Department of Applied Mathematics
Faculty of EEMCS

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands

Phone: +31-53-4893400
Fax: +31-53-4893114

Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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F. Izsák[1] and J.J.W. van der Vegt

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[1]Department of Applied Analysis, Loránd Eötvös University, P.O. Box 120, 1518 Budapest, Hungary

YOUNG INEQUALITY FOR SURFACE CONVOLUTIONS IN $\mathbf{R}^3$

FERENC IZSÁK[†‡] AND JAAP J.W. VAN DER VEGT[†]

Abstract. We prove a Young inequality for convolutions defined on a Lipschitz continuous surface in $\mathbf{R}^3$.

Key words. Young inequality, surface convolutions

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1. Introduction. In this paper we provide an estimate for the $L_p(\Omega)$ norm of a surface convolution defined at the boundary of a Lipschitz domain $\Omega \subset \mathbf{R}^3$. For $x \in \Omega$, the surface convolution is stated as:

$$x \to \int_{\partial\Omega} f(x-y)g(y)\,\mathrm{d}S_y,$$

where $f \in L_q(\mathbf{R}^3), g \in L_r(\mathbf{R}^3)$ with $q, r \geq 1$. For the proof of the main result (stated in Theorem 4) we use the classical results in [2] and [3], which are summarized in Section 2.

2. Preliminaries.

2.1. Young inequality in $\mathbf{R}^n$ with a sharp constant. Although the Young inequality has been known for a long time, the present form was only proved in the 70's, independently in [2] and [3]:

THEOREM 1. *For real numbers $p, q, r \geq 1$ with $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and functions $f \in L_q(\mathbf{R}^n)$, $g \in L_r(\mathbf{R}^n)$ we have the following inequality:*

$$\left(\int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} f(x-y)g(y)\,\mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} = \|f * g\|_p \leq \left(\frac{C_q C_r}{C_p} \right)^n \|f\|_q \|g\|_r, \tag{2.1}$$

where $C_p = \sqrt{\frac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}}$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $C_1 = C_\infty = 1$.

Remark: Taking the limit $p \to 1$ gives the value for C_1. This case is of a great importance since in many applications f is related to a fundamental solution of some differential operator (with a singularity) such that $f \notin L_q(\mathbf{R}^n)$ for $q > 1$.
For the proof of this theorem and further literature we refer [5] (pages 99-105).

2.2. The fully generalized Young inequality. In the meantime the result of Theorem 1 has been sharpened in two ways: the precise condition for equality in (2.1) has been provided and the inequality was generalized to a product of functions $f_1, f_2, \dots, f_k$ (not necessarily for a convolution) which are defined on $\mathbf{R}^{n_i}$ with different values of n_i instead of just taking f and g. We cite this general form of the inequality [4], [5] (page 100):

[†]Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands, [f.izsak, j.j.w.vandervegt]@math.utwente.nl

[‡]Department of Applied Analysis, Loránd Eötvös University, P.O. Box 120, 1518 Budapest, Hungary, izsakf@cs.elte.hu

Theorem 2. *Fix $k > 1$, integers $n_1, n_2, \dots, n_k$ and numbers $p_1, p_2, \dots, p_k \geq 1$. Let $M \geq 1$ and let B_i (for $i = 1, 2, \dots k$) be a linear mapping from $\mathbf{R}^M$ to $\mathbf{R}^{n_i}$. Let $Z : \mathbf{R}^M \to \mathbf{R}^+$ be some fixed Gaussian function,*

$$Z(x) = e^{-(x,Ax)}$$

with A a real, positive semidefinite $M \times M$ matrix (possibly zero) and $(\cdot,\cdot)$ the Euclidian inner product in $\mathbf{R}^M$. For functions $f_i \in L_{p_i}(\mathbf{R}^{n_i})$ consider the integral

$$I_Z(f_1, f_2, \dots, f_k) = \int_{\mathbf{R}^M} Z(x) \prod_{i=1}^{k} f_i(B_i x)\,\mathrm{d}x \tag{2.2}$$

and define

$$C_Z := \sup\{I_Z(f_1, f_2, \dots, f_k) : \ \|f_i\|_{p_i} = 1 \quad \text{for} \quad i = 1, 2, \dots, k\}. \tag{2.3}$$

