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Cyclic machine scheduling with tool
transportation - additional calculations

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Cyclic Machine Scheduling with Tool Transportation – Additional Calculations

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Abstract

In the PhD Thesis of Kuijpers a cyclic machine scheduling problem with tool transportation is considered. For the problem with two machines, it is shown that there always exists an optimal schedule with a certain structure. This is done by means of an elaborate case study. For a number of cases some calculations are only sketched in the thesis, but not explicitly given. This paper contains a complete elaboration of the calculations that are not explicitly given in the thesis.

Keywords: scheduling, cyclic, tools

Mathematics Subject Classification: 90B35

1 Introduction

In Kuijpers[1] a certain machine scheduling problem is examined. Most results concern the problem for two machines. For this two-machine problem, it turns out that there always exists an optimal schedule with certain structure properties. The proof of this result can be found in Chapter 5 and Appendix A of [1]. On the last four pages of Appendix A some calculations are described, but not explicitly given. A complete elaboration of these calculations is given below.

(For the definitions of the used variables, see Kuijpers[1].)

We prove that for every pair $j^*, j \in \bigcup_{a=1}^3 (P_a \cup Q_a)$ we have that $I(j^*) + II(j) < c$ in all of the following cases:

$$\text{Case 2(b)} \quad I(j^*) > p_{j^*} + 3d \text{ and } II(j) \leq p_j + 3d + \delta(j), \quad (1)$$

$$\text{Case 2(c)} \quad I(j^*) \leq p_{j^*} + 3d \text{ and } II(j) > p_j + 3d + \delta(j), \quad (2)$$

$$\text{Case 2(d)} \quad I(j^*) > p_{j^*} + 3d \text{ and } II(j) > p_j + 3d + \delta(j). \quad (3)$$

For proving this we have the following formulas at our disposal (see [1]).

General formulas

$$I(j^*) \in [p_{j^*} + 2kd + d, p_{j^*} + 2(k+1)d + d] \quad (4)$$

$$II(j) \in [p_j + 2l_j d + d + \delta(j), p_j + 2(l_j + 1)d + d + \delta(j)]. \quad (5)$$

$$c = \sum_{a=1}^3 l(P_a) + \sum_{a=1}^3 l(Q_a). \quad (6)$$

$$\delta(j) = \begin{cases} \delta_2 & \text{if } j \in Q_1 \text{ and } T_j^2(4) \geq b(P_2)^1 + d, \\ \delta_3 & \text{if } j \in Q_2 \text{ and } T_j^2(4) \geq b(P_3)^1 + d, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

$$\delta(j) < 2d. \quad (8)$$

$$l(P_a) \geq 2d \quad (a = 1, 2, 3). \quad (9)$$

$$l(Q_a) \geq 2d \quad (a = 1, 2, 3). \quad (10)$$

$$2d(\lfloor \frac{p_j}{2d} \rfloor + 1) > p_j. \quad (11)$$

If j^* is the first task of Q_1 or Q_2 :

$$I(j^*) = p_{j^*} + 2kd + d. \quad (12)$$

If $I(j^*) > p_{j^*} + 3d$:

$$k > 0 \quad (13)$$

$$p_{j^*} \geq 2d. \quad (14)$$

If $I(j^*) > p_{j^*} + 3d$ and $j^* \in P_a$ ($a = 1, 2, 3$):

$$l(P_a | < j^*) \geq 2kd + \delta_a. \quad (15)$$

$$l(P_a | > j^*) + l(Q_a) + l(P_{a \oplus 31}) \geq 2d(k + \lfloor \frac{p_{j^*}}{2d} \rfloor). \quad (16)$$

$$p_{j^*} < \frac{1}{2}l(P_a) + \frac{1}{2}l(Q_a) + \frac{1}{2}l(P_{a \oplus 31}) - 2kd + d - \frac{1}{2}\delta_a. \quad (17)$$

If $I(j^*) > p_{j^*} + 3d$, $j^* \in P_a$ and j^* is not the last task of P_a ($a = 1, 2, 3$):

$$l(P_a | > j^*) \geq 2d(k + \lfloor \frac{p_{j^*}}{2d} \rfloor) \quad (18)$$

If $I(j^*) > p_{j^*} + 3d$ and $j^* \in Q_a$ ($a = 1, 2, 3$):

$$l(P_a) + l(Q_a | < j^*) \geq 2kd + \delta_a. \quad (19)$$

$$l(Q_a | > j^*) + l(P_{a \oplus 31}) \geq 2d(k + \lfloor \frac{p_{j^*}}{2d} \rfloor). \quad (20)$$

$$p_{j^*} < \frac{1}{2}l(P_a) + \frac{1}{2}l(Q_a) + \frac{1}{2}l(P_{a \oplus 31}) - 2kd + d - \frac{1}{2}\delta_a. \quad (21)$$

If $I(j^*) > p_{j^*} + 3d$, $j^* \in Q_a$ ($a = 1, 2, 3$) and j^* is not the first task of Q_a ($a = 1, 2, 3$):

$$p_{j^*} < \frac{1}{2}l(P_a) + \frac{1}{2}l(Q_a) + \frac{1}{2}l(P_{a\oplus 31}) - 2kd - \frac{1}{2}\delta_a. \quad (22)$$

If $II(j) > p_j + 3d + \delta(j)$:

$$l_j > 0 \quad (23)$$

$$j \text{ is not the last task of } P_a \text{ (} a = 1, 2, 3 \text{)} \quad (24)$$

