Faculty of Mathematical Sciences

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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# On the formalism of local variational differential operators

S. Igonin, A. Verbovetsky[1]
and R. Vitolo[2]

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[1]Independent University of Moscow, B. Vlasevsky 11, 121002 Moscow, Russia
[2]Math. Dept. "E. De Giorgi", Via per Arnesano, 73100 Lecce, Italy

# ON THE FORMALISM OF LOCAL VARIATIONAL DIFFERENTIAL OPERATORS

SERGEI IGONIN, ALEXANDER VERBOVETSKY, AND RAFFAELE VITOLO

Abstract. The calculus of local variational differential operators introduced by B. L. Voronov, I. V. Tyutin, and Sh. S. Shakhverdiev is studied in the context of jet super space geometry. In a coordinate-free way, we relate these operators to variational multivectors, for which we introduce and compute the variational Poisson and Schouten brackets by means of a unifying algebraic scheme. We give a geometric definition of the algebra of multilocal functionals and prove that local variational differential operators are well defined on this algebra. To achieve this, we obtain some analytical results on the calculus of variations in smooth vector bundles, which may be of independent interest. In addition, our results give a new a new efficient method for finding Hamiltonian structures of differential equations.

## Introduction

In this paper we study the calculus of *local variational differential operators* introduced by B. L. Voronov, I. V. Tyutin, and Sh. S. Shakhverdiev [31]. The advantage of this calculus is that it allows to avoid $\delta(0)$-terms which appear in many computations in local quantum field theory (see, for example, [12, 31]).

Local variational differential operators arise as follows [31]. In local field theories formulas are often derived from finite-dimensional analogies. In some situations this approach works well, for example the expression of the antibracket

$$(1)\qquad (F_1, F_2) = \int dx \left( \frac{\delta F_1}{\delta u^i(x)} \frac{\delta F_2}{\delta u_i^*(x)} + (-1)^{|F_1|} \frac{\delta F_1}{\delta u_i^*(x)} \frac{\delta F_2}{\delta u^i(x)} \right),$$

where $u^i(x)$ are fields, $u_i^*(x)$ are antifields, $F_l = \int dx f_l(x^i, u^j_\sigma, u^*_{j,\sigma})$ are actions ($l = 1, 2$), $dx = dx^1 \wedge \dots \wedge dx^n$, can be obtained from the expression of the Poisson bracket

$$\{F_1, F_2\} = \frac{\partial F_1}{\partial q^i} \frac{\partial F_2}{\partial p_i} - \frac{\partial F_1}{\partial p_i} \frac{\partial F_2}{\partial q^i}$$

by replacing

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- functions by functionals,
- discrete indexes by pairs of indexes and points $i \mapsto (i, x_i)$, $x_i = (x_i^1, \dots, x_i^n)$,
- summation over discrete indexes $i$ by the summation and integration over $x_i$,
- partial derivatives by variational derivatives.

(The antifields $u_i^*$ are odd variables, hence the signs in (1)).

However, using this analogy leads to the 'problem of $\delta(0)$' [12, 31] when one considers higher order variational differential operators of the form

$$\Delta = \sum_{j_s} \int dx f_{j_1,\dots,j_k}(x^i, u^j_\sigma) \frac{\delta^k}{\delta u^{j_1}(x) \cdots \delta u^{j_k}(x)}. \tag{2}$$

These operators are not defined on actions (local functionals)

$$S = \int dx L(x^i, u^j_\sigma),$$

since $\Delta(S)$ contains $\delta(0)$-terms (see [31] for details). In local quantum field theory we encounter such higher order operators when we consider, for example, the change of variables in path integrals, or the quantum master equation (see, e.g., [12]). In [31] the authors suggest to redefine operators of the type of (2) to get rid of $\delta(0)$-terms. These new operators, called *local variational differential operators*, are defined by the formula

$$\begin{split} \nabla(F(S_1, \dots, S_N)) &= \sum_{j_s, \sigma_r} \left[ \int dx\, A^{j_1,\dots,j_k}_{\sigma_1,\dots,\sigma_k}(x^i, u^j_\sigma) D_{\sigma_1}\left(\frac{\delta S_{i_1}}{\delta u^{j_1}}\right) \cdots D_{\sigma_k}\left(\frac{\delta S_{i_k}}{\delta u^{j_k}}\right)\right] \\ &\qquad \frac{\partial^k F}{\partial t_{i_1} \cdots \partial t_{i_k}}(S_1, \dots, S_N). \end{split} \tag{3}$$

where

$$\nabla = \sum_{j_s, \sigma_r} \int dx \wedge dx_1 \wedge \dots \wedge dx_n\, A^{j_1,\dots,j_k}_{\sigma_1,\dots,\sigma_k}(x^i, u^j_\sigma) \delta^{(\sigma_1)}(x - x_1) \cdots \delta^{(\sigma_k)}(x - x_k) \frac{\delta}{\delta u^{j_1}} \cdots \frac{\delta}{\delta u^{j_k}},$$

$F(t_1, \dots, t_N) \in C^\infty(\mathbb{R}^N)$, $S_l = \int dx L_l(x^i, u^j_\sigma)$, $D_\sigma$ are total derivatives.

In [31] this formalism is applied to the BV formulation of general gauge field theories.

One of the tasks of this paper is to prove that (3) is well defined, i.e., if $F(S_1, \dots, S_N) = 0$ then $\nabla(F(S_1, \dots, S_N)) = 0$. This property is essential and was not addressed in [31]. To do this, we would like to understand (3) in the context of finite-dimensional calculus as described above. To this end, we observe that the operator $\nabla$ in (3) acts in a way analogous to the action of multivectors (i.e., antisymmetric contravariant tensors) on linear functions on a vector space. This suggests to add two further correspondences between field theory and finite-dimensional calculus:

- we regard Voronov-Tyutin-Shakhverdiev operators as symmetric multivectors;
- we regard actions as linear functions.

It is remarkable that the latter correspondence makes sense, as it follows from the analytical results of Section 7.

Basing on the above considerations we prove that local variational differential operators are well defined. Moreover we develop a theory of symmetric and skew-symmetric variational multivectors and variational Poisson and Schouten brackets. Note that variational one-vectors coincide with evolutionary fields [3, 14] and skew-symmetric variational bivectors whose variational Schouten bracket with itself vanishes are just Hamiltonian operators [15, 3, 14].

At present there is a large amount of publications on Hamiltonian operators and the rich theory that surrounds it. We can mention only a limited number of works that contain ideas and results close to ours (see, e.g., [7, 16, 6, 8, 24] for more references).

In paper [9] by Gel′fand and Dorfman (see [7] for details) it has been recognized that an algebraic framework is useful to deal with an infinite-dimensional version of the Schouten bracket. The authors developed such a framework based on the notion of a complex over a Lie algebra (formal differential geometry) and studied the case that the complex replaced by the coordinate version of the variational complex with one independent variable. The resulting Schouten bracket is of great importance in the Hamiltonian theory of the integrable evolution equations in one space variable (see, e.g., [7, 23, 24]).

We also start our study with an algebraic model, which, however, differs essentially from that of [9]. Approaches nearer to ours are carried out, for example, in [5, 22]. Being applied to variational multivectors, this model gives a geometric coordinate-free definition of the variational Schouten bracket in the case of an arbitrary base manifold.

The two more brackets—Poisson bracket on cotangent bundle to a bundle by Kupershmidt [15] and the antibracket (see, e.g., [12, 10] and references therein)—have also found coordinate-free interpretations in the present paper. We shall discuss relations to them in Sections 5 and 6 below.

Our results on the variational Schouten bracket made it possible to develop a new efficient method for searching for Hamiltonian structures of differential equations [13].

We adopt the jet bundle approach which is now widely used in local field theory [1, 25, 21]. This means that fields are regarded as sections of a vector bundle $\pi\colon E \to M$ over the spacetime manifold $M$, $\dim M =$

$n$. In fact, we assume $\pi$ to be a superbundle so that bosonic fields are sections of the even part of $\pi$ and fermionic fields are sections of the odd part. Local functions (i.e., smooth functions of fields and a finite number of their derivatives) are functions on the infinite order jet space $J^\infty(\pi)$ of $\pi$. A local functional is an $n$-cohomology class of the horizontal de Rham complex consisting of differential forms on $M$ with coefficients in local functions. In other words, local functionals are Lagrangian densities modulo total divergences.

Our variational multivectors turn out to be dual to the variational forms which are elements of the well-known variational complex [26, 27, 28] (see, for example, [3, 14] for an introductory treatment).

Note that the correspondences between field theory and finite-dimensional calculus described above is closely related to the reasoning behind the secondary calculus by Vinogradov (see [29] and references therein). The above form-multivector duality is in good agreement with this calculus, however, our results are valid for jet spaces only, while the secondary calculus is considered on more general spaces.

The paper is organized as follows.

In Sections 1 and 2, we construct an algebraic model, which will be used later to develop the formalism of variational brackets. In Section 1, we fix notation and terminology related to graded vector spaces, graded algebras and differential operators over such algebras. In Section 2, we study a Lie algebra of multilinear mappings which is a generalization of Nijenhuis-Richardson-type brackets [5, 22]. As a result we obtain an algebraic counterpart of formula (3).

In Section 3, we collect basic facts on the geometry of jet spaces of vector superbundles. Our coordinate-free, but explicit definition of superjets may be of independent interest.

In Section 4, we apply the algebraic model of Section 2 to a special class of multilinear maps on local functionals in order to obtain variational multivectors and variational Poisson brackets. From Theorem 4.2 we see that variational multivectors are dual to variational forms encountered in the variational complex [3, 14].

In Section 5, we relate the constructions of Section 4 to Kupershmidt's construction of cotangent bundle to a bundle [15].

In Section 6, we construct the variational Schouten bracket as the odd counterpart of the variational Poisson bracket.

In the last Section 7 we study the algebra of multilocal functionals $F(S_1, \dots, S_N)$. In order to confirm the above-mentioned correspondence between actions and linear functions, we prove the crucial Theorem 7.4, which says, roughly speaking, that locally all relations $F(S_1, \dots, S_N) = 0$ in the algebra of multilocal functionals arise from linear relations between the actions $S_1, \dots, S_n$. This result allows to prove that operators (3) are well defined on this algebra.

## 1. Graded algebras: conventions and definitions

In this section we set up notation and terminology related to graded vector spaces, graded algebras, and differential operators over such algebras. Full details can be found, e.g., in [4]. We consider vector spaces over a ground field $\Bbbk$ of characteristic zero.

Let $G$ be an Abelian group (written additively). A vector space $V$ is called *$G$-graded* if $V = \bigoplus_{g\in G} V_g$ for some vector spaces $V_g$. Elements $v \in V_g$ are called *homogeneous of degree $g$*. In what follows, whenever we consider the degree of an element $v \in V$ it is always assumed that $v$ is homogeneous.

*Superspaces* are $\mathbb{Z}_2$-graded vector spaces.

Every $G$-graded vector space $V$ can be equipped with a $G \oplus H$-grading, where $H$ is one more Abelian group, by putting $V_{(g,h_0)} = V_g$, $V_{(g,h)} = 0$ for $g \in G$, $h \in H \smallsetminus \{h_0\}$, where $h_0$ is a fixed element of the group $H$. In this cases we shall say that the space $V$ is *graded by the degree* $h_0 \in H$ and write $V_{[h_0]}$.

A subspace $W$ of a $G$-graded vector space $V$ is called *graded* if $W = \bigoplus_{g\in G} W_g$, where $W_g = W \cap V_g$. If $W$ is a graded subspace of $V$, then $V/W = \bigoplus_{g\in G} V_g/W_g$, so that the quotient space has a natural $G$-grading.

Consider two $G$-graded vector spaces $V$ and $U$. The spaces $V \oplus U$, $V \otimes_{\Bbbk} U$, and $\mathrm{Hom}_{\Bbbk}(V, U)$ are endowed with natural $G$-gradings defined by the formulas:

$$(V \oplus U)_g = V_g \oplus U_g,$$

$$(V \otimes_{\Bbbk} U)_g = \bigoplus_{g'+g''=g} V_{g'} \otimes_{\Bbbk} U_{g''}, \qquad g',\ g'' \in G,$$

$$\mathrm{Hom}_{\Bbbk}(V, U)_g = \{\, f \in \mathrm{Hom}_{\Bbbk}(V, U) \mid f(V_{g'}) \subset U_{g'+g} \text{ for all } g' \in G\,\}.$$

A $\Bbbk$-algebra $A$ is called *$G$-graded* if $A$ is a $G$-graded vector space and

$$A_{g_1} A_{g_2} \subset A_{g_1+g_2}, \qquad g_1,\ g_2 \in G.$$

If $A$ has the unity 1, then it is assumed that $1 \in A_0$.

Pick up a *commutation factor* $\{\cdot\,,\cdot\}$, i.e., a pairing $G \times G \to \Bbbk \smallsetminus \{0\}$, $(g_1, g_2) \mapsto \{g_1, g_2\}$, such that

(1) $\{g_1, g_2\}^{-1} = \{g_2, g_1\}$,
(2) $\{g_1 + g_2, g_3\} = \{g_1, g_3\}\{g_2, g_3\}$.

From the definition it readily follows that

$$\{g_1, g_2 + g_3\} = \{g_1, g_2\}\{g_1, g_3\},$$
$$\{g, 0\} = \{0, g\} = 1,$$
$$\{-g_1, g_2\} = \{g_1, -g_2\} = \{g_1, g_2\}^{-1}.$$

**Example 1.1.** Let $G = \mathbb{Z}$ or $\mathbb{Z}_2$. In this case there is only one nontrivial commutation factor, namely, the *super-commutation factor* $\{g_1, g_2\} = (-1)^{g_1 g_2}$.

**Example 1.2.** If $G = \mathbb{Z}^n = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ then commutation factors on $G$ are of the form

$$\{g, h\} = \prod_{1 \le i,j \le n} q_{ij}^{g_i h_j},$$

where $g = (g_1, \dots, g_n)$, $h = (h_1, \dots, h_n) \in \mathbb{Z}^n$ and the parameters $q_{ij} \in \Bbbk \smallsetminus \{0\}$ satisfy the conditions $q_{ij} q_{ji} = 1$ for all $i$, $j = 1, \dots, n$. In particular, $q_{ii} = \pm 1$.

Given two Abelian groups $G$ and $H$ with commutation factors $\{\cdot\,,\cdot\}_G$ and $\{\cdot\,,\cdot\}_H$ on them, we can define a commutation factor on $G \oplus H$ through the formula $\{(g_1, h_1), (g_2, h_2)\} = \{g_1, g_2\}_G \{h_1, h_2\}_H$.