Then C_Z is determined by restricting the functions f_i to be Gaussian functions, i.e.,

$$\begin{aligned} C_Z := \sup\{I_Z(f_1, f_2, \dots, f_k) : \ \|f_i\|_{p_i} = 1 \quad \text{and} \quad f_i(x) = e^{-(x,A_i x)} \\ \text{with } A_i \text{ a real, symmetric, positive definite } n_i \times n_i \text{ matrix}\}. \end{aligned} \tag{2.4}$$

To get an alternative form of the classical Young inequality [5] take $A = 0$, $k = 3$, $B_1 = (1, 0)$, $B_2 = (1, -1)$ and $B_3 = (0, 1)$.

3. Young inequality for convolutions on manifolds.

3.1. Special case: Young inequality for integrals on $\mathbf{R}^n \times \mathbf{R}^{n-1}$. We first prove the following upper estimate for an integral related to the left hand side of (2.1):

$$\left(\int_{\mathbf{R}^n} \left|\int_{\mathbf{R}^{n-1}} f(x - y(u,0)) g(u)\,\mathrm{d}u\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq C\|f\|_{L_q(\mathbf{R}^n)}\|g\|_{L_r(\mathbf{R}^{n-1})}, \tag{3.1}$$

where $f \in L_q(\mathbf{R}^n)$, $g \in L_r(\mathbf{R}^{n-1})$, $x \in \mathbf{R}^n$, $u = (u_1, u_2, \dots, u_{n-1}) \in \mathbf{R}^{n-1}$ and $y(u, 0) = y(u_1, u_2, \dots, u_{n-1}, 0) \in \mathbf{R}^n$, while $C \in \mathbf{R}$ is a positive constant.

In order to apply the standard estimate (2.1) we introduce a nonnegative, real-valued function J belonging to $C_0^\infty(\mathbf{R}^n)$ having the properties:

$$J(x) = 0, \qquad |x| \geq 1,$$
$$\int_{\mathbf{R}^n} J = 1.$$

For $\delta > 0$ we also define the mollifier $J_\delta = \delta^{-n} J(x/\delta)$ and $g_\delta : \mathbf{R}^n \to \mathbf{R}$ with $g_\delta(u, u_n) = g(u) J_\delta(u_n)$ for which $\int_{\mathbf{R}} g_\delta(u, u_n)\,\mathrm{d}u_n = g(u)$. Using these relations we rewrite the integral on the left hand side of (3.1) and obtain the following inequality; where in all formulae we use $u = (u_1, u_2, \dots, u_{n-1}) \in \mathbf{R}^{n-1}$ and $(u, u_n) =$

$(u_1, u_2, \ldots, u_{n-1}, u_n) \in \mathbf{R}^n$:

$$\Big(\int_{\mathbf{R}^n}\Big|\int_{\mathbf{R}^{n-1}} f(x-y(u,0))g(u)\,\mathrm{d}u\Big|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} \tag{3.2}$$
$$=\Big(\int_{\mathbf{R}^n}\Big|\int_{\mathbf{R}^n} f(x-y(u,0))g_\delta(u,u_n)\,\mathrm{d}u\,\mathrm{d}u_n\Big|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}$$
$$=\Big(\int_{\mathbf{R}^n}\Big|\int_{\mathbf{R}^n} f(x-y(u,u_n))g_\delta(u,u_n)\,\mathrm{d}u\,\mathrm{d}u_n +$$
$$\int_{\mathbf{R}^n}\big(f(x-y(u,0))-f(x-y(u,u_n))\big)g_\delta(u,u_n)\,\mathrm{d}u\,\mathrm{d}u_n\Big|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}$$
$$\le\Big(\int_{\mathbf{R}^n}\Big|\int_{\mathbf{R}^n} f(x-y(u,u_n))g_\delta(u,u_n)\,\mathrm{d}u\,\mathrm{d}u_n\Big|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} +$$
$$\Big(\int_{\mathbf{R}^n}\Big|\int_{\mathbf{R}^n}\big(f(x-y(u,0))-f(x-y(u,u_n))\big)g_\delta(u,u_n)\,\mathrm{d}u\,\mathrm{d}u_n\Big|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} \tag{3.3}$$

using Minkowski's inequality. Following [1] (page 242) we define the p-mean modulus of continuity ω_p^* as:

$$\omega_p^*(v,t) = \sup_{|h|\le t}\Big(\int_{\mathbf{R}^n}|v(x+h)-v(x)|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}, \qquad \text{for } t>0 \text{ and } 1\le p<\infty,$$

then for $1 \le p < \infty$ we have the limit $\omega_p^*(v,t) \to 0$ as $t \downarrow 0$ for all $v \in L_p(\mathbf{R}^n)$.
Our aim is to take the limit $\delta \downarrow 0$ in (3.3) and apply (2.1). For this we prove the following:

LEMMA 1. *For real numbers $p, q, r \ge 1$, with $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $p < \infty$, and functions $f \in L_q(\mathbf{R}^n)$ and $g_\delta \in L_r(\mathbf{R}^n)$ we have*

$$\lim_{\delta\downarrow 0}\Big(\int_{\mathbf{R}^n}\Big|\int_{\mathbf{R}^n}\big(f(x-y(u,0))-f(x-y(u,u_n))\big)g_\delta(u,u_n)\,\mathrm{d}u\,\mathrm{d}u_n\Big|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} = 0$$

Proof: Applying Minkowski's inequality for integrals (Theorem 2.9 in [1]) and the convergence result for the p-mean modulus of continuity we obtain:

$$\Big(\int_{\mathbf{R}^n}\Big|\int_{\mathbf{R}^n}\big(f(x-y(u,0))-(x-y(u,u_n))\big)g_\delta(u,u_n)\,\mathrm{d}u\,\mathrm{d}u_n\Big|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}$$
$$\le \int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n}\Big|\big(f(x-y(u,0))-f(x-y(u,u_n))\big)g_\delta(u,u_n)\Big|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\,\mathrm{d}u\,\mathrm{d}u_n$$
$$= \int_{\mathbf{R}^{n-1}}|g(u)|\Big(\int_{-\delta}^{\delta}J_\delta(u_n)\Big(\int_{\mathbf{R}^n}|f(x-y(u,0))-f(x-y(u,u_n))|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\,\mathrm{d}u_n\Big)\,\mathrm{d}u$$
$$\le \int_{\mathbf{R}^{n-1}}|g(u)|\Big(\int_{-\delta}^{\delta}J_\delta(u_n)\mathrm{d}u_n\,\omega_p^*(f,\delta)\Big)\,\mathrm{d}u$$
$$= \omega_p^*(f,\delta)\|g\|_{L_1(\mathbf{R}^{n-1})} \to 0 \qquad \text{as } \delta\downarrow 0. \quad \square$$

We can now prove (3.1) and give an upper bound for the constant C.

Theorem 3. *For real numbers $p, q, r \geq 1$, with $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $p < \infty$, and functions $f \in L_q(\mathbf{R}^n)$ and $g \in L_r(\mathbf{R}^{n-1})$ the following inequality holds:*

$$\left(\int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^{n-1}} f(x - y(u,0)) g(u) \, \mathrm{d}u \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \leq \left(\frac{C_q C_r}{C_p} \right)^n \|f\|_{L_q(\mathbf{R}^n)} \|g\|_{L_r(\mathbf{R}^{n-1})}, \tag{3.4}$$

where $C_p = \sqrt{\frac{p^{\frac{1}{p}}}{p'^{\frac{1}{p'}}}}$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $C_1 = C_\infty = 1$.

Proof: Taking the limit $\delta \downarrow 0$ and applying the result of Lemma 1, then using Theorem 1 and the relation $\|g_\delta\|_{L_r(\mathbf{R}^n)} = \|g\|_{L_r(\mathbf{R}^{n-1})}$, we can estimate (3.2) as follows:

$$\begin{aligned} &\left(\int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^{n-1}} f(x - y(u,0)) g(u) \, \mathrm{d}u \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &= \lim_{\delta \downarrow 0} \left(\int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} f(x - y(u,u_n)) g_\delta(u,u_n) \, \mathrm{d}u \, \mathrm{d}u_n \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leq \lim_{\delta \downarrow 0} \left(\frac{C_q C_r}{C_p} \right)^n \|f\|_{L_q(\mathbf{R}^n)} \|g_\delta\|_{L_r(\mathbf{R}^n)} \\ &= \left(\frac{C_q C_r}{C_p} \right)^n \|f\|_{L_q(\mathbf{R}^n)} \|g\|_{L_r(\mathbf{R}^{n-1})}. \quad \square \end{aligned} \tag{3.5}$$

Remark: Since we proved inequality (3.1) for functions g_δ which are never Gaussian functions, Theorem 2 shows that one can expect a sharper constant than the one given by Theorem 3.