If $II(j) > p_j + 3d + \delta(j)$ and $j \in P_a$ ($a = 1, 2, 3$):

$$l(P_a | > j) > 2l_j d. \quad (25)$$

$$l(P_{a\oplus 32}) + l(Q_{a\oplus 32}) + l(P_a | < j) \geq 2d(l_j + \lfloor \frac{p_j}{2d} \rfloor + 1). \quad (26)$$

$$p_j < \frac{1}{2}l(P_{a\oplus 32}) + \frac{1}{2}l(Q_{a\oplus 32}) + \frac{1}{2}l(P_a) - 2l_j d. \quad (27)$$

If $II(j) > p_j + 3d + \delta(j)$ and $j \in Q_a$ ($a = 1, 2$):

$$l(Q_a | > j) + l(P_{a\oplus 31}) > 2l_j d + \delta(j). \quad (28)$$

$$l(P_a) + l(Q_a | < j) \geq 2d(l_j + \lfloor \frac{p_j}{2d} \rfloor + 1). \quad (29)$$

$$p_j < \frac{1}{2}l(P_a) + \frac{1}{2}l(Q_a) + \frac{1}{2}l(P_{a\oplus 31}) - 2l_j d - \frac{1}{2}\delta(j). \quad (30)$$

If $II(j) > p_j + 3d + \delta(j)$ and $j \in Q_3$:

$$l(Q_3 | > j) + l(P_1) \geq 2l_j d + 2d + \delta(j). \quad (31)$$

$$l(P_3) + l(Q_3 | < j) \geq 2d(l_j + \lfloor \frac{p_j}{2d} \rfloor). \quad (32)$$

$$p_j < \frac{1}{2}l(P_3) + \frac{1}{2}l(Q_3) + \frac{1}{2}l(P_1) - 2l_j d - \frac{1}{2}\delta(j). \quad (33)$$

2 Calculations

2.1 $j^* \in P_a, j \in P_a$ ($a = 1, 2, 3$)

2.1.1 Case 2(b)

$j^* = j$

$$\begin{aligned} I(j^*) + II(j) &\stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\ &\stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 6d \\ &\stackrel{j^*=j}{=} 2p_{j^*} + 2kd + 6d \end{aligned}$$

$$\begin{aligned}
& \stackrel{(17), \delta_a \geq 0}{<} l(P_a) + l(Q_a) + l(P_{a \oplus 31}) - 2kd + 8d \\
& \stackrel{(13)}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus 31}) + 6d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus 32}) - l(Q_{a \oplus 31}) - l(Q_{a \oplus 32}) + 6d \\
& \stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

$j^* < j$

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 6d \\
& \stackrel{(15), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a | < j^*) + 6d \\
& \stackrel{j^*, j \in P_a; j^* < j}{\leq} l(P_a) + 6d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus 31}) - l(P_{a \oplus 32}) - \sum_{a=1}^3 l(Q_a) + 6d \\
& \stackrel{(9)(10)}{\leq} c - 4d.
\end{aligned}$$

$j^* > j$

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 6d \\
& \stackrel{(16)}{\leq} p_{j^*} + p_j + l(P_a | > j^*) + l(Q_a) + l(P_{a \oplus 31}) - \\
& \quad 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 6d \\
& \stackrel{j^*, j \in P_a; j^* > j}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus 31}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 6d \\
& \stackrel{(14)}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus 31}) + 4d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus 32}) - l(Q_{a \oplus 31}) - l(Q_{a \oplus 32}) + 4d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.1.2 Case 2(c)

$j^* = j$

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2l_j d + 6d \\
& \stackrel{j^* = j}{=} 2p_j + 2l_j d + 6d
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(27)}{<} l(P_{a\oplus_3 2}) + l(Q_{a\oplus_3 2}) + l(P_a) - 2l_j d + 6d \\
& \stackrel{(23)}{\leq} l(P_{a\oplus_3 2}) + l(Q_{a\oplus_3 2}) + l(P_a) + 4d \\
& \stackrel{(6)}{=} c - l(P_{a\oplus_3 1}) - l(Q_a) - l(Q_{a\oplus_3 1}) + 4d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

$\mathbf{j^* < j}$:

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2l_j d + 6d \\
& \stackrel{(25)}{<} p_{j^*} + p_j + l(P_a | > j) + 6d \\
& \stackrel{j^*, j \in P_a; j^* < j}{\leq} l(P_a) + 6d \\
& \stackrel{(6)}{=} c - l(P_{a\oplus_3 1}) - l(P_{a\oplus_3 2}) - \sum_{a=1}^3 l(Q_a) + 6d \\
& \stackrel{(9)(10)}{\leq} c - 4d.
\end{aligned}$$

$\mathbf{j^* > j}$

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2l_j d + 6d \\
& \stackrel{(26)}{\leq} p_{j^*} + p_j + l(P_{a\oplus_3 2}) + l(Q_{a\oplus_3 2}) + l(P_a | < j) - \\
& \quad 2d(\lfloor \frac{p_j}{2d} \rfloor + 1) + 6d \\
& \stackrel{j^*, j \in P_a; j^* > j}{\leq} l(P_{a\oplus_3 2}) + l(Q_{a\oplus_3 2}) + l(P_a) - 2d\lfloor \frac{p_j}{2d} \rfloor + 4d \\
& \stackrel{p_j > 0}{\leq} l(P_{a\oplus_3 2}) + l(Q_{a\oplus_3 2}) + l(P_a) + 4d \\
& \stackrel{(6)}{=} c - l(P_{a\oplus_3 1}) - l(Q_a) - l(Q_{a\oplus_3 1}) + 4d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.1.3 Case 2(d)