For a fixed element $g = (g_1, \dots, g_n) \in G^n = G \oplus \cdots \oplus G$ there is a unique function $\epsilon_g \colon S_n \to \Bbbk \smallsetminus \{0\}$ on the permutation group $S_n$ such that

(1) $\epsilon_g(\sigma_i) = \{g_i, g_{i+1}\}$ for a transposition $\sigma_i = (i, i+1)$;
(2) $\epsilon_g(\sigma' \circ \sigma'') = \epsilon_{\sigma''(g)}(\sigma')\epsilon_g(\sigma'')$, where $\sigma(g) = (g_{\sigma(1)}, \dots, g_{\sigma(n)})$.

The proof of this fact is straightforward. For the super-commutation factor the function $\epsilon_{(1,\dots,1)}$ coincides with the standard sign of permutations. We shall denote it by $\epsilon$.

For the sake of convenience, we adopt the following notation: if $v \in V_{g_1}$ and $w \in W_{g_2}$ are two homogeneous elements of arbitrary $G$-graded vector spaces $V$ and $W$, then we write $\{v, w\}$ rather than $\{g_1, g_2\}$, i.e., the symbol of a graded object used as an argument of the commutation factor denotes the degree of this object. Similarly, if $v = (v_1, \dots, v_n) \in V_{g_1} \oplus \cdots \oplus V_{g_n}$, we let $\epsilon_v$ stand for $\epsilon_g$, where $g = (g_1, \dots, g_n)$.

In formulas commutation factors are used in accordance with the following 'generalized rule of signs': whenever an object (i.e., an element of a $G$-graded vector space) of degree $g_1$ is passed through an object of degree $g_2$, the multiplier $\{g_1, g_2\}$ is introduced.

A $G$-graded algebra $A$ is called *commutative* if for all $a$, $b \in A$ we have

$$ab = \{a, b\} ba.$$

Clearly, $a_1 \cdots a_n = \epsilon_a(\sigma) a_{\sigma(1)} \cdots a_{\sigma(n)}$, where $a_i \in A$ and $\sigma \in S_n$.

**Example 1.3.** Let $V$ be a $G$-graded vector space and

$$T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

be the tensor algebra of $V$. Consider the ideal $I$ in $T(V)$ generated by the elements $(v \otimes w - \{v, w\} w \otimes v) \in V^{\otimes 2}$ for $v$, $w \in V$. The algebra

$S(V) = T(V)/I$ is a $G$-graded associative commutative algebra with unity called the *symmetric algebra* of $V$. Similarly to the case of non-graded spaces, the *symmetrization* $\mathrm{Sym}\colon T(V) \to T(V)$

$$\mathrm{Sym}(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon_v(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

where $v = (v_1, \ldots, v_k)$, permits $S(V)$ to be identified with a direct summand of $T(V)$. Obviously, $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$, where $S^k(V) = S(V) \cap V^{\otimes k}$.

A $\mathbb{Z}_2$-graded algebra is *supercommutative* if it is commutative with respect to the super-commutation factor $\{g_1, g_2\} = (-1)^{g_1 g_2}$, where $g_1, g_2 \in \mathbb{Z}_2$.

A left module $M$ over a $G$-graded commutative algebra $A$ is called *$G$-graded* if $M = \bigoplus_{g \in G} M_g$ and $A_{g_1} M_{g_2} \subset M_{g_1+g_2}$. Such a module has also a right module structure defined by the equality $ma = \{m, a\} am$ for $m \in M$, $a \in A$. Obviously, $M_{g_1} A_{g_2} \subset M_{g_1+g_2}$.

A *$G$-graded Lie algebra* is a $G$-graded algebra $A$ such that the multiplication in $A$, denoted by $[\cdot\,,\cdot]\colon A \otimes_{\Bbbk} A \to A$, satisfies the properties:

$$[a, b] = -\{a, b\}[b, a],$$
$$[[a, b], c] = [a, [b, c]] + \{b, c\}[[a, c], b],$$

for all $a$, $b$, $c \in A$.

**Example 1.4.** Given two linear transformations $f_1, f_2 \in \mathrm{Hom}_{\Bbbk}(V, V)$ of a $G$-graded vector space $V$, their *commutator* $[f_1, f_2]$ is defined by the formula

$$[f_1, f_2] = f_1 \circ f_2 - \{f_1, f_2\} f_2 \circ f_1.$$

Obviously, the vector space $\mathrm{Hom}_{\Bbbk}(V, V)$, equipped with the commutator, is a $G$-graded Lie algebra.

A *Lie superalgebra* is $\mathbb{Z}_2$-graded Lie algebra with respect to the super-commutation factor.

Further on $A$ will be an associative commutative $G$-graded algebra with unity.

**Definition 1.5.** A $\Bbbk$-homomorphism $\Delta \in \mathrm{Hom}_{\Bbbk}(A, A)$ is called a *scalar $G$-graded differential operator* of order $k$, if for all $a_0, \ldots, a_k \in A$ we have

$$[a_0, [a_1, \ldots, [a_k, \Delta] \ldots]] = 0. \tag{1.1}$$

In this equality $a_i$ are the operators of left multiplication.

Denote by $\mathrm{Diff}_k(A)$ the set of all scalar differential operators of order $k$. It is clear that $\mathrm{Diff}_0(A) = A$ and $\mathrm{Diff}_k(A) \subset \mathrm{Diff}_l(A)$ for $k \leq l$.

The obvious equality

$$[a, \Delta_1 \circ \Delta_2] = [a, \Delta_1] \circ \Delta_2 + \{a, \Delta_1\} \Delta_1 \circ [a, \Delta_2] \tag{1.2}$$

implies the following result.

**Proposition 1.6.** *If $\Delta_1 \in \mathrm{Diff}_k(A)$ and $\Delta_2 \in \mathrm{Diff}_l(A)$, then*
(1) $\Delta_1 \circ \Delta_2 \in \mathrm{Diff}_{k+l}(A)$;
(2) $[\Delta_1, \Delta_2] \in \mathrm{Diff}_{k+l-1}(A)$.

## 2. The Lie algebra of multilinear mappings: an algebraic model

Here, we introduce a graded Lie algebra structure on a space of multilinear maps on a vector space. Such a structure generalizes the Nijenhuis-Richardson and Schouten brackets. Then, by interpreting multilinear maps as differential operators on polynomial functions, we provide an algebraic counterpart for the local variational differential operators.

Let $V$ be a $G$-graded vector space. Suppose that for all integers $k \geq 0$ we have chosen spaces $\mathcal{M}_k(V)$ of $k$-linear mappings $V \times \cdots \times V \to V$ such that

(1) $\mathcal{M}_0(V) = V$;
(2) $\mathcal{M}_k(V)$ is a graded subspace of $\mathrm{Hom}_{\Bbbk}(V^{\otimes k}, V)$;
(3) for all $f \in \mathcal{M}_k(V)$, $g \in \mathcal{M}_l(V)$, and $1 \leq i \leq k$ the $(k+l-1)$-linear map $h$ defined by the formula
$$h(v_1, \dots, v_{k+l-1}) = f(v_1, \dots, v_{i-1}, g(v_i, \dots, v_{i+l-1}), v_{i+l}, \dots, v_{k+l-1})$$
belongs to $\mathcal{M}_{k+l-1}(V)$.

The space $\mathcal{M}_1(V) \subset \mathrm{Hom}_{\Bbbk}(V, V)$ is a graded Lie algebra with respect to the commutator. Let $\mathfrak{g}$ be a graded Lie subalgebra of $\mathcal{M}_1(V)$. For each $k \geq 1$ define $G$-graded vector space $\mathfrak{g}^{(k)}$ to be the graded subspace of $\mathcal{M}_k(V)$ such that $f \in \mathfrak{g}^{(k)}$ if

(1) $f$ is symmetric, i.e., $f(v_1, \dots, v_k) = \epsilon_v(\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$ for all $\sigma \in S_k$;
(2) for all $v_1, \dots, v_{k-1}$ the maps $v \mapsto f(v_1, \dots, v_{i-1}, v, v_i, \dots, v_{k-1})$ belong to $\mathfrak{g}$.

Obviously, $\mathfrak{g}^{(1)} = \mathfrak{g}$. By definition, we put $\mathfrak{g}^{(0)} = V$ and $\mathfrak{g}^{(k)} = 0$ for $k < 0$.

Let us introduce the following notation. If $f \in \mathrm{Hom}_{\Bbbk}(V^{\otimes k}, V)$ and $v_i \in V$, then $f(v_1, \dots, v_l)$ for $l \leq k$ will stand for the multilinear map belonging to $\mathrm{Hom}_{\Bbbk}(V^{\otimes (k-l)}, V)$ such that
$$f(v_1, \dots, v_l)(v_{l+1}, \dots, v_k) = f(v_1, \dots, v_k).$$
It is obvious that if $f \in \mathfrak{g}^{(k)}$ then $f(v_1, \dots, v_l) \in \mathfrak{g}^{(k-l)}$.

**Theorem 2.1.** *On the space $\mathfrak{g}^{(*)} = \bigoplus_k \mathfrak{g}^{(k)}$ there exists a unique $G$-graded Lie algebra structure $[\![\cdot\,,\cdot]\!]$ such that*
(1) $[\![\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}]\!] \subset \mathfrak{g}^{(k+l-1)}$;
(2) $[\![f, v]\!] = f(v)$ *for* $v \in \mathfrak{g}^{(0)} = V$, $f \in \mathfrak{g}^{(k)}$;

*Proof.* From condition (1) it follows that

$$[\![v_1, v_2]\!] = 0, \qquad v_1,\ v_2 \in V.$$

Further, using condition (2) and the Jacobi identity we get

$$[\![f_1, f_2]\!](v) = [\![f_1, f_2(v)]\!] + \{f_2, v\}[\![f_1(v), f_2]\!], \tag{2.1}$$

where $f_1,\ f_2 \in \mathfrak{g}^{(*)}$, $v \in V$. These two equations and condition (2) define a bilinear map $\mathfrak{g}^{(k)} \times \mathfrak{g}^{(l)} \to \mathrm{Hom}_{\Bbbk}(V^{\otimes(k+l-1)}, V)$. A trivial verification shows that this map is a Lie algebra structure and fulfills conditions (1) and (2). □

Note that the bracket $[\![\cdot\,,\cdot]\!]$ is an extension of the commutator:

$$[\![f_1, f_2]\!] = [f_1, f_2], \qquad f_1,\ f_2 \in \mathfrak{g}^{(1)} = \mathfrak{g}.$$

Using the induction, we get the following explicit formula for this bracket:

$$\begin{aligned} &[\![f_1, f_2]\!](v_1, \ldots, v_{k+l-1}) \\ &\quad = \sum_{\sigma \in S^l_{k+l-1}} \epsilon_v(\sigma) f_1\big(f_2(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(k+l-1)}\big) \\ &\quad - \{f_1, f_2\} \sum_{\sigma \in S^k_{k+l-1}} \epsilon_v(\sigma) f_2\big(f_1(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l-1)}\big), \end{aligned} \tag{2.2}$$

where $S^i_n \subset S_n$ is the set of all $(i, n-i)$-*unshuffles*[1], that is, all permutations $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$.

**Example 2.2.** Let $W$ be a vector space. Take the grading group $G = \mathbb{Z}_2$ with the super-commutation factor on it. Consider the space $V = W_{[1]}$. Recall that by this is meant that $V$ is a $\mathbb{Z}_2$-graded space such that $V_0 = 0$ and $V_1 = W$. Put $\mathcal{M}_k(V) = \mathrm{Hom}_{\Bbbk}(V^{\otimes k}, V)$ and $\mathfrak{g} = \mathcal{M}_1(V) = \mathrm{Hom}_{\Bbbk}(V, V)$. This gives $\mathfrak{g}^{(k)} = \mathrm{Hom}_{\Bbbk}(S^k(V), V)$. As vector spaces, with no regard for grading, $\mathfrak{g}^{(k)}$ is isomorphic to $\mathrm{Hom}_{\Bbbk}(\Lambda^k(W), W)$. It can easily be checked that the bracket $[\![\cdot\,,\cdot]\!]$ is thus identified with the *Nijenhuis-Richardson bracket*.

**Example 2.3.** Let $M$ be a smooth manifold. Again take the grading group $G = \mathbb{Z}_2$ with the super-commutation factor. Set $V = C^\infty(M)_{[1]}$, $\mathcal{M}_k(V) = \mathrm{Hom}_{\Bbbk}(V^{\otimes k}, V)$ and $\mathfrak{g} = \mathrm{D}(M)$ the Lie algebra of vector fields on $M$. Then $\mathfrak{g}^{(k)} = \mathrm{Hom}_{C^\infty(M)}(\Lambda^k(M), C^\infty(M))$ the space of all skew-symmetric $k$-vector fields. It is readily seen that the bracket $[\![\cdot\,,\cdot]\!]$ coincides with the *Schouten bracket* (cf. [30, Theorem 2.1]). Note that if we consider $V = C^\infty(M)$ as an even space, the construction

[1] The term *unshuffle* is borrowed from [17] and means separating an ordered set into two subsets, the order within each subset being as in the original set.

yields a bracket on symmetric multivector fields. This bracket, sometimes called the *symmetric Schouten concomitant*, is the restriction of the Poisson bracket on the space $T^*(M)$ to the fiberwise polynomial functions on $T^*(M)$.

Now to multilinear maps $f \in \mathfrak{g}^{(k)}$ we assign differential operators on the algebra $S(V)$.

**Proposition 2.4.** *For each $f \in \mathfrak{g}^{(k)}$ and $k > 0$ there exists a unique differential operator $\nabla_f \in \mathrm{Diff}_k(S(V))$ such that*

$$\nabla_f|_{S^l(V)} = 0, \quad 0 \le l < k,$$
$$\nabla_f(v_1 \cdots v_k) = f(v_1, \dots, v_k).$$

*Proof.* Let $\Delta \in \mathrm{Hom}_{\Bbbk}(A, A)$ and $n > k$. The equality

$$[a_1, [a_2, \dots, [a_n, \Delta] \dots]] = 0, \quad a_i \in A,$$

can be written as

$$\Delta(a_1 \cdots a_n) = \sum_{0 \le i < n} (-1)^{n-i+1} \sum_{\sigma \in S_n^i} \epsilon_a(\sigma) \Delta(a_{\sigma(1)} \cdots a_{\sigma(i)}) a_{\sigma(i+1)} \cdots a_{\sigma(n)}.$$

For $A = S(V)$, $\Delta = \nabla_f$, and $a_i \in V$, this formula defines a unique extension of $\nabla_f$ to $S(V)$. The map $\nabla_f \colon S(V) \to S(V)$ thus constructed fulfills the definition of differential operator (1.1) for $a_i \in V$. The general case follows from the obvious formula

$$[ab, \Delta] = a[b, \Delta] + \{b, \Delta\}[a, \Delta]b, \quad a,\ b \in A,$$

and equality (1.2). □

For $k = 0$ we define $\nabla_f$ to be the operator of multiplication by $f \in V$. It is clear from the definition that $\nabla_f(S^l(V)) \subset S^{l-k+1}(V)$.