3.2. General case: Young inequality for surface convolutions in $\mathbf{R}^3$. In the following we investigate the case when g is defined on the surface $\Gamma_i \subset \partial\Omega$ of a bounded domain Ω determined by $l_i : \mathbf{R}^2 \to \mathbf{R}$, a Lipschitz continuous function with Lipschitz constant L_i, such that

$$\Gamma_i = \{(u, l_i(u)) : u \in G_i\} \tag{3.6}$$

for some open set $G_i \subset \mathbf{R}^2$. We make use of the following definitions:

Definition 3.1. *For $f \in L_q(\Omega \dotplus (-\Omega))$, with $q \geq 1$ and a domain $\Omega \subset \mathbf{R}^3$, we introduce $f_\Omega \in L_q(\mathbf{R}^3)$ with:*

$$f_\Omega(x) = \begin{cases} f(x), & x \in \Omega \dotplus (-\Omega) \\ 0, & x \notin \Omega \dotplus (-\Omega), \end{cases} \tag{3.7}$$

where $\Omega \dotplus (-\Omega) = \{x - y : \; x, y \in \Omega\}$.

Definition 3.2. *For $g \in L_r(S)$, with $r \geq 1$ and $S \subset \mathbf{R}^3$, we introduce $g_S \in L_r(\mathbf{R}^3)$ with:*

$$g_S(x) = \begin{cases} g(x), & x \in S \\ 0, & x \notin S. \end{cases} \tag{3.8}$$

Definition 3.3. *For $l_i \in Lip(G_i)$, with $r \geq 1$ and $G_i \subset \mathbf{R}^2$, we introduce $m_i \in L_\infty(\mathbf{R}^2)$ with:*

$$m_i(u) = \begin{cases} \sqrt{1+|\mathrm{grad}\, l_i(u)|^2}, & u \in G_i \\ 0, & u \notin G_i. \end{cases} \tag{3.9}$$

Using (3.7) and (3.8) for the case when $S \subset \overline{\Omega}$ we obtain that

$$f(x-y)g(y) = f_\Omega(x-y)g(y) = f_\Omega(x-y)g_S(y) \tag{3.10}$$

holds for any $x \in \Omega$ and $y \in S$.
Using these results we formalize the estimate for surface integrals in $\mathbf{R}^3$.

Theorem 4. *Consider the boundary surface $\Gamma \subset \partial\Omega$ of a domain $\Omega \subset \mathbf{R}^3$, which is covered almost everywhere with a finite number of disjoint surfaces Γ_i (each of them is the graph of a Lipschitz function given in (3.6)), i.e. $\lambda(\Gamma \setminus \cup_{i=1}^N \Gamma_i) = 0$, where λ is the Lebesgue measure on the surface $\partial\Omega$. Let $f \in L_q(\Omega \dot{+} -\Omega)$ and $g \in L_r(\Gamma)$, then for real numbers $p, q, r \geq 1$, with $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $p < \infty$, we have the relation:*

$$\left\| \int_\Gamma f(x-y)g(y)\, \mathrm{d}S_y \right\|_{L_p(\Omega)} \leq \left(\frac{C_q C_r}{C_p} \right)^3 \|f\|_{L_q(\Omega \dot{+} (-\Omega))} \sum_{i=1}^N (1+L_i^2)^{(r-1)/2r} \|g\|_{L_r(\Gamma_i)}. \tag{3.11}$$

