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A discounted model for a repairable system with continuous state space

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We examine repairable systems with a continous state space and partial repair options, carried out at fixed times $n = 1, 2, ...$ Every time interval $[n, n+1)$ there is a manufacturing cost and a repair cost. These cost functions are not restricted to the class of bounded functions in this study. Conditions are found under which a control-limit replacement policy minimizes the discounted cost. Hence these conditions guarantee that there is an optimal policy under the discounted cost criterion which does not use partial repairs. We explicitly explain how to derive this optimal policy.

Keywords and phrases: reliability, availability, maintenance, inspection, control-limit policy, partial repair, discounted cost, unbounded cost.

Subject Classification: 93E20, 90B25

1 Introduction

1.1 Model-description and definitions

This paper deals with a system which is inspected at discrete time instants $n \in I\!N$. After every inspection a state x of the state space $I = I\!R^+$ or $I = [0, N]$, $N \in I\!R^+$ is given to the system. If $I = I\!R^+$ we set $N = \infty$. X_n^- will denote the state of the system just before time n. After the $n-$th inspection a repair action $A_n \in [0, X_n^-]$ is taken at time n which improves the state of the system to state $X_n = X_n^- - A_n$. We define the random variable X_0 as the initial state. The length of time needed for inspection and repair is negligible. The deterioration from state $X_n = x$ to state X_{n+1}^- is subject to the distribution function $F(.|x)$. There exists a density $f(y|x)$ with $f(y|x) = 0$ for $y < x$, so $F(y|x) = \int_0^y f(z|x)\, dz = \int_x^y f(z|x)\, dz$. A possibly randomized strategy δ is identified by the set of random variables $\{\delta_n(x);\, x \in I,\, n \in I\!N\}$ with image set I. If the strategy is stationary we use $\delta(x)$ instead of $\delta_n(x)$. We also define the distribution functions $G_x^{\delta_n}(a) := P(\delta_n(x) \leq a)$. Obviously $G_x^{\delta_n}(a) = 1$ for all $a \geq x$ holds. $\delta_n(x) = 0$ for a $n \in I\!N$, $x \in I$ means that if the system's state at time n is x, a minimal repair action is chosen. The opposite of a minimal repair is a maximal repair ($\delta_n(x) = x$) which stands for complete repair or replacement by a new system, so the state after that repair is zero.

Now we describe the cost functions. In the nth interval $(n-1, n]$, $n \in I\!N$, there is a manufacturing cost $r(X_{n-1})$ and the cost of repair will be $d(X_n^-, \delta_n(X_n^-))$. The cost of repair depending on the first component might also be interpreted as cost for production loss for a *bad* system. The cost of repair (energy, personnel, etc) are paid after each period together with the cost of manufacturing. These cost functions are not restricted to the class of bounded functions. Conditions are found under which a control-limit replacement policy minimizes the discounted cost. Hence these conditions guarantee that there is an optimal policy under the discounted cost criterion which does not use partial repairs. We explicitly explain how to derive this optimal policy.

Obviously the random variables X_n and X_n^- depend on the strategy δ, but we will not use the notations X_n^δ or $X_n^{-,\delta}$ in this paper.

1.2 History

Many authors have considered stochastically deteriorating systems. Very often it is only allowed to repair after a failure. A lot of papers are written dealing with replacement strategies, where there are two different repair actions: to replace or not to replace the system. Brown and Proschan [3] used an imperfect repair first. With probability p an imperfect repair is a perfect repair/replacement, and with probability $1-p$ an imperfect repair is a minimal repair. A minimal repair restores the failed system to its condition just prior to failure. Several kinds of minimal repair (black-box minimal repair, physical minimal repair) are explained by Aven and Jensen in their recent book [2].

A paper which deals with general degree of repair and uses a general state space has been written by Stadje and Zuckerman [13]. In their model the state defines the virtual age. Thus state 0 stands for a new system and the higher the state the worse the system. This fact holds for our model, too. In their model a maintenance action which reduces the virtual age from $x \in I\!R^+$ to some $y \in [0, x]$ can be taken. Since the state of the system is found out after an inspection in our model, the system state is not the virtual age. The cost functions may not be bounded, either. Taking into account the possibility of unbounded cost functions became quite fashionable during the last years, e.g. in the book of Hernandez-Lerma, Lassarre [8]. Recently Hordijk and Yushkevich [10], [11] looked for general Blackwell optimality if the cost functions are unbounded.

Discrete repair models with the option of general degree of repair have been, among others, investigated by Bruns [4], Douer and Yechiali [7] and Stadje and Zuckerman [14].

1.3 Preliminaries

Before presenting the model we give some general lemmas. First we define the measurable space $\Omega := ([0,N], \mathcal{B}_{[0,N]})$. Now we define a stochastical kernel

$$K : ([0,N], \mathcal{B}_{[0,N]}) \to [0,1]$$

in the following way:

$$K(x,B) = P(X_1^- \in B | X_0 = x) \; \forall x \in [0,N], \text{ for all } B \in \mathcal{B}_{[0,N]}.$$

Furthermore we define the transition law $P_x : \mathcal{B}_{[0,N]} \to [0,1]$ with $P_x(B) := K(x,B) = P(X_1^- \in B | X_0 = x)$ for $B \in \mathcal{B}_{[0,N]}$. Hence the identity $P_x[0,y] = F(y|x)$ holds. Finally we define the multi-function $A: \; [0,N] \to \mathcal{B}_{[0,N]}$ which gives to every state x the action set $A(x)$. In our model we have $A(x) = [0,x]$. As an extension $A(x)$ can be chosen as any closed subset of $[0,x]$ which includes the values 0 and x.

$A(x) \subset [0,x]$, $A(x)$ closed, $\{0,x\} \subset A(x)$ can also be taken.

Lemma 1 *The multi-function $A: \; (I,d) \to (2^I, \rho_d)$, $A(x) = [0,x]$ is compact-valued and continuous for every metric d.*

From $A(x) = [0,x]$ for every $x \in I$ it follows that A is compact-valued and the continuity we get from the identity

$$\rho_d\left(A(x), A(y)\right) = \rho_d\left([0,x],[0,y]\right) = d(x,y). \quad \blacksquare$$

Lemma 2 *Let X be a metric space, $f, g: \; X \to \mathbb{R}^+$ and $h : X \to X$ be lower semicontinuous functions. Let $u : X^2 \to \mathbb{R}^+$ denote a function being lower semicontinuous in the second component. We have*

1. *$f \circ h$ is lower semicontinuous.*
2. *The function $v(x) := \min_{\{a \in A(x)\}} u(x,a)$ exists and is measurable.*

1. holds since $(f \circ h)^{-1}(M)$ is a closed set if M is closed. 2. has already been proven by Himmelberg et al [9] in their Theorem 2 under the condition that A is Borel-measurable. This condition holds by lemma 1. $\blacksquare$

Lemma 3 *Let the function $g : I \to \mathbb{R}$ be non-decreasing. Then there exists a sequence of step-functions*

$$g_n(x) = \frac{1}{2^n} \sum_{k=1}^{\infty} 1_{[(a_{k,n},N]} + \beta \textit{ with } [(a_{k,n},N] := \left\{ x \in \mathbb{R} \middle| g(x) \in \left[g(0) + \frac{k}{2^n}, N \right] \right\}$$

and a value $\beta \in \mathbb{R}$, such that $g_n \uparrow g$ and $||g - g_n||_\infty \to 0$ as $n \to \infty$.
If g is bounded only a finite number of summands differ from zero. If g is non-negative, then so is β.

