in practice, we now impose strong conditions on the system, especially condition (4), according to which the difference between the cost of maximum repair and minimum repair is non-increasing in the state variable. Since this condition may not hold in certain 'real world' applications, we give an alternative condition in theorem 7. The sixth condition means that the increase of the probability of reaching state k or a higher one from state i during a period decreases in i. Condition (7) is e.g. satisfied for $d(i,a) = C_0 + C_1 1_{\{a>0\}}(a) + C_2(a)$, $C_0, C_1, C_2 \in I\!R^+$; it requires that the marginal repair cost is decreasing, which is a reasonable assumption.
Next we present a basic lemma:

Lemma 4 *Condition (6) is valid iff $\sum_{j=i}^{N}(p_{i+1,j} - p_{ij})f(j)$ is non-increasing in i, $i \in I$ for all non-decreasing $f: I \rightarrow I\!R$.*

'$\Leftarrow$' For the functions $f_k(i) = 1_{\{i \geq k\}}(i)$, $k \in I$, on I it follows that
$\sum_{j=0}^{N}(p_{i+1,j} - p_{ij})f_k(j) = \sum_{j=k}^{N}(p_{i+1,j} - p_{ij})$ is non-increasing in $i \in I$ for every $k \in I$.
'$\Rightarrow$' We can write $f = \sum_{k=0}^{N} c_k f_k - |f(0)|$ with $c_0 = f(0) + |f(0)|$, $c_k = f(k) - f(k-1) \geq 0$. Therefore

$$\begin{aligned}\sum_{j=0}^{N}(p_{i+1,j} - p_{ij})f(j) &= \sum_{j=0}^{N} p_{i+1,j}f(j) - \sum_{j=0}^{N} p_{ij}f(j) \\ &= \sum_{k=0}^{N} c_k \sum_{j=k}^{N} p_{i+1,j} - \sum_{k=0}^{N} c_k \sum_{j=k}^{N} p_{ij} = \sum_{k=0}^{N} c_k \sum_{j=k}^{N}(p_{i+1,j} - p_{ij}). \quad (19)\end{aligned}$$

So $\sum_{j=0}^{N}(p_{i+1,j} - p_{ij})f(j)$ is non-increasing in i. ■

Theorem 6 *If the conditions $(1),...,(7)$ are valid, $V_\alpha(i)$ and $V_\alpha^b(i)$ are concave in $i \in I$ for all $\alpha \in (0,1)$.*

Proof: We have to prove that $V_\alpha(i+1) - V_\alpha(i)$ is non-increasing in i, $i \in I$ for all $\alpha \in (0,1)$:

$$V_\alpha(i+1) - V_\alpha(i) = r(i+1) - r(i) + \sum_{j=0}^{N}(p_{i+1,j} - p_{ij}) \min_{a \in \{0,\dots,j\}}\{d(j,a) + \alpha V_\alpha(j-a)\}.$$

As in the proof of theorem 3 it is seen that $\min_{a\in\{0,\dots,j\}}\{d(j,a) + \alpha V_\alpha(j-a)\}$ is non-decreasing in j. The result follows from assumption (5) and the last lemma. After replacing the expressions $\min_{\{0,\dots,k\}}$ by $\min_{\{0,k\}}$ and V_α by V_α^b above we get again that

V_α^b is concave. ■

Obviously, it follows that if V_α is concave and g is linear then the composition of V_α and g is also concave. We can now conclude that under the conditions (1), ..., (7) $d(i,a)+\alpha V_\alpha(i-a)$ is, for every fixed i, a concave function of $a \in \{0,...,i\}$ and therefore attains its minimum at one of the boundary points 0 and i. Thus there is an optimal strategy δ with $\delta(i) \in \{0,i\}$. We have proved

