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Space-time discontinuous Galerkin method for parabolic problems in time-dependent domains
Part II. Analysis

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Space-Time Discontinuous Galerkin Method for Parabolic Problems in Time-Dependent Domains Part II. Analysis

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Abstract

In this report we analyze further the space-time discontinuous Galerkin (DG) finite element method for the solution of the advection-diffusion-reaction equation in time-dependent domains. We prove that the method is consistent, stable, coercive, and gives a unique solution. We also analyze the error estimates and *hp*-convergence of the method. The analysis is completed by analyzing the corresponding dual problems.

Keywords: discontinuous Galerkin, parabolic problems, time-dependent domain.
Mathematics Subject Classification: 65M60, 76M10, 35K20

1 Introduction

In the previous report [16] we developed a space-time discontinuous Galerkin (DG) finite element method for linear advection-diffusion-reaction equations. In that report, we extended the space-time DG formulation proposed in [17, 18] to include second-order partial differential equations. The space-time DG method has as key feature that time is treated as an extra dimension which makes the method particularly useful for problems with time-dependent flow domains. For the numerical flux related to second-order partial differential equations, we follow the same approach as in [3].

In this paper, we further analyze some important properties of the new technique in the finite element framework, such as consistency, orthogonality, coercivity and stability. Using this coercivity property, we can prove the existence of a unique DG numerical solution. In order to achieve these properties, we extend the analysis given in [9, 10, 11, 12] to the space-time domain. We also analyze the error estimate of the DG solution and the *hp* convergence. To complete the analysis, we consider the corresponding dual problem and analyze the error estimate and *hp* convergence of the solution of the dual problem.

To have a complete description, this report is organized as follows. First, a model problem for time-dependent parabolic partial differential equations is introduced in Section 2, followed by a discussion of the geometry of the space-time domain and elements. Next, the definitions of the finite element spaces and the trace operators related to the problem are given. Then the transformation of the model problem to the space-time framework is introduced. Section 2 is completed with the variational formulation obtained in [16]. All definitions and results in this section follow the same lines as in [16]. Then, Section 3 starts with some norms related to our analysis. First, we prove the consistency of the DG formulation. After that, we prove the coercivity of the method, followed by the existence of a unique DG solution. We also prove the stability property of the DG method. In Section 4 we prove the error estimate and the *hp* convergence of the method. The analysis of the corresponding dual problems is presented in Section 5. Concluding remarks are drawn in Section 6.

2 Space-Time DG Formulation of Parabolic Problems

2.1 Model problem

Let Ω_t be an open, bounded, time-dependent domain in $\mathbb{R}^d$, where d is the number of space dimensions. The closure of Ω_t is $\bar{\Omega}_t$ and the boundary of Ω_t is denoted by $\partial\Omega_t$. Denoting $\bar{x} = (x_1, \ldots, x_d)$ as the space variables, we consider the time-dependent advection-diffusion-reaction equation in the domain Ω_t:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i(\bar{x})u) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(D_{ij}(\bar{x})\frac{\partial u}{\partial x_i}\right) + c(\bar{x})u = f(t,\bar{x}), \quad t \in [t_0, T], \tag{2.1}$$

where $f \in L^2(\Omega_t)$ and $c \in L^\infty(\Omega_t)$, $c \geq 0$ are real-valued functions, $b = \{b_i\}_{i=1}^d$ a vector function whose entries are Lipschitz continuous real-valued functions on $\bar{\Omega}_t$ and $D = \{D_{ij}\}_{i,j=1}^d$ a symmetric positive definite matrix on $\bar{\Omega}_t$ whose entries are bounded, piecewise continuous real-valued functions. We denote by $\bar{n} = \{n_i\}_{i=1}^d$ the normal vector to $\partial\Omega_t$. Using the same argument as in [12], we define

$$\partial_0\Omega_t = \{\bar{x} \in \partial\Omega_t : \bar{n}^T D\bar{n} > 0\},$$
$$\partial_-\Omega_t = \{\bar{x} \in \partial\Omega_t \setminus \partial_0\Omega_t : b\cdot\bar{n} < 0\}, \quad \partial_+\Omega_t = \{\bar{x} \in \partial\Omega_t \setminus \partial_0\Omega_t : b\cdot\bar{n} \geq 0\}.$$

We assign the sets $\partial_-\Omega_t$ and $\partial_+\Omega_t$ as the inflow and outflow boundary, respectively. Clearly, $\partial\Omega_t = \partial_0\Omega_t \cup \partial_-\Omega_t \cup \partial_+\Omega_t$. If $\partial_0\Omega_t$ is nonempty, we further divide it into disjoint subsets $\partial_D\Omega_t$ and $\partial_M\Omega_t$ whose union is $\partial_0\Omega_t$, with $\partial_D\Omega_t$ having a non-zero measure. The disjoint sets $\partial D\Omega_t$ and $\partial_M\Omega_t$ are related to the Dirichlet and mixed or Robin boundary conditions, respectively. We supplement (2.1) with the initial condition

$$u = u_0 \quad \text{at } t = t_0, \tag{2.2}$$

with u_0 a real-valued function on $\Omega(t_0)$ and the boundary conditions:

$$u = g_D \quad \text{on} \quad \partial_D\Omega_t, \qquad \alpha u + \sum_{i,j=1}^{d} n_j D_{ij}\frac{\partial u}{\partial x_i} = g_M \quad \text{on} \quad \partial_M\Omega_t, \tag{2.3}$$

where g_D, g_N are given functions on $\partial_D\Omega_t$ and on $\partial_M\Omega_t$, respectively, and $\alpha \geq 0$ a continuous function on $\partial_M\Omega_t$. We adopt the (physically reasonable) hypothesis [12] that $b\cdot\bar{n} \geq 0$ on $\partial_M\Omega_t$ whenever $\partial_M\Omega_t$ is nonempty.

2.2 Geometry of space-time domain and elements

In the space-time discontinuous Galerkin discretization we do not make a distinction between space and time variables and directly consider a domain in $\mathbb{R}^{d+1}$. Let $\mathcal{E} \subset \mathbb{R}^{d+1}$ be an open domain. A point $x \in \mathbb{R}^{d+1}$ has coordinates $(x_0, \bar{x}) = (x_0, x_1, \ldots, x_d)$, with $t = x_0$ representing time. The space domain Ω_t is redefined as the space-time domain $\Omega_t := \{\bar{x} \in \mathbb{R}^d \mid (t, \bar{x}) \in \mathcal{E}\}$ for $t \in [t_0, T]$, where t_0 and T represent the initial and final time of the evolution of the domain. The space-time domain boundary $\partial\mathcal{E}$ consists of the hypersurfaces $\Omega_{t_0} := \{x \in \partial\mathcal{E} \mid x_0 = t_0\}$, $\Omega_T := \{x \in \partial\mathcal{E} \mid x_0 = T\}$, and $\mathcal{Q} := \{x \in \partial\mathcal{E} \mid t_0 < x_0 < T\}$.

Next, we consider the time interval $\mathcal{I} = (t_0, T)$, partitioned by an ordered series of time levels $t_0 < t_1 < \ldots < t_{N_t} = T$. Denoting the nth time interval as $I_n = (t_n, t_{n+1})$, we have $\mathcal{I} = \cup_{n=0}^{N_t-1} I_n$. The length of each time interval is defined as $\Delta_n t = t_{n+1} - t_n$. The space-time domain $\mathcal{E}$ is then divided into N_t space-time slabs $\mathcal{E}^n = \mathcal{E} \cap I_n$. Each space-time slab $\mathcal{E}^n$ is bounded by Ω_{t_n}, $\Omega_{t_{n+1}}$, and $\mathcal{Q}^n = \partial\mathcal{E}^n \setminus (\Omega_{t_n} \cup \Omega_{t_{n+1}})$.

We describe now the construction of the space-time elements K_j^n in $\mathcal{E}^n$. Let Ω_{h,t_n} be an approximation to Ω_{t_n} at time level t_n, with $\Omega_{h,t_n} \to \Omega_{t_n}$ as $h \to 0$. Similarly, $\Omega_{h,t_{n+1}}$ is an approximation to $\Omega_{t_{n+1}}$ at time level t_{n+1}. The domain Ω_{h,t_n} is divided into N_n non-overlapping spatial elements $K_j^n = K_j(t_n)$. At time level t_{n+1} the spatial elements $K_j^{n+1} = K_j(t_{n+1})$ are obtained by mapping K_j^n to their new position at $t = t_{n+1}$. Each element $\mathcal{K}_j^n$ is now obtained by connecting elements K_j^n and K_j^{n+1} using linear interpolation in time. The element boundary $\partial\mathcal{K}_j^n$ is denoted as the union of open faces of $\mathcal{K}_j^n$, which contains three parts $K_j(t_n^+) = \lim_{\epsilon\downarrow 0} K_j(t_n + \epsilon)$, $K_j(t_{n+1}^-) = \lim_{\epsilon\downarrow 0} K_j(t_{n+1} - \epsilon)$, and $\mathcal{Q}_j^n = \partial\mathcal{K}_j^n \setminus (K_j(t_n^+) \cup K_j(t_{n+1}^-))$. The definitions are completed with the tessellation $\mathcal{T}_h^n$, which consists of all space-time elements in the space-time slab $\mathcal{E}_h^n$, an approximation to $\mathcal{E}^n$, and $\mathcal{T}_h = \cup_{n=0}^{N_t-1} \mathcal{T}_h^n$, the union of all space-time elements in the space-time domain $\mathcal{E}_h$, which is an approximation to $\mathcal{E}$.