$\mathbf{j^* = j}$

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 2l_j d + 6d \\
& \stackrel{j^* = j}{=} 2p_{j^*} + 2kd + 2l_j d + 6d
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(24)(18)}{\leq} 2p_{j^*} + l(P_a | > j^*) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 2l_j d + 6d \\
& \stackrel{(26)}{\leq} 2p_{j^*} + l(P_a | > j^*) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + l(P_{a \oplus_3 2}) + l(Q_{a \oplus_3 2}) + \\
& \quad l(P_a | < j) - 2d(\lfloor \frac{p_j}{2d} \rfloor + 1) + 6d \\
& \stackrel{j^*, j \in P_a; j^* = j}{=} p_{j^*} + l(P_a) + l(P_{a \oplus_3 2}) + l(Q_{a \oplus_3 2}) - 4d \lfloor \frac{p_{j^*}}{2d} \rfloor + 4d \\
& \stackrel{(11)}{<} l(P_a) + l(P_{a \oplus_3 2}) + l(Q_{a \oplus_3 2}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 6d \\
& \stackrel{(14)}{\leq} l(P_a) + l(P_{a \oplus_3 2}) + l(Q_{a \oplus_3 2}) + 4d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus_3 1}) - l(Q_a) - l(Q_{a \oplus_3 1}) + 4d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

$j^* < j$

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 2l_j d + 6d \\
& \stackrel{(15), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a | < j^*) + 2l_j d + 6d \\
& \stackrel{(25)}{<} p_{j^*} + p_j + l(P_a | < j^*) + l(P_a | > j) + 6d \\
& \stackrel{j^*, j \in P_a; j^* < j}{\leq} l(P_a) + 6d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus_3 1}) - l(P_{a \oplus_3 2}) - \sum_{a=1}^3 l(Q_a) + 6d \\
& \stackrel{(9)(10)}{\leq} c - 4d.
\end{aligned}$$

$j^* > j$

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 2l_j d + 6d \\
& \stackrel{(16)}{\leq} p_{j^*} + p_j + l(P_a | > j^*) + l(Q_a) + l(P_{a \oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + \\
& \quad 2l_j d + 6d \\
& \stackrel{(26)}{\leq} p_{j^*} + p_j + l(P_a | > j^*) + l(Q_a) + l(P_{a \oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + \\
& \quad l(P_{a \oplus_3 2}) + l(Q_{a \oplus_3 2}) + l(P_a | < j) - 2d(\lfloor \frac{p_j}{2d} \rfloor + 1) + 6d \\
& \stackrel{j^*, j \in P_a; j^* > j}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) + l(P_{a \oplus_3 2}) + l(Q_{a \oplus_3 2})
\end{aligned}$$

$$\begin{aligned}
& -2d\lfloor \frac{p_{j^*}}{2d} \rfloor - 2d\lfloor \frac{p_j}{2d} \rfloor + 4d \\
\stackrel{(14), p_j > 0}{\leq} & l(P_a) + l(Q_a) + l(P_{a\oplus_3 1}) + l(P_{a\oplus_3 2}) + l(Q_{a\oplus_3 2}) + 2d \\
\stackrel{(6)}{=} & c - l(Q_{a\oplus_3 1}) + 2d \\
\stackrel{(10)}{\leq} & c.
\end{aligned}$$

2.2 $j^* \in P_a, j \in P_b$ ($a, b \in \{1, 2, 3\}, a \neq b$)

2.2.1 Case 2(b)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 6d \\
& \stackrel{(15), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a | < j^*) + 6d \\
& \stackrel{j^* \in P_a, j \in P_b}{\leq} l(P_a) + l(P_b) + 6d \\
& \stackrel{(6), a \neq b}{=} c - \sum_{i \neq a, b} l(P_i) - \sum_{a=1}^3 l(Q_a) + 6d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.2.2 Case 2(c)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2l_j d + 6d \\
& \stackrel{(25)}{<} p_{j^*} + p_j + l(P_b | > j) + 6d \\
& \stackrel{j^* \in P_a, j \in P_b}{\leq} l(P_a) + l(P_b) + 6d \\
& \stackrel{(6), a \neq b}{=} c - \sum_{i \neq a, b} l(P_i) - \sum_{a=1}^3 l(Q_a) + 6d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.2.3 Case 2(d)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 2l_j d + 6d \\
& \stackrel{(15), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a | < j^*) + 2l_j d + 6d \\
& \stackrel{(25)}{<} p_{j^*} + p_j + l(P_a | < j^*) + l(P_b | > j) + 6d
\end{aligned}$$

$$\begin{aligned}
& \stackrel{j^* \in P_a, j \in P_b}{\leq} l(P_a) + l(P_b) + 6d \\
& \stackrel{(6), a \neq b}{=} c - \sum_{i \neq a, b} l(P_i) - \sum_{a=1}^3 l(Q_a) + 6d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.3 $j^* \in P_a, j \in Q_b$ ($a \in \{1, 2, 3\}, b \in \{a, a \oplus_3 1\}$)