Theorem 7 *Under the conditions* (1), ..., (7) *and* B, *the bang-bang-strategies* δ_k^*, $k \in \{j_\alpha^*, j_\alpha^*+1, ..., i_\alpha^*\}$ *defined in theorem 4 are optimal. This is also true if condition* (4) *is replaced by condition*
(4a) : *There are* $\rho \in I\!R$ *and* $i_0 \in I$ *such that*

$$d(i,i) - d(i,0) = \rho \cdot \alpha \cdot (i-1) + d(1,1) - d(1,0), \ i \in \{1,...,N\} \tag{20}$$

that is, $d(i,i) - d(i,0)$ *linear in* $i \in I$ *and*

$$r(i) - r(0) > \rho \cdot (i-1) + \frac{1}{\alpha}(d(1,1) - d(1,0)), \ i \in \{i_0,...,N\}. \tag{21}$$

Proof: It remains to prove the assertion under condition 4a. By $(4a)$,

$$\begin{aligned} \alpha\,(V_\alpha(i) - V_\alpha(0)) &\geq \alpha(r(i) - r(0)) \\ &> \alpha\rho \cdot (i-1) + (d(1,1) - d(1,0)) = d(i,i) - d(i,0) \ \forall i \geq i_0. \end{aligned}$$

Thus, the concave function $\alpha(V_\alpha(i) - V_\alpha(0))$ crosses the straight line $d(i,i) - d(i,0)$ at most once. ■

Hence, if the function $d(i,i) - d(i,0)$ is linear on I, it need not be non-increasing to get a concave non-increasing value function.
If the state space is $\{0,...,N\}$, the threshold i_α^* can be computed by solving the following equation system of range $N+1$ for every threshold $i^* \in \{0,...,N\}$:

$$V_{\delta_{i^*},\alpha}(i) = r(i) + \sum_{j=i}^{i^*-1} p_{ij}(d(j,0) + V_{\delta_{i^*},\alpha}(j)) + \sum_{j=i^*}^{N} p_{ij}(d(j,j) + V_{\delta_{i^*},\alpha}(0)) \ \forall i \in \{0,...,N\}.$$

The solution is $(V_{\delta_{i^*},\alpha}(0), ..., V_{\delta_{i^*},\alpha}(N))$. The optimal threshold i_α^* is that one fulfilling $V_{\delta_{i_\alpha^*},\alpha}(i) \leq V_{\delta_{i^*},\alpha}(i)$ for every state $i \in \{0,...,N\}$.

3 Average cost

The average cost function is defined by

$$\phi_\delta(i) = \limsup_{m\to\infty} \frac{1}{m+1} E_\delta\left(\sum_{n=0}^{m}\left(r(X_n) + \sum_{j=0}^{N} P\left(X_{n+1}^- = j | X_n\right) d(j, f(j))\right)\middle| X_0 = i\right), \tag{22}$$

$$i \in I \text{ if the mean } E_\delta\left(r(X_n) + \sum_{j=i}^{N} P(X_{n+1}^- = j|X_n) d(j,\delta(j))\right) \text{ exists for all } n \in I\!N.$$

If this mean does not exist for at least one $n \in I\!N$, let $\phi_\delta(i) = \infty$.
In this chapter we want to find an average-cost-optimal strategy, that is, a strategy δ^* satisfying $\phi_{\delta^*}(i) = \inf_{\delta\in\Pi} \phi_\delta(i)$. First we impose the following

AC-condition: The conditions A and $(1),(2),(3)$ of the discounted model are fulfilled, the cost-functions r and d are non-negative and the sum $\sum_{j=i}^{N} p_{ij} d(j,j)$ is finite for every $i \in I$.

This condition is always fulfilled in this chapter.
The following lemma proves the boundness of the function $\alpha \rightarrow (1-\alpha)V_\alpha(0)$ on $[0,1)$:

Lemma 5 *The following inequality holds for all $\alpha \in (0,1)$:*

$$\left(r(0) + \sum_{j=0}^{N} p_{0j} d(j,0) \right) \leq (1-\alpha)V_\alpha(0) \leq \left(r(0) + \sum_{j=0}^{N} p_{0j} d(j,j) \right).$$

Proof: The optimality equation of the discounted model yields to

$$V_\alpha(0) \;\leq\; r(0) + \sum_{j=0}^{N} p_{0j}(d(j,j) + \alpha V_\alpha(0)). \tag{23}$$