All the faces S in the space-time discretization are grouped into the set $\mathcal{F}$, which is the union of two disjoint sets: the set $\mathcal{F}_{\text{int}}$, which consists of all faces in $\mathcal{E}_h$ shared by two elements, and the set $\mathcal{F}_{\text{bnd}}$, which consists of all faces at the boundary of $\mathcal{E}_h$. We also consider the faces in the space-time slab $\mathcal{E}_h^n$. We denote by $\mathcal{S}^n$ the set of open faces in $\mathcal{E}_h^n$. First, we define the set $\mathcal{S}_I^n \subset \mathcal{S}^n$. Each face $S \in \mathcal{S}_I^n$ is connected to two space-time elements within the same slab. At the space-time slab boundary $\mathcal{Q}^n$, we define two sets of boundary faces; the set $\mathcal{S}_D^n$ with a Dirichlet boundary condition and the set $\mathcal{S}_M^n$ with a mixed boundary condition. The sets $\mathcal{S}_I^n$ and $\mathcal{S}_D^n$ are grouped into the set $\mathcal{S}_{ID}^n$.

2.3 Function spaces and trace operators

In this section, we give the standard definitions of the Sobolev spaces for real-valued functions in the domains Ω_t and $\mathcal{E}$, taken from [13]. Although the definition of the Sobolev space in [13] is for a fixed space domain, by a change of variables, the definition also holds for a time-dependent domain Ω_t.

First, in the domain Ω_t we introduce the standard definition of the Sobolev space $H^s(\Omega_t)$ for real-valued functions, with $s \in \mathbb{R}$. We refer to [4] for more details. When $s = 0$, the space $H^0(\Omega_t)$ is denoted as $L^2(\Omega_t)$, equipped with standard inner-product and norm

$$(w, v)_{L^2(\Omega_t)} := \int_{\Omega_t} wv \, \mathrm{d}\mathcal{K}, \qquad \|v\|_{L^2(\Omega_t)} := (v, v)_{L^2(\Omega_t)}^{1/2}, \tag{2.4}$$

and for nonnegative integer m, the Sobolev norm and semi-norm are defined as

$$\|v\|_{H^m(\Omega_t)} := \Big(\sum_{|\gamma|\le m} \|D^\gamma v\|^2_{L^2(\Omega_t)}\Big)^{\frac{1}{2}}, \qquad |v|_{H^m(\Omega_t)} := \Big(\sum_{|\gamma|=m} \|D^\gamma v\|^2_{L^2(\Omega_t)}\Big)^{\frac{1}{2}}, \tag{2.5}$$

where $D^\gamma = (\partial/\partial x_1)^{\gamma_1}\dots(\partial/\partial x_d)^{\gamma_d}$ denotes the usual partial derivative with multi-index $\gamma = (\gamma_1,\dots,\gamma_d)$, γ_i non- negative integers, and the length of γ given by $|\gamma| := \sum_{i=1}^d \gamma_i$.

The standard definition of the Sobolev space $H^s(\mathcal{E})$, with $s \in \mathbb{R}$, is similar as the definition of the Sobolev space in Ω_t, except with the extension of one dimension. For $s = 1$, we also introduce the space $H^{1,0}(\mathcal{E}) = L^2((t_0,T); H^1(\Omega_t))$ which is the space consisting of the elements of the space $L^2(\mathcal{E})$ having partial derivatives $\partial/\partial x_i, i = 1,\dots,d$, square summable on $\mathcal{E}$.

Now we introduce the finite element space associated with the tessellation $\mathcal{T}_h$. For simplicity of notation, in the remaining part of this section we denote the space-time element with $\mathcal{K}$. We assume that each element $\mathcal{K}$ is an image of a fixed master element $\hat{\mathcal{K}}$, i.e. $\mathcal{K} = G_{\mathcal{K}}(\hat{\mathcal{K}})$ for all $\mathcal{K} \in \mathcal{T}_h$, where $\hat{\mathcal{K}}$ is the open unit hypercube in $\mathbb{R}^{d+1}$. Analogously, for $k \ge 1$, $\mathcal{Q}_k(\hat{\mathcal{K}})$ is defined as the set of all tensor-product polynomials on $\hat{\mathcal{K}}$ of degree k in each coordinate direction.

To each element $\mathcal{K}$ we assign a nonnegative integer $p_{\mathcal{K}}$ (local polynomial degree) and a nonnegative integer $s_{\mathcal{K}}$ (local Sobolev index), and collect $p_{\mathcal{K}}$ and $s_{\mathcal{K}}$ in the vectors: $\mathbf{p} = \{p_{\mathcal{K}} : \mathcal{K} \in \mathcal{T}_h\}$ and $\mathbf{s} = \{s_{\mathcal{K}} : \mathcal{K} \in \mathcal{T}_h\}$. We consider the finite element space

$$V_h := \{v \in L^2(\mathcal{E}_h) : v|_{\mathcal{K}} \circ G_{\mathcal{K}} \in \mathcal{Q}_{p_{\mathcal{K}}}(\hat{\mathcal{K}}), \forall \mathcal{K} \in \mathcal{T}_h\}. \tag{2.6}$$

Further, we assign to $\mathcal{T}_h$ the broken Sobolev space $H^{\mathbf{s}}(\mathcal{E}_h,\mathcal{T}_h) := \{u \in L^2(\mathcal{E}_h) : u|_{\mathcal{K}} \in H^{s_{\mathcal{K}}}(\mathcal{K}), \forall \mathcal{K} \in \mathcal{T}_h\}$, equipped with the broken Sobolev norm and corresponding semi-norm , respectively,

$$\|u\|_{\mathbf{s},\mathcal{T}_h} := \Big(\sum_{\mathcal{K}\in\mathcal{T}_h} \|u\|^2_{H^{s_{\mathcal{K}}}(\mathcal{K})}\Big)^{\frac{1}{2}}, \qquad |u|_{\mathbf{s},\mathcal{T}_h} := \Big(\sum_{\mathcal{K}\in\mathcal{T}_h} |u|^2_{H^{s_{\mathcal{K}}}(\mathcal{K})}\Big)^{\frac{1}{2}}. \tag{2.7}$$

For $u \in H^1(\mathcal{E}_h,\mathcal{T}_h)$, we define the broken gradient $\nabla_h u$ of u by $(\nabla_h u)|_{\mathcal{K}} := \nabla(u|_{\mathcal{K}}), \forall \mathcal{K} \in \mathcal{T}_h$. In the derivation and analysis of the numerical discretization we will also make use of the auxiliary space Σ_h:

$$\Sigma_h = \{\tau \in L^2(\mathcal{E}_h)^{d+1} : \tau|_{\mathcal{K}} \circ G_{\mathcal{K}} \in \mathcal{Q}_{p_{\mathcal{K}}}(\hat{\mathcal{K}})^{d+1}, \forall \mathcal{K} \in \mathcal{T}_h\}.$$

For consistency reasons, we require $\nabla_h V_h \subset \Sigma_h$. The trace of functions $v \in V_h$ at the boundary $\partial\mathcal{K}$ is defined as:

$$v^{\pm}_{\mathcal{K}} = \lim_{\epsilon\downarrow 0} v(x \pm \epsilon n_{\mathcal{K}}),$$

with $n_{\mathcal{K}}$ the unit outward space-time normal vector at $\partial\mathcal{K}$. The trace of functions $\tau \in \Sigma_h$ is defined similarly.

Next, we define the *average* $\{\!\{\cdot\}\!\}$ and *jump* $[\![\cdot]\!]$ operators as trace operators for the sets $\mathcal{F}_{\text{int}}$ and $\mathcal{F}_{\text{bnd}}$. Note that functions $v \in V_h$ and $\tau \in \Sigma_h$ are multivalued at internal faces $S \in \mathcal{F}_{\text{int}}$. Introducing the functions $v_i := v|_{\mathcal{K}_i}$, $\tau_i := \tau|_{\mathcal{K}_i}, n_i := n|_{\mathcal{K}_i}$, the average operator is defined as:

$$\{\!\{v\}\!\} = (v_i^- + v_j^-)/2, \quad \text{on } S \in \mathcal{F}_{\text{int}}, \qquad \{\!\{v\}\!\} = v^-, \quad \text{on } S \in \mathcal{F}_{\text{bnd}}, \tag{2.8a}$$

$$\{\!\{\tau\}\!\} = (\tau_i^- + \tau_j^-)/2, \quad \text{on } S \in \mathcal{F}_{\text{int}}, \qquad \{\!\{\tau\}\!\} = \tau^-, \quad \text{on } S \in \mathcal{F}_{\text{bnd}}, \tag{2.8b}$$

while the jump operator is defined as:

$$[\![v]\!] = v_i^- n_i + v_j^- n_j, \quad \text{on } S \in \mathcal{F}_{\text{int}}, \qquad [\![v]\!] = v^- n, \quad \text{on } S \in \mathcal{F}_{\text{bnd}}, \tag{2.8c}$$

$$[\![\tau]\!] = \tau_i^- \cdot n_i + \tau_j^- \cdot n_j, \quad \text{on } S \in \mathcal{F}_{\text{int}}, \qquad [\![\tau]\!] = \tau^- \cdot n, \quad \text{on } S \in \mathcal{F}_{\text{bnd}}, \tag{2.8d}$$

with i and j the indices of the two elements $\mathcal{K}_i$ and $\mathcal{K}_j$ which connect to the face S. The unit normal vectors $n|_{\mathcal{K}_i}$ and $n|_{\mathcal{K}_j}$ are defined pointing exterior to $\mathcal{K}_i$ and $\mathcal{K}_j$, respectively. Note that the jump $[\![v]\!]$ is a vector parallel to the normal and the jump $[\![\tau]\!]$ is a scalar quantity. We will also need the spatial jump operator $\langle\!\langle \cdot \rangle\!\rangle$ for functions $v \in V_h$, which is defined as:

$$\langle\!\langle v \rangle\!\rangle = v_i^- \bar{n}_i + v_j^- \bar{n}_j, \quad \text{on } S \in \mathcal{F}_{\text{int}}, \qquad \langle\!\langle v \rangle\!\rangle = v^- \bar{n}, \quad \text{on } S \in \mathcal{F}_{\text{bnd}}. \tag{2.9}$$