2.3.1 Case 2(b)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
& \stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 8d \\
& \stackrel{(15), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a | < j^*) + 8d \\
& \stackrel{j^* \in P_a, j \in Q_b}{\leq} l(P_a) + l(Q_b) + 8d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus_3 1}) - l(P_{a \oplus_3 2}) - l(Q_{b \oplus_3 1}) - l(Q_{b \oplus_3 2}) + 8d \\
& \stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.3.2 Case 2(c)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
& \stackrel{(28)/(31)}{<} p_{j^*} + p_j + l(Q_b | > j) + l(P_{b \oplus_3 1}) + 6d \\
& \stackrel{j^* \in P_a, j \in Q_b}{\leq} l(P_a) + l(Q_b) + l(P_{b \oplus_3 1}) + 6d \\
& \stackrel{(6), b \in \{a, a \oplus_3 1\}}{=} c - \sum_{i \neq a, b \oplus_3 1} l(P_i) - l(Q_{b \oplus_3 1}) - l(Q_{b \oplus_3 2}) + 6d \\
& \stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.3.3 Case 2(d)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
& \stackrel{(15), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a | < j^*) + 2l_j d + 6d + \delta(j) \\
& \stackrel{(28)/(31)}{<} p_{j^*} + p_j + l(P_a | < j^*) + l(Q_b | > j) + l(P_{b \oplus_3 1}) + 6d \\
& \stackrel{j^* \in P_a, j \in Q_b}{\leq} l(P_a) + l(Q_b) + l(P_{b \oplus_3 1}) + 6d \\
& \stackrel{(6), b \in \{a, a \oplus_3 1\}}{=} c - \sum_{i \neq a, b \oplus_3 1} l(P_i) - l(Q_{b \oplus_3 1}) - l(Q_{b \oplus_3 2}) + 6d \\
& \stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.4 $j^* \in P_a, j \in Q_{a \oplus_3 2}$ ($a = 1, 2, 3$)

2.4.1 Case 2(b)

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
&\stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 8d \\
&\stackrel{(15), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a | < j^*) + 8d \\
&\stackrel{j^* \in P_a, j \in Q_{a \oplus_3 2}}{\leq} l(P_a) + l(Q_{a \oplus_3 2}) + 8d \\
&\stackrel{(6)}{=} c - l(P_{a \oplus_3 1}) - l(P_{a \oplus_3 2}) - l(Q_a) - l(Q_{a \oplus_3 1}) + 8d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.4.2 Case 2(c)

$a = 1$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
&\stackrel{(7)}{=} p_{j^*} + p_j + 2l_j d + 6d \\
&\stackrel{(32)}{\leq} p_{j^*} + p_j + l(P_3) + l(Q_3 | < j) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
&\stackrel{j^* \in P_1, j \in Q_3}{\leq} l(P_1) + l(P_3) + l(Q_3) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
&\stackrel{p_j > 0}{\leq} l(P_1) + l(P_3) + l(Q_3) + 6d \\
&\stackrel{(6)}{=} c - l(P_2) - l(Q_1) - l(Q_2) + 6d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

$a = 2, 3$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
&\stackrel{(8)}{<} p_{j^*} + p_j + 2l_j d + 8d \\
&\stackrel{(29)}{\leq} p_{j^*} + p_j + l(P_{a \oplus_3 2}) + l(Q_{a \oplus_3 2} | < j) - 2d(\lfloor \frac{p_j}{2d} \rfloor + 1) + 8d \\
&\stackrel{j^* \in P_a, j \in Q_{a \oplus_3 2}}{\leq} l(P_a) + l(P_{a \oplus_3 2}) + l(Q_{a \oplus_3 2}) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
&\stackrel{p_j > 0}{\leq} l(P_a) + l(P_{a \oplus_3 2}) + l(Q_{a \oplus_3 2}) + 6d \\
&\stackrel{(6)}{=} c - l(P_{a \oplus_3 1}) - l(Q_a) - l(Q_{a \oplus_3 1}) + 6d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.4.3 Case 2(d)

$a = 1$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
&\stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 2l_j d + 6d \\
&\stackrel{(15), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_1 | < j^*) + 2l_j d + 6d \\
&\stackrel{(32)}{\leq} p_{j^*} + p_j + l(P_1 | < j^*) + l(P_3) + l(Q_3 | < j) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
&\stackrel{j^* \in P_1, j \in Q_3}{\leq} l(P_1) + l(P_3) + l(Q_3) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
&\stackrel{p_j > 0}{\leq} l(P_1) + l(P_3) + l(Q_3) + 6d \\
&\stackrel{(6)}{=} c - l(P_2) - l(Q_1) - l(Q_2) + 6d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

$a = 2, 3$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
&\stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 8d \\
&\stackrel{(15), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a | < j^*) + 2l_j d + 8d \\
&\stackrel{(29)}{\leq} p_{j^*} + p_j + l(P_a | < j^*) + l(P_{a \oplus 3 2}) + l(Q_{a \oplus 3 2} | < j) - \\
&\quad 2d(\lfloor \frac{p_j}{2d} \rfloor + 1) + 8d \\
&\stackrel{j^* \in P_a, j \in Q_{a \oplus 3 2}}{\leq} l(P_a) + l(P_{a \oplus 3 2}) + l(Q_{a \oplus 3 2}) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
&\stackrel{p_j > 0}{\leq} l(P_a) + l(P_{a \oplus 3 2}) + l(Q_{a \oplus 3 2}) + 6d \\
&\stackrel{(6)}{=} c - l(P_{a \oplus 3 1}) - l(Q_a) - l(Q_{a \oplus 3 1}) + 6d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.5 $j^* \in Q_a, j \in P_a$ ($a = 1, 2, 3$)