Hence $\quad (1-\alpha)V_\alpha(0) \;\leq\; \left(r(0) + \sum_{j=0}^{N} p_{0j} d(j,j) \right).$

The monotonocity of d and V_α yields to

$$V_\alpha(0) \;\geq\; r(0) + \sum_{j=0}^{N} p_{0j} \left(d(j,0) + \alpha V_\alpha(0) \right). \tag{24}$$

Hence $\quad (1-\alpha)V_\alpha(0) \;\geq\; \left(r(0) + \sum_{j=0}^{N} p_{0j} d(j,0) \right).$ ■

Lemma 6 *There exist $g \in I\!R^+$ and $(\alpha_n)_{n=1}^{\infty} \subset (0,1)$ fulfilling the equations $\lim_{n\to\infty} \alpha_n = 1$ and $\lim_{n\to\infty}(1-\alpha_n)V_{\alpha_n}(i) = g$ for all states $i \in I$.*

Proof: Lemma 5 guarantees the existence of a $g \in I\!R^+$ and $(\alpha_n)_{n=1}^{\infty} \subset (0,1)$ fulfilling the equations $\lim_{n\to\infty} \alpha_n = 1$ and $\lim_{n\to\infty}(1-\alpha_n)V_{\alpha_n}(0) = g$. The optimality equation of the discounted model yields to the following inequation:

$$\begin{aligned} |(1-\alpha_n)V_{\alpha_n}(i) - g| &\overset{(11)}{=} (1-\alpha_n)|V_{\alpha_n}(i) - V_{\alpha_n}(0)| \\ &\leq (1-\alpha_n)(r(i) + \sum_{j=i}^{N} p_{ij} d(j,j) + \alpha V_{\alpha_n}(0) - V_{\alpha_n}(0) \\ &\leq (1-\alpha_n)\left(r(i) + \sum_{j=i}^{N} p_{ij} d(j,j) \right) \quad \text{for all } i \in I. \end{aligned} \tag{25}$$

Then

$$|\lim_{n\to\infty}(1-\alpha_n)V_{\alpha_n}(i) - g)| \leq \lim_{n\to\infty}(1-\alpha_n)\left(r(i) + \sum_{j=i}^{N} p_{ij} d(j,j) \right) = 0 \text{ for all } i \in I. \;\blacksquare$$

Apart from the variables defined in this lemma the following variables will also be used in the following theorems and lemmas: Let $g(\alpha) = (1-\alpha)V_\alpha(0)$, $h_\alpha(i) = V_\alpha(i) - V_\alpha(0)$, $h = \liminf_{n\to\infty} h_{\alpha_n}$ and the strategy δ^* is defined by the following identity:

$$d(j,\delta^*(j)) + h(j - \delta^*(j)) = \min_{\{0\leq a\leq j\}} \{d(j,a) + h(j-a)\}. \tag{26}$$

We prove the following lemma by induction:

Lemma 7 *For all $n \in I\!N$ the following identity holds:*

$$ng + h(i) = E_{\delta^*}\left(\sum_{k=0}^{n-1}\left(r(X_k) + d(X_{k+1}^-, \delta^*(X_{k+1}^-))\right)\Big| X_0 = i\right) + E_{\delta^*}(h(X_n)|X_0 = i).$$

Proof: We prove this equation via induction by n:
$n = 1$: The optimality equation (11) yields for $i \in \{0, ..., N\}$ the following identity:

$$g(\alpha) + h_\alpha(i) \quad = \quad r(i) + \sum_{j=i}^{N} p_{ij} \min_{\{0\le a\le j\}} \{d(j,a) + \alpha h_\alpha(j-a)\} \tag{27}$$

which yields to

$$\liminf_{n\to\infty} (g(\alpha_n) + h_{\alpha_n}(i)) \quad = r(i) + \sum_{j=i}^{N} p_{ij} \min_{\{0\le a\le j\}} \left\{d(j,a) + \liminf_{n\to\infty} \alpha_n h_{\alpha_n}(j-a)\right\}.$$