2.4 Lifting operators

The derivation of the primal space-time DG formulation requires several trace lifting operators. First, for each face $S \in \mathcal{S}_{ID}^n$ we define the local lifting operator $r_S : (L^2(S))^{d+1} \to \Sigma_h$ as

$$\sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} r_S(\phi) \cdot q \, \mathrm{d}\mathcal{K} = -\int_S \phi \cdot \{\!\{q\}\!\} \, \mathrm{d}S, \ \forall q, \phi \in \Sigma_h, \quad \text{on } S \in \mathcal{S}_{ID}^n. \tag{2.10}$$

The support of the operator r_S is limited to the element(s) that share the face S. Next, we define the global lifting operator $R : (L^2(\mathcal{S}_{ID}^n))^{d+1} \to \Sigma_h$ as

$$\begin{aligned} \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} R(\phi) \cdot q \, \mathrm{d}\mathcal{K} &= \sum_{n=0}^{N_t-1} \sum_{S \in \mathcal{S}_{ID}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} r_S(\phi) \cdot q \, \mathrm{d}\mathcal{K}, \\ &= -\sum_{n=0}^{N_t-1} \sum_{S \in \mathcal{S}_{ID}^n} \int_S \phi \cdot \{\!\{q\}\!\} \, \mathrm{d}S, \ \forall q, \phi \in \Sigma_h. \end{aligned} \tag{2.11}$$

We also define the global lifting operator $R_D : (L^2(\mathcal{S}_D^n))^{d+1} \to \Sigma_h$ as:

$$\begin{aligned} \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} R_D(\mathcal{P} g_D n) \cdot q \, \mathrm{d}\mathcal{K} &= \sum_{n=0}^{N_t-1} \sum_{S \in \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} r_S(\mathcal{P} g_D n) \cdot q \, \mathrm{d}\mathcal{K} \\ &= -\sum_{n=0}^{N_t-1} \sum_{S \in \mathcal{S}_D^n} \int_S \mathcal{P} g_D n \cdot q \, \mathrm{d}S \\ &= -\sum_{n=0}^{N_t-1} \sum_{S \in \mathcal{S}_D^n} \int_S g_D n \cdot q \, \mathrm{d}S, \ \forall q \in \Sigma_h, \end{aligned} \tag{2.12}$$

since $\mathcal{P} : (L^2(S))^{d+1} \to \Sigma_h$ is the L^2 projection on Σ_h. Finally, we define the global lifting operator $R_{g_D} : (L^2(\mathcal{S}_{ID}^n))^{d+1} \to \Sigma_h$ as

$$\begin{aligned}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} R_{g_D}(\phi)\cdot q \,\mathrm{d}\mathcal{K} &= \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_{ID}^n}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} r_{S,g_D}(\phi)\cdot q\,\mathrm{d}\mathcal{K},\\ &= \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} R(\phi)\cdot q\,\mathrm{d}\mathcal{K} - \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} R_D(\mathcal{P}g_D n)\cdot q\,\mathrm{d}\mathcal{K}\\ &= -\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_{ID}^n}\int_S \phi\cdot\{\!\{q\}\!\}\,\mathrm{d}S + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_D^n}\int_S g_D n\cdot q\,\mathrm{d}S,\ \forall q,\phi\in\Sigma_h.\end{aligned} \tag{2.13}$$

We will also use the spatial part of the lifting operators R, r_S , denoted by $\bar{R}$ and $\bar{r}_S$, which is obtained by setting the first component of R resp. r_S equal to zero.

2.5 Space-time formulation of parabolic equations

In this section, first we will reformulate problem (2.1)- (2.3) in the space-time framework. We introduce the vector function $B\in\mathbb{R}^{d+1}$ and the symmetric matrix $A\in\mathbb{R}^{(d+1)\times(d+1)}$ as:

$$B=\begin{pmatrix}1\\ b\end{pmatrix},\qquad A=\begin{pmatrix}0 & 0\\ 0 & D\end{pmatrix},$$

with D the symmetric positive definite matrix defined in Section 2.1, which admits a unique square root $D^{1/2}$.

The parabolic partial differential equation (2.1) can now be transformed into a space-time formulation as:

$$-\nabla\cdot(A\nabla u - Bu) + cu = f,\quad \text{in}\quad \mathcal{E}, \tag{2.14}$$

where $\nabla=\left(\frac{\partial}{\partial x_0},\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_d}\right)^T$ denotes the gradient operator in $\mathbb{R}^{d+1}$. Later we will also use the notation $\overline{\nabla}$ to denote the spatial gradient operator in $\mathbb{R}^d$, defined as $\overline{\nabla}=\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_d}\right)^T$. The unit outward normal vector at $\partial\mathcal{E}$ is denoted with n. The domain boundary $\partial\mathcal{E}$ is divided into disjoint subsets $\partial\mathcal{E}=\Gamma_A\cup\Gamma_0\cup\Gamma_-\cup\Gamma_+$, where:

$$\begin{aligned}\Gamma_A &:= \{x\in\partial\mathcal{E} : n^T A n > 0\}, & \Gamma_0 &:= \{x\in\partial\mathcal{E}\setminus\Gamma_D : B\cdot n = 0\},\\ \Gamma_- &:= \{x\in\partial\mathcal{E}\setminus\Gamma_D : B\cdot n < 0\}, & \Gamma_+ &:= \{x\in\partial\mathcal{E}\setminus\Gamma_D : B\cdot n > 0\}.\end{aligned}$$

Further, we divide Γ_A into disjoint subsets Γ_D and Γ_M, with Γ_D nonempty and relatively open in $\partial\mathcal{E}$. The initial and boundary conditions in the space-time formulation are written as

$$u=u_0\quad\text{on }\Gamma_-,\qquad u=g_D\quad\text{on }\Gamma_D,\qquad \alpha u + n\cdot(A\nabla u) = g_M\quad\text{on }\Gamma_M. \tag{2.15}$$

The parabolic partial differential equation (2.14) with initial and boundary conditions (2.15) has a unique solution $u\in H^{1,0}(\mathcal{E})$ [13].

Now we introduce the space-time DG variational formulation of (2.14). Before that, we introduce the element boundaries decomposition as in [10]. Each element boundary $\partial\mathcal{K}$ can be decomposed into the union of disjoint boundaries

$$\begin{aligned}\partial\mathcal{K} &\equiv \partial_0\mathcal{K}\cup\partial_+\mathcal{K}\cup(\partial_-\mathcal{K}\setminus\Gamma_{-,D})\cup(\partial_-\mathcal{K}\cap\Gamma_-)\cup(\partial_-\mathcal{K}\cap\Gamma_D)\\ &\equiv \partial_0\mathcal{K}\cup(\partial_+\mathcal{K}\cap\Gamma_{+,D,M})\cup(\partial_+\mathcal{K}\setminus\Gamma_{+,D,M})\cup(\partial_-\mathcal{K}\cap\Gamma_{-,D})\cup(\partial_-\mathcal{K}\setminus\Gamma_{-,D})\end{aligned} \tag{2.16}$$

with $\Gamma_{+,D,M} = \Gamma_+ \cup \Gamma_D \cup \Gamma_M$, $\Gamma_{-,D} = \Gamma_- \cup \Gamma_D$ and

$$\partial_0\mathcal{K} := \{x \in \partial\mathcal{K} : B\cdot n_{\mathcal{K}} = 0\}, \quad \partial_-\mathcal{K} := \{x \in \partial\mathcal{K} : B\cdot n_{\mathcal{K}} < 0\}, \quad \partial_+\mathcal{K} := \{x \in \partial\mathcal{K} : B\cdot n_{\mathcal{K}} > 0\}.$$