2.5.1 Case 2(b)

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
&\stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 6d \\
&\stackrel{(20)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus 3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 6d
\end{aligned}$$

$$\begin{aligned}
& \stackrel{j^* \in Q_a, j \in P_a}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 6d \\
& \stackrel{(14)}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) + 4d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus_3 2}) - l(Q_{a \oplus_3 1}) - l(Q_{a \oplus_3 2}) + 4d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.5.2 Case 2(c)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2l_j d + 6d \\
& \stackrel{(25)}{<} p_{j^*} + p_j + l(P_a | > j) + 6d \\
& \stackrel{j^* \in Q_a, j \in P_a}{\leq} l(P_a) + l(Q_a) + 6d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus_3 1}) - l(P_{a \oplus_3 2}) - l(Q_{a \oplus_3 1}) - l(Q_{a \oplus_3 2}) + 6d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.5.3 Case 2(d)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 2l_j d + 6d \\
& \stackrel{(20)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 2l_j d + 6d \\
& \stackrel{(25)}{<} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + l(P_a | > j) + 6d \\
& \stackrel{j^* \in Q_a, j \in P_a}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 6d \\
& \stackrel{(14)}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) + 4d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus_3 2}) - l(Q_{a \oplus_3 1}) - l(Q_{a \oplus_3 2}) + 4d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.6 $j^* \in Q_a, j \in P_{a \oplus_3 1}$ ($a = 1, 2, 3$)

2.6.1 Case 2(b)

$$I(j^*) + II(j) \stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j)$$

$$\begin{aligned}
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 6d \\
& \stackrel{(19), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a) + l(Q_a | < j^*) + 6d \\
& \stackrel{j^* \in Q_a, j \in P_{a \oplus 31}}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus 31}) + 6d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus 32}) - l(Q_{a \oplus 31}) - l(Q_{a \oplus 32}) + 6d \\
& \stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.6.2 Case 2(c)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2l_j d + 6d \\
& \stackrel{(25)}{<} p_{j^*} + p_j + l(P_{a \oplus 31} | > j) + 6d \\
& \stackrel{j^* \in Q_a, j \in P_{a \oplus 31}}{\leq} l(Q_a) + l(P_{a \oplus 31}) + 6d \\
& \stackrel{(6)}{=} c - l(P_a) - l(P_{a \oplus 32}) - l(Q_{a \oplus 31}) - l(Q_{a \oplus 32}) + 6d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.6.3 Case 2(d)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 2l_j d + 6d \\
& \stackrel{(19), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a) + l(Q_a | < j^*) + 2l_j d + 6d \\
& \stackrel{(25)}{<} p_{j^*} + p_j + l(P_a) + l(Q_a | < j^*) + l(P_{a \oplus 31} | > j) + 6d \\
& \stackrel{j^* \in Q_a, j \in P_{a \oplus 31}}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus 31}) + 6d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus 32}) - l(Q_{a \oplus 31}) - l(Q_{a \oplus 32}) + 6d \\
& \stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.7 $j^* \in Q_a, j \in P_{a \oplus 32}$ ($a = 1, 2, 3$)

2.7.1 Case 2(b)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 6d \\
& \stackrel{(19), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a) + l(Q_a | < j^*) + 6d
\end{aligned}$$

$$\begin{aligned}
& j^* \in Q_a, j \in P_{a \oplus 32} \\
& \leq l(P_a) + l(Q_a) + l(P_{a \oplus 32}) + 6d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus 31}) - l(Q_{a \oplus 31}) - l(Q_{a \oplus 32}) + 6d \\
& \stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.7.2 Case 2(c)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2l_j d + 6d \\
& \stackrel{(25)}{<} p_{j^*} + p_j + l(P_{a \oplus 32} | > j) + 6d \\
& j^* \in Q_a, j \in P_{a \oplus 32} \\
& \leq l(Q_a) + l(P_{a \oplus 32}) + 6d \\
& \stackrel{(6)}{=} c - l(P_a) - l(P_{a \oplus 31}) - l(Q_{a \oplus 31}) - l(Q_{a \oplus 32}) + 6d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.7.3 Case 2(d)

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 2l_j d + 6d \\
& \stackrel{(20)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus 31}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor \\
& \quad + 2l_j d + 6d \\
& \stackrel{(25)}{<} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus 31}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + \\
& \quad l(P_{a \oplus 32} | > j) + 6d \\
& j^* \in Q_a, j \in P_{a \oplus 32} \\
& \leq l(Q_a) + l(P_{a \oplus 31}) + l(P_{a \oplus 32}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 6d \\
& \stackrel{(14)}{\leq} l(Q_a) + l(P_{a \oplus 31}) + l(P_{a \oplus 32}) + 4d \\
& \stackrel{(6)}{=} c - l(P_a) - l(Q_{a \oplus 31}) - l(Q_{a \oplus 32}) + 4d \\
& \stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.8 $j^* \in Q_a, j \in Q_a$ ($a = 1, 2, 3$)

2.8.1 Case 2(b)

$j^* = j$ If j^* is not a task of Q_1 or Q_2 :

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
& \stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 6d
\end{aligned}$$

$$\begin{aligned}
& \stackrel{j^*=j}{=} 2p_{j^*} + 2kd + 6d \\
(21), \delta_a \geq 0 & < l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) - 2kd + 8d \\
(13) & \leq l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) + 6d \\
(6) & \stackrel{=}{=} c - l(P_{a \oplus_3 2}) - l(Q_{a \oplus_3 1}) - l(Q_{a \oplus_3 2}) + 6d \\
(9)(10) & \leq c.
\end{aligned}$$