**Lemma 2.5.** *For $f \in \mathfrak{g}^{(k)}$, $n > k$, $v_i \in V$, we have*

$$\nabla_f(v_1 \cdots v_n) = \sum_{\sigma \in S_n^k} \epsilon_v(\sigma) f(v_{\sigma(1)} \cdots v_{\sigma(k)}) v_{\sigma(k+1)} \cdots v_{\sigma(n)}.$$

*Proof.* The proof is by induction on $n$. For $n = 1$, there is nothing to prove. For general $n$ we get:

$$\begin{aligned}
\nabla_f(v_1 \cdots v_n) &= [\nabla_f, v_1](v_2 \cdots v_n) + \{f, v_1\} v_1 \nabla_f(v_2 \cdots v_n) \\
&= \sum_{\substack{\sigma \in S_n^k \\ \sigma(1)=1}} \epsilon_v(\sigma) [\nabla_f, v_1](v_{\sigma(2)} \cdots v_{\sigma(k)}) v_{\sigma(k+1)} \cdots v_{\sigma(n)} \\
&+ \sum_{\substack{\sigma \in S_n^{k+1} \\ \sigma(1)=1}} \epsilon_v(\sigma) \{f, v_1\} v_1 f(v_{\sigma(2)} \cdots v_{\sigma(k+1)}) v_{\sigma(k+2)} \cdots v_{\sigma(n)} \\
&= \sum_{\substack{\sigma \in S_n^k \\ \sigma(1)=1}} \epsilon_v(\sigma) f(v_1 v_{\sigma(2)} \cdots v_{\sigma(k)}) v_{\sigma(k+1)} \cdots v_{\sigma(n)}
\end{aligned}$$

$$+ \sum_{\substack{\sigma\in S_n^k \\ \sigma(k+1)=1}} \epsilon_v(\sigma) f(v_{\sigma(1)}\cdots v_{\sigma(k)}) v_1 v_{\sigma(k+2)}\cdots v_{\sigma(n)}$$

$$= \sum_{\sigma\in S_n^k} \epsilon_v(\sigma) f(v_{\sigma(1)}\cdots v_{\sigma(k)}) v_{\sigma(k+1)}\cdots v_{\sigma(n)}. \qquad \square$$

The next result, which follows directly from the last Lemma and formula (2.2), gives the interpretation of the bracket $[\![\cdot\,,\cdot]\!]$ as the graded commutator of differential operators.

**Theorem 2.6.** *If $f_1 \in \mathfrak{g}^{(k)}$ and $f_2 \in \mathfrak{g}^{(l)}$ then*

$$[\nabla_{f_1}, \nabla_{f_2}] = \nabla_{[\![f_1,f_2]\!]}.$$

Now we extend the operators $\nabla_f$ to a bigger algebra. For this purpose, let us represent the graded space $V$ in the form $V = V_0 \oplus V_+$, where $V_+ = \bigoplus_{g\in G\smallsetminus\{0\}} V_g$. We have $S(V) = S(V_0) \otimes_{\Bbbk} S(V_+)$. The algebra $S(V_0)$ can be identified with an algebra of (polynomial) functions on the dual space $V_0^*$. Let us extend this algebra to the algebra $\mathcal{A}_0$ of functions on $V_0^*$ of the form $F(v_1, \ldots, v_N)$, where $F \in C^\infty(\mathbb{R}^N)$ is a smooth function in many arguments and $v_i \in V_0$. Assign to all elements of $\mathcal{A}_0$ the degree 0. Denote by $\mathcal{A}(V)$ the graded algebra $\mathcal{A}_0 \otimes_{\Bbbk} S(V_+)$.

**Proposition 2.7.** *For each $f \in \mathfrak{g}^{(k)}$ the operator $\nabla_f$ has a unique extension to the algebra $\mathcal{A}(V)$.*

*Proof.* The uniqueness is obvious, let us prove the existence. Define the required extension by the formula

$$\nabla_f(F(v_1, \ldots, v_N) \otimes v_+) = \sum_{\substack{0\le l\le k \\ \sigma\in S_N^l}} \frac{\partial^l F}{\partial t_{\sigma(1)} \cdots \partial t_{\sigma(l)}}(v_1, \ldots, v_N) \otimes \nabla_{f(v_{\sigma(1)},\ldots,v_{\sigma(l)})}(v_+),$$

where $v_+ \in S(V_+)$ and $t_i$ stand for arguments of $F$. We are to prove that the operator is well defined. Assume that $F(v_1, \ldots, v_N)$ vanishes. Let $w_1, \ldots, w_L$ be a basis in the linear span of $v_1, \ldots, v_N$ and $v_i = \sum_j a_{ij} w_j$, where $a_{ij} \in \mathbb{R}$. The function

$$G(s_1, \ldots, s_L) = F(\sum_j a_{1j} s_j, \ldots, \sum_j a_{Nj} s_j)$$

is zero, so that by the chain rule we easily obtain

$$\nabla_f(F(v_1, \ldots, v_N) \otimes v_+) = \sum_{\substack{0\le l\le k \\ \sigma\in S_N^l}} \frac{\partial^l F}{\partial t_{\sigma(1)} \cdots \partial t_{\sigma(l)}}(v_1, \ldots, v_N) \otimes \nabla_{f(v_{\sigma(1)},\ldots,v_{\sigma(l)})}(v_+)$$

$$= \sum_{\substack{0\le l\le k\\ \sigma\in S_N^l}} \frac{\partial^l G}{\partial s_{\sigma(1)}\cdots\partial s_{\sigma(l)}}(w_1,\dots,w_L)\otimes\nabla_{f(w_{\sigma(1)},\dots,w_{\sigma(l)})}(v_+)=0.$$

□

If $V$ is a non-graded space then $\mathcal{A}(V)=\mathcal{A}_0$ and for $f\in\mathfrak{g}^{(k)}$ we have (2.3)

$$\nabla_f(F(v_1,\dots,v_N))=\sum_{\sigma\in S_N^k} f(v_{\sigma(1)},\dots,v_{\sigma(k)})\frac{\partial^k F}{\partial t_{\sigma(1)}\cdots\partial t_{\sigma(k)}}(v_1,\dots,v_N).$$

Next, if $V$ is a superspace (i.e., $G=\mathbb{Z}_2$, $V=V_0\oplus V_1$, and $\{g_1,g_2\}=(-1)^{g_1g_2}$) then each element of $\mathcal{A}(V)$ can be written as

$$F(v_1,\dots,v_{N_0},v_{N_0+1},\dots,v_N),$$

where $v_i\in V_0$ if $1\le i\le N_0$ and $v_i\in V_1$ if $i>N_0$, for a function $F\in C^\infty(\mathbb{R}^{N_0|N_1})$ in $N_0$ even variables and $N_1=N-N_0$ odd variables. Formula (2.3) holds true in this case.

## 3. The jet bundle setting: preliminaries

In this paper we deal with the infinite jet bundles of vector *superbundles*. While the jets of purely even bundles have been detailed extensively in the literature (see, e.g., [3, 14] and the references given there), we think that the graded setting needs to be described here. We assume that all manifolds and maps are $C^\infty$.

We start with fixing some notation related to jet bundles of *purely even* vector bundles (see [3, 14] for details). Let $\alpha\colon N\to M$ be such a vector bundle over a smooth manifold $M$ of dimension $n$, $\Gamma(\alpha)$ be the set of its sections, and $\alpha_k\colon J^k(\alpha)\to M$ be jet bundles of $\alpha$. For $s\in\Gamma(\alpha)$ we denote by $j_k(s)\in\Gamma(\alpha_k)$ the *$k$-jet* of $s$. Obviously, for $k>l$ there are vector bundles $\alpha_{k,l}\colon J^k(\alpha)\to J^l(\alpha)$ defined by the conditions $\alpha_{k,l}\circ j_k=j_l$.

If $x^1,\dots,x^n,u^1,\dots,u^m$ is a coordinate system on $N$ such that $x^i$ are base coordinates and $u^j$ are fiber ones, then $x^i$, $u^j_\sigma$, where $\sigma=i_1i_2\dots i_r$ are symmetric multiindexes of length $|\sigma|=r\le k$, $0\le k\le\infty$, are local coordinates on $J^k(\alpha)$. Here $u^j_\sigma$ are defined by

$$u^j_\sigma\circ j_k(s)=\frac{\partial^{|\sigma|}(u^j\circ s)}{\partial x^{i_1}\partial x^{i_2}\cdots\partial x^{i_r}},\qquad s\in\Gamma(\alpha).$$

*Remark* 3.1. Denote by $\mathrm{diff}(\alpha)$ the algebra of nonlinear differential operators $\Gamma(\alpha)\to C^\infty(M)$. Recall that there is one-to-one correspondence between $C^\infty(J^\infty(\alpha))$ and $\mathrm{diff}(\alpha)$ that takes an operator $\delta\in\mathrm{diff}(\alpha)$ to the function $f_\delta\in C^\infty(J^\infty(\alpha))$ such that $\delta(s)=j_k^*(s)(f_\delta)$ for $s\in\Gamma(\alpha)$.

Now let us turn to the case of superbundles. We say that a vector bundle $\pi\colon E\to M$ is a *superbundle* if it is the direct sum $\pi=\pi^0\oplus\pi^1$

of two vector bundles $\pi^0\colon E_0 \to M$ and $\pi^1\colon E_1 \to M$. When it comes to field theory, the sections of $\pi^0$ are regarded as bosonic fields, while the sections of $\pi^1$ as fermionic ones.

To define the jet space of a superbundle $\pi$ we shall use the following well-known fact.

**Proposition 3.2.** [2, 18, 19] *Suppose that $\alpha\colon N \to M$ is a (purely even) vector bundle with $m$-dimensional fiber; then the sheaf $\mathcal{S}_\alpha$ of sections of the exterior algebra $\Lambda^*(N^*)$ of $\alpha^*$ is the structure sheaf of an $n|m$-dimensional supermanifold. The correspondence $\alpha \mapsto \mathcal{S}_a$ is a functor.*

*Remark* 3.3. Note that the algebra $\mathcal{S}_\alpha(M)$ of functions on the supermanifold can be identified with the algebra of skew-symmetric $C^\infty(M)$-linear maps $\Gamma(\alpha) \times \cdots \times \Gamma(\alpha) \to C^\infty(M)$.

Consider the pullback $(\pi^0_k)^*(\pi^1_k)\colon J^k(\pi^0) \times_M J^k(\pi^1) \to J^k(\pi^0)$ of the bundle $\pi^1_k\colon J^k(\pi^1) \to M$ along the projection $\pi^0_k\colon J^k(\pi^0) \to M$. We define the *$k$-jet space* of $\pi$ to be the supermanifold that by virtue of Proposition 3.2 corresponds to the bundle $(\pi^0_k)^*(\pi^1_k)$. It will be denoted by $J^k(\pi)$. Clearly, the underlying even manifold of $J^k(\pi)$ is $J^k(\pi^0)$.

If $u^1, \ldots, u^{m_0}$ and $u^{m_0+1}, \ldots, u^m$ are fiber coordinates in $E_0$ and $E_1$ respectively, then $x^i$, $u^j_\sigma$ are local coordinates on $J^k(\pi)$, with $x^i$ and $u^j_\sigma$ for $j \le m_0$ being even coordinates, while $u^j_\sigma$ for $j > m_0$ being odd ones.

The reader will have no difficulty in constructing the natural projections $\pi_k\colon J^k(\pi) \to M$ and $\pi_{k,l}\colon J^k(\pi) \to J^l(\pi)$ for $k > l$. The inverse limit of the chain of projections

$$\cdots \longrightarrow J^k(\pi) \xrightarrow{\pi_{k,k-1}} J^{k-1}(\pi) \longrightarrow \cdots \longrightarrow J^1(\pi) \xrightarrow{\pi_{1,0}} J^0(\pi)$$

is said to be the *infinite jet space* and is denoted by $J^\infty(\pi)$. Obviously, the algebra $\mathcal{F}(\pi)$ of smooth functions on $J^\infty(\pi)$ is the direct limit of the algebra homomorphisms $\pi^*_{k,k-1}\colon C^\infty(J^{k-1}(\pi)) \to C^\infty(J^k(\pi))$. Note also that the algebra $\mathcal{F}(\pi)$ is filtered by its subalgebras $\mathcal{F}_k(\pi) = \pi^*_{\infty,k}(C^\infty(J^k(\pi))) \subset \mathcal{F}(\pi)$, since $\mathcal{F}_l(\pi) \subset \mathcal{F}_k(\pi)$ for $l < k$ and $\mathcal{F}(\pi) = \bigcup_k \mathcal{F}_k(\pi)$.

From Remarks 3.1 and 3.3 it is readily seen that every skew-symmetric multilinear differential operator $\delta\colon \Gamma(\pi^1) \times \cdots \times \Gamma(\pi^1) \to \operatorname{diff}(\pi^0)$ on $M$ can be interpreted as a function $f_\delta \in \mathcal{F}(\pi)$. Moreover, every function $f \in \mathcal{F}(\pi)$ can be written as a sum of the form $f = f_{\delta_1} + f_{\delta_2} + \cdots + f_{\delta_N}$.

Now we define the differential forms and differential operators over $J^\infty(\pi)$. The $\mathcal{F}(\pi)$-module $\Lambda^i(\pi)$ of differential forms of degree $i$ on $J^\infty(\pi)$ is naturally defined as the direct limit of the homomorphisms $\pi^*_{k,k-1}\colon \Lambda^i(J^{k-1}(\pi)) \to \Lambda^i(J^k(\pi))$.

Below we omit the letter $\pi$ and write simply $\mathcal{F}$, $\Lambda^k$, and so on, when no confusion can arise.