Thus

$$\begin{aligned} g + h(i) \quad &= \quad r(i) + \sum_{j=i}^{N} p_{ij} \min_{\{0\le a\le j\}} \{d(j,a) + h(j-a)\} \\ &= E_{\delta^*}\left(\sum_{k=0}^{0}\left(r(X_k) + d(X_{k+1}^-, \delta^*(X_{k+1}^-))\right)\Big| X_0 = i\right) + E_{\delta^*}(h(X_1)|X_0 = i). \end{aligned}$$

$n-1 \to n:$ $\quad g + h(X_{n-1}) = E_{\delta^*}\left(r(X_{n-1}) + d(X_n^-, \delta^*(X_n^-)\Big| X_{n-1}\right) + E_{\delta^*}(h(X_n)|X_{n-1}),$

so

$$\begin{aligned} E_{\delta^*} h(X_{n-1}|X_0 = i) + g \quad &= \quad E_{\delta^*}\left(r(X_{n-1}) + d(X_n^-, \delta^*(X_n^-))\Big| X_0 = i\right) \\ &\quad + E_{\delta^*}(h(X_n)|X_0 = i). \end{aligned} \tag{28}$$

This equality we will use now:

$$\begin{aligned} ng + h(i) \quad &= \quad (n-1)g + h(i) + g \\ &= \quad E_{\delta^*}\left(\sum_{k=0}^{n-2} r(X_k) + d(X_{k+1}^-, \delta^*(X_{k+1}^-))\Bigg| X_0 = i\right) + E_{\delta^*}\left(h(X_{n-1}|X_0 = i\right) + g \\ &\overset{(28)}{=} \quad E_{\delta^*}\left(\sum_{k=0}^{n-1} r(X_k) + d(X_{k+1}^-, \delta^*(X_{k+1}^-))\Bigg| X_0 = i\right) + E_{\delta^*}\left(h(X_n)|X_0 = i\right). \quad \blacksquare \end{aligned}$$

Theorem 8 *For all $i \in \{0, 1, ..., N\}$ we have $\phi(i) = \phi_{\delta^*}(i) = g$ where $g \equiv \lim_{\alpha\to 1}(1-\alpha)V_\alpha(0)$.*

Proof: The last lemma yields the subsequent identity for the value g:

$$g = \frac{1}{n} E_{\delta^*}\left(\sum_{k=0}^{n-1} r(X_k) + d(X_{k+1}^-, \delta^*(X_{k+1}^-))\Bigg| X_0 = i\right) + \frac{1}{n} E_{\delta^*}(h(X_n)|X_0 = i) - \frac{1}{n} h(i).$$

Thus

$$\begin{aligned} g \quad &\ge \quad \limsup_{n\to\infty} \frac{1}{n} E_{\delta^*}\left(\sum_{k=0}^{n-1} r(X_k) + d(X_{k+1}^-, \delta^*(X_{k+1}^-))\Bigg| X_0 = i\right) = \phi_{\delta^*}(i) \ge \phi(i) \\ &\ge \quad \lim_{\alpha\uparrow 1} (1-\alpha)V_\alpha(0) = g. \end{aligned}$$

The last inequality follows from Hernandez-Lerma [7], Lemma 5.3.1. ■

Corollary 2 *There is a sequence* $(\alpha_n^b) \subset (0,1)$ *which holds* $\lim_{n\to\infty} \alpha_n^b = 1$ *and* $\phi_{\delta_b^*}(i) = \lim_{n\to\infty}(1-\alpha_n^b)V_{\alpha_n^b}(i)$ *for all* $i \in I$

The **proof** is identical to that one of theorem 8 and lemma 6, we just have to exchange the expressions V_α by V_α^b and $\min_{\{0\le a\le j\}}$ by $\min_{\{0,j\}}$ in the theorems and lemmas of this chapter.