If j^* is the first task of Q_1 or Q_2 :

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(1)(12)}{\leq} p_{j^*} + p_j + 2kd + 4d + \delta(j) \\
& \stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 6d \\
& \stackrel{j^*=j}{=} 2p_{j^*} + 2kd + 6d \\
(21), \delta_a \geq 0 & < l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) - 2kd + 8d \\
(13) & \leq l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) + 6d \\
(6) & \stackrel{=}{=} c - l(P_{a \oplus_3 2}) - l(Q_{a \oplus_3 1}) - l(Q_{a \oplus_3 2}) + 6d \\
(9)(10) & \leq c.
\end{aligned}$$

If j^* is a task of Q_1 or Q_2 , but not the first task of Q_1 or Q_2 :

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
& \stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 8d \\
& \stackrel{j^*=j}{=} 2p_{j^*} + 2kd + 8d \\
(22), \delta_a \geq 0 & < l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) - 2kd + 8d \\
(13) & \leq l(P_a) + l(Q_a) + l(P_{a \oplus_3 1}) + 6d \\
(6) & \stackrel{=}{=} c - l(P_{a \oplus_3 2}) - l(Q_{a \oplus_3 1}) - l(Q_{a \oplus_3 2}) + 6d \\
(9)(10) & \leq c.
\end{aligned}$$

$j^* < j$

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
& \stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 8d \\
& \stackrel{(19), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a) + l(Q_a | < j^*) + 8d \\
& \stackrel{j^*, j \in Q_a; j^* < j}{\leq} l(P_a) + l(Q_a) + 8d
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(6)}{=} c - l(P_{a\oplus_3 1}) - l(P_{a\oplus_3 2}) - l(Q_{a\oplus_3 1}) - l(Q_{a\oplus_3 2}) + 8d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

$\mathbf{j^* > j}$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
&\stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 8d \\
&\stackrel{(20)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a\oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 8d \\
&\stackrel{j^*, j \in Q_a; j^* > j}{\leq} l(Q_a) + l(P_{a\oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 8d \\
&\stackrel{(14)}{\leq} l(Q_a) + l(P_{a\oplus_3 1}) + 6d \\
&\stackrel{(6)}{=} c - l(P_a) - l(P_{a\oplus_3 2}) - l(Q_{a\oplus_3 1}) - l(Q_{a\oplus_3 2}) + 6d \\
&\stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

2.8.2 Case 2(c)

$\mathbf{j^* = j}$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
&\stackrel{j^* = j}{=} 2p_j + 2l_j d + 6d + \delta(j) \\
&\stackrel{(30)/(33)}{<} l(P_a) + l(Q_a) + l(P_{a\oplus_3 1}) - 2l_j d + 6d \\
&\stackrel{(23)}{\leq} l(P_a) + l(Q_a) + l(P_{a\oplus_3 1}) + 4d \\
&\stackrel{(6)}{=} c - l(P_{a\oplus_3 2}) - l(Q_{a\oplus_3 1}) - l(Q_{a\oplus_3 2}) + 4d \\
&\stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

$\mathbf{j^* < j}$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
&\stackrel{(28)/(31)}{<} p_{j^*} + p_j + l(Q_a | > j) + l(P_{a\oplus_3 1}) + 6d \\
&\stackrel{j^*, j \in Q_a; j^* < j}{\leq} l(Q_a) + l(P_{a\oplus_3 1}) + 6d \\
&\stackrel{(6)}{=} c - l(P_a) - l(P_{a\oplus_3 2}) - l(Q_{a\oplus_3 1}) - l(Q_{a\oplus_3 2}) + 6d \\
&\stackrel{(9)(10)}{\leq} c - 2d.
\end{aligned}$$

$\mathbf{j^* > j}$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
&\stackrel{(8)}{<} p_{j^*} + p_j + 2l_j d + 8d \\
&\stackrel{(29)/(32)}{\leq} p_{j^*} + p_j + l(P_a) + l(Q_a | < j) - 2d \lfloor \frac{p_j}{2d} \rfloor + 8d \\
&\stackrel{j^*, j \in Q_a; j^* > j}{\leq} l(P_a) + l(Q_a) - 2d \lfloor \frac{p_j}{2d} \rfloor + 8d \\
&\stackrel{p_j > 0}{\leq} l(P_a) + l(Q_a) + 8d \\
&\stackrel{(6)}{=} c - l(P_{a \oplus 3 1}) - l(P_{a \oplus 3 2}) - l(Q_{a \oplus 3 1}) - l(Q_{a \oplus 3 2}) + 8d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.8.3 Case 2(d)

$\mathbf{j^* = j}$

$a = 1, 2$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
&\stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 8d \\
&\stackrel{j^* = j}{=} 2p_{j^*} + 2kd + 2l_j d + 8d \\
&\stackrel{(20)}{\leq} 2p_{j^*} + l(Q_a | > j^*) + l(P_{a \oplus 3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 2l_j d + 8d \\
&\stackrel{(29)}{\leq} 2p_{j^*} + l(Q_a | > j^*) + l(P_{a \oplus 3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + l(P_a) + \\
&\quad l(Q_a | < j) - 2d(\lfloor \frac{p_j}{2d} \rfloor + 1) + 8d \\
&\stackrel{j^*, j \in Q_a; j^* = j}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus 3 1}) + p_{j^*} - 4d \lfloor \frac{p_{j^*}}{2d} \rfloor + 6d \\
&\stackrel{(11)(14)}{<} l(P_a) + l(Q_a) + l(P_{a \oplus 3 1}) + 6d \\
&\stackrel{(6)}{=} c - l(P_{a \oplus 3 2}) - l(Q_{a \oplus 3 1}) - l(Q_{a \oplus 3 2}) + 6d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