This lemma will be proven in the Appendix.

2 About the conditions of the model

The following boundary assumption holds throughout this paper.
It is called **condition (A)**.
There exists $K_x \in I\!R$ and $\kappa_x \in I\!N$ such that

$$\int_x^N \left(\sup_{\{0 \le x_1 \le y\}} \{|r(x_1)|\} + \sup_{\{0 \le x_3 \le x_2 \le y\}} \{|d(x_2, x_3)|\} \right) dF^{(n)}(y|x) < K_x n^{\kappa_x} \qquad \forall x \in I.$$

δ_∞ denotes the strategy using only minimal repair. We denote $F^{(n)}(y|x)$ as the probability $P_{\delta_\infty}(X_n \le y \,|X_0 = x)$, so $F^{(1)}(y|x) = F(y|x)$ and $F^{(n)}(y|x) = \int_x^y F^{(n-1)}(y|z)\, dF(z|x)$.

Furthermore we assume the following condition which will be called **condition (3)** later.

$$F(.|x) \text{ is non-increasing for all } x \in I.$$

Thus the probability $P(X_{n+1}^- \ge x | X_n = a) = 1 - F(x|a)$ is non-decreasing.

Lemma 4 *If $X_0^{\delta_\infty} = X_0^\delta$ then for all n, $X_n^{\delta_\infty} \overset{st}{\ge} X_n^\delta$ and $X_n^{\delta_\infty,-} \overset{st}{\ge} X_n^{\delta,-}$.*

Proof: This lemma will be proven by induction.

$$\begin{aligned} P\left(X_{n+1}^{\delta_\infty,-} \ge y\right) &= \int_{x=0}^N P\left(X_{n+1}^{\delta_\infty,-} \ge y | X_n^{\delta_\infty} = x\right) dF\left(x | X_n^{\delta_\infty}\right) \\ &= \int_{x=0}^N (1 - F(y|x))\, dF\left(x | X_n^{\delta_\infty}\right). \end{aligned}$$

Let $f_y(x) = 1 - F(y|x) = P(X_1^- > y | X_0 = x)$, then

$$P\left(X_{n+1}^{\delta_\infty,-} \ge y\right) = E\left(f_y\left(X_n^{\delta_\infty}\right)\right) \ge E\left(f_y\left(X_n^\delta\right)\right) = P\left(X_{n+1}^{\delta,-} \ge y\right).$$

The inequality holds since each function f_x is non-decreasing. ■

The mean discounted cost function $V_{\delta,\alpha}(x)$ fulfilles the subsequent identity. The existence will be proven in the next theorem.

$$V_{\delta,\alpha}(x) = E_\delta\left(\sum_{n=0}^\infty \alpha^n c_\delta(n) \middle| X_0 = x\right)$$

with a countable set of random variables $\{c_\delta(n),\, n \in I\!N\}$ where

$$\begin{aligned} c_\delta(n) &:= r(X_n) + d(X_{n+1}^-, \delta(X_{n+1}^-)), \\ &= r(X_n) + \int_x^N \int_0^y d(y,a)\, dG_y^\delta(a)\, dF(y|X_n) \quad \forall x \in I. \end{aligned}$$

Hence,

$$V_{\delta,\alpha}(x) = E_\delta\left(\sum_{n=0}^\infty \alpha^n \left(r(X_n) + \int_0^N \int_0^y d(y,a)\, dG_y^\delta(a)\, dF(y|X_n) \middle| X_0 = x \right)\right). \quad (1)$$

Theorem 1 $|V_{\delta,\alpha}(x)| \le |r(x)| + \tilde{\kappa}_x^\alpha < \infty$ *where* $\tilde{\kappa}_x^\alpha := K_x \sum_{n=1}^\infty \alpha^{n-1}(n+1)^{\kappa_x}$.

Proof:

$$
\begin{aligned}
& |V_{\delta,\alpha}(x)| \\
\leq\ & |r(x)| + \sum_{n=1}^{\infty} \alpha^{n-1} E_\delta \left(\sup_{\{0 \leq x_3 \leq x_2 \leq X_n\}} \{|d(x_2,x_3)|\} + \sup_{\{0 \leq x_1 \leq X_n\}} \{|r(x_1)|\} \middle| X_0 = x \right) \\
\leq\ & |r(x)| + \sum_{n=1}^{\infty} \alpha^{n-1} E_{\delta_\infty} \left(\sup_{\{0 \leq x_3 \leq x_2 \leq X_n\}} \{|d(x_2,x_3)|\} + \sup_{\{0 \leq x_1 \leq X_n\}} \{|r(x_1)|\} \middle| X_0 = x \right) \quad (2) \\
\leq\ & |r(x)| + \sum_{n=1}^{\infty} \alpha^{n-1} \int_x^N \left(\sup_{\{0 \leq x_3 \leq x_2 \leq y\}} \{|d(x_2,x_3)|\} + \sup_{\{0 \leq x_1 \leq y\}} \{|r(x_1)|\} \right) dF^{(n)}(y|x) (3) \\
\leq\ & |r(x)| + \kappa_x \sum_{n=1}^{\infty} \alpha^{n-1}(n+1)^{\kappa_i} = |r(x)| + \tilde{\kappa}_x^\alpha < \infty.
\end{aligned}
$$

The inequality (2) is valid because of lemma (4). ■

Now we present further conditions to our model.

(1) The cost function r is measurable, non-decreasing and left semicontinuous; the cost-function d is measurable, non-decreasing in both components and left semicontinuous in the second component.
(2) The function $x \to P_x(B)$ is continuous in x on I for every set $B \in \mathcal{B}_{[0,N]}$.

Hence r is lower semicontinuous and d is lower semicontinous in the second component. The class of functions on I being lower semicontinuous will be denoted as $C_u(I)$ from now on. Condition (2) is valid if the family $\{f(.|x),\ x \in I\}$ is continuous in x with respect to the L_1-norm, i.e. for all $x \in I$ and $\epsilon > 0$ exists a $\delta > 0$ with $\int_I |f(y|z) - f(y|x)|\, dy < \epsilon\ \forall z \in [x-\delta, x+\delta]$. Then we have $|P_z(B) - P_x(B)| \leq \int_B |f(y|z) - f(y|x)|\ dy < \epsilon\ \forall z \in [x-\delta, x+\delta]$.