Theorem 9 *There exists an increasing sequence* $(\alpha_n)_{n\in I\!N}$, $\lim_{n\to\infty}\alpha_n = 1$ *such that the limits* $i^* := \lim_{n\to\infty} i^*_{\alpha_n}$ *and* $\lim_{n\to\infty}\left(V^b_{\alpha_n}(j) - V^b_{\alpha_n}(0)\right)$ *exist. Moreover,* $i^* \le \tilde{i}^* := \min\left\{j \in I : \lim_{n\to\infty}\left(V^b_{\alpha_n}(j) - V^b_{\alpha_n}(0)\right) > d(j,j) - d(j,0)\right\}$ $(\min\emptyset := \infty)$. *In the subclass of bang-bang strategies those with thresholds in* $\{i^*, i^*+1, ..., \tilde{i}^*\}$ *minimize the average cost. If* $i^* = \infty$*, this is also valid if* r *and* d *are bounded and the following condition holds for every* $i \in I$*:* $p_{ii} \neq 1$ *and there exists a finite set* A_i *such that* $p_{ij} = 0\ \forall j \notin A_i$.

Proof: Since the conditions (1), ..., (3) are valid (AC-condition), the strategy with threshold $i^*_\alpha := \min\{j \in I : \alpha(V^b_\alpha(j) - V^b_\alpha(0)) > d(j,j) - d(j,0)\}$ minimizes the $\alpha-$discounted cost in the subclass of bang-bang strategies. By using the Cantor diagonalization method we can construct a function $h(j) := \lim_{n\to\infty}\left(V^b_{\alpha_n}(j) - V^b_{\alpha_n}(0)\right)$ for some sequence $(\alpha_n)_{n\in I\!N} \subset (0,1)$ with $\lim_{n\to\infty}\alpha_n = 1$. Let $g(j) := h(j) - (d(j,j) - d(j,0))$, $g_n(j) := h_n(j) - (d(j,j) - d(j,0))$. First consider what happens if $\tilde{i}^* < \infty$. Then $g(\tilde{i}^*) > 0$; $g(0), ..., g(\tilde{i}^* - 1) \le 0$. The relation $\lim_{n\to\infty} g_n(\tilde{i}^*) = g(\tilde{i}^*)$ yields $g_n(\tilde{i}^*) > 0\ \forall n \ge n_0$, $n_0 \in I\!N$. Thus $i^*_{\alpha_n} \le \tilde{i}^*\ \forall n \ge n_0$. Hence a subsequence of (α_n) exists such that $\lim_{n\to\infty} i_{\alpha_n}$ exists along this subsequence. Without loss of generality, $i^* = \lim_{n\to\infty} i_{\alpha_n} (\le \tilde{i}^*)$ exists. Now take the bang-bang strategy with threshold j^*. Then by Theorem 2(4),

$$\phi_{\delta_{j^*}} = \lim_{n\to\infty}(1-\alpha_n)V_{\delta_{j^*},\alpha_n}(0) \ge \lim_{n\to\infty}(1-\alpha_n)V_{\delta_{i^*_{\alpha_n}},\alpha_n}(0) = \lim_{n\to\infty}(1-\alpha_n)V_{\delta_{i^*},\alpha_n}(0) = \phi_{\delta_{i^*}}.$$

The identity $g(i) = 0$ yields to

$$\lim_{n\to\infty}\left(\left(d(i,0) + \alpha_n V^b_{\alpha_n}(i)\right) - \left(d(i,i) + \alpha_n V^b_{\alpha_n}(0)\right)\right)$$
$$= \lim_{n\to\infty}\left(\alpha_n\left(V^b_{\alpha_n}(i) - V^b_{\alpha_n}(0)\right)\right) - (d(i,i) - d(i,0))) = 0.$$

By Theorem 4, the bang-bang-strategies with thresholds i and $i+1$ yield the same

α_n-discounted cost if we let $n \to \infty$. Thus $\lim_{n\to\infty}\left(V_{\delta_i,\alpha_n} - V_{\delta_{i+1},\alpha_n}\right) = 0$

implies that $\phi_{\delta_i} = \lim_{n\to\infty}(1-\alpha_n)V_{\delta_i,\alpha_n} = \lim_{n\to\infty}(1-\alpha_n)V_{\delta_{i+1},\alpha_n} = \phi_{\delta_{i+1}}$.