$a = 3$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
&\stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 2l_j d + 6d \\
&\stackrel{j^* = j}{=} 2p_{j^*} + 2kd + 2l_j d + 6d
\end{aligned}$$

$$\begin{aligned}
& \stackrel{(20)}{\leq} 2p_{j^*} + l(Q_3 | > j^*) + l(P_1) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 2l_j d + 6d \\
& \stackrel{(32)}{\leq} 2p_{j^*} + l(Q_3 | > j^*) + l(P_1) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + l(P_3) \\
& \quad + l(Q_3 | < j) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
& \stackrel{j^*, j \in Q_3; j^* = j}{\leq} l(P_3) + l(Q_3) + l(P_1) + p_{j^*} - 4d \lfloor \frac{p_{j^*}}{2d} \rfloor + 6d \\
& \stackrel{(11)(14)}{<} l(P_3) + l(Q_3) + l(P_1) + 6d \\
& \stackrel{(6)}{=} c - l(P_2) - l(Q_1) - l(Q_2) + 6d \\
& \stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

$j^* < j$

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
& \stackrel{(19), \delta_a \geq 0}{\leq} p_{j^*} + p_j + l(P_a) + l(Q_a | < j^*) + 2l_j d + 6d + \delta(j) \\
& \stackrel{(28)/(31)}{<} p_{j^*} + p_j + l(P_a) + l(Q_a | < j^*) + l(Q_a | > j) + l(P_{a \oplus 31}) + 6d \\
& \stackrel{j^*, j \in Q_a; j^* < j}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus 31}) + 6d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus 32}) - l(Q_{a \oplus 31}) - l(Q_{a \oplus 32}) + 6d \\
& \stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

$j^* > j$

$$\begin{aligned}
I(j^*) + II(j) & \stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
& \stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 8d \\
& \stackrel{(20)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus 31}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 2l_j d + 8d \\
& \stackrel{(29)/(32)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus 31}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + l(P_a) \\
& \quad + l(Q_a | < j) - 2d \lfloor \frac{p_j}{2d} \rfloor + 8d \\
& \stackrel{j^*, j \in Q_a; j^* > j}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus 31}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor - 2d \lfloor \frac{p_j}{2d} \rfloor + 8d \\
& \stackrel{(14), p_j > 0}{\leq} l(P_a) + l(Q_a) + l(P_{a \oplus 31}) + 6d \\
& \stackrel{(6)}{=} c - l(P_{a \oplus 32}) - l(Q_{a \oplus 31}) - l(Q_{a \oplus 32}) + 6d \\
& \stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.9 $j^* \in Q_a, j \in Q_{a \oplus_3 1}$ ($a = 1, 2, 3$)

2.9.1 Case 2(b)

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
&\stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 8d \\
&\stackrel{(20)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 8d \\
&\stackrel{j^* \in Q_a, j \in Q_{a \oplus_3 1}}{\leq} l(Q_a) + l(P_{a \oplus_3 1}) + l(Q_{a \oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 8d \\
&\stackrel{(14)}{\leq} l(Q_a) + l(P_{a \oplus_3 1}) + l(Q_{a \oplus_3 1}) + 6d \\
&\stackrel{(6)}{=} c - l(P_a) - l(P_{a \oplus_3 2}) - l(Q_{a \oplus_3 2}) + 6d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.9.2 Case 2(c)

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
&\stackrel{(28)/(31)}{<} p_{j^*} + p_j + l(Q_{a \oplus_3 1} | > j) + l(P_{a \oplus_3 2}) + 6d \\
&\stackrel{j^* \in Q_a, j \in Q_{a \oplus_3 1}}{\leq} l(Q_a) + l(Q_{a \oplus_3 1}) + l(P_{a \oplus_3 2}) + 6d \\
&\stackrel{(6)}{=} c - l(P_a) - l(P_{a \oplus_3 1}) - l(Q_{a \oplus_3 2}) + 6d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.9.3 Case 2(d)

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
&\stackrel{(20)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 2l_j d + 6d + \delta(j) \\
&\stackrel{(28)/(31)}{<} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus_3 1}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + l(Q_{a \oplus_3 1} | > j) + l(P_{a \oplus_3 2}) + 6d \\
&\stackrel{j^* \in Q_a, j \in Q_{a \oplus_3 1}}{\leq} l(Q_a) + l(P_{a \oplus_3 1}) + l(Q_{a \oplus_3 1}) + l(P_{a \oplus_3 2}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 6d \\
&\stackrel{(14)}{\leq} l(Q_a) + l(P_{a \oplus_3 1}) + l(Q_{a \oplus_3 1}) + l(P_{a \oplus_3 2}) + 4d \\
&\stackrel{(6)}{=} c - l(P_a) - l(Q_{a \oplus_3 2}) + 4d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

2.10 $j^* \in Q_a, j \in Q_{a \oplus 32}$ ($a = 1, 2, 3$)