Then, introducing the bilinear forms $a : V_h \times V_h \to \mathbb{R}$, $a_a : V_h \times V_h \to \mathbb{R}$, $a_d : V_h \times V_h \to \mathbb{R}$ as

$$a(u_h, v) = a_a(u_h, v) + a_d(u_h, v), \tag{2.17}$$

defined by

$$\begin{aligned} a_a(u_h, v) = &\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} (-Bu_h \cdot \nabla_h v + cu_h v)\, \mathrm{d}\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_+\mathcal{K}} B\cdot n u_h^- v^- \,\mathrm{d}\partial\mathcal{K} \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} B\cdot n u_h^+ v^- \,\mathrm{d}\partial\mathcal{K}, \end{aligned} \tag{2.18a}$$

$$\begin{aligned} a_d(u_h, v) = &\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} D\overline{\nabla}_h u_h \cdot \overline{\nabla}_h v \,\mathrm{d}\mathcal{K} - \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_{ID}^n} \int_S \big(\langle\!\langle u_h\rangle\!\rangle \cdot D\{\!\!\{\overline{\nabla}_h v\}\!\!\} + D\{\!\!\{\overline{\nabla}_h u_h\}\!\!\} \cdot \langle\!\langle v\rangle\!\rangle\big)\,\mathrm{d}S \\ &+ \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \int_S \alpha u_h^- v^- \,\mathrm{d}S + \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_0 \int_{\mathcal{K}} D\bar{r}_S([\![u_h]\!]) \cdot \bar{r}_S([\![v]\!])\,\mathrm{d}\mathcal{K}, \end{aligned} \tag{2.18b}$$

and the functional $\ell : V_h \to \mathbb{R}$, defined by

$$\begin{aligned} \ell(v) = &\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} fv\,\mathrm{d}\mathcal{K} - \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_D^n} \int_S g_D \bar{n} \cdot D\overline{\nabla}_h v^- \,\mathrm{d}S + \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \int_S g_M v^- \,\mathrm{d}S \\ &- \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_D^n} \eta_0 \int_S g_D D\,\bar{n} \cdot \bar{r}_S([\![v]\!])\,\mathrm{d}S - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\cap\Gamma_-} B\cdot n u_0 v^- \,\mathrm{d}\partial\mathcal{K} \\ &- \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\cap\Gamma_D} B\cdot n g_D v^- \,\mathrm{d}\partial\mathcal{K}, \end{aligned} \tag{2.18c}$$

we consider the following space-time DG method:

Find a $u_h \in V_h$ such that:

$$a(u_h, v) = \ell(v) \quad \forall v \in V_h. \tag{2.19}$$

In the bilinear forms a_d and a_a and the functional ℓ, we use the spatial gradient operator $\overline{\nabla}$, the spatial jump operator $\langle\!\langle\cdot\rangle\!\rangle$ and spatial lifting operator $\bar{r}_S$ defined earlier in this report.

3 Analysis

3.1 Preliminaries

Before starting the analysis of variational formulation (2.19), we introduce some related definitions. First, we define the boundary norm and the DG norm for the bilinear form (2.17).

Definition 1 *Define $\|\cdot\|_\tau$, $\tau \subset \partial\mathcal{K}$ as the (semi)-norm associated with the (semi)-inner-product*

$$(v, w)_\tau = \int_\tau |B \cdot n| vw \ dS.$$

Definition 2 *Define the DG norm $|||\cdot|||_{DG}$ corresponding to the bilinear form* (2.17) *as*

$$\begin{aligned}|||v|||_{DG}^2 &= \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|_{L^2(\mathcal{K})}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\overline{\nabla}_h v\|_{L^2(\mathcal{K})}^2 + \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S([\![v]\!])\|_{L^2(\mathcal{K})}^2 \\ &+ \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \|\sqrt{\alpha} v^-\|_{L^2(S)}^2 + \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h} \|v^-\|_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}}^2 + \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h} \|v^-\|_{\partial_-\mathcal{K}\cap\Gamma_{-,D}}^2 \\ &+ \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h} \|v^- - v^+\|_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}}^2. \end{aligned} \tag{3.1}$$

We also define a function c_0 by

$$c_0(x) = c(x) + \frac{1}{2}\nabla \cdot B(x), \ \text{a.e. } x \in \mathcal{E}. \tag{3.2}$$

Then for the error analysis later in this report, we define the following norm on the element boundary $\partial_-\mathcal{K} \setminus \Gamma_{-,D}$.

Definition 3 *Define a norm on $\partial_-\mathcal{K} \setminus \Gamma_{-,D}$ as*

$$\|v\|_\star^2 = \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h} \|v^+\|_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}}^2. \tag{3.3}$$

3.2 Consistency and Orthogonality

In following lemma we prove that the variational formulation (2.19) is consistent.

Lemma 1 *Let u solve the parabolic problems* (2.14)- (2.15)*. Then*

$$a(u, v) = \ell(v) \quad \forall v \in H^2(\mathcal{E}_h, \mathcal{T}_h). \tag{3.4}$$

Proof. First, we substitute u_h with u in bilinear forms (2.18a)-(2.18b) to obtain

$$\begin{aligned} a(u, v) &= \sum_{\mathcal{K}\in\mathcal{T}_h} \int_\mathcal{K} (-Bu \cdot \nabla_h v + cuv) \,\mathrm{d}\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_\mathcal{K} A\nabla_h u \cdot \nabla_h v \,\mathrm{d}\mathcal{K} \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_+\mathcal{K}} B \cdot n u^- v^- \,\mathrm{d}\partial\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} B \cdot n u^+ v^- \,\mathrm{d}\partial\mathcal{K} \\ &- \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_{ID}^n} \int_S \left([\![u]\!] \cdot A\{\!\{\nabla_h v\}\!\} + A\{\!\{\nabla_h u\}\!\} \cdot [\![v]\!]\right) \mathrm{d}S \\ &+ \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_0 \int_\mathcal{K} Ar_S([\![u]\!]) \cdot r_S([\![v]\!]) \,\mathrm{d}\mathcal{K} + \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \int_S \alpha u^- v^- \,\mathrm{d}S. \end{aligned} \tag{3.5}$$

Since u solves the problems (2.14)- (2.15), we have $u^- = u^+ = u$ on the element boundary and $\nabla_h u = \nabla u$, which means $[\![u]\!] = 0$ on $\mathcal{S}_I^n$, $[\![u]\!] = g_D n$ on $\mathcal{S}_D^n$, and $\{\!\{\nabla_h u\}\!\} = \nabla u$ on $\mathcal{S}_{ID}^n$. Substituting these relations into (3.5), we have

$$\begin{aligned}a(u,v) = &\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\big((A\nabla u - Bu)\cdot\nabla_h v + cuv\big)\,\mathrm{d}\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial_+\mathcal{K}\cup(\partial_-\mathcal{K}\setminus\Gamma_{-,D})} B\cdot nuv^-\,\mathrm{d}\partial\mathcal{K}\\&-\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_{ID}^n}\int_S A\nabla u\cdot[\![v]\!]\,\mathrm{d}S - \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_D^n}\int_S g_D n\cdot A\nabla_h v^-\,\mathrm{d}S\\&+\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_D^n}\sum_{\mathcal{K}\in\mathcal{T}_h}\eta_0\int_{\mathcal{K}} Ar_S(\mathcal{P}g_D n)\cdot r_S([\![v]\!])\,\mathrm{d}\mathcal{K} + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_M^n}\int_S \alpha uv^-\,\mathrm{d}S.\end{aligned}\tag{3.6}$$

By integration by parts, we have

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}(A\nabla u - Bu)\cdot\nabla_h v\,\mathrm{d}\mathcal{K} = -\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\nabla\cdot(A\nabla u - Bu)v\,\mathrm{d}\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}(A\nabla u - Bu)\cdot nv^-\,\mathrm{d}\partial\mathcal{K}.\tag{3.7}$$

Substituting (3.7) into (3.6) and after rearranging the terms the bilinear form $a(u,v)$ then becomes

$$\begin{aligned}a(u,v) = &\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\big(-\nabla\cdot(A\nabla u - Bu) + cu\big)v\,\mathrm{d}\mathcal{K}\\&-\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_D^n}\int_S g_D n\cdot A\nabla_h v^-\,\mathrm{d}S + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_M^n}\int_S(\alpha u + A\nabla u\cdot n)v^-\,\mathrm{d}S\\&+\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_D^n}\sum_{\mathcal{K}\in\mathcal{T}_h}\eta_0\int_{\mathcal{K}} Ar_S(\mathcal{P}g_D n)\cdot r_S([\![v]\!])\,\mathrm{d}\mathcal{K} - \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial_-\mathcal{K}\cap\Gamma_-} B\cdot nu_0 v^-\,\mathrm{d}\partial\mathcal{K}\\&-\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial_-\mathcal{K}\cap\Gamma_D} B\cdot ng_D v^-\,\mathrm{d}\partial\mathcal{K},\end{aligned}\tag{3.8}$$

since on Γ_- and Γ_D the solution u is equal to u_0 and g_D, respectively (see (2.15)). Since $-\nabla\cdot(A\nabla u - Bu) + cu = f$ (see (2.14)) and $\alpha u + A\nabla u\cdot n = g_M$ on $\mathcal{S}_M^n$ (see (2.15)), the bilinear form $a(u,v)$ can be written further as

$$\begin{aligned}a(u,v) = &\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} fv\,\mathrm{d}\mathcal{K} - \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_D^n}\int_S g_D n\cdot A\nabla_h v^-\,\mathrm{d}S + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_M^n}\int_S g_M v^-\,\mathrm{d}S\\&+\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_D^n}\sum_{\mathcal{K}\in\mathcal{T}_h}\eta_0\int_{\mathcal{K}} Ar_S(\mathcal{P}g_D n)\cdot r_S([\![v]\!])\,\mathrm{d}\mathcal{K} - \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial_-\mathcal{K}\cap\Gamma_-} B\cdot nu_0 v^-\,\mathrm{d}\partial\mathcal{K}\\&-\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial_-\mathcal{K}\cap\Gamma_D} B\cdot ng_D v^-\,\mathrm{d}\partial\mathcal{K}.\end{aligned}\tag{3.9}$$