If $g(\tilde{i}^* - j) < 0$ that means $g_n(\tilde{i}^* - j) < 0\,\forall n \ge n_1$, $n_1 \in I\!N$ then $i^*_{\alpha_n} > \tilde{i}^* - j\ \forall n \ge n_1$

and so $i^* > \tilde{i}^* - j$. Thus $g(\tilde{i}^* - 1) < 0$ entails $i^* \ge \tilde{i}^*$ and hence $i^* = \tilde{i}^*$, which yields

the implication $i^* < \tilde{i}^* \Rightarrow g(\tilde{i}^* - 1) = 0 \Rightarrow \exists j \in \{2, ..., \tilde{i} - 1\} : g(\tilde{i}^* - 1) = ... = g(\tilde{i}^* - j) = 0$,

$g(\tilde{i}^* - j - 1) < 0,\ g(i) = 0\,\forall i < \tilde{i}^* \ \Rightarrow \phi_{\delta_{\tilde{i}^*}} = \phi_{\delta_{\tilde{i}^*-1}} = ... = \phi_{\delta_{\tilde{i}^*-j}}$ or $\left(\ \phi_{\delta_{\tilde{i}^*}} = ... = \phi_{\delta_1}\right)$.

In other words $\phi_{\delta_{i^*}} = \phi_{\delta_{i^*+1}} = ... = \phi_{\delta_{\tilde{i}}}$. So besides the bang-bang-strategy with threshold i^* the bang-bang-strategies with threshold in $\{i^*,\ i^*+1, ..., \tilde{i}^*\}$ are optimal, too.

Finally we prove the Theorem in the case $N = \tilde{i}^* = i^* = \infty$: As the cost functions r and d are bounded and non-decreasing, there exist real constants $\tilde{r}$ and $\tilde{d}$ and a function $\epsilon : I \to I\!R^+$ with $0 \le \tilde{r} - r(j) \le \frac{1}{2}\epsilon(j),\ 0 \le \tilde{d} - d(j,0) \le \frac{1}{2}\epsilon(j),\ \epsilon(j) \ge \epsilon(j+1)$ and $\lim_{j\to\infty} \epsilon(j) = 0$. Now compute

$$\lim_{\alpha\to\infty} \left(1 - \alpha \sum_{j=i}^{i_\alpha^*-1} p_{ij}\right) V_\alpha(i) - \phi = \lim_{\alpha\to\infty} \left(1 - \alpha \sum_{j=i}^{i_\alpha^*-1} p_{ij}\right) V_\alpha(i) - \lim_{\alpha\to\infty} (1-\alpha)V_\alpha(i)$$

$$= \lim_{\alpha\to 1} \alpha \left(1 - \sum_{j=i}^{i_\alpha^*-1} p_{ij}\right) V_\alpha(i) = 0.$$ Note that there exists $\alpha_0 \in (0,1)$ such that

$i_\alpha^* > \max A_i$ for every $\alpha \in (\alpha_0, 1)$, which yields $\sum_{j=i}^{i_\alpha^*-1} p_{ij} = 1$ for every $\alpha \in (\alpha_0, 1)$. Furthermore, the inequality

$$\begin{aligned} V_\alpha(i) &= r(i) + \sum_{j=i}^{i_\alpha^*-1} p_{ij}\left(d(j,0) + \alpha V_\alpha(j)\right) + \sum_{j=i_\alpha^*}^{N} p_{ij}\left(d(j,j) + \alpha V_\alpha(0)\right) \\ &\ge r(i) + \sum_{j=i}^{i_\alpha^*-1} p_{ij}\left(d(j,0) + \alpha V_\alpha(i)\right) \\ &\ge \tilde{r} - \frac{1}{2}\epsilon(i) + \sum_{j=i}^{i_\alpha^*-1} p_{ij}\left(\tilde{d} - \frac{1}{2}\epsilon(j) + \alpha V_\alpha(i)\right),\ i \in I\ \forall \alpha \in (0,1), \end{aligned}$$

implies that $\tilde{r} - \frac{1}{2}\epsilon(i) \le \ V_\alpha(i) - \sum_{j=i}^{i_\alpha^*-1} p_{ij}\left(\tilde{d} - \frac{1}{2}\epsilon(i) + \alpha V_\alpha(i)\right)$, since $e(i) \ge e(j)$,

$$= \left(1 - \alpha \sum_{j=i}^{i_\alpha^*-1} p_{ij}\right) V_\alpha(i) - \sum_{j=i}^{i_\alpha^*-1} p_{ij}\left(\tilde{d} - \frac{1}{2}\epsilon(i)\right),\ i \in I,\ \alpha \in (0,1).$$