Let $\alpha_k \colon N_k \to J^k(\pi)$ be vector (super)bundles, $\alpha_{k+1,k} \colon N_{k+1} \to N_k$ be their morphisms, i.e., $\alpha_k \circ \alpha_{k+1,k} = \pi_{k+1,k} \circ \alpha_{k+1}$, and $N$ be the projective limit of $\alpha_{k+1,k}$. Consider the $\mathcal{F}$-module $P = \Gamma(\alpha_\infty)$ of sections of the vector bundle $\alpha_\infty \colon N \to J^\infty(\pi)$. From now on the term *module* will always mean an $\mathcal{F}$-module of this kind. For example, $\Lambda^i$ is a module. Note that every module $P$ is filtered by the submodules $P^k = \alpha^*_{\infty,k}(\Gamma(\alpha_k)) \subset P$, with $P_k$ being an $\mathcal{F}_k$-module.

If all the bundles $\alpha_k$ are pullbacks of a bundle $\alpha\colon N \to M$, that is, $\alpha_k = \pi^*_k(\alpha) \colon N_k = \pi^*_k(N) \to J^k(\pi)$, then the module $P = \Gamma(\alpha_\infty)$ is called *horizontal.* Clearly, in this case we have $P = \Gamma(\alpha) \otimes_{C^\infty(M)} \mathcal{F}$.

The correspondence $\delta \mapsto f_\delta$ between functions and differential operators can be generalized to the case of horizontal modules. Let $\operatorname{diff}(\beta_1, \beta_2)$ be the set of nonlinear differential operators $\Gamma(\beta_1) \to \Gamma(\beta_2)$, where $\beta_i$, $i = 1, 2$, are even vector bundles over $M$. We identify every skew-symmetric multilinear differential operator $\delta\colon \Gamma(\pi^1) \times \cdots \times \Gamma(\pi^1) \to \operatorname{diff}(\pi^0, \beta)$ with an element $p_\delta$ of the horizontal module $P = \Gamma(\pi^*_{\infty,0}(\beta))$. Clearly, every element $p \in P$ can be written as a sum of the form $p = p_{\delta_1} + p_{\delta_2} + \cdots + p_{\delta_N}$.

Let $P$ and $Q$ be two modules. A map $\Delta\colon P \to Q$ is called a *linear differential operator* if $\Delta(P^i) \subset Q^{i+j}$, where $j$ can depend on $i$, and all restrictions $\Delta|_{P^i}$ are linear differential operators.

*Remark* 3.4. An operator has a finite order if and only if there is an integer $l$ such that all the operators $\Delta|_{P^i}$ are of order less than $l$. Of course, operators of order zero are filtered module homomorphisms. First order scalar operators have the form $X + f$, where $X$ is a vector field and $f \in \mathcal{F}$.

In coordinates a scalar operator $\Delta\colon \mathcal{F} \to \mathcal{F}$ looks like

$$\Delta = \sum_{ij\sigma} a^{\sigma_1\ldots\sigma_s}_{j_1\ldots j_s i_1\ldots i_r} \frac{\partial^{s+r}}{\partial x^{i_1}\cdots\partial x^{i_r}\partial u^{j_1}_{\sigma_1}\cdots\partial u^{j_s}_{\sigma_s}}, \tag{3.1}$$

where $a^{\sigma_1\ldots\sigma_s}_{j_1\ldots j_s i_1\ldots i_r} \in \mathcal{F}$ and for each integer $k$ there is only a finite number of nonzero coefficients $a^{\sigma_1\ldots\sigma_s}_{j_1\ldots j_s i_1\ldots i_r}$ with $|\sigma_l| \le k$.

Let $\alpha_1$ and $\alpha_2$ be two vector (super)bundles over $M$ and $\Delta\colon \Gamma(\alpha_1) \to \Gamma(\alpha_2)$ be a linear differential operator (over $M$). Consider the horizontal modules $P_i = \Gamma(\pi^*_{\infty,0}(\alpha_i))$, $i = 1, 2$. The equality

$$\hat{\Delta}(p_\delta) = p_{\Delta\circ\delta}, \qquad p_\delta \in P_1,$$

where the operator $\Delta \circ \delta\colon \Gamma(\pi^0) \times \cdots \times \Gamma(\pi^0) \to \operatorname{diff}(\pi^0, \alpha)$ is given by $(\Delta \circ \delta)(s_1, \ldots, s_N) = \Delta \circ (\delta(s_1, \ldots, s_N))$, defines a linear differential operator $\hat{\Delta}\colon P_1 \to P_2$ called the *lifting* of $\Delta$.

Note that the parity of the operator $\hat{\Delta}$ is equal to the parity of $\Delta$.

The coordinate expression of $\hat{\Delta}$ is obtained from that of $\Delta$ by replacing partial derivatives $\partial/\partial x^i$ by the *total derivatives*

$$D_i = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}.$$

Let $P_1$ and $P_2$ be horizontal modules. A linear differential operator $\Delta\colon P_1 \to P_2$ is called *$\mathcal{C}$-differential* if it is of the form

$$\Delta = \sum_i f_i \hat{\Delta}_i, \qquad f_i \in \mathcal{F}. \tag{3.2}$$

It is obvious that the set of all $\mathcal{C}$-differential operators from $P_1$ to $P_2$, denoted by $\mathcal{C}\mathrm{Diff}(P_1, P_2)$, is a horizontal module.

In local coordinates, $\mathcal{C}$-differential operators have the form of a matrix $\left\|a^\sigma_{ij} D_\sigma\right\|$, where $a^\sigma_{ij} \in \mathcal{F}$, $D_\sigma = D_{i_1} \circ \cdots \circ D_{i_r}$ for $\sigma = i_1 \dots i_r$.

A differential operator $\nabla\colon P_1 \to P_2$ on $J^\infty(\pi)$ is said to be *vertical* if $[\nabla, f] = 0$ for all $f \in C^\infty(M)$. In coordinates, this is equivalent to say that a vertical operator does not contain derivatives with respect to $x^i$ and has no free term, i.e., $\nabla = \sum_{s\geq 1} a^{\sigma_1\dots\sigma_s}_{j_1\dots j_s} \partial^s/\partial u^{j_1}_{\sigma_1} \cdots \partial u^{j_s}_{\sigma_s}$.

**Lemma 3.5.** *Each differential operator $\Delta$ can be written as*

$$\Delta = \sum_i \square_i \circ \nabla_i$$

*where $\square_i$ are $\mathcal{C}$-differential operators and $\nabla_i$ are vertical operators. The sum may be infinite.*

*Proof.* It suffices to consider the case of scalar operator $\Delta$. In coordinate expression (3.1) we substitute operators $(D_i - \sum_{j\sigma} u^j_{\sigma i}\partial/\partial u^j_\sigma)$ for $\partial/\partial x^i$, obtaining the expression $\Delta = \sum_\sigma D_\sigma \circ \nabla_\sigma$, where $\nabla_\sigma$ are vertical operators. The proof is completed by using the partition of unity technique. □

*Remark* 3.6. The coordinate expression $\Delta = \sum_\sigma D_\sigma \circ \nabla_\sigma$ for a scalar operator is unique.

*Remark* 3.7. The above lemma shows that differential operators considered in our paper coincide with 'local differential operators' from [20].

*Remark* 3.8. It is clear that each scalar vertical operator $\nabla\colon \mathcal{F} \to \mathcal{F}$ induces a well-defined map $\nabla\colon P \to P$ for any horizontal module $P$. In coordinate language, this is the component-wise action.

A vector field $X$ on $J^\infty(\pi)$ is said to be a *Cartan vector field* if it has the form

$$X = \sum_i f_i \hat{X}_i, \qquad f_i \in \mathcal{F}, \tag{3.3}$$

where $X_i \in \mathrm{D}(M)$ are vector fields on $M$. In other words, the set of Cartan vector fields is the intersection of the sets of all vector fields and $\mathcal{C}$-differential operators. In coordinates, $X = \sum_i a_i D_i$, where $a_i \in \mathcal{F}$.

A vertical vector field $Y$ on $J^\infty(\pi)$ is called an *evolutionary field* if it commutes with all vector fields lifted from the base $M$, that is, $[Y, \hat{X}] = 0$ for all vector fields $X$ on $M$.

Denote the Lie algebra of all evolutionary field by $\varkappa$. From the definition it immediately follows that each evolutionary field $Y$ is uniquely determined by its restriction $Y|_{\mathcal{F}_0} : \mathcal{F}_0 \to \mathcal{F}$. Conversely, any $C^\infty(M)$-linear map $Y\colon \mathcal{F}_0 \to \mathcal{F}$ satisfying the Leibniz rule

$$Y(fg) = Y(f)g + (-1)^{Yf} fY(g)$$

gives rise to an evolutionary field. Here and subsequently, symbols used as the exponents of $(-1)$ stand for the corresponding parities, similarly to our convention on commutation factors.

Next, note that the sections of the bundle $\pi^*$ dual to $\pi$ can be identified with fiberwise linear functions on $J^0(\pi)$, thus $\Gamma(\pi^*) \subset \mathcal{F}_0$, and evolutionary fields are uniquely determined by their restrictions to $\Gamma(\pi^*)$. Moreover, it is easy to see that each $C^\infty(M)$-linear mapping $\Gamma(\pi^*) \to \mathcal{F}$ yields an evolutionary field, so that $\varkappa = \mathrm{Hom}_{C^\infty(M)}(\Gamma(\pi^*), \mathcal{F}) = \Gamma(\pi) \otimes_{C^\infty(M)} \mathcal{F} = \Gamma({\pi_\infty}^*(\pi))$. In particular, $\varkappa$ is a horizontal module.

In coordinate language, the evolutionary field $E_\varphi$ that corresponds to an element $\varphi \in \Gamma(\pi_\infty^*(\pi))$ has the form

$$E_\varphi = \sum_{i,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma},$$

where $\varphi^1, \ldots, \varphi^m$ are components of $\varphi$, $\varphi^j \in \mathcal{F}$.

Remark that there is one canonical evolutionary field $E_{\varphi_{\mathrm{id}}}$ defined by the condition that the corresponding map $\Gamma(\pi^*) \to \mathcal{F}$ (i.e., the restriction of $E_{\varphi_{\mathrm{id}}}$ to fiberwise linear functions on $J^0(\pi)$) is the inclusion. In coordinates, we have $\varphi_{\mathrm{id}} = (u^1, \ldots, u^m)$, whence $E_\varphi(\varphi_{\mathrm{id}}) = \varphi$ for all $\varphi \in \varkappa$. The element $E_\varphi(\varphi_{\mathrm{id}})$ is well-defined due to Remark 3.8.

Obviously, the module $\varkappa$ is a Lie algebra over $C^\infty(M)$ with respect to the commutator of evolutionary fields. Denote the bracket on $\varkappa$ by $[\![\cdot\,,\cdot]\!]$. By definition, $[E_{\varphi_1}, E_{\varphi_2}] = E_{[\![\varphi_1,\varphi_2]\!]}$, where $\varphi_i \in \varkappa$.

**Lemma 3.9.** *For $\varphi_1$, $\varphi_2 \in \varkappa$ we have*

$$[\![\varphi_1, \varphi_2]\!] = E_{\varphi_1}(\varphi_2) - (-1)^{\varphi_1\psi_2} E_{\varphi_2}(\varphi_1).$$

*Proof.* $[\![\varphi_1, \varphi_2]\!] = [E_{\varphi_1}, E_{\varphi_2}](\varphi_{\mathrm{id}}) = E_{\varphi_1}(\varphi_2) - (-1)^{\varphi_1\varphi_2} E_{\varphi_2}(\varphi_1)$. □

Let $P$ be a horizontal module. For each element $p \in P$ there is a $\mathcal{C}$-differential operator $\ell_p\colon \varkappa \to P$ called the *universal linearization* of $p$ and defined by

$$\ell_p(\varphi) = (-1)^{p\varphi} E_\varphi(p), \qquad \varphi \in \varkappa.$$

A differential form $\omega \in \Lambda^k$ on $J^\infty(\pi)$ is called a *Cartan form* if it is annihilated by Cartan fields, so that $\omega(X_1, \ldots, X_k) = 0$ for Cartan vector fileds $X_i$. In coordinates, every Cartan one-form is a linear combination of the basic Cartan forms

$$\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma i}\, dx^i.$$

Denote the module of all Cartan $q$-forms by $\mathcal{C}\Lambda^q$. It is not hard to prove that $d(\mathcal{C}\Lambda^q) \subset \mathcal{C}\Lambda^{q+1}$. Therefore the quotient $\bar{d}\colon \bar{\Lambda}^q \to \bar{\Lambda}^{q+1}$ of the de Rham differential $d\colon \Lambda^q \to \Lambda^{q+1}$, where $\bar{\Lambda}^q = \Lambda^q/\mathcal{C}\Lambda^q$, is well defined. Elements of $\bar{\Lambda}^q$ can be identified with pullbacks of $q$-forms on $M$ along the projection $\pi_\infty$ and are called *horizontal forms*. Hence, $\bar{\Lambda}^q$ is a horizontal module. In terms of coordinates, $\bar{\Lambda}^q$ is generated by the forms $f\, dx^{i_1} \wedge \cdots \wedge dx^{i_q}$, where $f \in C^\infty(J^\infty(\pi))$.

The operator $\bar{d}\colon \bar{\Lambda}^q \to \bar{\Lambda}^{q+1}$ is called the *horizontal differential*. It is easily shown that $\bar{d} = \hat{d}$. In coordinates $\bar{d}$ looks like

$$\bar{d}(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_q}) = \sum_{i=1}^{n} D_i(f)\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q}.$$

The complex

$$0 \to \mathcal{F} \xrightarrow{\bar{d}} \bar{\Lambda}^1 \to \cdots \to \bar{\Lambda}^q \xrightarrow{\bar{d}} \bar{\Lambda}^{q+1} \to \cdots \to \bar{\Lambda}^n \to 0,$$

is called the *horizontal de Rham complex* on $J^\infty(\pi)$.

From Vinogradov's 'one-line theorem' [26, 27, 28] it follows that the cohomology of this complex, called the *horizontal cohomology*, at the terms $\bar{\Lambda}^q$ for $q \le n-1$ coincides with the de Rham cohomology of the base manifold $M$. Of interest is the cohomology group $\bar{H}^n$ at the term $\bar{\Lambda}^n$: forms belonging to $\bar{\Lambda}^n$ can be regarded as Lagrangian densities, while the subset $\bar{d}\bar{\Lambda}^{n-1} \subset \bar{\Lambda}^n$ consists of total divergences, so that the elements of $\bar{H}^n = \bar{\Lambda}^n/\bar{d}\bar{\Lambda}^{n-1}$ are Lagrangians (also called 'actions' or 'local functionals').