Next, using the lifting operator R_D (2.12)and the structure of matrix A, we have

$$\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_D^n}\sum_{\mathcal{K}\in\mathcal{T}_h}\eta_0\int_{\mathcal{K}} Ar_S(\mathcal{P}g_D n)\cdot r_S([\![v]\!])\,\mathrm{d}\mathcal{K} = -\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_D^n}\eta_0\int_S g_D n\cdot Ar_S([\![v]\!])\,\mathrm{d}S.\tag{3.10}$$

Substituting (3.10) into (3.9) and considering the structure of matrix A, we have

$$\begin{aligned}
a(u,v) = &\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} fv \, \mathrm{d}\mathcal{K} - \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_D^n} \int_S g_D \bar{n} \cdot D\overline{\nabla}_h v^- \, \mathrm{d}S + \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \int_S g_M v^- \, \mathrm{d}S \\
&- \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_D^n} \eta_0 \int_S g_D D\bar{n} \cdot \bar{r}_S([\![v]\!]) \, \mathrm{d}S - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\cap\Gamma_-} B \cdot n u_0 v^- \, \mathrm{d}\partial\mathcal{K} \\
&- \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\cap\Gamma_D} B \cdot n g_D v^- \, \mathrm{d}\partial\mathcal{K} \\
= &\ \ell(v).
\end{aligned} \tag{3.11}$$

This completes the proof of consistency of variational formulation (2.19). □

Combining (2.19) and (3.4) yields the Galerkin orthogonality property

$$a(u - u_h, v) = 0 \quad \forall v \in V_h. \tag{3.12}$$

3.3 Coercivity of the DG variational formulation

In following lemma we prove that the bilinear form (2.17) is coercive with respect to the DG-norm (3.1). Here, we use or modify some relations from [9, 10, 11, 12]. In addition, ss we use similar lifting operators R and r_S as in [6], the proof involving these terms follows the same lines as used in [6].

Lemma 2 *If $\eta_0 > N_f$ with N_f the number of faces of each element $\mathcal{K} \in \mathcal{T}_h$ and $\bar{c}_0 = \inf_{x\in\mathcal{E}} c_0(x) > 0$, then there exists a constant $\beta > 0$, independent of the meshsize h, such that*

$$a(v,v) \geq \beta |||v|||_{DG}^2, \ \forall v \in V_h.$$

Proof. First, we replace u_h by v in the bilinear forms (2.18a) and (2.18b). Then we start to prove the coercivity for bilinear form a_a. Using the following relation

$$vB \cdot \nabla_h v = -\frac{1}{2}(\nabla_h \cdot B)v^2 + \frac{1}{2}\nabla_h \cdot (Bv^2),$$

then applying Gauss' Theorem and using the boundary decomposition (2.16), we can write the bilinear form $a_a(v,v)$ as

$$\begin{aligned}
a_a(v,v) = &\sum_{\mathcal{K}\in\mathcal{T}_h} \big(\frac{1}{2}\nabla_h \cdot B + c\big)v^2 \, \mathrm{d}\mathcal{K} + \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} B \cdot n(v^-)^2 \, \mathrm{d}\partial\mathcal{K} \\
&+ \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_+\mathcal{K}\setminus\Gamma_{+,D,M}} B \cdot n(v^-)^2 \, \mathrm{d}\partial\mathcal{K} - \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\cap\Gamma_{-,D}} B \cdot n(v^-)^2 \, \mathrm{d}\partial\mathcal{K} \\
&+ \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} B \cdot n(v^-)^2 \, \mathrm{d}\partial\mathcal{K} - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} B \cdot n(v^- - v^+)v^- \, \mathrm{d}\partial\mathcal{K}.
\end{aligned} \tag{3.13}$$

Using the relation

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \frac{1}{2} \int_{\partial_+\mathcal{K}\setminus\Gamma_{+,D,M}} B \cdot n u_h^- v^- \, \mathrm{d}\partial\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{1}{2} \int_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} B \cdot n u_h^+ v^+ \, \mathrm{d}\partial\mathcal{K} = 0, \tag{3.14}$$

with u_h is replaced with v and

$$(v^- - v^+)v^- = \frac{1}{2}(v^-)^2 + \frac{1}{2}(v^- - v^+)^2 - \frac{1}{2}(v^+)^2, \tag{3.15}$$

the bilinear form $a_a(v,v)$ can be rewritten further as

$$\begin{aligned} a_a(v,v) = &\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\Big(\frac{1}{2}\nabla_h\cdot B + c\Big)v^2 \,\mathrm{d}\mathcal{K} + \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} B\cdot n(v^-)^2 \,\mathrm{d}\partial\mathcal{K} \\ &- \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial_-\mathcal{K}\cap\Gamma_{-,D}} B\cdot n(v^-)^2 \,\mathrm{d}\partial\mathcal{K} - \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} B\cdot n(v^- - v^+)^2 \,\mathrm{d}\partial\mathcal{K}. \end{aligned} \tag{3.16}$$

Using Definition 1 and function c_0 (3.2), we can write $a_a(v,v)$ as

$$\begin{aligned} a_a(v,v) = &\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} c_0(x)v^2 \,\mathrm{d}\mathcal{K} + \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\|v^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} + \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\|v^-\|^2_{\partial_-\mathcal{K}\cap\Gamma_{-,D}} \\ &+ \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\|v^- - v^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}}. \end{aligned} \tag{3.17}$$

Next, we consider the bilinear form $a_d(v,v)$. Using the fact that D is a symmetric positive definite matrix we write the bilinear form $a_d(v,v)$ as

$$a_d(v,v) = \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} D^{1/2}\overline{\nabla}_h v\cdot D^{1/2}\overline{\nabla}_h v \,\mathrm{d}\mathcal{K} - 2\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\int_S D^{1/2}\langle\!\langle v\rangle\!\rangle\cdot D^{1/2}\{\!\{\overline{\nabla}_h v\}\!\} \,\mathrm{d}S \tag{3.18}$$

$$+ \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{M}}\int_S \alpha(v^-)^2 \,\mathrm{d}S + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\eta_0\int_{\mathcal{K}} D^{1/2}\bar{r}_S([\![v]\!])\cdot D^{1/2}\bar{r}_S([\![v]\!]) \,\mathrm{d}\mathcal{K}. \tag{3.19}$$

Then, using the definition of the global lifting operator $\bar{R}$, which is the spatial part of the lifting operator R defined in (2.11), we obtain

$$\begin{aligned} a_d(v,v) \;=\; &\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\,\overline{\nabla}_h v\|^2_{L^2(\mathcal{K})} + 2\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} D^{1/2}\overline{\nabla}_h v\cdot D^{1/2}\bar{R}([\![v]\!]) \,\mathrm{d}\mathcal{K} \\ &+ \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{M}}\|\sqrt{\alpha}v^-\|^2_{L^2(S)} + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\eta_0\|D^{1/2}\bar{r}_S([\![v]\!])\|^2_{L^2(\mathcal{K})}. \end{aligned} \tag{3.20}$$

Using the Schwarz' inequality and the arithmetic-geometric mean inequality we obtain

$$-2\int_{\mathcal{K}} D^{1/2}\overline{\nabla}_h v\cdot D^{1/2}\bar{R}([\![v]\!]) \,\mathrm{d}\mathcal{K} \le \epsilon\|D^{1/2}\overline{\nabla}_h v\|^2_{L^2(\mathcal{K})} + \frac{1}{\epsilon}\|D^{1/2}\bar{R}([\![v]\!])\|^2_{L^2(\mathcal{K})}, \tag{3.21a}$$

with $\epsilon > 0$. As a consequence of (2.11), we also have

$$\|D^{1/2}\bar{R}([\![v]\!])\|^2_{L^2(\mathcal{K})} = \|\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}} D^{1/2}\bar{r}_S([\![v]\!])\|^2_{L^2(\mathcal{K})} \le N_f\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\|D^{1/2}\bar{r}_S([\![v]\!])\|^2_{L^2(\mathcal{K})}, \tag{3.21b}$$

with N_f the number of faces of each element $\mathcal{K} \in \mathcal{T}_h$. Introducing (3.21a) and (3.21b) into (3.20) and combining with (3.17), we deduce

$$\begin{aligned} a(v,v) \geq\ & \bar{c}_0 \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|^2_{L^2(\mathcal{K})} + (1-\epsilon) \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\overline{\nabla}_h v\|^2_{L^2(\mathcal{K})} \\ & + \big(\eta_0 - \frac{N_f}{\epsilon}\big) \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}^n_{ID}} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S([\![v]\!])\|^2_{L^2(\mathcal{K})} + \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}^n_M} \|\sqrt{\alpha} v^-\|^2_{L^2(S)} \\ & + \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|v^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} + \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|v^-\|^2_{\partial_-\mathcal{K}\cap\Gamma_{-,D}} + \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|v^- - v^+\|^2_{\partial_-\mathcal{K}\backslash\Gamma_{-,D}}, \end{aligned} \tag{3.22}$$

with η_0 defined as $\eta_0 = \min_{\mathcal{K}\in\mathcal{T}_h} \eta_\mathcal{K}$ and $\bar{c}_0 = \inf_{x\in\mathcal{E}} c_0(x)$. If $\bar{c}_0 > 0$ and we define the parameters $\eta_0 > N_f$ and $\epsilon \in (\frac{N_f}{\eta_0}, 1)$, then for $\beta = \min(1-\epsilon, \eta_0 - \frac{N_f}{\epsilon}, \bar{c}_0) > 0$, we obtain the relation

$$a(v,v) \geq \beta |\!|\!| v |\!|\!|^2_{DG},$$

which completes the proof of coercivity. □

If $\bar{c}_0 = \inf_{x\in\mathcal{E}} c_0(x) = 0$, we do not have the standard coercivity. Instead, we use a Garding inequality to show the coercivity.