After letting $\alpha \uparrow 1$(that is $i_\alpha^* \to \infty$) and using the first part of the proof,

we get $\quad \tilde{r} - \frac{1}{2}\epsilon(i) \quad \le \quad \phi - \left(\tilde{d} - \frac{1}{2}\epsilon(i)\right) \quad$ for every $i \in I$.

Since $p_{ii} < 1$ for every $i \in I$ we have $\phi_{\delta_\infty} = \tilde{r} + \tilde{d} \le \phi + \epsilon(i)$ for all $i \in I$ and thus $\phi_{\delta_\infty} \le \phi$ (recall $\lim_{i\to\infty} \epsilon(i) = 0$), so that δ_∞ minimizes the average cost.

Theorem 10 *Under the additional conditions* (5), (6), (7), *the bang-bang strategies with thresholds* i^*, $i^*+1, ..., \tilde{i}^*$ *minimize the average cost, where*
$\tilde{i}^* = \min\{j \in I : \lim_{n\to\infty} (V_{\alpha_n}(j) - V_{\alpha_n}(0)) > d(j,j) - d(j,0)\}$. ($\min\emptyset := \infty$)
If $i^* = \infty$ *this is also valid if the following condition holds for every* $i \in I$: $p_{ii} \ne 1$ *and there exists a finite set* A_i *such that* $p_{ij} = 0\ \forall j \notin A_i$.

Proof: The proof follows the same lines as that of Theorem 9 apart from the beginning: Under the conditions (1), ..., (7) the bang-bang strategy with threshold i_α minimizes the $\alpha-$discounted cost in the class of all strategies. For every strategy δ we have

$$\phi_\delta = \lim_{n\to\infty} (1-\alpha_n)V_{\delta,\alpha_n}(0) \ge \lim_{n\to\infty} (1-\alpha_n)V_{\delta_{i_{\alpha_n}^*},\alpha_n}(0) = \lim_{n\to\infty} (1-\alpha_n)V_{\delta_{i^*},\alpha_n}(0) = \phi_{\delta_{i^*}}.$$

Now use Theorem 7 instead of Theorem 4 and replace V_α^b by V_α (they are equal) in the proof of Theorem 9. ■

4 Examples

In this section we present some examples in which we use the policy iteration explained in Chapter 3, incorporated in a C-program. The results for $N = 50$ ($I = \{0, ..., N\}$) are approximately confirmed by a computer simulation, in which all possible stationary policies were tested. We choose the following parametric class of cost functions and transition probabilities ($a, i, j \in I$, $a \leq i$):

$$r(i) = r_0 + i \cdot \gamma$$

$$\text{case (a)}: \quad d(0,0) = 0 \qquad d(1,0) = d_{1,0} \qquad d(1,1) = d_{1,1}.$$

$$d(i,a) = \kappa \left(\frac{a}{i-1}\right)^{\beta} + \delta_0 i^{\lambda}, \text{ for other values of } (i,a)$$

$$\text{case (b)}: \quad d(i,a) = \beta \cdot i + 1_{\{a>0\}} \left(\kappa\sqrt{i \cdot a} + \delta_0\right),$$

$$p_{ij} = \begin{cases} \left(\frac{i+1}{j+1}\right)^{\epsilon} - \left(\frac{i+1}{j+2}\right)^{\epsilon} & N > j \geq i \geq 0, \\ \left(\frac{i+1}{N+1}\right)^{\epsilon} & j = N, \\ 0 & \text{otherwise.} \end{cases}$$