If $P$ is a module, we write $\hat{P} = \mathrm{Hom}_{\mathcal{F}}(P, \bar{\Lambda}^n)$ and use the angular brackets $\langle\cdot\,,\cdot\rangle$ to denote the natural pairing $\hat{P} \times P \to \bar{\Lambda}^n$. The module $\hat{\hat{P}}$ will be routinely identified with $P$ by means of the isomorphism $i_P\colon P \to \hat{\hat{P}}$ given by $\langle i_P(p), \hat{p}\rangle = (-1)^{p\hat{p}}\langle \hat{p}, p\rangle$, where $p \in P$ and $\hat{p} \in \hat{P}$. In particular, we shall consider both $\langle \hat{p}, p\rangle$ and $\langle p, \hat{p}\rangle = (-1)^{p\hat{p}}\langle \hat{p}, p\rangle$.

Let us recall the following fact.

**Proposition 3.10.** *For each operator $\Delta \in \mathcal{C}\mathrm{Diff}(P,Q)$ there exists a unique operator $\Delta^* \in \mathcal{C}\mathrm{Diff}(\hat{Q}, \hat{P})$ such that*

$$[\langle \hat{q}, \Delta(p)\rangle] = (-1)^{\Delta\hat{q}}[\langle \Delta^*(\hat{q}), p\rangle], \qquad \hat{q} \in \hat{Q}, \quad p \in P, \tag{3.4}$$

*where $[\omega]$ denotes the horizontal cohomology class of $\omega \in \bar{\Lambda}^n$.*

The operator $\Delta^*$ is called *adjoint* to $\Delta$.

From (3.4) it immediately follows that

(1) $\Delta$ and $\Delta^*$ are of equal parity;
(2) $(\Delta_1 \circ \Delta_2)^* = (-1)^{\Delta_1\Delta_2}\Delta_2^* \circ \Delta_1^*$;

(3) $\Delta^{**} = \Delta$.

In coordinates, we have

$$\Big\| \sum_\sigma a_{ij}^\sigma D_\sigma \Big\|^* = \Big\| \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_{ij}^\sigma \Big\|^{\mathrm{st}},$$

where $a_{ij}^\sigma \in \mathcal{F}$ and the symbol 'st' denotes the supertransposition.

Since evolutionary fields commute with the horizontal differential, the cohomology class $[E_\varphi(\omega)]$ for $\omega \in \bar{\Lambda}^n$ is well defined by $[\omega]$; denote it by $E_\varphi([\omega])$. By (3.4) we have

$$(3.5)\quad E_\varphi([\omega]) = [E_\varphi(\omega)] = (-1)^{\varphi\omega}[\ell_\omega(\varphi)] = [\langle \varphi, \ell_\omega^*(1)\rangle] = [\langle \varphi, \mathcal{E}(\omega)\rangle],$$

where $\mathcal{E}\colon \bar{\Lambda}^n \to \hat{\varkappa}$, $\mathcal{E}(\omega) = \ell_\omega^*(1)$, is the *Euler operator*, which takes Lagrangians to the corresponding Euler-Lagrange equations. Of course, the value $\mathcal{E}(\omega)$ is completely determined by the cohomology class $[\omega]$.

In coordinates, $\mathcal{E}(L\,dx^1 \wedge \cdots \wedge dx^n) = (\delta L/\delta u^1, \dots, \delta L/\delta u^m)$, where $\delta L/\delta u^j = \sum_\sigma (-1)^{|\sigma|} D_\sigma(\partial L/\partial u_\sigma^j)$.

*Remark* 3.11. From Vinogradov's 'one-line theorem' [26, 27, 28] it follows that

(1) $\ker \mathcal{E}/\bar{d}(\bar{\Lambda}^{n-1}) = H^n(M)$;
(2) $\psi \in \operatorname{im} \mathcal{E}$ if and only if $\ell_\psi^* = \ell_\psi$.

From Remark 3.6 we easily get the following statement.

**Lemma 3.12.** *If $\nabla\colon \bar{H}^n \to \bar{H}^n$ is a vertical operator and $\mathcal{E} \circ \nabla = 0$, then $\nabla = 0$.*

**Corollary 3.13.** *If two vertical operators induce the same map on $\bar{H}^n$ then they coincide.*

**Lemma 3.14.** *Let $\Delta \in \mathcal{C}\mathrm{Diff}(P, Q)$ and $p \in P$. Define an operator $\ell_{\Delta,p} \in \mathcal{C}\mathrm{Diff}(\varkappa, Q)$ by the formula*

$$\ell_{\Delta,p}(\varphi) = (-1)^{p\varphi} \ell_\Delta(\varphi)(p), \qquad \varphi \in \varkappa.$$

*Then*

$$\ell_{\Delta,p}^*(\hat{q}) = (-1)^{p\hat{q}} \ell_{\Delta^*,\hat{q}}^*(p), \qquad \hat{q} \in \hat{Q}.$$

*Proof.* By formula (3.4),

$$[\langle \Delta(p), \hat{q}\rangle] = (-1)^{\Delta p}[\langle p, \Delta^*(\hat{q})\rangle].$$

Applying $E_\varphi$ to both sides, we get

$$[\langle E_\varphi(\Delta)(p), \hat{q}\rangle] = (-1)^{p(\Delta+\varphi)}[\langle p, E_\varphi(\Delta^*)(\hat{q})\rangle],$$

and so

$$[\langle \ell_{\Delta,p}(\varphi), \hat{q}\rangle] = (-1)^{\Delta p + \varphi\hat{q}}[\langle p, \ell_{\Delta^*,\hat{q}}(\varphi)\rangle].$$

Again (3.4) yields

$$[\langle \varphi, \ell_{\Delta,p}^*(\hat{q})\rangle] = (-1)^{\varphi(\Delta+p+\hat{q})+p\hat{q}}[\langle \ell_{\Delta^*,\hat{q}}^*(p), \varphi\rangle]$$

and the lemma is proved. □

**Lemma 3.15.** *If* $\Delta \in \mathcal{C}\mathrm{Diff}(P, \bar{\Lambda}^n)$ *and* $p \in P$ *then*

$$\mathcal{E}(\Delta(p)) = \ell^*_{\Delta^*(1)}(p) + (-1)^{\Delta p}\ell^*_p(\Delta^*(1)).$$

*Proof.* By the definition of the Euler operator we have $\mathcal{E}(\Delta(p)) = \ell^*_{\Delta(p)}(1)$. Next, $\ell_{\Delta(p)} = \ell_{\Delta,p} + \Delta \circ \ell_p$, so that

$$\begin{aligned}\ell^*_{\Delta(p)}(1) &= \ell^*_{\Delta,p}(1) + (-1)^{\Delta p}\ell^*_p(\Delta^*(1)) \\ &= \ell^*_{\Delta^*,1}(p) + (-1)^{\Delta p}\ell^*_p(\Delta^*(1)) = \ell^*_{\Delta^*(1)}(p) + (-1)^{\Delta p}\ell^*_p(\Delta^*(1)). \quad \square\end{aligned}$$

**Corollary 3.16.** *If* $p \in P$ *and* $\hat{p} \in \hat{P}$ *then*

$$\mathcal{E}(\langle \hat{p}, p\rangle) = \ell^*_{\hat{p}}(p) + (-1)^{p\hat{p}}\ell^*_p(\hat{p}).$$

## 4. Variational Poisson bracket

In this section we introduce a (super)symmetric bracket on multilinear maps of the form $\bar{H}^n \times \cdots \times \bar{H}^n \to \bar{H}^n$. Then, we provide a description of maps and the bracket in terms of selfadjoint operators. To define the bracket we apply the general construction of Section 2.

Correspondingly, as the grading group we take $G = \mathbb{Z}_2$ with the super-commutation factor. Next, we put $V = \bar{H}^n$. The space $\mathcal{M}_k(\bar{H}^n)$ is defined to be the set of multilinear maps

$$f\colon \underbrace{\bar{H}^n \times \cdots \times \bar{H}^n}_{k \text{ times}} \to \bar{H}^n$$

such that $f$ has the form

$$f([\omega_1], \dots, [\omega_k]) = [\tilde{f}(\omega_1, \dots, \omega_k)] \tag{4.1}$$

where $\tilde{f}\colon \bar{\Lambda}^n \times \cdots \times \bar{\Lambda}^n \to \bar{\Lambda}^n$ is a multilinear differential operator.

From Corollary 3.13 it follows that the natural mapping $i\colon \varkappa \to \mathcal{M}_1(\bar{H}^n)$ given by the equality $i(\varphi)([\omega]) = E_\varphi([\omega]) = [E_\varphi(\omega)]$, where $\omega \in \bar{\Lambda}^n$, is an inclusion. Hence we can take $\varkappa$ for the algebra $\mathfrak{g}$. The spaces $\mathfrak{g}^{(k)}$, thus constructed, we shall denote by $\varkappa^{(k)}$. Elements of $\varkappa^{(k)}$ will be referred to as *variational (super)symmetric multivectors.*

**Definition 4.1.** The bracket $[\![\cdot\,,\cdot]\!]$ on $\varkappa^{(k)}$ defined above is said to be the *variational Poisson bracket.*

Notice that on $\varkappa$ the Poisson bracket coincide with the bracket defined just before Lemma 3.9.

Let us denote by $\mathcal{C}\mathrm{Diff}^{\mathrm{self}}_{(k)}(P, \hat{P})$ the module of $k$-linear $\mathcal{C}$-differential operators $\Delta\colon P \times \cdots \times P \to \hat{P}$ which are symmetric and selfadjoint in each argument, that is,

$$\Delta(p_1, \dots, p_i, p_{i+1}, \dots, p_k) = (-1)^{p_i p_{i+1}}\Delta(p_1, \dots, p_{i+1}, p_i, \dots, p_k)$$

and $\Delta^{*_i} = \Delta$, where ${}^{*_i}$ means adjoining in $i$th argument.

**Theorem 4.2.** *For each $f \in \varkappa^{(k)}$ there exists a unique $\mathcal{C}$-differential operator $\Delta_f \in \mathcal{C}\mathrm{Diff}^{\mathrm{self}}_{(k-1)}(\hat{\varkappa}, \varkappa)$ such that*

$$f(\omega_1, \dots, \omega_k) = [\langle \Delta_f(\mathcal{E}(\omega_1), \dots, \mathcal{E}(\omega_{k-1})), \mathcal{E}(\omega_k)\rangle]. \tag{4.2}$$

*Proof.* We start with existence of $\Delta_f$.

*Step* 1. Using Lemma 3.5 we can write the operator $\tilde{f}$ as follows:

$$\tilde{f}(\omega_1, \dots, \omega_k) = \sum_i \square_i(\nabla^i_1(\omega_1), \dots, \nabla^i_k(\omega_k)),$$

where $\nabla^i_j$ are vertical scalar operators and $\square_i$ are $\mathcal{C}$-differential scalar operators. Then by formula (3.4) we get

$$f(\omega_1, \dots, \omega_k) = [\sum_i (\square_i(\nabla^i_1(\omega_1), \dots, \nabla^i_{k-1}(\omega_{k-1})))^*(1)\nabla^i_k(\omega_k)].$$

But from the definition of variational multivectors we have

$$f(\omega_1, \dots, \omega_k) = [E_\varphi(\omega_k)].$$

Thus,

$$\varphi = \sum_i (\square_i(\nabla^i_1(\omega_1), \dots, \nabla^i_{k-1}(\omega_{k-1})))^*(1)\nabla^i_k(\varphi_{\mathrm{id}}).$$

By (3.5) we obtain

$$\begin{aligned} f(\omega_1, \dots, \omega_k) \\ = [\langle \sum_i (\square_i(\nabla^i_1(\omega_1), \dots, \nabla^i_{k-1}(\omega_{k-1})))^*(1)\nabla^i_k(\varphi_{\mathrm{id}}), \mathcal{E}(\omega_k)\rangle]. \end{aligned} \tag{4.3}$$

Let us define $\mathcal{C}$-differential operators

$$\square'_i \colon \underbrace{\bar{\Lambda}^n \times \dots \times \bar{\Lambda}^n}_{(k-1)\text{ times}} \to \varkappa$$

by the formula

$$\square'_i(\omega_1, \dots, \omega_{k-1}) = (\square_i(\omega_1, \dots, \omega_{k-1}))^*(1)\nabla^i_k(\varphi_{\mathrm{id}}).$$

Now we can rewrite (4.3) as follows

$$f(\omega_1, \dots, \omega_k) = [\langle \sum_i \square'_i(\nabla^i_1(\omega_1), \dots, \nabla^i_{k-1}(\omega_{k-1})), \mathcal{E}(\omega_k)\rangle].$$

*Step* 2. Formula (3.4) yields

$$\begin{aligned} f(\omega_1, \dots, \omega_k) = [\sum_i (-1)^{\omega_k(\omega_{k-1}+\nabla^i_{k-1})} \\ (\square'_i(\nabla^i_1(\omega_1), \dots, \nabla^i_{k-2}(\omega_{k-2})))^*(\mathcal{E}(\omega_k))\nabla^i_{k-1}(\omega_{k-1})]. \end{aligned}$$

As before, from this equality we obtain

$$\begin{aligned} f(\omega_1, \dots, \omega_k) = [\langle \sum_i (-1)^{\omega_k(\omega_{k-1}+\nabla^i_{k-1})} \\ (\square'_i(\nabla^i_1(\omega_1), \dots, \nabla^i_{k-2}(\omega_{k-2})))^*(\mathcal{E}(\omega_k))\nabla^i_{k-1}(\varphi_{\mathrm{id}}), \mathcal{E}(\omega_{k-1})\rangle]. \end{aligned} \tag{4.4}$$

Introducing a new operator

$$\square_i''\colon \underbrace{\bar\Lambda^n \times \cdots \times \bar\Lambda^n}_{(k-2)\text{ times}} \times \hat{\varkappa} \to \varkappa,$$

$$\square_i''(\omega_1,\dots,\omega_{k-2},\psi) = (\square_i'(\omega_1,\dots,\omega_{k-2}))^*(\psi)\nabla_{k-1}^i(\varphi_{\mathrm{id}}),$$

we rewrite formula (4.4) as

$$f(\omega_1,\dots,\omega_k) = [\langle \sum_i (-1)^{\omega_k(\omega_{k-1}+\nabla_{k-1}^i)} \square_i''(\nabla_1^i(\omega_1),\dots,\nabla_{k-2}^i(\omega_{k-2}),\mathcal{E}(\omega_k)),\mathcal{E}(\omega_{k-1})\rangle].$$

Repeating Step 2 $(k-2)$ times, we obtain the desired operator $\Delta_f$. The uniqueness of $\Delta_f$ is evident from the following statement.