Lemma 3 *If $\eta_0 > N_f$, with N_f the number of faces of each element $\mathcal{K} \in \mathcal{T}_h$, then there exists a constant $\beta_c > 0$, independent of the meshsize h , such that*

$$a(v,v) + \gamma_c \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|^2_{L^2(\mathcal{K})} \geq \beta_c |\!|\!| v |\!|\!|^2_{DG}, \ \forall v \in V_h, \tag{3.23}$$

with $\beta_c = \min(1-\epsilon, \eta_0 - \frac{N_f}{\epsilon}, \gamma_c + \bar{c}_0)$, $\epsilon \in (\frac{N_f}{\eta_0}, 1)$, and γ_c is defined such as $\gamma_c + \bar{c}_0 > 0$.

Proof. Using (3.22), we obtain:

$$\begin{aligned} a(v,v) + \gamma_c \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|^2_{L^2(\mathcal{K})} \geq & (\bar{c}_0 + \gamma_c) \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|^2_{L^2(\mathcal{K})} + (1-\epsilon) \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\overline{\nabla}_h v\|^2_{L^2(\mathcal{K})} \\ & + \big(\eta_0 - \frac{N_f}{\epsilon}\big) \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}^n_{ID}} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S([\![v]\!])\|^2_{L^2(\mathcal{K})} \\ & + \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}^n_M} \|\sqrt{\alpha} v^-\|^2_{L^2(S)} + \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|v^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} \\ & + \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|v^-\|^2_{\partial_-\mathcal{K}\cap\Gamma_{-,D}} + \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|v^- - v^+\|^2_{\partial_-\mathcal{K}\backslash\Gamma_{-,D}}. \end{aligned} \tag{3.24}$$

If we define γ_c such as $\gamma_c + \bar{c}_0 > 0$, then by choosing $\beta_c = \min(1-\epsilon, \eta_0 - \frac{N_f}{\epsilon}, \gamma_c + \bar{c}_0)$, with $\epsilon \in (\frac{N_f}{\eta_0}, 1)$, we completes the proof of Garding inequality. Note that γ_c need not to be positive, if $\bar{c}_0 > 0$. □

3.4 Existence of a unique DG solution

In this section we prove the existence of a unique solution of (2.19).

Theorem 1 *If $\eta_0 > N_f$, with N_f the number of faces of each element $\mathcal{K} \in \mathcal{T}_h$, and $\bar{c}_0 = \inf_{x\in\mathcal{E}} c_0(x) > 0$, then there exists a unique solution $u_h \in V_h$ for the variational problem (2.19).*

Proof. To show the uniqueness of the DG solution for (2.19) it is sufficient to prove that the homogeneous equation:

Find a $u_h \in V_h$ such that:

$$a(u_h, v) = 0, \quad \forall v \in V_h, \qquad \text{with } u_h(t_0, \bar{x}) = 0, \tag{3.25}$$

only has the trivial solution $u_h = 0$ for all $t > t_0$.

Assume u_h is a solution and choose $v = u_h$ in the bilinear form $a(u_h, v)$. Then we rewrite the coercivity statement as:

$$\begin{aligned} a(u_h, u_h) \geq \beta |||u_h|||_{DG}^2 = \beta \sum_{n=0}^{N_t-1} \Bigg(&\sum_{\mathcal{K}\in\mathcal{T}_h^n} \|u_h\|_{L^2(\mathcal{K})}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h^n} \|D^{1/2}\overline{\nabla}_h u_h\|_{L^2(\mathcal{K})}^2 \\ &+ \sum_{S\in\mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h^n} \|D^{1/2}\bar{r}_S([\![u_h]\!])\|_{L^2(\mathcal{K})}^2 + \sum_{S\in\mathcal{S}_M^n} \|\sqrt{\alpha} u_h^-\|_{L^2(S)}^2 \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h^n} \frac{1}{2}\|u_h^-\|_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h^n} \frac{1}{2}\|u_h^-\|_{\partial_-\mathcal{K}\cap\Gamma_{-,D}}^2 \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h^n} \frac{1}{2}\|u_h^- - u_h^+\|_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}}^2 \Bigg). \end{aligned} \tag{3.26}$$

First, consider the time slab for $n = 0$, then the coercivity condition in combination with u_h^+ at $t = 0$ implies $u_h = 0$ in the first time slab. We can continue this argument to other time slabs and we obtain that $u_h = 0$ is the only solution possible for the homogeneous equation. Hence the DG algorithm has a unique solution for linear basis functions in time. □

3.5 Stability Analysis of DG discretization

Lemma 4 *The solution u_h to (2.19) satisfies the following bound:*

$$\begin{aligned} \beta_s |||u_h|||_{DG}^2 \leq & \frac{1}{2\epsilon_1} \sum_{\mathcal{K}\in\mathcal{T}_h} \|f\|_{L^2(\mathcal{K})}^2 + \Big(\eta_0 + \frac{N_f}{2(1-\epsilon)}\Big) \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S(\mathcal{P}g_D n)\|_{L^2(\mathcal{K})}^2 \\ &+ \frac{1}{2} \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \|\alpha^{-1/2} g_M\|_{L^2(S)}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_0\|_{\partial_-\mathcal{K}\cap\Gamma_-}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \|g_D\|_{\partial_-\mathcal{K}\cap\Gamma_D}^2, \end{aligned}$$

with $0 < \beta_s = \min(\frac{1}{2}, \frac{1}{2} - \frac{\epsilon}{2}, \frac{3}{4}\eta_0 - \frac{N_f}{\epsilon}, \bar{c}_0 - \frac{\epsilon_1}{2})$, *for* $\epsilon \in (\frac{4N_f}{3\eta_0}, 1)$ *and* $\epsilon_1 \in (0, 2\bar{c}_0)$.

Proof. Consider $u_h \in V_h$ the solution of the variational formulation (2.19). Using the lifting operator R_D (2.12), the variational formulation (2.19) can be written as

$$\begin{aligned} a(u_h,u_h) =& \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} f u_h \,\mathrm{d}\mathcal{K} + \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \int_S g_M u_h^- \,\mathrm{d}S + \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_D^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \bar{r}_S(\mathcal{P} g_D n)\cdot D\overline{\nabla}_h u_h \,\mathrm{d}\mathcal{K} \\ &+ \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_D^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_0 \int_{\mathcal{K}} \bar{r}_S(\mathcal{P} g_D n)\cdot D\bar{r}_S([\![u_h]\!]) \,\mathrm{d}\mathcal{K} - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\cap\Gamma_-} B\cdot n u_0 u_h^- \,\mathrm{d}\partial\mathcal{K} \\ &- \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\cap\Gamma_D} B\cdot n g_D u_h^- \,\mathrm{d}\partial\mathcal{K}. \end{aligned} \tag{3.27}$$