Lemma 5 *Condition* (2) *is equivalent to the strong continuity of the stochastical kernel K.*

Proof: If the kernel is strong continuous the probability $P_x(B) = \int 1_B dF(y|x)$ is continuous with respect to x for all $B \in \mathcal{B}$. The other statement will be proven now:
(i) $h = \sum_{i=1}^n c_i 1_{B_i}$ with $c_i \in I\!R^+$ and $B_i \in \mathcal{B}$: $\int h\ dF(y|x) = \sum_{i=1}^n c_i \int 1_{B_i}\, dF(y|x)$
$= \sum_{i=1}^n c_i P_x(B_i)$ is continuous on x und bounded above by $\max_{i \in \{1,\ldots,n\}}\{c_i\}$.
(ii) h measurable, bounded and positive: There exists a sequence $(h_n)_{n \in \boldsymbol{N}}$ with $h_n \uparrow h$, where h_n is a function as defined in (i) and $||h - h_n||_\infty < \frac{1}{n}$ holds. Thus

$$
\int h\ dF(y|x) = \int \lim_{n\to\infty} h_n\, dF(y|x) = \lim_{n\to\infty} \int h_n\, dF(y|x)
$$

is the limit of continuous and bounded functions. It is also continuous and bounded since

$$
\left|\left| \int h(y)\ dF(y|x) - \int h_n(y)\ dF(y|x) \right|\right|_\infty \leq \int ||h(y) - h_n(y)||_\infty\ dF(y|x) < \frac{1}{n}.
$$

(iii) h measurable and bounded: from the equality $h = h^+ - h^-$ where h^+ and h^- are measurable, bounded and positive, we receive the strong continuity of the stochastical kernel K. ■

Since the cost function r is bounded below by $r(0)$ and the cost function d is bounded below by $d(0,0)$ without loss of generality we will just deal with non-negative cost functions from now on.

We define $\mathcal{M}(X)$ as the set of measurable functions on X and $\mathcal{M}_b(X)$ as the subset of bounded functions of $\mathcal{M}(X)$. Since the cost functions might not be bounded from above we need the result of the subsequent lemma:

Lemma 6 *The function $h(x) := \int g(y)\, dF(y|x)$ defined on I is lower semicontinuous for every function $g \in \{\tilde{g} : I \to \mathbb{R}^+$ measurable, $\tilde{g}(.)f(.|x)$ integrable for every $x \in I\}$.*

Proof: For $n \in \mathbb{N}$ we define $g_n(y) = \min\{g(y), n\}$ and $h_n(x) = \int g_n(x,y)\, dF(y|x)$. The strong continuity of the stochastic kernel yields continuity and boundness of h_n since they hold for g_n. From $g_n \uparrow g$ follows that for every fixed $x \in I$ we have $g_n f(.|x) \uparrow gf(.|x)$, so the integrability of $gf(.|x)$ yields $h_n(x) \uparrow h(x)$. Hence the continuity of h_n yields to the result that the function h is lower semicontinuous. ■

Theorem 2
Under the conditions (1) *and* (2) *the subsequent statements are fulfilled:*

1. $\min_{\{a\in[0,x]\}}\{d(x,a)+V_\alpha(x-a)\}$ *exists and is measurable where* $V_\alpha(x) := \inf_{\{\delta\in\Pi\}} V_{\delta,\alpha}(x)$.
2. $d(x,\delta(x)) + V_{\delta,\alpha}(x-\delta(x))$ *is measurable for every* $\delta \in \Pi$.
3. *The function* $f_\alpha :\ I \to I$, *which is defined by the subsequent identity, is measurable.*
   $d(x, f_\alpha(x)) + V_\alpha(x - f_\alpha(x)) = \min_{\{a\in[0,x]\}}\{d(x,a) + V_\alpha(x-a)\}$.

Furthermore, the functions V_α *and* $V_{\delta,\alpha}$ *are lower semicontinuous for each* $\delta \in \Pi$.

Obviously the function f_α stands for a stationary deterministic strategy.

Proof: We denote by $V_{\delta,\alpha}^{(n)}(x)$ as the $\alpha-$discounted cost up to timepoint n using strategy δ so $V_{\delta,\alpha}^{(n)}(x) = E_\delta\left(\sum_{t=0}^{n} c_\delta(t) \middle|\, X_0 = x\right)$ and $V_\alpha^{(n)}(x) = \inf_{\{\delta\in\Pi\}} V_{\delta,\alpha}^{(n)}(x)$. The function $V_{\delta,\alpha}^{(n)}$ exists since the function $V_{\delta,\alpha}$ exists. For a system with finite horizon the following results will be proven by induction on $n \in \mathbb{N}$.

(i) $\quad y \to \min_{a\in[0,y]} \left\{d(y,a) + \alpha V_\alpha^{(n)}(y-a)\right\}$ exists and is measurable.

(ii) $V_\alpha^{(0)}(x) \equiv 0,\ V_\alpha^{(n)}(x) = r(x) + \int \min_{a\in[0,y]} \left\{d(y,a) + \alpha V_\alpha^{(n-1)}(y-a)\right\} dF(y|x)\ \forall n \in \mathbb{N}$.

(iii) $\quad V_\alpha^{(n)}$ is lower semicontinuous.

$n = 0$: (i) holds since $\min_{\{a\in[0,y]\}} d(y,a)$ exists and is measurable by lemma 2. (ii) and (iii) are obvious.
$n-1 \to n$: starting with part (ii):

$$\begin{aligned} V_{\delta,\alpha}^{(n)}(x) &= r(x) + \int_x^N \int_0^y d(y,\delta(y)) + \alpha V_{\delta,\alpha}^{(n-1)}(y-\delta(y))dG_y^\delta(a)dF(y|x) \\ &\geq r(x) + \int_x^N \min_{a\in[0,y]}\{d(y,a) + \alpha V_\alpha^{(n-1)}(y-a)\}dF(y|x) \quad \forall \delta \in \Pi. \end{aligned}$$

Hence,

$$V_\alpha^{(n)}(x) \;\; \geq \;\; r(x) + \int_x^N \min_{a\in[0,y]}\{d(y,a) + \alpha V_\alpha^{(n-1)}(y-a)\} dF(y|x).$$

The minimum exists and is integrable because of part (i) of the induction condition. For $\epsilon > 0$ let δ_ϵ be the strategy satisfying $V_{\delta_\epsilon,\alpha}^{(n-1)}(x) \leq V_\alpha^{(n-1)}(x) + \epsilon \;\forall x \in I$ and choosing the action a_0 at time-point n and state y such that

$$d(y,a_0) + \alpha V_\alpha^{(n-1)}(y-a_0) = \min_{a\in[0,y]} \left\{ d(y,a) + \alpha V_\alpha^{(n-1)}(y-a) \right\},$$

so

$$\begin{aligned} V_{\delta_\epsilon,\alpha}^{(n)}(x) &= r(x) + \int_x^N \left(d(y,\delta_\epsilon(y)) + \alpha V_{\delta_\epsilon,\alpha}^{(n-1)}(y - \delta_\epsilon(y)) \right) dF(y|x) \\ &\leq r(x) + \int_x^N \left(d(y,\delta_\epsilon(y)) + \alpha V_\alpha^{(n-1)}(y - \delta_\epsilon(y) \right) dF(y|x) + \alpha\epsilon \\ &= r(x) + \int_x^N \min_{a\in[0,y]} \left\{ d(y,a) + \alpha V_\alpha^{(n-1)}(y-a) \right\} dF(y|x) + \alpha\epsilon. \end{aligned}$$