Proof: Using Definitions 3.1, 3.2 and 3.3 we can estimate the norm of the surface integral on Γ_i (given in (3.6)) as follows:

$$\begin{aligned} &\left\| \int_{\Gamma_i} f(x-y)g(y)\, \mathrm{d}S_y \right\|_{L_p(\Omega)} = \left(\int_\Omega \left| \int_{\Gamma_i} f(x-y)g(y)\, \mathrm{d}S_y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbf{R}^3} \left| \int_{\Gamma_i} f_\Omega(x-y)g(y)\, \mathrm{d}S_y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbf{R}^3} \left| \int_{G_i} f_\Omega(x-y(u,0))g(y(u,0))\sqrt{1+|\mathrm{grad}\, l_i(u)|^2}\, \mathrm{d}u \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbf{R}^3} \left| \int_{\mathbf{R}^2} f_\Omega(x-y(u,0))\tilde{g}_{\Gamma_i}(u) m_i(u)\, \mathrm{d}u \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leq \left(\frac{C_q C_r}{C_p} \right)^3 \|f_\Omega\|_{L_q(\mathbf{R}^3)} \|\tilde{g}_{\Gamma_i} m_i\|_{L_r(\mathbf{R}^2)} \\ &= \left(\frac{C_q C_r}{C_p} \right)^3 \|f\|_{L_q(\Omega \dot{+} (-\Omega))} \left(\int_{G_i} |g(u, l_i(u))|^r \left| \sqrt{1+|\mathrm{grad}\, l_i(u)|^2} \right|^r \mathrm{d}u \right)^{\frac{1}{r}} \\ &\leq (1+L_i^2)^{(r-1)/2r} \left(\frac{C_q C_r}{C_p} \right)^3 \|f\|_{L_q(\Omega \dot{+} (-\Omega))} \|g\|_{L_r(\Gamma_i)}, \end{aligned} \tag{3.12}$$

where in the third inequality we applied Theorem 3. In addition, we have used the relations $y(u,0) = T(u, l_i(u))$, with $T : \mathbf{R}^3 \to \mathbf{R}^3$ a linear mapping, $\tilde{g}_{\Gamma_i}(u) = g_{\Gamma_i}(y(u,0))$ and L_i denotes the Lipschitz constant of l_i. Next we consider the decomposition Γ into subdomains:

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_N \cup R, \tag{3.13}$$

where $\lambda(R) = 0$ with the surface measure λ according to (3.6) and to the assumptions in the theorem. Taking the sum of the inequalities in (3.12) for $i = 1, 2, \ldots, N$ we obtain that

$$\begin{aligned}\left\|\int_\Gamma f(x-y)g(y)\,\mathrm{d}S_y\right\|_{L_p(\Omega)} &\le \sum_{i=1}^N \left\|\int_{\Gamma_i} f(x-y)g(y)\,\mathrm{d}S_y\right\|_{L_p(\Omega)} \\ &\le \left(\frac{C_q C_r}{C_p}\right)^3 \|f\|_{L_q(\Omega\dot{+}(-\Omega))} \sum_{i=1}^N (1+L_i^2)^{(r-1)/2r} \|g\|_{L_r(\Gamma_i)}.\end{aligned} \tag{3.14}$$

□

3.3. Examples.

1. Ω is a bounded polyhedron with closed faces $\overline{F_1}, \overline{F_2}, \ldots, \overline{F_N} \subset \partial\Omega$ and we define $\Gamma_i = \mathrm{int}(F_i)$. Then clearly, we can define $l_i : \mathbf{R}^2 \to \mathbf{R}^3$ as a constant function such that the right hand side in (3.14) is $\left(\frac{C_q C_r}{C_p}\right)^3 \|f\|_{L_q(\Omega\dot{+}(-\Omega))}\|g\|_{L_r(\Gamma)}$.
2. Ω is a curvilinear polyhedron with closed faces $\overline{F_1}, \overline{F_2}, \ldots, \overline{F_N} \subset \partial\Omega$ such that $\mathrm{int}(F_i)$ is the map of a Lipschitz function with $\mathrm{int}\, F_i = \{(y, l_i(y)) : y \in G_i\}$, where $\lambda(\partial G_i) = 0$ with λ the Lebesgue measure in $\mathbf{R}^2$.
   Using the definition of the surface measure an easy calculation gives that the surface measure of the boundaries ∂F_i is zero, as well. According to the assumptions in Theorem 4 this gives the decomposition

$$\partial\Omega = F_1 \cup F_2 \cup \cdots \cup F_N \cup \bigcup_{i=1}^N \partial F_i.$$

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