We set $r_0 = 10$. The parameters are subject to the restrictions

$$\alpha \in (0,1), \qquad \beta \in [0,1], \qquad \gamma \in I\!R^+, \qquad \delta_0 \in I\!R^+, \qquad \epsilon \in (0,1], \qquad \mu \in I\!R^+,$$

$$\text{additionally for case } (a): \qquad \delta_0 \lambda > \kappa\beta 2^{\beta+1}, \quad r_0 \in I\!R, \quad d_0 \in I\!R;$$

$$\text{additionally for case } (b): \qquad 0 < \kappa < \gamma\alpha.$$

It is an easy computation to check that the conditions (1) to (7) are valid in both cases (a) and b with (4) replaced by $(4a)$ in case (b). For the computer program we take the finite state space $\{0, ..., N\}$. As expected, the policy iteration always leads to bang-bang strategies. In the following table, examples are shown with threshold x_0 and the corresponding cost computed by the program. Let us look at case (a): The values in the central column have to be non-negative for the conditions to be valid. But these conditions are not necessary. So the optimal strategy may be a bang-bang strategy, even if the value is negative. In the following table such examples occur. First consider case (a):

N	α	β	γ	δ_0	ϵ	κ	λ	d_{10}	d_{11}	$\delta_0\lambda-\kappa\beta 2^{\beta+1}$	x_0	$V_\alpha(0)$	$(1-\alpha)V_\alpha(0)$
1000	0.9	0.001	2	21	0.99	1000	0.1	20	1021	0.099	126	1801	181
1000	0.9	0.001	10	21	0.99	1000	0.1	20	1021	0.099	1	48301	4830
50	0.9	0.001	10	21	0.99	1000	0.1	20	1021	0.099	32	2398	240
50	0.995	0.001	10	21	0.99	1000	0.1	20	1021	0.099	25	54880	274.4
50	0.9999	0.001	10	21	0.99	1000	0.1	20	1021	0.099	25	$2754 \cdot 10^3$	275.4
50	0.9	0.001	20	21	0.99	1000	0.1	20	1021	0.099	18	2751	275
50	0.995	0.001	20	21	0.99	1000	0.1	20	1021	0.099	15	61531	307.7
50	0.9999	0.001	20	21	0.99	1000	0.1	20	1021	0.099	14	$3083 \cdot 10^3$	308.3
50	0.9	0.00001	10	21	0.99	1000	0.1	20	1021	0.099	32	2398	240
50	0.9	0.001	10	21	0.5	1000	0.1	20	1021	0.099	47	3911	391
50	0.9	0.001	10	21	0.99	1000	1.0	20	1021	2.08	8	4167	417
50	0.9	0.001	10	21	0.99	100	0.1	20	1021	19	4	544	54
50	0.9	0.001	10	40	0.99	1000	0.1	40	1041	2	31	2582	258
50	0.9	10	1	21	0.99	1000	0.1	20	1021	$-2 \cdot 10^8$	$(*)$	384	38

$(*)$ is an example where no bang-bang strategy is optimal, the optimal strategy is $\delta_0^*(i) := \lfloor \frac{29i}{50} \rfloor$. Examples for case (b) are displayed in the following table:

N	α	β	γ	δ_0	ϵ	κ	$\gamma\alpha - \kappa$	x_0	$V_\alpha(0)$	$(1-\alpha)V_\alpha(0)$
50	0.9	1.0	2.5	100	0.99	0.2	2.05	9	385	39
50	0.9	1.0	2.5	100	0.99	3.0	-0.75	5	540	54
50	0.995	1.0	2.5	100	0.99	3.0	-0.5	4	11353	56.8
50	0.9999	1.0	2.5	100	0.99	3.0	-0.5	4	565987	56.6
50	0.9	1.0	2.5	200	0.99	3.0	-0.75	11	795	80
50	0.9	1.0	2.5	500	0.99	3.0	-0.75	∞	1172	540
50	0.9	5.0	2.5	200	0.99	3.0	-0.75	5	1315	132
1000	0.9	1.0	2.5	100	0.99	3.0	-0.75	3	724	72
1000	0.9	1.0	2.5	500	0.99	3.0	-0.75	11	1815	182
1000	0.9	1.0	25	100	0.99	3.0	19.5	2	769	77
1000	0.9	1.0	25	500	0.99	3.0	19.5	6	2075	208
1000	0.9	1.0	2.5	500	0.5	3.0	-0.75	6	5370	537

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