By applying Schwarz' and arithmetic-geometric mean inequalities on each term, we obtain

$$\begin{aligned} a(u_h,u_h) \le& \frac{1}{2\epsilon_1} \sum_{\mathcal{K}\in\mathcal{T}_h} \|f\|^2_{L^2(\mathcal{K})} + \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \epsilon_1 \|u_h\|^2_{L^2(\mathcal{K})} + \frac{1}{2\epsilon_2} \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \|\alpha^{-1/2} g_M\|^2_{L^2(S)} \\ &+ \frac{1}{2}\epsilon_2 \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \|\sqrt{\alpha} u_h^-\|^2_{L^2(S)} + \frac{1}{2\epsilon_3} \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_D^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S(\mathcal{P} g_D n)\|^2_{L^2(\mathcal{K})} \\ &+ \frac{1}{2}\epsilon_3 \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\overline{\nabla}_h u_h\|^2_{L^2(\mathcal{K})} + \frac{\eta_0}{2\epsilon_4} \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_D^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S(\mathcal{P} g_D n)\|^2_{L^2(\mathcal{K})} \\ &+ \frac{\eta_0}{2}\epsilon_4 \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S([\![u_h]\!])\|^2_{L^2(\mathcal{K})} + \frac{1}{2\epsilon_5} \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_0\|^2_{\partial_-\mathcal{K}\cap\Gamma_-} + \frac{1}{2}\epsilon_5 \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_h^-\|^2_{\partial_-\mathcal{K}\cap\Gamma_-} \\ &+ \frac{1}{2\epsilon_5} \sum_{\mathcal{K}\in\mathcal{T}_h} \|g_D\|^2_{\partial_-\mathcal{K}\cap\Gamma_D} + \frac{1}{2}\epsilon_5 \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_h^-\|^2_{\partial_-\mathcal{K}\cap\Gamma_D}, \end{aligned} \tag{3.28}$$

with $\epsilon_1, \dots, \epsilon_5 > 0$. If we combine (3.28) with (3.22), we obtain

$$\begin{aligned} &\big(\bar{c}_0 - \frac{1}{2}\epsilon_1\big) \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_h\|^2_{L^2(\mathcal{K})} + \big(1-\epsilon-\frac{1}{2}\epsilon_3\big) \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\overline{\nabla}_h u_h\|^2_{L^2(\mathcal{K})} \\ &+ \big(\eta_0 - \frac{N_f}{\epsilon} - \frac{\eta_0}{2}\epsilon_4\big) \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S([\![u_h]\!])\|^2_{L^2(\mathcal{K})} + \big(1-\frac{1}{2}\epsilon_2\big) \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \|\sqrt{\alpha} u_h^-\|^2_{L^2(S)} \\ &+ \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_h^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} + \big(\frac{1}{2} - \frac{1}{2}\epsilon_5\big) \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_h^-\|^2_{\partial_-\mathcal{K}\cap\Gamma_{-,D}} + \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_h^- - u_h^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} \\ &\le \frac{1}{2\epsilon_1} \sum_{\mathcal{K}\in\mathcal{T}_h} \|f\|^2_{L^2(\mathcal{K})} + \big(\frac{1}{2\epsilon_3} + \frac{\eta_0}{2\epsilon_4}\big) \sum_{\mathcal{K}\in\mathcal{T}_h} \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_D^n} \|D^{1/2}\bar{r}_S(\mathcal{P} g_D n)\|^2_{L^2(\mathcal{K})} \\ &+ \frac{1}{2\epsilon_2} \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \|\alpha^{-1/2} g_M\|^2_{L^2(S)} + \frac{1}{2\epsilon_5} \sum_{\mathcal{K}\in\mathcal{T}_h} \big(\|u_0\|^2_{\partial_-\mathcal{K}\cap\Gamma_-} + \|g_D\|^2_{\partial_-\mathcal{K}\cap\Gamma_D}\big). \end{aligned} \tag{3.29}$$

Substituting the following coefficients

$$\epsilon_1 < 2\bar{c}_0, \qquad \epsilon_2 = 1, \qquad \epsilon_3 = 1-\epsilon, \qquad \epsilon_4 = \frac{1}{2}, \qquad \epsilon_5 = \frac{1}{2},$$

into (3.29), we then obtain

$$
\begin{aligned}
&\big(\bar{c}_0 - \frac{1}{2}\epsilon_1\big) \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_h\|^2_{L^2(\mathcal{K})} + \frac{1}{2}(1-\epsilon) \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\overline{\nabla}_h u_h\|^2_{L^2(\mathcal{K})} \\
&+ \big(\frac{3}{4}\eta_0 - \frac{N_f}{\epsilon}\big) \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}^n_{ID}} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S([\![u_h]\!])\|^2_{L^2(\mathcal{K})} + \frac{1}{2} \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}^n_M} \|\sqrt{\alpha} u_h^-\|^2_{L^2(S)} \\
&+ \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_h^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} + \frac{1}{4} \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_h^-\|^2_{\partial_-\mathcal{K}\cap\Gamma_{-,D}} + \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_h^- - u_h^+\|^2_{\partial_-\mathcal{K}\backslash\Gamma_{-,D}} \\
&\leq \frac{1}{2\epsilon_1} \sum_{\mathcal{K}\in\mathcal{T}_h} \|f\|^2_{L^2(\mathcal{K})} + \big(\frac{1}{2-2\epsilon} + \eta_0\big) \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}^n_D} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S(\mathcal{P}g_D n)\|^2_{L^2(\mathcal{K})} \\
&\quad + \frac{1}{2} \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}^n_M} \|\alpha^{-1/2} g_M\|^2_{L^2(S)} + \sum_{\mathcal{K}\in\mathcal{T}_h} \|u_0\|^2_{\partial_-\mathcal{K}\cap\Gamma_-} + \sum_{\mathcal{K}\in\mathcal{T}_h} \|g_D\|^2_{\partial_-\mathcal{K}\cap\Gamma_D}.
\end{aligned}
\tag{3.30}
$$

Choosing $\beta_s = \min(\frac{1}{2}, \frac{1}{2} - \frac{\epsilon}{2}, \frac{3}{4}\eta_0 - \frac{N_f}{\epsilon}, \bar{c}_0 - \frac{\epsilon_1}{2})$, for $\epsilon \in (\frac{4N_f}{3\eta_0}, 1)$ and $\epsilon_1 \in (0, 2\bar{c}_0)$ completes the proof for the upper bound. □

4 Error Estimate and Convergence

We start this section by defining the projection $P : L^2(\mathcal{E}) \to V_h$ as

$$\sum_{\mathcal{K}\in\mathcal{T}_h} (Pu, v)_{\mathcal{K}} = \sum_{\mathcal{K}\in\mathcal{T}_h} (u, v)_{\mathcal{K}}, \quad \forall v \in V_h, \tag{4.1}$$

with $(a, b)_{\mathcal{K}} = \int_{\mathcal{K}} ab \,\mathrm{d}\mathcal{K}$. Then we have the following orthogonality relation:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} (u - Pu, v)_{\mathcal{K}} = 0, \quad \forall v \in V_h. \tag{4.2}$$

We may then decompose global error $u - u_h$ as

$$u - u_h = (u - Pu) + (Pu - u_h) \equiv \rho + \theta. \tag{4.3}$$

We discuss about the behaviour of θ during time evolution. From the Garding inequality (3.23), we have

$$\beta_c |\!|\!|\theta|\!|\!|^2_{DG} - \gamma_c \sum_{\mathcal{K}\in\mathcal{T}_h} \|\theta\|^2_{L^2(\mathcal{K})} \leq a(\theta, \theta) \leq |a(\rho, \theta)|. \tag{4.4}$$

If $\gamma_c > 0$, we introduce $\tilde{\theta} = \exp(-\gamma_c t)\theta(t, \bar{x})$. This new variable satisfies the following equation

$$\frac{\partial\tilde{\theta}}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i(\bar{x})\tilde{\theta}) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(D_{ij}(\bar{x}) \frac{\partial\tilde{\theta}}{\partial x_i}\right) + (c(\bar{x}) + \gamma_c)\tilde{\theta} = \tilde{f}(t, \bar{x}),$$

with $\tilde{f} = \exp(-\gamma_c t) f$. Hence for $\tilde{\theta}$ the bilinear form then satisfies the coercivity estimate

$$\beta_c |\!|\!|\tilde{\theta}|\!|\!|^2_{DG} \leq a(\tilde{\theta}, \tilde{\theta}). \tag{4.5}$$

In the remainder of the report, we skip therefore the term $\gamma_c \sum_{\mathcal{K}\in\mathcal{T}_h} \|\theta\|^2_{L^2(\mathcal{K})}$, since in $\tilde{\theta}$ the bilinear form is always coercive. Note however, boundedness in $\tilde{\theta}$ implies exponential growth in $\theta = \exp(\gamma_c t)\tilde{\theta}$, depending on the righthand side and boundary conditions.

Lemma 5 *There exists a constant $\beta_c > 0$, independent of the meshsize h, such that the functions ρ and θ defined in* (4.3) *satisfy the inequality*

$$\begin{aligned}
\frac{1}{4}\beta_c |\!|\!|\theta|\!|\!|^2_{DG} \leq & \frac{1}{\beta_c} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{-1/2}b\|^2_{L^\infty(\mathcal{K})} \|\rho\|^2_{L^2(\mathcal{K})} + \frac{1}{\beta_c} \sum_{\mathcal{K}\in\mathcal{T}_h} \|c\|^2_{L^\infty(\mathcal{K})}) \|\rho\|^2_{L^2(\mathcal{K})} \\
& + \frac{2}{\beta_c} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\overline{\nabla}_h \rho\|^2_{L^2(\mathcal{K})} + \frac{1}{\beta_c}(\eta_0 + 1) \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}^n_{ID}} \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{K})} \\
& + \frac{1}{\beta_c} \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}^n_M} \|\sqrt{\alpha}\rho^-\|^2_{L^2(S)} + \frac{1}{\beta_c} \sum_{\mathcal{K}\in\mathcal{T}_h} \|\rho^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} \\
& + \frac{1}{\beta_c} \sum_{\mathcal{K}\in\mathcal{T}_h} \|\rho^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}},
\end{aligned}$$

with $\beta_c = \min(1-\epsilon, \eta_0 - \frac{N_f}{\epsilon})$, $\epsilon \in (\frac{N_f}{\eta_0}, 1)$.