Hence

$$V_\alpha^{(n)}(x) \;\; \leq \;\; \lim_{\epsilon\downarrow 0} V_{\delta_\epsilon,\alpha}^{(n)}(x) \leq r(x) + \int \min_{a\in[0,y]} \left\{ \{ d(y,a) + \alpha V_\alpha^{(n-1)}(y-a) \right\} dF(y|x).$$

Both inequations yield part (ii). It follows from lemma 5 that the continuity of $F(y|.)$ for every $y \in I$ yields to the strong continuity of the stochastic kernel K. Hence the integral of (ii) is lower semicontinuous by lemma 6 and part (i) of the induction condition. Thus we get part (iii) from the lower semicontinuity of r. Finally we proof part (i) using part (iii).
We define the continuous function $h : I^2 \to I$, $h(x,a) = x-a$. The induction condition, part (iii) and part (1) of lemma 2 yield to that the function
$u(y,a) := d(y,a) + \alpha V_\alpha^{(n)}(y-a) = d(y,a) + (V_\alpha^{(n)} \circ h)(y,a))$ is lower semicontinuous. Furthermore part (3) of lemma 2 yields that the function $\min_{\{a\in A(y)\}} u(y,a)$ is measurable. This proves part (i) and completes the induction.
The function V_α is also lower semicontinuous because of $V_\alpha^{(n)}(x) \uparrow V_\alpha(x)$. Thus as in the induction proof of (i) it can be shown that the function
$\min_{\{a\in[0,y]\}}\{d(y,a) + \alpha V_\alpha(y-a)\}$ exists and is measurable. It has been proven by Himmelberg et al [9] that f_α is measurable. The results from Himmelberg et al, used in this paper, are also mentioned by Hernandez-Lerma, Lasserre [8], in Proposition D.5.
$d(x,\delta(x)) + V_{\delta,\alpha}(x - \delta(x))$ is measurable in x since d, δ und $V_{\delta,\alpha}$ are measurable. We get the lower continuity of $V_{\delta,\alpha}$ from $V_{\delta,\alpha}^{(n)}(x) \uparrow V_{\delta,\alpha}(x)$ and the lower continuity of $V_{\delta,\alpha}^{(n)}(x)$ for all $n \in I\!N$, which can be proven by a similar induction using the recursive equation system

$$V_{\delta,\alpha}(0) \equiv 0,\; V_{\delta,\alpha}^{(n)}(x) = r(x) + \int_x^N \int_0^y \left(d(y,a) + \alpha V_{\delta,\alpha}^{(n-1)}(y-a) \right) dG_y^\delta(a)\, dF(y|x) \;\forall n \in I\!N.$$

This theorem the Proof of the Theorem. ■

3 Some functional results

The principle of dynamic programming yields the following functional equation:

$$V_{\delta,\alpha}(x) = r(x) + \int_x^N \int_0^y (d(y,a) + \alpha V_{\delta,\alpha}(y-a))\, dG_y^\delta(a) dF(y|x).$$

If the policy δ is deterministic, the subsequent identity holds.

$$V_{\delta,\alpha}(x) = r(x) + \int_x^N (d(y,\delta(y)) + \alpha V_{\delta,\alpha}(y-\delta(y),\alpha))\, dF(y|x).$$

Next we present the **optimality equation** of our model:

Theorem 3 *The function $V_\alpha(x)$ fulfilles the following identity.*

$$V_\alpha(x) = r(x) + \int_{y=x}^N \min_{a\in[0,y]} \{d(y,a) + \alpha V_\alpha(y-a)\}\; dF(y|x).$$

The **proof** of this theorem is standard.

Definition 1 *We define the operator $T_\alpha : C_u(I) \to C_u(I)$ as follows.*

$$(T_\alpha(u))(x) \;\; := \;\; r(x) + \int_x^N \min_{a\in[0,y]} \{d(y,a) + \alpha u(y-a)\}\; dF(y|x).$$

Lemma 2 guarantees the existence of T_α.

Now we formulate a condition which shall only be assumed when mentioned explicitly.

Condition B: $\lim_{n\to\infty} \alpha^n \int_{y=x}^N \left(\sup_{z\in[0,y]} \{|r(z)| + \tilde{\kappa}_z^\alpha\} \right) dF^{(n)}(y|x) = 0.$

Lemma 7 *Condition B leads to the following identity.*

$$\lim_{n\to\infty} \alpha^n E\left(V_{\delta,\alpha}(X_n^\delta) \middle| X_0 = i \right) = 0 \;\forall x \in I \;\forall \delta \in \Pi.$$

The **proof** is identical to the proof of lemma 2 of Bruns [4].

Theorem 4 *Let δ_α be a stationary deterministic strategy with*

$$d(x,\delta_\alpha(x) + \alpha V_\alpha(x-\delta_\alpha(x)) = \min_{a\in[0,x]} \{d(x,a) + \alpha V_\alpha(x-a)\} \quad \forall x \in I.$$

If condition B holds we have $V_{\delta_\alpha,\alpha}(x) = V_\alpha(x) \quad \forall x \in I$, so $V_{\delta_\alpha,\alpha} = V_\alpha$. Hence δ_α is optimal.

Proof: Similarly to the proof of Theorem II 2.2 of Ross [12], we get the following identity.

$$V_\alpha(x) = V_{\delta_\alpha,\alpha}^{(n)}(x) + \alpha^n E_{\delta_\alpha}\left(V_\alpha(X_n) \middle| X_0 = x \right) \quad \forall n \in I\!N.$$

Lemma (7) yields the result of the Theorem. ■

Lemma 8 *Let $u \in C_u(I)$. Then*

(a) $\quad u \geq T_\alpha u$ *and* $\quad \left(\lim_{n\to\infty} \alpha^n E_\delta(u(X_n) \mid X_0 = x) = 0 \;\forall x \in I \;\forall \delta \in \Pi_2 \text{ or } u \geq 0 \right)$

yield $u \geq V_\alpha$.