**Lemma 4.3.** *If for any $\omega_i \in \bar\Lambda^n$ an $l$-linear $\mathcal{C}$-differential operator $A\colon \hat\varkappa \times \cdots \times \hat\varkappa \to P$ satisfies the condition $A(\mathcal{E}(\omega_1),\dots,\mathcal{E}(\omega_l)) = 0$, then $A = 0$.*

*Proof.* This results from the fact that any element of $\hat\varkappa$ whose components belong to $C^\infty(M)$ can be written as $\mathcal{E}(\omega)$ for an $\omega \in \bar\Lambda^n$. □

To complete the proof, it remains to note that the uniqueness of the operator $\Delta_f$ implies that it is symmetric and selfadjoint. □

From now on, we identify variational multivectors with the corresponding $\mathcal{C}$-differential operators.

**Theorem 4.4.** *If $\Delta_1 \in \varkappa^{(k)}$ and $\Delta_2 \in \varkappa^{(l)}$ then for all $\psi_1,\dots,\psi_{k+l-2} \in \hat\varkappa$ we have*

$$\begin{aligned}(4.5)\quad & [\![\Delta_1,\Delta_2]\!](\psi_1,\dots,\psi_{k+l-2}) \\ &= (-1)^{\Delta_1\Delta_2}\Big(\sum_{\sigma\in S_{k+l-2}^{l-1}} \epsilon_\psi(\sigma)(-1)^{\Delta_1\psi_{\sigma(1,l-1)}}\ell_{\Delta_2,\psi_{\sigma(1,l-1)}}(\Delta_1(\psi_{\sigma(l,k+l-2)})) \\ &\qquad - \sum_{\sigma\in S_{k+l-2}^{k}} \epsilon_\psi(\sigma)\Delta_2(\ell^*_{\Delta_1,\psi_{\sigma(1,k-1)}}(\psi_{\sigma(k)}),\psi_{\sigma(k+1,k+l-2)})\Big) \\ &\quad - \sum_{\sigma\in S_{k+l-2}^{k-1}} \epsilon_\psi(\sigma)(-1)^{\Delta_2\psi_{\sigma(1,k-1)}}\ell_{\Delta_1,\psi_{\sigma(1,k-1)}}(\Delta_2(\psi_{\sigma(k,k+l-2)})) \\ &\qquad\qquad + \sum_{\sigma\in S_{k+l-2}^{l}} \epsilon_\psi(\sigma)\Delta_1(\ell^*_{\Delta_2,\psi_{\sigma(1,l-1)}}(\psi_{\sigma(l)}),\psi_{\sigma(l+1,k+l-2)}),\end{aligned}$$

*where $\ell_{\Delta,\psi_1,\dots,\psi_s}$ is the $\mathcal{C}$-differential operator $\varkappa \to \varkappa$ defined by*

$$\ell_{\Delta,\psi_1,\dots,\psi_s}(\varphi) = (-1)^{(\psi_1+\cdots+\psi_s)\varphi}\ell_\Delta(\varphi)(\psi_1,\dots,\psi_s), \qquad \varphi \in \varkappa,$$

*and $\psi_{\sigma(k_1,k_2)}$ is the notation for the vector $\psi_{\sigma(k_1)},\dots,\psi_{\sigma(k_2)}$. When $\psi_{\sigma(k_1,k_2)}$ is used as the exponent of $(-1)$ it means the sum of degrees of $\psi_{\sigma(k_1)},\dots,\psi_{\sigma(k_2)}$.*

*Proof.* Assume that $\psi_i \in \operatorname{im}\mathcal{E}$. Then equality (2.2) yields

$$
\begin{aligned}
(4.6)\quad [\langle [\![\Delta_1,\Delta_2]\!](\psi_1,\dots,\psi_{k+l-2}),\psi_{k+l-1}\rangle] &= \sum_{\sigma\in S^l_{k+l-1}} \epsilon_\psi(\sigma) \\
&[\langle \Delta_1(\mathcal{E}(\langle \Delta_2(\psi_{\sigma(1,l-1)}),\psi_{\sigma(l)}\rangle),\psi_{\sigma(l+1,k+l-2)}),\psi_{\sigma(k+l-1)}\rangle] \\
&- (-1)^{\Delta_1\Delta_2} \sum_{\sigma\in S^k_{k+l-1}} \epsilon_\psi(\sigma) \\
&[\langle \Delta_2(\mathcal{E}(\langle \Delta_1(\psi_{\sigma(1,k-1)}),\psi_{\sigma(k)}\rangle),\psi_{\sigma(k+1,k+l-2)}),\psi_{\sigma(k+l-1)}\rangle] \\
&= \sum_{\sigma\in S^l_{k+l-1}} \epsilon_\psi(\sigma)(-1)^{\Delta_1(\Delta_2+\psi_{\sigma(1,l)})} \\
&[\langle \mathcal{E}(\langle \Delta_2(\psi_{\sigma(1,l-1)}),\psi_{\sigma(l)}\rangle),\Delta_1(\psi_{\sigma(l+1,k+l-1)})\rangle] \\
&- \sum_{\sigma\in S^k_{k+l-1}} \epsilon_\psi(\sigma)(-1)^{\Delta_2\psi_{\sigma(1,k)}} \\
&[\langle \mathcal{E}(\langle \Delta_1(\psi_{\sigma(1,k-1)}),\psi_{\sigma(k)}\rangle),\Delta_2(\psi_{\sigma(k+1,k+l-1)})\rangle].
\end{aligned}
$$

It follows easily that for any operator $\Delta \in \mathcal{C}\mathrm{Diff}^{\mathrm{self}}_{(s)}(\hat{\varkappa},\varkappa)$ and $\psi_i \in \operatorname{im}\mathcal{E}$ we have

$$
\begin{aligned}
\mathcal{E}(\langle \Delta(\psi_1,\dots,\psi_s),\psi_{s+1}\rangle) &= \ell^*_{\Delta,\psi_1,\dots,\psi_s}(\psi_{s+1}) \\
&+ \sum_{i=1}^{s+1}(-1)^{\psi_i(\Delta+\psi_1+\dots+\psi_{i-1})}\ell_{\psi_i}(\Delta(\psi_1,\dots,\psi_{i-1},\psi_{i+1},\dots,\psi_{s+1})).
\end{aligned}
$$

If we combine this with (4.6) and take into account that all terms of the form $[\langle \Delta.(\psi_{\sigma(\dots)}),\ell_{\psi_i}\Delta.(\psi_{\sigma(\dots)})\rangle]$ cancel, we get

$$
\begin{aligned}
(4.7)\quad & \\
[\langle [\![\Delta_1,\Delta_2]\!](\psi_1,\dots,\psi_{k+l-2}),\psi_{k+l-1}\rangle] &= \sum_{\sigma\in S^l_{k+l-1}} \epsilon_\psi(\sigma)(-1)^{\Delta_1(\Delta_2+\psi_{\sigma(1,l)})} \\
&[\langle \ell^*_{\Delta_2,\psi_{\sigma(1,l-1)}}(\psi_{\sigma(l)}),\Delta_1(\psi_{\sigma(l+1,k+l-1)})\rangle] \\
- \sum_{\sigma\in S^k_{k+l-1}} \epsilon_\psi(\sigma)(-1)^{\Delta_2\psi_{\sigma(1,k)}} &[\langle \ell^*_{\Delta_1(\psi_{\sigma(1,k-1)}}(\psi_{\sigma(k)}),\Delta_2(\psi_{\sigma(k+1,k+l-1)})\rangle].
\end{aligned}
$$

The sums in the right-hand side of the last equality can be split into two parts depending on whether $\sigma(k+l-1)=k+l-1$ or not:

$$
\begin{aligned}
(4.8)\quad & \\
[\langle [\![\Delta_1,\Delta_2]\!](\psi_1,\dots,\psi_{k+l-2}),\psi_{k+l-1}\rangle] &= \sum_{\sigma\in S^l_{k+l-2}} \epsilon_\psi(\sigma)(-1)^{\Delta_1(\Delta_2+\psi_{\sigma(1,l)})} \\
&[\langle \ell^*_{\Delta_2,\psi_{\sigma(1,l-1)}}(\psi_{\sigma(l)}),\Delta_1(\psi_{\sigma(l+1,k+l-2)},\psi_{k+l-1})\rangle] \\
&+ \sum_{\sigma\in S^{l-1}_{k+l-2}} \epsilon_\psi(\sigma)(-1)^{\Delta_1(\Delta_2+\psi_{\sigma(1,l-1)})+\psi_{k+l-1}(\Delta_1+\psi_{\sigma(l,k+l-2)})}
\end{aligned}
$$

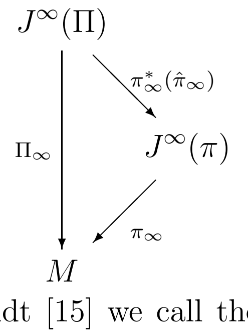
$$
\begin{aligned}
&[\langle \ell^*_{\Delta_2,\psi_{\sigma(1,l-1)}}(\psi_{k+l-1}), \Delta_1(\psi_{\sigma(l,k+l-2)})\rangle] \\
&\quad - \sum_{\sigma\in S^k_{k+l-2}} \epsilon_\psi(\sigma)(-1)^{\Delta_2\psi_{\sigma(1,k)}} \\
&[\langle \ell^*_{\Delta_1(\psi_{\sigma(1,k-1)}}(\psi_{\sigma(k)}), \Delta_2(\psi_{\sigma(k+1,k+l-2)}, \psi_{k+l-1})\rangle] \\
&- \sum_{\sigma\in S^{k-1}_{k+l-2}} \epsilon_\psi(\sigma)(-1)^{\Delta_2\psi_{\sigma(1,k-1)}+\psi_{k+l-1}(\Delta_1+\psi_{\sigma(k,k+l-2)})} \\
&[\langle \ell^*_{\Delta_1,\psi_{\sigma(1,k-1)}}(\psi_{k+l-1}), \Delta_2(\psi_{\sigma(k,k+l-2)})\rangle] = \sum_{\sigma\in S^l_{k+l-2}} \epsilon_\psi(\sigma) \\
&[\langle \Delta_1(\ell^*_{\Delta_2,\psi_{\sigma(1,l-1)}}(\psi_{\sigma(l)}), \psi_{\sigma(l+1,k+l-2)}), \psi_{k+l-1}\rangle] \\
&\quad + \sum_{\sigma\in S^{l-1}_{k+l-2}} \epsilon_\psi(\sigma)(-1)^{\Delta_1(\Delta_2+\psi_{\sigma(1,l-1)})} \\
&[\langle \ell_{\Delta_2,\psi_{\sigma(1,l-1)}}(\Delta_1(\psi_{\sigma(l,k+l-2)})), \psi_{k+l-1}\rangle] \\
&\quad - (-1)^{\Delta_1\Delta_2} \sum_{\sigma\in S^k_{k+l-2}} \epsilon_\psi(\sigma) \\
&[\langle \Delta_2(\ell^*_{\Delta_1,\psi_{\sigma(1,k-1)}}(\psi_{\sigma(k)}), \psi_{\sigma(k+1,k+l-2)}), \psi_{k+l-1}\rangle] \\
&- \sum_{\sigma\in S^{k-1}_{k+l-2}} \epsilon_\psi(\sigma)(-1)^{\Delta_2\psi_{\sigma(1,k-1)}}[\langle \ell_{\Delta_1,\psi_{\sigma(1,k-1)}}(\Delta_2(\psi_{\sigma(k,k+l-2)})), \psi_{k+l-1}\rangle].
\end{aligned}
$$

Since the right-hand side of (4.5) is a $\mathcal{C}$-differential operator in $\psi_i$, Lemma 4.3 shows that (4.5) holds for arbitrary $\psi_i \in \hat{\varkappa}$, and not just for $\psi_i \in \operatorname{im}\mathcal{E}$. This concludes the proof. □

## 5. Cotangent bundle to a bundle and the variational Poisson bracket

Here we compare our variational Poisson brackets with the Poisson brackets constructed by Kupershmidt [15].

Let us consider the bundles $\hat{\pi}\colon \hat{E} = E^* \otimes_M \bigwedge^n(T^*M) \to M$ and $\Pi = \pi \oplus \hat{\pi}\colon E \oplus \hat{E} \to M$. We have the commutative diagram




Following Kupershmidt [15] we call the bundle $\Pi_\infty\colon J^\infty(\Pi) \to M$ the *cotangent bundle* of the bundle $\pi$.

Let us denote by $p^j$, $j = 1, \dots, m$, the fiber coordinates in $\hat{E}$ dual to $u^j$ with respect to a volume form on $M$. Then coordinates in $J^\infty(\Pi)$ are $x^i$, $u^j_\sigma$, $p^j_\sigma$.

We see that $\varkappa(\Pi) = \varkappa_\Pi \oplus \hat{\varkappa}_\Pi$, where $\varkappa_\Pi = \Gamma(\Pi^*(\pi))$.

On the space $\bar{H}^n(\Pi)$ there exists a natural *Poisson bracket* [15]

$$[\![F, H]\!] = [\langle A(\mathcal{E}(F)), \mathcal{E}(H)\rangle], \qquad F,\ H \in \bar{H}^n(\Pi), \tag{5.1}$$

where $A\colon \hat{\varkappa}(\Pi) \to \varkappa(\Pi)$, $A(\psi, \varphi) = (\varphi, -\psi)$ for $\varphi \in \varkappa_\Pi$ and $\psi \in \hat{\varkappa}_\Pi$. In coordinates we have

$$[\![F, H]\!] = \sum_j \Big(\frac{\delta F}{\delta p^j}\frac{\delta H}{\delta u^j} - \frac{\delta F}{\delta u^j}\frac{\delta H}{\delta p^j}\Big).$$

Let us consider the $\mathbb{Z}_2$-graded commutative algebra $\mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(*)}(\hat{\varkappa}, \mathcal{F})$ with the multiplication

$$\Delta_1 \cdot \Delta_2(\psi_1, \dots, \psi_{k+l}) = \sum_{\sigma \in S^k_{k+l}} \epsilon_\psi(\sigma)(-1)^{\Delta_2 \psi_{\sigma(1,k)}} \Delta_1(\psi_{\sigma(1,k)})\Delta_2(\psi_{\sigma(k+1,k+l)}),$$

where $\Delta_1 \in \mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(k)}(\hat{\varkappa}, \mathcal{F})$, $\Delta_2 \in \mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(l)}(\hat{\varkappa}, \mathcal{F})$, and $\Delta_1 \cdot \Delta_2 \in \mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(k+l)}(\hat{\varkappa}, \mathcal{F})$.