2.10.1 Case 2(b)

$$\begin{aligned}
 I(j^*) + II(j) & \stackrel{(1)(4)}{<} p_{j^*} + p_j + 2kd + 6d + \delta(j) \\
 & \stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 8d \\
 & \stackrel{(20)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus 31}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 8d \\
 & \stackrel{j^* \in Q_a, j \in Q_{a \oplus 32}}{\leq} l(Q_a) + l(P_{a \oplus 31}) + l(Q_{a \oplus 32}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 8d \\
 & \stackrel{(14)}{\leq} l(Q_a) + l(P_{a \oplus 31}) + l(Q_{a \oplus 32}) + 6d \\
 & \stackrel{(6)}{=} c - l(P_a) - l(P_{a \oplus 32}) - l(Q_{a \oplus 31}) + 6d \\
 & \stackrel{(9)(10)}{\leq} c.
 \end{aligned}$$

2.10.2 Case 2(c)

$a = 1$

$$\begin{aligned}
 I(j^*) + II(j) & \stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
 & \stackrel{(7)}{=} p_{j^*} + p_j + 2l_j d + 6d \\
 & \stackrel{(32)}{\leq} p_{j^*} + p_j + l(P_3) + l(Q_3 | < j) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
 & \stackrel{j^* \in Q_1, j \in Q_3}{\leq} l(Q_1) + l(P_3) + l(Q_3) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
 & \stackrel{p_j > 0}{\leq} l(Q_1) + l(P_3) + l(Q_3) + 6d \\
 & \stackrel{(6)}{=} c - l(P_1) - l(P_2) - l(Q_2) + 6d \\
 & \stackrel{(9)(10)}{\leq} c.
 \end{aligned}$$

$a = 2, 3$

$$\begin{aligned}
 I(j^*) + II(j) & \stackrel{(2)(5)}{<} p_{j^*} + p_j + 2l_j d + 6d + \delta(j) \\
 & \stackrel{(8)}{<} p_{j^*} + p_j + 2l_j d + 8d \\
 & \stackrel{(29)}{\leq} p_{j^*} + p_j + l(P_{a \oplus 32}) + l(Q_{a \oplus 32} | < j) - 2d(\lfloor \frac{p_j}{2d} \rfloor + 1) + 8d \\
 & \stackrel{j^* \in Q_a, j \in Q_{a \oplus 32}}{\leq} l(Q_a) + l(P_{a \oplus 32}) + l(Q_{a \oplus 32}) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
 & \stackrel{p_j > 0}{\leq} l(Q_a) + l(P_{a \oplus 32}) + l(Q_{a \oplus 32}) + 6d \\
 & \stackrel{(6)}{=} c - l(P_a) - l(P_{a \oplus 31}) - l(Q_{a \oplus 31}) + 6d
 \end{aligned}$$

$$\stackrel{(9)(10)}{\leq} c.$$

2.10.3 Case 2(d)

$a = 1$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
&\stackrel{(7)}{=} p_{j^*} + p_j + 2kd + 2l_j d + 6d \\
&\stackrel{(20)}{\leq} p_{j^*} + p_j + l(Q_1 | > j^*) + l(P_2) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + 2l_j d + 6d \\
&\stackrel{(32)}{\leq} p_{j^*} + p_j + l(Q_1 | > j^*) + l(P_2) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + l(P_3) + \\
&\quad l(Q_3 | < j) - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
&\stackrel{j^* \in Q_1, j \in Q_3}{\leq} l(Q_1) + l(P_2) + l(P_3) + l(Q_3) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
&\stackrel{(14), p_j > 0}{\leq} l(Q_1) + l(P_2) + l(P_3) + l(Q_3) + 4d \\
&\stackrel{(6)}{=} c - l(P_1) - l(Q_2) + 4d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

$a = 2, 3$

$$\begin{aligned}
I(j^*) + II(j) &\stackrel{(4)(5)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 6d + \delta(j) \\
&\stackrel{(8)}{<} p_{j^*} + p_j + 2kd + 2l_j d + 8d \\
&\stackrel{(20)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus 31}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor + \\
&\quad 2l_j d + 8d \\
&\stackrel{(29)}{\leq} p_{j^*} + p_j + l(Q_a | > j^*) + l(P_{a \oplus 31}) - 2d \lfloor \frac{p_{j^*}}{2d} \rfloor \\
&\quad + l(P_{a \oplus 32}) + l(Q_{a \oplus 32} | < j) - 2d (\lfloor \frac{p_j}{2d} \rfloor + 1) + 8d \\
&\stackrel{j^* \in Q_a, j \in Q_{a \oplus 32}}{\leq} l(Q_a) + l(P_{a \oplus 31}) + l(P_{a \oplus 32}) + l(Q_{a \oplus 32}) - \\
&\quad 2d \lfloor \frac{p_{j^*}}{2d} \rfloor - 2d \lfloor \frac{p_j}{2d} \rfloor + 6d \\
&\stackrel{(14), p_j > 0}{\leq} l(Q_a) + l(P_{a \oplus 31}) + l(P_{a \oplus 32}) + l(Q_{a \oplus 32}) + 4d \\
&\stackrel{(6)}{=} c - l(P_a) - l(Q_{a \oplus 31}) + 4d \\
&\stackrel{(9)(10)}{\leq} c.
\end{aligned}$$

References

- [1] C.M.H. Kuijpers. *Cyclic Machine Scheduling with Tool Transportation*. PhD thesis, University of Twente, Enschede, the Netherlands, 2001. To appear.