(b) $u \leq T_\alpha u$ *and* $\quad \lim_{n\to\infty} \alpha^n E_\delta(u(X_n) \mid X_0 = x) = 0 \quad \forall x \in I \;\forall \delta \in \Pi$ *yield $u \leq V_\alpha$.*

Proof: For $u \in C_u(I)$, $y \in I$ the function $d(y,a) + \alpha u(y-z)$ will be maximized in $a = \delta_u(y) \in [0,y]$. Then

$$T_\alpha u(x) = r(x) + \int_{y=x}^{N} \left(d(y, \delta_u(y)) + \alpha u(y - \delta_u(y))\right) dF_x(y)$$

holds. Similarly to the proof of lemma 4.2.7 (a) in Hernandez-Lerma, Lasserre [8] we get

$$u(x) \geq E_{\delta_u}\left[\sum_{t=0}^{n-1} \alpha^t c_{\delta_u}(t) \middle| X_0 = x \right] + \alpha^n E_{\delta_u}\left[u(X_n) | X_0 = x \right] \quad \forall n \in I\!N,$$

so $n \to \infty$ yields to $u(i) \geq V_{\delta_u}(i) \geq V_\alpha(i)$.
The proof of (b) is identical to the proof of lemma 4.2.7 (b) of Hernandez-Lerma and Lasserre [8]. ■

From $V_\delta \in C_u(I)$ for every $\delta \in \Pi$ and the second part of lemma 7 we get the subsequent result.

Theorem 5
If condition B holds, V_α is the only function fulfilling the optimality equation.

Now we look at the policy iteration method. We choose a stationary strategy $\delta^0 \in \Pi$. Inductively the strategy δ^n is defined by the following equality.

$$d(x, \delta^n(x)) + \alpha V_{\delta^{n-1},\alpha}(x - \delta^n(x)) = \min_{a \in [0,x]} \left\{ d(x,a) + \alpha V_{\delta^{n-1},\alpha}(x-a) \right\}.$$

Theorem 6 *If condition B holds the equality $\lim_{n\to\infty} V_{\delta_n,\alpha}(x) = V_\alpha(x)$ is valid for $x \in I$. Thus the sequence $(\delta_n)_{n=1}^\infty$ converges to the $\alpha-$optimal strategy.*

Proof:

$$\begin{aligned} V_{\delta_n,\alpha}(x) &= r(x) + \int_{y=x}^{N} \left(d(y,\delta(y)) + \alpha V_{\delta,\alpha}(y - \delta(y))\right) dF(y|x) \\ &\leq r(x) + \int_{y=x}^{N} \Big(d(y, \delta_{n-1}(y)) + \alpha V_{\delta_{n-1},\alpha}(y - \delta_{n-1}(y))\Big) dF(y|x) \\ &= V_{\delta_{n-1},\alpha}(x). \end{aligned} \qquad (4)$$

Hence $v := \lim_{n\to\infty} V_{\delta_n,\alpha}$ exists and is clearly contained in $C_u(I)$. Equation (4) yields to $V_{\delta_n,\alpha} = T_\alpha V_{\delta_{n-1},\alpha}$ $\forall n \in I\!N$ thus $v = T_\alpha v$ holds. Hence $\lim_{n\to\infty} V_{\delta_n,\alpha} = v = V_\alpha$. ■
Next we want to prove the monotonocity of V_α. First we need the subsequent lemma.

Lemma 9 *$F(a|x)$ is non-decreasing in $x \in I$ for every $a \in I$ if and only if the function $\int_0^N g(y)\, dF(y|x)$ is non-increasing in $x \in I$ for every non-decreasing function g on I for that the integral $\int_0^N g(y)\, dF(y|x)$ exists.*

Proof of '⇒': From the last lemma we know that for all $k, n \in I\!N$ there exist $\alpha_{k,n} \in I\!R^+$, $\beta \in I\!R$ such that the subsequent identity holds.

$$g_n(x) = \frac{1}{2^n} \sum_{k=1}^{\infty} 1_{[(a_{k,n}\infty)}(x) + \beta$$

and $g_n \uparrow g$. This yields to the following identity.

$$\begin{aligned}
\int_0^N g(y)dF(y|x) &= \int_0^N \lim_{n\to\infty} \frac{1}{2^n}\left(\sum_{k=1}^{\infty} 1_{[(a_{k,n},\infty)}(y) + \beta\right) dF(y|x) \\
&= \lim_{n\to\infty}\left(\int_I \frac{1}{2^n}\sum_{k=1}^{\infty} 1_{[(a_{k,n},\infty)}(y)\, dF(y|x) + \beta\right) \\
&= \lim_{n\to\infty} \frac{1}{2^n}\sum_{k=1}^{\infty}\int_{a_{k,n}}^{\infty} 1\, dF(y|x) + \beta = \lim_{n\to\infty}\frac{1}{2^n}\sum_{k=1}^{\infty}(1 - F(a_{k,n}|x)) + \beta.
\end{aligned}$$

Since the terms $1 - F(a_{k,n}|x)$ are non-decreasing this holds for the entire sum.
To prove '$\Leftarrow$' we choose $g := 1_{(a,\infty)}$. Then the function $\int g(y)\, dF(y|x) = P_x((a,\infty))$ $= 1 - F(a|x)$ is non-decreasing in $x \in I$, so $F(a|x)$ is non-increasing in $x \in I$. ■

Theorem 7 *V_α is non-increasing on I for every $\alpha \in (0,1)$.*

Proof: Using induction we show the monotonocity of $\psi_\alpha(x,M) = \inf_{\{\delta\in\Pi\}}\left\{V_{\delta,\alpha}^{(M)}(x)\right\}$.
$M = 0$: $\psi_\alpha(x,0) \equiv 0$ is non-increasing in $x \in I$.

$$M \to M{+}1:\ \psi_\alpha(x, M{+}1) = r(x) + \int_{y=0}^N \inf_{a\in[0,y]}\{d(y,a) + \alpha\psi_\alpha(y{-}a, M)\}dF(y|x).$$

For $\epsilon \in [0,y]$ the subsequent inequality holds.

$$\begin{aligned}
&\inf_{a\in[0,y]}\{d(y,a) + \alpha\psi_\alpha(y{-}a,M)\} \\
&= \min\left\{\inf_{a\in[0,y]}\{d(y,a){+}\alpha\psi_\alpha(y{-}a,M)\},\ \inf_{a\in[y-\epsilon,y]}\{d(y,a){+}\alpha\psi_\alpha(y{-}a,M)\}\right\} \\
&\le \min\left\{\inf_{a\in[0,y]}\{d(y{+}\epsilon,a) + \alpha\psi_\alpha(y{+}\epsilon{-}a,M)\},\ \inf_{a\in[y-\epsilon,y]}\{d(y{+}\epsilon,a{+}\epsilon) + \alpha\psi_\alpha(y{-}a,M)\}\right\} \\
&= \min\left\{\inf_{a\in[0,y]}\{d(y{+}\epsilon,a) + \alpha\psi_\alpha(y{+}\epsilon{-}a,M)\},\ \inf_{a\in[y,y+\epsilon]}\{d(y{+}\epsilon,a) + \alpha\psi_\alpha(y{+}\epsilon{-}a,M)\}\right\} \\
&= \inf_{a\in[0,y+\epsilon]}\{d(y+\epsilon,a) + \alpha\psi_\alpha(y+\epsilon-a,M)\}. \qquad (5)
\end{aligned}$$

Now we use lemma 9 with $g(y) = \inf_{\{a\in[0,y]\}}\{d(y,a) + \alpha\psi(y{-}a,\alpha,M)\}$. Since $\psi_\alpha(x,M)$ is non-decreasing for every $M \in I\!N$, the induction is completed and we have proved that $V_\alpha = \lim_{m\to\infty}\psi_\alpha(x,M)$ in non-decreasing. ■

Now we look at the derivative of V_α. First we look for a weaker condition than the condition usually used to be able to interchange differentiation and integration.