Now, since by definition elements of $\mathcal{F}(\hat{\pi})$ are identified with differential operators from $\Gamma(\hat{\pi})$ to $C^\infty(M)$, we have the natural inclusion $\mathcal{C}\mathrm{Diff}(\hat{\varkappa}, \mathcal{F}) \to \mathcal{F}(\Pi)$, which uniquely prolongs to the inclusion of algebras

$$\mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(*)}(\hat{\varkappa}, \mathcal{F}) \to \mathcal{F}(\Pi).$$

This allows us to consider the commutative diagram

$$\begin{array}{ccc} \mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(*)}(\hat{\varkappa}, \bar{\Lambda}^n) & \longrightarrow & \bar{\Lambda}^n(\Pi) \\ {\scriptstyle \mu_{(*)}}\downarrow & & \downarrow \\ \mathcal{C}\mathrm{Diff}^{\mathrm{self}}_{(*)}(\hat{\varkappa}, \varkappa) = \varkappa^{(*)} & \xrightarrow{\ \iota\ } & \bar{H}^n(\Pi) \end{array}$$

where $\mu_{(k)}\colon \mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(k)}(\hat{\varkappa}, \bar{\Lambda}^n) \to \mathcal{C}\mathrm{Diff}^{\mathrm{self}}_{(k-1)}(\hat{\varkappa}, \varkappa) = \varkappa^{(k)}$ is defined by the formula $\mu_{(k)}(\Delta)(\psi_1, \dots, \psi_{k-1}) = (\Delta(\psi_1, \dots, \psi_{k-1}))^*(1)$. The map $\iota$ is uniquely defined by this diagram.

Let us show that $\iota$ is an injection. Indeed, assume that $\iota(\square) = 0$, $\square \in \mathcal{C}\mathrm{Diff}^{\mathrm{self}}_{(k-1)}(\hat{\varkappa}, \varkappa)$. Then $\nu = \langle \square(\psi_1, \dots, \psi_{k-1}), \psi_k\rangle$ belongs to $\bar{d}\bar{\Lambda}^{n-1}$ for all $\psi_1, \dots, \psi_k \in \hat{\varkappa}$, which is impossible because $\nu$ is linear in $\psi_k$.

**Theorem 5.1.** *The Poisson bracket* (5.1) *extends the variational Poisson bracket* (*Definition* 4.1) *to* $\bar{H}^n(\Pi)$.

*Proof.* From the definition of the Euler operator it follows that

$$\mathcal{E}|_{\mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(k)}(\hat{\varkappa},\bar{\Lambda}^n)} : \mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(k)}(\hat{\varkappa},\bar{\Lambda}^n) \to \mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(k)}(\hat{\varkappa},\hat{\varkappa}) \oplus \mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(k-1)}(\hat{\varkappa},\varkappa)$$

has the form

$$\mathcal{E}|_{\mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(k)}(\hat{\varkappa},\bar{\Lambda}^n)}(\Delta) = (\eta_{(k)}(\Delta), \mu_{(k)}(\Delta)),$$

where $\eta_{(k)}(\Delta)(\psi_1,\dots,\psi_k) = \ell^*_{\Delta,\psi_1,\dots,\psi_k}(1)$ and the operator $\ell_{\Delta,\psi_1,\dots,\psi_k}\colon \varkappa \to \bar{\Lambda}^n$ is defined by $\ell_{\Delta,\psi_1,\dots,\psi_k}(\varphi) = (-1)^{\varphi(\psi_1,\dots,\psi_k)}\ell_\Delta(\varphi)(\psi_1,\dots,\psi_k)$.

*Remark* 5.2. In coordinates, $\eta_{(k)} = (\delta/\delta u^1,\dots,\delta/\delta u^m)$ and $\mu_{(k)} = (\delta/\delta p^1,\dots,\delta/\delta p^m)$.

Now if we recall (4.7), we see that the proof is completed by proving the formula

$$\eta_{(k)}(\Delta)(\psi_1,\dots,\psi_k) = \ell^*_{\Box,\psi_1,\dots,\psi_{k-1}}(\psi_k),$$

where $\Delta \in \mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(k)}(\hat{\varkappa},\bar{\Lambda}^n)$ and $\Box = \mu_{(k)}(\Delta)$. To this end, let us consider the equality

$$[\Delta(\psi_1,\dots,\psi_k)] = [\langle \Box(\psi_1,\dots,\psi_{k-1}),\psi_k\rangle].$$

Applying $E_\varphi$ to both sides, we get

$$[E_\varphi(\Delta)(\psi_1,\dots,\psi_k)] = [\langle E_\varphi(\Box)(\psi_1,\dots,\psi_{k-1}),\psi_k\rangle],$$

and so

$$[\langle \varphi, \ell^*_{\Delta,\psi_1,\dots,\psi_k}(1)\rangle] = [\langle \varphi, \ell^*_{\Box,\psi_1,\dots,\psi_{k-1}}(\psi_k)\rangle]. \qquad \Box$$

## 6. Variational Schouten bracket, Antibracket, Hamiltonian formalism

The purpose of this section is to construct an odd counterpart of the Poisson bracket discussed above. We shall again make use of the general scheme of Section 2.

To this end, we take $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to be the grading group, the commutation factor being the product of two super-commutation factors. Put $V = \bar{H}^n_{[1]}$, i.e., $V_{i,0} = 0$, $V_{i,1} = \bar{H}^n_{(i)}$, $i \in \mathbb{Z}_2$. The spaces $\mathcal{M}_k(V)$ and the algebra $\mathfrak{g}$ are defined in the same way as in Section 4.

The resulting spaces $\mathfrak{g}^{(k)}$ we denote by $\varkappa^{(k)}_{\mathrm{odd}}$ and call their elements *variational multivectors*. The bracket $[\![\cdot\,,\cdot]\!]$ on $\varkappa^{(k)}_{\mathrm{odd}}$ is called the *variational Schouten bracket*.

Let us denote by $\mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(k)}(P,\hat{P})$ the module of $k$-linear $\mathcal{C}$-differential operators $\Delta\colon P \times \dots \times P \to \hat{P}$ which are skewsymmetric and skew-adjoint in each argument, that is,

$$\Delta(p_1,\dots,p_i,p_{i+1},\dots,p_k) = -(-1)^{p_ip_{i+1}}\Delta(p_1,\dots,p_{i+1},p_i,\dots,p_k).$$

The following theorem is proved analogously to the proof of Theorem 4.2.

**Theorem 6.1.** *For each $f \in \varkappa^{(k)}_{\mathrm{odd}}$ there exists a unique $\mathcal{C}$-differential operator $\Delta_f \in \mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(k-1)}(\hat{\varkappa}, \varkappa)$ such that*

$$f(\omega_1, \dots, \omega_k) = [\langle \Delta_f(\mathcal{E}(\omega_1), \dots, \mathcal{E}(\omega_{k-1})), \mathcal{E}(\omega_k)\rangle].$$

Computation similar to that in the Proof of Theorem 4.4 shows that formula (4.5) holds for the variational Schouten bracket:

$$\begin{aligned}
&[\![\Delta_1, \Delta_2]\!](\psi_1, \dots, \psi_{k+l-2}) \\
&= (-1)^{\Delta_1\Delta_2}\Big( \sum_{\sigma \in S^{l-1}_{k+l-2}} \epsilon_\psi(\sigma)(-1)^\sigma (-1)^{\Delta_1 \psi_{\sigma(1,l-1)}} \ell_{\Delta_2, \psi_{\sigma(1,l-1)}}(\Delta_1(\psi_{\sigma(l,k+l-2)})) \\
&-(-1)^{(k-1)(l-1)} \sum_{\sigma \in S^{k}_{k+l-2}} \epsilon_\psi(\sigma)(-1)^\sigma \Delta_2(\ell^*_{\Delta_1, \psi_{\sigma(1,k-1)}}(\psi_{\sigma(k)}), \psi_{\sigma(k+1,k+l-2)})\Big) \\
&-(-1)^{(k-1)(l-1)} \sum_{\sigma \in S^{k-1}_{k+l-2}} \epsilon_\psi(\sigma)(-1)^\sigma (-1)^{\Delta_2 \psi_{\sigma(1,k-1)}} \ell_{\Delta_1, \psi_{\sigma(1,k-1)}}(\Delta_2(\psi_{\sigma(k,k+l-2)})) \\
&\qquad + \sum_{\sigma \in S^{l}_{k+l-2}} \epsilon_\psi(\sigma)(-1)^\sigma \Delta_1(\ell^*_{\Delta_2, \psi_{\sigma(1,l-1)}}(\psi_{\sigma(l)}), \psi_{\sigma(l+1,k+l-2)}),
\end{aligned}$$

A bivector $\Delta_f \in \varkappa^{(2)}_{\mathrm{odd}}$ is called *Hamiltonian* if $f$ fulfills the Jacobi identity (see, e.g., [3]).

The following statement is obvious.

**Proposition 6.2.** *If $A \in \varkappa^{(2)}_{\mathrm{odd}}$ is an even bivector, i.e., $\{A, A\} = 1$ and satisfies the condition $[\![A, A]\!] = 0$, then $A$ is Hamiltonian.*

From formula (4.5) for an even Hamiltonian bivector $A$ we have

$$\begin{aligned}
&0 = [\![A, A]\!](\psi_1, \psi_2) = -\{A, \psi_1\} A(\ell_{\psi_1}(A(\psi_2))) + \{A + \psi_1, \psi_2\} A(\ell_{\psi_2}(A(\psi_1))) \\
&+ A(\ell^*_{A,\psi_1}(\psi_2)) - \{A + \psi_1, \psi_2\} \ell_{A(\psi_2)}(A(\psi_1)) + \{A, \psi_1\} \ell_{A(\psi_1)}(A(\psi_2)) \\
&= \{A, \psi_1\} \ell_{A,\psi_1}(A(\psi_2)) - \{A + \psi_1, \psi_2\} \ell_{A,\psi_2}(A(\psi_1)) + A(\ell^*_{A,\psi_1}(\psi_2)).
\end{aligned}$$

Thus, we obtained the well-known criterion for a bivector to be Hamiltonian (see, e.g., [3]).

We can construct an odd counterpart of Poisson bracket of the above section. To this end, we take $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to be the grading group. The additional component of the grading is said to be *parity*. We endow the fibers of the bundle $\pi$ with parity 0 and the fibers of the bundle $\hat{\pi}$ with parity 1. Consider the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-graded bundle $\Pi_{\mathrm{odd}} = \pi \oplus \hat{\pi}$. Similarly to the previous section, we construct an odd Hamiltonian bivector

$$A \colon \varkappa(\Pi_{\mathrm{odd}}) \to \varkappa(\Pi_{\mathrm{odd}})$$

on $J^\infty(\Pi_{\mathrm{odd}})$: $A(\psi, \varphi) = (-1)^{p(\psi)p(\varphi)}\{\psi, \varphi\}(\varphi, -\psi)$, where $p$ is the parity. The corresponding bracket is called the *Schouten bracket* on

$\bar{H}^n(\Pi_{\mathrm{odd}})$, or *antibracket* (see, e.g., [12]). Its coordinate expression is

$$\sum_j \frac{\delta F}{\delta p^j}\frac{\delta H}{\delta u^j} - (-1)^{p(F)p(H)}\{F,H\}\frac{\delta H}{\delta p^j}\frac{\delta F}{\delta u^j}.$$

As above, $\varkappa^{(*)}_{\mathrm{odd}}$ is a subset of $\bar{H}^n(\Pi_{\mathrm{odd}})$ and the two Schouten brackets coincide on $\varkappa^{(*)}_{\mathrm{odd}}$.

## 7. Local variational differential operators

7.1. **Horizontal cohomology classes are local functionals.** We want to identify the space $\bar{H}^n$ with a subspace of functionals on $\Gamma(\pi)$. Consider first the case of a purely even vector bundle $\pi\colon E \to M$.

If $M$ is compact and oriented, the standard way to do this is the following. Each horizontal cohomology class $[\omega]$ gives a functional on the space of global sections $\Gamma(\pi)$

$$\omega(s) = \int_M j_\infty(s)^*(\omega), \qquad s \in \Gamma(\pi). \tag{7.1}$$

For general $M$ we suggest the following definition of functionals. Below a *domain* is an oriented open subset $U$ of $M$ with compact closure $\bar{U}$ and smooth boundary $\partial U$.

Let $U_1$, $U_2 \subset M$ be two domains such that $B = \partial U_1 = \partial U_2$ and the orientations of $U_1$, $U_2$ induce opposite orientations on $B$. Let

$$s_i \in \Gamma(\pi, \bar{U}_i),\ i = 1, 2, \quad [s_1]^\infty_x = [s_2]^\infty_x \ \forall\, x \in B. \tag{7.2}$$

A horizontal cohomology class $[\omega]$ gives a well-defined functional on such 4-tuples $(U_1, U_2, s_1, s_2)$ as follows

$$\omega(U_1, U_2, s_1, s_2) = \int_{\bar{U}_1} j_\infty(s_1)^*(\omega) + \int_{\bar{U}_2} j_\infty(s_2)^*(\omega). \tag{7.3}$$

It is easily seen that each nonzero class determines a nonzero functional.

*Remark* 7.1. Physically, the section $s_2$ plays the role of the *ground* (or *reference*) *field.* In concrete models it is possible to have $s_2 = 0$ or other kinds of ansatz depending on the physics of the theory.

If $M$ is compact and orientable then there are two types of such collections $(U_1, U_2, s_1, s_2)$

$$U_1 \cap U_2 = \emptyset, \quad U_1 \cup \partial U_1 \cup U_2 = M, \tag{7.4}$$

$$U_1 = U_2 \ \text{with opposite orientations.} \tag{7.5}$$

In case (7.4) sections (7.2) constitute one global section $s$ over $M$, and (7.3) coincides with (7.1). If $M$ is not compact and orientable then only type (7.5) is possible. However, with each collection $(U_1, U_2, s_1, s_2)$ of type (7.5) we can associate one of type (7.4), but for a different vector bundle.