Lemma 10 *Let $h:\ I\times I \to I\!R^+$ such that for fixed $x \in I$ $h(x,y)$ is integrable in y on I. If for fixed $y \in I$ the following holds:*
(i) $h(z,y)$ is partially differentiable in z, (ii) this partial derivation $\frac{\partial h(z,y)}{\partial z}$ is continuous, (iii) for all $x \in I$ there exists an $\epsilon(x) > 0$ and a function g_x being integrable on I with

$$\left|\frac{\partial}{\partial z}h(z,y)\right| \le g_x(y) \quad \forall (z,y) \in ([x-\epsilon(x), x+\epsilon(x)]^+ \times I),$$

then we have

$$\frac{d}{dx}\int_I h(x,y)\, dy = \int_I \frac{\partial}{\partial x}h(x,y)\, dy \quad \forall x \in I.$$

This lemma is proven at the Appendix. Using the function $h(x,y) = f(y|x) \cdot 1_{(-\infty,a]}(y)$ for any fixed $a \in I$, we receive the subsequent corollary.

Corollary 1 *If (i) $f(y|.)$ is differentable in I for every $y \in I$ and (ii) for every $x \in I$ there exists a $\epsilon(x) > 0$ such that the function $\tilde{g}_x(y) := \sup_{\{|z-x|<\epsilon(x)\}^+} \left|\frac{\partial f(y|z)}{\partial z}\right|$ is integrable, then $F(y|.)$ is differentiable for any $y \in I$ and the identity $\frac{\partial F(a|x)}{\partial x} = \int_0^a \frac{\partial f(y|x)}{\partial x}\, dy\ \forall a \in I$ holds. ($\{...\}^+$ stands for $\{...\} \cap I\!R^+$)*

Theorem 8 *If the function $f(y|.)$ is continously differentable for every $y \in I$ and if for every $x \in I$ there exists a $\epsilon(x) > 0$ such that the function*

$$g_x(y) := \tilde{K}^\alpha_y \sup_{\{|z-x|<\epsilon(x)\}^+} \left\{\left|\frac{\partial f(y|z)}{\partial z}\right|\right\}$$

is integrable on I and if r is differentable then the function V_α is differentable and the following identity holds.

$$\frac{d\, V_\alpha(x_0)}{d\, x_0} = \frac{d\, r(x_0)}{d\, x_0} + \int_{x_0}^N \min_{a\in[0,y]} \{d(y,a) + \alpha V_\alpha(y-a)\} \frac{\partial f(y|x)}{\partial x}(x_0)\, dy.$$

Proof: The monotonocity of $\min_{a\in[0,y]}\{d(y,a)+\alpha V_\alpha(y-a)\}$ in y can be proven similarly to the monotonocity of $\inf_{a\in[0,y]} \{d(y,a) + \alpha\psi(y-a,a,M)\}$ in x, as we have done in (5). Hence

$$\begin{aligned} V_\alpha(y) &= r(y) + \int \min_{a\in[0,z]} \{d(z,a) + \alpha V_\alpha(z-a)\}\, dF(z|y) \\ &\geq r(y) + \min_{a\in[0,y]} \{d(y,a) + \alpha V_\alpha(y-a)\}. \end{aligned}$$

Since the cost-functions are non-negative we have

$$\min_{a\in[0,y]} \{d(y,a) + \alpha V_\alpha(y-a)\} \leq V_\alpha(y) - r(y) \leq V_\alpha(y) \leq \tilde{K}^\alpha_y. \tag{6}$$

Lemma (10) can be taken using g_x and $h(z,y) := \min_{a\in[0,y]}\{d(y,a)+\alpha V_\alpha(y-a)\} f(y|z)$, since for $z \in [x-\epsilon(x),\, x+\epsilon(x)]$ we have:

$$\left|\frac{\partial}{\partial z} h(z,y)\right| = \min_{a\in[0,y]} \{d(y,a) + \alpha V_\alpha(y-a)\} \left|\frac{\partial f(y|z)}{\partial z}\right| \leq \tilde{K}^\alpha_y \sup_{\{|z-x|<\epsilon(x)\}} \left\{\left|\frac{\partial f(y|z)}{\partial z}\right|\right\} = g_x(y).$$

Hence

$$\begin{aligned} \frac{d\, V_\alpha}{d\, x}(x) &= \frac{d\, r(x)}{d\, x} + \frac{d}{dx} \int_0^N \min_{a\in[0,y]} \{d(y,a) + \alpha V_\alpha(y-a)\}\, dF(y|x) \\ &= \frac{d\, r}{d\, x}(x) + \int_0^N \min_{a\in[0,y]} \{d(y,a) + \alpha V_\alpha(y-a)\} \frac{\partial f(y|x)}{\partial x}\, dy. \blacksquare \end{aligned}$$

4 Optimization

We denote by Π^b the class of the bang-bang strategies. A bang-bang strategy is a strategy δ with $\delta(x) \in \{0, x\}$ for every $x \in I$. If $\delta(x) = x \cdot 1_{[z,\infty)}(x)$ holds for a $z \in I$, δ is denoted as bang-bang strategy with threshold z. Furthermore, we define V^b_α as $\inf_{\{\delta\in\Pi^b\}} V_{\delta,\alpha}$.

The subsequent lemma gives some properties for the function V^b_α. We do not present the proof since it is similar to the corresponding properties of the function V_α.

Lemma 11

$$V_\alpha^b(x) = r(x) + \int_{y=x}^{N} \min\left\{d(y,0) + \alpha V_\alpha^b(y),\, d(y,y) + \alpha V_\alpha^b(0)\right\}\, dF(y|x).$$

If condition B holds and for $\delta_\alpha \in \Pi^b$ the subsequent identity is valid

$$d(y,\delta_\alpha(y)) + \alpha V_\alpha^b(y - \delta_\alpha(y)) = \min\left\{d(y,0) + \alpha V_\alpha^b(y),\, d(y,y) + \alpha V_\alpha^b(0)\right\} \quad \forall y \in I$$

then $V_{\delta_\alpha,\alpha}$ equals V_α^b. Furthermore, V_α^b is non decreasing.

Now we identity an optimal strategy in the class Π^b.

Theorem 9 (1) *If condition B is valid strategy δ_α^* minimizes the discounted cost within the class Π^b, where*

$$\delta_\alpha^*(j) = \begin{cases} 0 & \alpha(V_\alpha^b(x) - V_\alpha^b(0)) \le d(x,x) - d(x,0), \\ x & \alpha(V_\alpha^b(x) - V_\alpha^b(0)) > d(x,x) - d(x,0). \end{cases}$$

If $d(x,x) - d(x,0)$ is non-increasing in x, the following bang-bang strategies with threshold z are optimal within the subclass of bang-bang strategies:

$$\delta_{z,\alpha}^*(y) = \begin{cases} 0 & y < z, \\ y & y \ge z, \end{cases} \quad z \in [y_\alpha^*, x_\alpha^*]$$

with

$$x_\alpha^* = \inf A_\alpha,\ A_\alpha := \{x \in I :\ \alpha(V_\alpha^b(x) - V_\alpha^b(0)) > d(x,x) - d(x,0)\}$$

and

$$y_\alpha^* = \inf \tilde{A}_\alpha,\ \tilde{A}_\alpha := \{y \in I :\ \alpha(V_\alpha^b(y) - V_\alpha^b(0)) \ge d(y,y) - d(y,0)\},$$

$$(x_\alpha^* = \infty \text{ if } A_\alpha = \emptyset \text{ and } y_\alpha^* = \infty \text{ if } \tilde{A}_\alpha = \emptyset).$$

The **proof** is similar to the proof of Theorem 3 in Stadje, Zuckerman [13]. The subsequent lemma will be proven in the appendix. We use the following notation:
$X := \Big\{g : I \to I\!R^+ | \forall x \exists \epsilon(x) > 0,\ \sup_{\{|z-x|<\epsilon(x)\}^+} \left\{\left|\frac{\partial f(.|z)}{\partial z}\right|\right\} g(.)$ and $f(.|x)g(.)$ integrable on $I\ \forall x \in I\Big\} \cup \{1_I\}$.

Lemma 12
Let $f(y|.)$ be continuously differentiable for every fixed $y \in I$. Then the function $\frac{\partial F(y|x)}{\partial x}$ is non-decreasing in $x \in I$ for every fixed $y \in I$ if and only if $\int_I \frac{\partial f(y|x)}{\partial x} g(y)\, dy$ is non-increasing in x is for every function g in X.

Proof: Since $1_L \in X$ the function $\sup_{\{|z-x|<\epsilon(x)\}^+} \left\{\left|\frac{\partial f(y|z)}{\partial z}\right|\right\}$ is integrable. For the function g let $(g_n)_{n\in I\!N}$ be the sequence defined in lemma 3. Since g is non-negative this also holds for the constant β. The function $\sup_{\{|z-x|<\epsilon(x)\}^+} \left\{\left|\frac{\partial f(y|z)}{\partial z}\right|\right\} |g(y)|$ is an majorant for $\frac{\partial f(y|x)}{\partial x} g$ and $\frac{\partial f(y|x)}{\partial x} g_n(y)$. So we can use lemma 10.

$$\begin{aligned}
\int_I \frac{\partial f(y|x)}{\partial x} g(y)\, dy &= \int_I \lim_{n\to\infty} \frac{\partial f(y|x)}{\partial x} g_n(y)\, dy = \lim_{n\to\infty} \int_I \frac{\partial f(y|x)}{\partial x} g_n(y)\, dy \\
&= \lim_{n\to\infty} \int_I \frac{\partial f(y|x)}{\partial x} \left(\frac{1}{2^n} \sum_{k=1}^{\infty} 1_{[(a_{k,n},\infty)}(y) + \beta\right) dy \\
&= \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \int_{a_{k,n}}^{N} \frac{\partial f(y|x)}{\partial x}\, dy + \beta \int_I \frac{\partial f(y|x)}{\partial x}\, dy \qquad (7) \\
&= \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{\partial}{\partial x} \int_{a_{k,n}}^{N} f(y|x)\, dy + \beta \frac{d}{dx} \int_I f(y|x)\, dy \qquad (8)
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{\partial}{\partial x}(1 - F(a_{k,n}|x)) + \beta \frac{d}{dx} 1$$

$$= \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \left(-\frac{\partial}{\partial x} F(a_{k,n}|x) \right). \quad (9)$$

Equation (8) holds since $1_L \in X$. Now we prove equation (7). For each natural number n the function $\psi_m(x) := \frac{\partial f(y|x)}{\partial x} \frac{1}{2^n} \left(\sum_{k=1}^{m} 1_{[(a_{k,n},N]}(y) + \beta \right)$ fulfilles the subsequent inequality on I.

$$|\psi_m(x)| = \left| \frac{\partial f(y|x)}{\partial x} \right| \frac{1}{2^n} \left(\sum_{k=1}^{m} 1_{[(a_{k,n},N]}(y) + \beta \right) \le \left| \frac{\partial f(y|x)}{\partial x} \right| \frac{1}{2^n} \left(\sum_{k=1}^{\infty} 1_{[(a_{k,n},N]}(y) + \beta \right).$$

Hence $\quad |\psi_m(x)| \quad \le \quad \left| \frac{\partial f(y|x)}{\partial x} \right| g_n(y).$

Since $\left| \frac{\partial f(y|x)}{\partial x} \right| g_n(y)$ is integrable and dominates all functions ψ_m, Lebesgue's convergence Theorem yields

$$\int_0^N \frac{\partial f(y|x)}{\partial x} \left(\sum_{k=1}^{\infty} 1_{[(a_{k,n},\infty)}(y) + \beta \right) = \lim_{m\to\infty} \int_0^N \left(\frac{\partial f(y|x)}{\partial x} \sum_{k=1}^{m} 1_{[(a_{k,n},\infty)}(y) + \beta \right) dy$$

$$= \sum_{k=1}^{\infty} \int_0^N \frac{\partial f(y|x)}{\partial x} \left(1_{[(a_{k,n},\infty)}(y) + \beta \right) dy.$$

Thus we have also proven (7). Now consider (9). Since $\frac{\partial F}{\partial x}(y|x)$ is non-decreasing in x, the entire integral is non-increasing in x which shows '$\Rightarrow$'. To prove '$\Leftarrow$' we use the function $h := 1_{[a,N]}$ which is an element of X since 1_I is. We used the same function h in the proof of lemma (9).

$$F(a|x) = 1 - \int_0^N f(y|x) h(y)\, dy.$$

Hence

$$\frac{\partial F(a|x)}{\partial x} = - \int_0^N \frac{\partial f(y|x)}{\partial x} h(y)\, dy$$

is non-decreasing. ■

Now we define some further conditions on our model:

(3) One of the two subsequent conditions holds:
(a) $d(x,x) - d(x,0)$ *is non-increasing in* x *on* I*;*
(b) there exist $\rho > 0$ *and* $x_0 > 0$ *such that*
$d(x,x) - d(x,0) = \alpha\rho x$ *and* $r(x) > r(0) + \rho x \ \forall x \ge x_0$.
(4) The function r *is differentiable and* $\frac{d\,r}{d\,x}(x)$ *is non-increasing.*
(5) For every fixed $y \in I$, $f(y|.)$ *is continuously differentiable and for all* $x \in I$*, there exists an* $\epsilon(x) > 0$ *such that* $g_x(y) := \max\left\{\tilde{K}_y^{\alpha}, 1\right\} \sup_{\{|z-x|<\epsilon(x)\}^+} \left| \frac{\partial f(y|z)}{\partial z} \right|$ *is integrable on* I.
(6) The cost function d *is concave in the second component.*
(7) The function $\frac{\partial F(y|.)}{\partial .}$ *is non-decreasing for every fixed* $y \in I$.