Namely, consider the compact oriented manifold $M'$ that consists of two copies of $U_1$ with opposite orientations glued naturally along $\partial U_1$. The algebra $C^\infty(M')$ is isomorphic to the algebra of pairs $f$, $g \in C^\infty(\bar{U}_1)$ such that $[f]_x^\infty = [g]_x^\infty$ for all $x \in \partial U_1$. The bundle $\pi$ as well as its jet bundles and the bundle of horizontal forms are naturally restricted to $M'$. If (7.5) holds then sections (7.2) determine one global section over $M'$.

Taking into account this construction, below we consider only the case of a compact oriented base $M$.

7.2. **Multilocal functionals and local variational differential operators.** Following [31] we call functionals of the form $F(\omega_1, \dots, \omega_N)$, where $F$ is a smooth function in many arguments, *multilocal functionals.* The set $\mathcal{F} = \mathcal{F}(\pi)$ of multilocal functionals is a commutative algebra.

A natural question arises: what are the relations in $\mathcal{F}$, i.e., for what nonzero smooth functions $F$ and horizontal forms $\omega_i$ the expression $F(\omega_1, \dots, \omega_N)$ induces an identically zero functional? Of course, there are relations of this type arising from linear relations in the space $\bar{H}^n$. Remarkably, it turns out (Theorem 7.4 below) that at least locally (in a certain sense) all relations in $\mathcal{F}$ are generated by linear relations in $\bar{H}^n$.

Let us first introduce some technical notions and notation. Preimages of open sets in finite jet spaces $J^k(\pi)$ under the mappings

$$\pi_{\infty,k}\colon J^\infty(\pi) \to J^k(\pi)$$

constitute the base of the topology on $J^\infty(\pi)$. We choose also a topology on $\Gamma(\pi)$ as follows: its base consists of sets of the form $\{s \in \Gamma(\pi) \mid j_\infty s(M) \subset U\}$, where $U$ is an open subset $J^\infty(\pi)$.

For a finite subset $V \subset M$ and $s_0 \in \Gamma(\pi)$ put

$$\Gamma_{s_0,V} = \{s \in \Gamma(\pi) \mid [s]_x^\infty = [s_0]_x^\infty \ \forall\, x \in V\}$$

Such subsets of $\Gamma(\pi)$ are endowed with the relative topology.

**Theorem 7.2.** *Let $s_0 \in \Gamma(\pi)$, $V$ be a finite subset of $M$, and $\omega_1, \dots, \omega_l \in \bar{\Lambda}^n$. Consider the linear map*

$$\Psi\colon \Gamma(\pi) \to \mathbb{R}^l, \qquad \Psi(s) = (\omega_1(s), \dots, \omega_l(s)).$$

*Suppose that there exists a neighborhood $\Gamma \subset \Gamma_{s_0,V}$ of $s_0$ such that $\Psi(\Gamma)$ does not contain any open subset of $\mathbb{R}^l$. Then there is a nontrivial linear combination $a_1\omega_1 + \dots + a_l\omega_l$ that induces a constant functional on a neighborhood $\Gamma' \subset \Gamma_{s_0,V\cup V'}$ of $s_0$, where $V' \subset M$ consists of $k < l$ points.*

*Proof.* For each $x \in M \setminus V$ consider a system of neighborhoods of $x$

$$U_x^1 \supset U_x^2 \supset \dots \supset U_x^i \supset \cdots \qquad \bigcap_{i=1}^\infty U_x^i = \{x\}.$$

Taking smaller $\Gamma$ if necessary, we can assume that for all $\alpha \in \Gamma(\pi)$ the relation $s_0 + \alpha \in \Gamma$ implies $s_0 + t\alpha \in \Gamma$, $0 \le t \le 1$. Set

$$\Gamma^i_x = \{\, s \in \Gamma(\pi) \mid \operatorname{supp}(s) \subset U^i_x,\ s_0 + s \in \Gamma \,\}.$$

Consider the following subsets of $\mathbb{R}^l$

$$L^i_x = \{\, \frac{d}{dt}\Psi(\psi + t\alpha)\Big|_{t=t_0} \in \mathbb{R}^l \mid s \in \Gamma^i_x,\ 0 \le t_0 \le 1 \,\}.$$

Suppose that there exist $l$ different points $x_1, \dots, x_l \in M \setminus V$ such that for any positive integer $i$ we can find $l$ linearly independent vectors $v^i_j \in L^i_{x_j}$, $j = 1, \dots, l$. Choose $i$ such that $U^i_{x_p} \cap U^i_{x_q} = \varnothing$ for all $1 \le p < q \le l$. There exist sections $s_j \in \Gamma^i_{x_j}$ such that for some $0 \le \bar{t}_1, \dots, \bar{t}_l \le 1$ the vectors

$$v^i_j = \frac{d}{dt}\Psi(s_0 + ts_j)\Big|_{\bar{t}=t_j}, \quad j = 1, \dots, l,$$

are linearly independent.

Using the condition $\operatorname{supp}(s_p) \cap \operatorname{supp}(s_q) = \varnothing$ for all $1 \le p < q \le l$ and the additivity of integration we easily obtain

$$\Psi(s_0 + t_1 s_1 + \dots + t_l s_l) = \Psi(s_0 + t_1 s_1) + \dots + \Psi(s_0 + t_l s_l) - (l-1)\Psi(s_0).$$

Therefore, the mapping

$$(t_1, \dots, t_l) \mapsto \Psi(s_0 + t_1 s_1 + \dots + t_l s_l)$$

is a local diffeomorphism on a neighborhood of the point $\bar{t}_1, \dots, \bar{t}_l$. Then $\Psi(\Gamma)$ contains an open subset of $\mathbb{R}^l$, which leads to a contradiction.

Let $k$ be the maximal number of different points $x_1, \dots, x_k \in M \setminus V$ such that for any positive integer $i$ we can find $k$ linearly independent vectors $v^i_j \in L^i_{x_j}$, $j = 1, \dots, k$. We proved that $k < l$. Define the subspaces $L^i \subset \mathbb{R}^l$

$$L^i = \bigcap_{\substack{v_j \in L^i_{x_j} \\ \dim\langle v_1, \dots, v_k\rangle = k}} \langle v_1, \dots, v_k \rangle,$$

where $\langle \dots \rangle$ stands for the linear span. We have $L^1 \subset L^2 \subset \dots L^i \subset \dots$. Put $L = \bigcup_{i=1}^{\infty} L^i$. Obviously, $\dim L < l$. Let $(a_1, \dots, a_l)$ be some nonzero vector from $\operatorname{Ann} L \subset (\mathbb{R}^l)^* \cong \mathbb{R}^l$. Set $V' = \{x_1, \dots, x_k\}$. By the definition of $k$, for any $x \in M \setminus (V \cup V')$ there exists a positive integer $i$ such that $L^i_x \subset L^i$. Hence, there is a neighborhood $V_x \subset J^\infty(\pi)$ of $[s_0]^\infty_x$ such that

$$a_1 \mathcal{E}(\omega_1) + \dots + a_l \mathcal{E}(\omega_l)|_{V_x} = 0.$$

From the properties of the Euler operator it follows that the sections whose infinite jets lie in

$$\bigcup_{x\in M\setminus(V\cup V')} V_x \subset J^\infty(\pi)$$

constitute the sought-for neighborhood $\Gamma'$. □

Let $F(\omega_1,\dots,\omega_N)$ be an identically zero functional. If all relations in $\mathcal{F}$ were literally arising from linear relations in $\bar{H}^n$ then for a basis $v_1,\dots,v_L$ in the linear span of $\omega_1,\dots,\omega_N$, $\omega_i=\sum_j a_{ij}v_j$, the function

$$G(s_1,\dots,s_L)=F(\sum_j a_{1j}s_j,\dots,\sum_j a_{Nj}s_j)$$

would be identically zero. The next theorem shows that this is maybe not precisely, but at least "almost" the case.

Let us explain what "almost" means here. First of all, there may be nontrivial horizontal cohomology classes that induce constant functionals. These are precisely those classes that belong to the image of the natural embedding $H^n(M)\to\bar{H}^n(\pi)$. We shall also consider horizontal cohomology classes $\omega$ that are constant on a neighborhood of a given section $s_0\in\Gamma(\pi)$. Clearly, this is equivalent to the fact that $\mathcal{E}(\omega)$ vanishes on a neighborhood of the set $j_\infty s_0(M)\subset J^\infty(\pi)$.

We say that horizontal cohomology classes $\omega_1,\dots,\omega_k$ are *strictly linearly independent around* $s_0$ if for each finite subset $V$ of $M$ and each neighborhood $\Gamma\subset\Gamma_{s_0,V}$ of $s_0$ there does not exist a linear combination of $\omega_1,\dots,\omega_k$ that induces a constant functional on $\Gamma$.

We say that a function $F\in C^\infty(\mathbb{R}^N)$ is *almost identically zero around* $x\in\mathbb{R}^N$ if for any neighborhood $U$ of $x$ there is a nonempty open subset $U'\subset U$ such that $F|_{U'}=0$. The following statement is obvious.

**Lemma 7.3.** *If $F$ is almost identically zero around $x$ then all the partial derivatives of $F$ vanish at $x$.*

Let $\omega_1,\dots,\omega_N$ be horizontal cohomology classes and $v_1,\dots,v_L$ be a maximal subset of them that is strictly linearly independent around $s_0$. By the definition, there exists a finite subset $V\subset M$ and a neighborhood $\Gamma\subset\Gamma_{s_0,V}$ of $s_0$ such that

$$\omega_i|_\Gamma=\sum_{j=1}^L a_{ij}\,v_j|_\Gamma+a_{i0},\qquad 1\le i\le N,\ a_{ij}\in\mathbb{R}.$$

This is equivalent to the fact that for some neighborhood $U\subset J^\infty(\pi)$ of the set $j_\infty s_0(M\setminus V)$ one has

$$\mathcal{E}(\omega_i)\big|_U=\sum_{j=1}^L a_{ij}\,\mathcal{E}(v_j)\big|_U\,. \tag{7.6}$$

**Theorem 7.4.** *Let $F(\omega_1, \dots, \omega_N)$ be a multilocal functional that is zero on a neighborhood of $s_0$. Then the function*

$$G(s_1, \dots, s_L) = F(\sum_j a_{1j}s_j + a_{10}, \dots, \sum_j a_{Nj}s_j + a_{N0}) \tag{7.7}$$

*is almost identically zero around $(v_1(s_0), \dots, v_L(s_0)) \in \mathbb{R}^L$.*

*Proof.* By assumption, $G$ vanishes on the subset

$$\{(v_1(s), \dots, v_L(s)) \mid s \in \Gamma\} \subset \mathbb{R}^L$$

By Theorem 7.2, since $v_1, \dots, v_L$ are strictly linearly independent, this guarantees that $G$ is almost identically zero around $(v_1(s_0), \dots, v_L(s_0))$. □

**Corollary 7.5.** *For any variational multivector $f \in \varkappa^{(k)}$ the formula*

$$\nabla_f(F(\omega_1, \dots, \omega_N)) = \sum_{1 \le i_1, \dots, i_k \le N} \frac{\partial^k F}{\partial t_{i_1} \dots \partial t_{i_k}}(\omega_1, \dots, \omega_N) \cdot f(\omega_{i_1}, \dots, \omega_{i_k}). \tag{7.8}$$

*determines a well-defined differential operator $\nabla_f \colon \mathcal{F} \to \mathcal{F}$ of order $k$.*

*Proof.* Suppose that a multilocal functional $F(\omega_1, \dots, \omega_N)$ is zero in $\mathcal{F}$. To establish that (7.8) gives a well-defined operator, we must prove that

$$\nabla_f(F(\omega_1, \dots, \omega_N))(s_0) = 0 \tag{7.9}$$

for any $s_0 \in \Gamma(\pi)$.

Let $v_1, \dots, v_L$ be as above. Combining (4.2) with (7.6), we obtain

$$f(\omega_{i_1}, \dots, \omega_{i_k})(s_0) = \sum_{j_1, \dots, j_k} a_{i_1 j_1} \dots a_{i_k j_k} f(v_{j_1}, \dots, v_{j_k})(s_0).$$

Using this and definition (7.7), similarly to the proof of Proposition 2.7, by the chain rule one gets

$$\sum_{1 \le i_1, \dots, i_k \le N} \frac{\partial^k F}{\partial t_{i_1} \dots \partial t_{i_k}}(\omega_1(s_0), \dots, \omega_N(s_0)) \cdot f(\omega_{i_1}, \dots, \omega_{i_k})(s_0)$$
$$= \sum_{1 \le i_1, \dots, i_k \le L} \frac{\partial^k G}{\partial t_{i_1} \dots \partial t_{i_k}}(v_1(s_0), \dots, v_L(s_0)) \cdot f(v_{i_1}, \dots, v_{i_k})(s_0).$$

By Theorem 7.4 and Lemma 7.3, this is zero, which proves that $\nabla_f$ is well defined. The fact that it is a differential operator of order $k$ is obvious. □

Following [31], we call (7.8) *local variational differential operators.*

7.3. **Generalization to the case of super-bundles.** If $\pi$ is a super vector bundle then the horizontal cohomology space is $\mathbb{Z}$-graded $\bar{H}^n = \oplus_{i\geq 0} H_i^n$, the space $H_0^n$ being isomorphic to the $n$-th horizontal cohomology space of the even component $\pi_{\bar{0}}$. In this case we define the algebra of multilocal functionals as follows

$$\mathcal{F}(\pi) = \mathcal{F}(\pi_0) \otimes_{\mathbb{R}} S(\oplus_{i\geq 1} H_i^n).$$

Choose a basis $\{h_\alpha\}$ in the space $S(\oplus_{i\geq 1} H_i^n)$. If an element

$$\sum_\alpha F_\alpha(\omega_1^\alpha, \dots, \omega_N^\alpha) \otimes h_\alpha$$

is zero in $\mathcal{F}(\pi)$ then all $F_\alpha(\omega_1^\alpha, \dots, \omega_N^\alpha)$ are zero in $\mathcal{F}(\pi_0)$. Because of this, it follows easily from Corollary 7.5 applied to $\pi_0$ that local variational differential operators are well defined on $\mathcal{F}(\pi)$.

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Sergei Igonin, Independent University of Moscow & University of Twente

*Current address*: University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

*E-mail address*: `igonin@mccme.ru`

Alexander Verbovetsky, Independent University of Moscow, B. Vlasevsky 11, 121002 Moscow, Russia

*E-mail address*: `verbovet@mccme.ru`

Raffaele Vitolo, Math. Dept. "E. De Giorgi", Via per Arnesano, 73100 Lecce, Italy

*E-mail address*: `Raffaele.Vitolo@unile.it`