The function $\frac{\partial F(y|x)}{\partial x}$ used in condition (7) is well defined since Corollary 1 and condition

(5) hold.

Now we are able to prove the following important theorem.

Theorem 10 *The function V_α is concave on I for every fixed $\alpha \in (0,1)$.*

Proof: The function $\tilde{g}_x$ defined in condition (5) is a majorant of the function g_x defined at Theorem 8. Hence g_x is measurable. For every $y \in I$ the function $f(y|.)$ is continuously differentiable by condition (5) and the function r is differentiable by condition (4). Theorem 8 yields the identity

$$\frac{d\,V_\alpha(x)}{d\,x}(x) = \frac{d\,r}{d\,x}(x) + \int_0^N \frac{\partial f(y|x)}{\partial x} \min_{a\in[0,y]} \{d(y,a) + \alpha V_\alpha(y-a)\}\ dy. \tag{10}$$

We have

$$I_x(y) := \frac{\partial f(y|x)}{\partial x} \min_{a\in[0,y]} \{d(y,a) + \alpha V_\alpha(y-a)\} \leq \sup_{\{|z-x|<\epsilon(x)\}^+} \left|\frac{\partial f(y|z)}{\partial z}\right| \cdot \tilde{K}_y^\alpha \leq \tilde{g}_x(y).$$

The last inequality holds by condition (5). So $I_x \in X$, and lemma 12 can be used. Both summands of (10) are non-increasing by conditions (4) and (5), so $\frac{d\,V_\alpha(x)}{d\,x}$ is non-increasing. Hence, V_α is concave.

Theorem 11 *If the conditions A, B, (1), (2), (4), ..., (7) are valid, the following bang-bang strategy minimizes the $\alpha-$discounted cost.*

$$\delta_\alpha^*(x) = \begin{cases} 0 & \alpha(V_\alpha(x) - V_\alpha(0)) \leq d(x,x) - d(x,0), \\ x & \alpha(V_\alpha(x) - V_\alpha(0)) > d(x,x) - d(x,0). \end{cases}$$

If furthermore condition (3) holds, the following strategies minimize the $\alpha-$discounted cost.

$$\delta_{z,\alpha}^*(y) = \begin{cases} 0 & y < z, \\ y & y \geq z, \end{cases} \quad z \in [y_\alpha^*, x_\alpha^*].$$

It remains to **prove** this theorem if the second part of condition (4) holds.

$$\begin{aligned} V_\alpha(x) - V_\alpha(0) &= r(x) - r(0) + \int_0^N \min_{z\in[0,y]} \{d(y,z) + V_\alpha(y-z))\}\ dF(y|x) \\ &\quad - \int_0^N \min_{z\in[0,y]} \{d(y,z) + V_\alpha(y-z))\}\ dF(y|0) && (11) \\ &\geq r(x) - r(0) && (12) \end{aligned}$$

so

$$\alpha\,(V_\alpha(x) - V_\alpha(0)) > \alpha\rho x = d(x,x) - d(x,0) \quad \forall x \geq x_0.$$

(12) follows from the monotonocity of the function $\min_{z\in[0,y]} \{d(y,z) + V_\alpha(y-z))\}$ (to be seen at the beginning of the proof of Theorem 8) and lemma 9. Thus, the concave function $\alpha\,(V_\alpha(x) - V_\alpha(0))$ crosses the straight line $d(x,x) - d(x,0)$ at most once. ■

5 Appendix

Now we present the **proof** of lemma 3:

Let $\beta = g(0)$ so that $g_n(x) = g(0) + \frac{1}{2^n}\sum_{k=1}^{\infty} 1_{\{g(x)-g(0)\geq \frac{k}{2^n}\}}(x)$ $\forall x \in I$. By the monotonocity of g the set

$$\left\{x \in I | g(x) - g(0) \geq \frac{k}{2^n}\right\} = \left\{x \in I | g(x) \in [g(0) + \frac{k}{2^n}, N]\right\}$$

is not bounded from above which justifies its definition. g_n is a step-function where the height of the steps is always $\frac{1}{2^n}$. So it is a non-decreasing function. From $g_n(x) = \frac{k}{2^n}$ we get $g(x) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$. Furthermore, for any $n \in I\!N$ and arbitrary $x \in I$, there exists a $k_0 \in I\!N$ with

$$g(x) \in \left[g(0) + \frac{k_0}{2^n}, g(0) + \frac{k_0+1}{2^n}\right).$$

This yields the identity

$$g_n(x) \quad = \quad g(0) + \frac{1}{2^n}\sum_{k=1}^{k_0} 1 + \sum_{k=k_0+1}^{\infty} 0 = g(0) + \frac{k_0}{2^n}.$$

So $g(x) - g_n(x) \in [0, \frac{1}{2^n})$, hence, $||g - g_n||_\infty < \frac{1}{2^n}$. ∎

Proof of lemma (10): Let $x_0 \in I$ and $(x_n)_{n\in I\!N} \subset (x_0 - \epsilon(x_0), x_0) \cup (x_0, x_0 + \epsilon(x_0))$ be a sequence with limit point x_0. Furthermore let $H(x) = \int h(x,y)\,dy$. Since h is partial differentiable in the first component we can use the mean value theorem. For every $n \in I\!N$ there is a $\chi_n \in [x_0, x_n] \cup [x_n, x_0]$ with $\frac{\partial h}{\partial x}(\chi_n, y) = \frac{h(x_n)-h(x_0)}{x_n - x_0}$. The continuity of $\frac{\partial h}{\partial x}(x,y)$ and the fact that $x_n \to x_0$, so $\chi_n \to x_0$ yields to the identity

$$\lim_{n\to\infty} \frac{\partial h}{\partial x}(\chi_n, y) = \frac{\partial h}{\partial x}(x_0, y).$$

As well the partial derivatives $\frac{\partial h}{\partial x}(\chi_n, y)$ as $\frac{\partial h}{\partial x}(x_0, y)$ are dominated by the integrable function g_x. Thus the Lebesgue convergence Theorem yields to

$$\begin{aligned}\int_0^N \frac{\partial h}{\partial x}(x_0, y)\,dy &= \lim_{n\to\infty}\int_0^N \frac{\partial h}{\partial x}(\chi_n, y)\,dy = \lim_{n\to\infty}\int_0^N \frac{h(x_n,y) - h(x_0,y)}{x_n - x_0}\,dy \\ &= \lim_{n\to\infty}\frac{H(x_n) - H(x_0)}{x_n - x_0} = \left(\frac{d}{dx}\int_0^N h(x,y)dy\right)(x_0). \blacksquare\end{aligned}$$

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