Proof. First, from the orthogonality property (3.12), we have

$$a(\theta + \rho, v) = 0, \quad \forall v \in V_h. \tag{4.6}$$

Taking $v = \theta$, then we obtain

$$a(\theta, \theta) = -a(\rho, \theta) \leq |a_a(\rho, \theta)| + |a_d(\rho, \theta)|. \tag{4.7}$$

First, we consider the bilinear form $a_a(\rho, \theta)$ of the form:

$$\begin{aligned}
a_a(\rho, \theta) = & - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \rho \frac{\partial\theta}{\partial t} \, \mathrm{d}\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} (-b\rho \cdot \overline{\nabla}_h \theta + c\rho\theta) \, \mathrm{d}\mathcal{K} \\
& + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_+\mathcal{K}} B \cdot n \rho^- \theta^- \, \mathrm{d}\partial\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} B \cdot n \rho^+ \theta^- \, \mathrm{d}\partial\mathcal{K}.
\end{aligned} \tag{4.8}$$

Since $\theta \in V_h$, which is polynomials, we have $\frac{\partial\theta}{\partial t} \in V_h$. Thus, we can use the orthogonality property (4.2) and relation (3.14) to obtain

$$\begin{aligned}
a_a(\rho, \theta) = & \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} (-D^{-1/2} b\rho \cdot D^{1/2} \overline{\nabla}_h \theta + c\rho\theta) \, \mathrm{d}\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} B \cdot n \rho^- \theta^- \, \mathrm{d}\partial\mathcal{K} \\
& + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} B \cdot n \rho^+ (\theta^- - \theta^+) \, \mathrm{d}\partial\mathcal{K}.
\end{aligned} \tag{4.9}$$

Using Schwarz' and arithmetic-geometric mean inequalities on each term, we have

$$
\begin{aligned}
|a_a(\rho,\theta)| \leq & \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{-1/2}b\|^2_{L^\infty(\mathcal{K})}\|\rho\|^2_{L^2(\mathcal{K})} + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|c\|^2_{L^\infty(\mathcal{K})}\|\rho\|^2_{L^2(\mathcal{K})} \\
& + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|\rho^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|\rho^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} \\
& + \frac{1}{4}\beta_c\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\overline{\nabla}_h\theta\|^2_{L^2(\mathcal{K})} + \frac{1}{4}\beta_c\sum_{\mathcal{K}\in\mathcal{T}_h}\|\theta\|^2_{L^2(\mathcal{K})} \\
& + \frac{1}{4}\beta_c\sum_{\mathcal{K}\in\mathcal{T}_h}\|\theta^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} + \frac{1}{4}\beta_c\sum_{\mathcal{K}\in\mathcal{T}_h}\|\theta^- - \theta^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} \\
\leq & \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{-1/2}b\|^2_{L^\infty(\mathcal{K})}\|\rho\|^2_{L^2(\mathcal{K})} + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|c\|^2_{L^\infty(\mathcal{K})}\|\rho\|^2_{L^2(\mathcal{K})} \\
& + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|\rho^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|\rho^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} + \frac{1}{4}\beta_c|||\theta|||^2_{DG}.
\end{aligned} \tag{4.10}
$$

Next, we consider the bilinear form $a_d(\rho,\theta)$ of the form

$$
\begin{aligned}
a_d(\rho,\theta) = & \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} D^{1/2}\overline{\nabla}_h\rho\cdot D^{1/2}\overline{\nabla}_h\theta \,\mathrm{d}\mathcal{K} + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} D^{1/2}\bar{r}_S([\![\rho]\!])\cdot D^{1/2}\overline{\nabla}_h\theta \,\mathrm{d}\mathcal{K} \\
& + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} D^{1/2}\overline{\nabla}_h\rho\cdot D^{1/2}\bar{r}_S([\![\theta]\!]) \,\mathrm{d}\mathcal{K} + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{M}}\int_S \alpha\rho^-\theta^- \,\mathrm{d}S \\
& + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\eta_0\int_{\mathcal{K}} D^{1/2}\bar{r}_S([\![\rho]\!])\cdot D^{1/2}\bar{r}_S([\![\theta]\!]) \,\mathrm{d}\mathcal{K}.
\end{aligned} \tag{4.11}
$$

Using the Schwarz' and arithmetic-geometric mean inequalities, we then obtain:

$$
\begin{aligned}
|a_d(\rho,\theta)| \leq & \frac{2}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\overline{\nabla}_h\rho\|^2_{L^2(\mathcal{K})} + \frac{1}{\beta_c}(\eta_0+1)\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{K})} \\
& + \frac{1}{\beta_c}\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_M}\|\sqrt{\alpha}\rho^-\|^2_{L^2(S)} + \frac{2}{4}\beta_c\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\overline{\nabla}_h\theta\|^2_{L^2(\mathcal{K})} \\
& + \frac{2}{4}\beta_c\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\bar{r}_S([\![\theta]\!])\|^2_{L^2(\mathcal{K})} + \frac{1}{4}\beta_c\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_M}\|\sqrt{\alpha}\theta^-\|^2_{L^2(S)} \\
\leq & \frac{2}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\overline{\nabla}_h\rho\|^2_{L^2(\mathcal{K})} + \frac{1}{\beta_c}(\eta_0+1)\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{K})} \\
& + \frac{1}{\beta_c}\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_M}\|\sqrt{\alpha}\rho^-\|^2_{L^2(S)} + \frac{2}{4}\beta_c|||\theta|||^2_{DG}.
\end{aligned} \tag{4.12}
$$

Adding (4.10) and (4.12), we have:

$$
\begin{aligned}
|a(\rho,\theta)| \leq & \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{-1/2}b\|^2_{L^\infty(\mathcal{K})}\|\rho\|^2_{L^2(\mathcal{K})} + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|c\|^2_{L^\infty(\mathcal{K})}\|\rho\|^2_{L^2(\mathcal{K})} \\
& + \frac{2}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\overline{\nabla}_h\rho\|^2_{L^2(\mathcal{K})} + \frac{1}{\beta_c}(\eta_0+1)\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{K})} \\
& + \frac{1}{\beta_c}\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_M}\|\sqrt{\alpha}\rho^-\|^2_{L^2(S)} + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|\rho^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|\rho^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} \\
& + \frac{3}{4}\beta_c|\!|\!|\theta|\!|\!|^2_{DG}. \qquad (4.13)
\end{aligned}
$$

Combining (4.13) and (4.4) (without the term with γ_c), we obtain

$$
\begin{aligned}
\beta_c|\!|\!|\theta|\!|\!|^2_{DG} \leq & \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{-1/2}b\|_{L^\infty(\mathcal{K})}\|\rho\|^2_{L^2(\mathcal{K})} + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|c\|_{L^\infty(\mathcal{K})}\|\rho\|^2_{L^2(\mathcal{K})} \\
& + \frac{2}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\overline{\nabla}_h\rho\|^2_{L^2(\mathcal{K})} + \frac{1}{\beta_c}(\eta_0+1)\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{K})} \\
& + \frac{1}{\beta_c}\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_M}\|\sqrt{\alpha}\rho^-\|^2_{L^2(S)} + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|\rho^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} + \frac{1}{\beta_c}\sum_{\mathcal{K}\in\mathcal{T}_h}\|\rho^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} \\
& + \frac{3}{4}\beta_c|\!|\!|\theta|\!|\!|^2_{DG}. \qquad (4.14)
\end{aligned}
$$

Placing all terms of θ in (4.14) to the left hand side completes the proof. □

By applying the triangle inequality to (4.3), we obtain the following bound on the global error $u-u_h$ as follows

$$|\!|\!|u-u_h|\!|\!|_{DG} \leq |\!|\!|\rho|\!|\!|_{DG} + |\!|\!|\theta|\!|\!|_{DG}. \qquad (4.15)$$

Using Lemma 5, the bound for $u-u_h$ in the DG norm can be defined in terms of the projection error ρ. Next we derive a bound on DG norm of ρ in terms of h and p.

To obtain the bounds of ρ in the DG norm and Lemma 5 in terms of h and p, we need to estimate the following norms:

$$\|\rho\|^2_{L^2(\mathcal{K})}, \|\overline{\nabla}_h\rho\|^2_{L^2(\mathcal{K})}, \|\bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{K})}, \|\rho\|^2_{L^2(\partial\mathcal{K})}. \qquad (4.16)$$

Following the similar argument as in [12], the terms of boundary norms $\|\rho^-\|_{\partial_-\mathcal{K}}, \|\rho^+\|_{\partial_-\mathcal{K}}$ and $\|\rho^-\|_{\partial_+\mathcal{K}}$ can be dealt by bounding them above by $\|B\|^{1/2}_{L^\infty(\mathcal{K})}\|\rho\|_{L^2(\partial\mathcal{K})}$.

We begin our analysis by recalling the result from [12]. The hp error estimates of steady state problems of the following terms: $\|\rho\|_{L^2(K)}, \|\nabla\rho\|_{L^2(K)}$ and $\|\rho\|_{L^2(\partial K)}$ have been derived in [12]. Here we write down the final result.

Lemma 6 *Suppose that K is the element in $\mathbb{R}^d$ of diameter h_K and $u|_K \in H^{k_K}(K), k_K \geq 0$. Suppose further P denotes the orthogonal projector in L^2 onto the suitable finite element* *space. Then the projection error* $\rho = u - Pu$ *on* K *and its boundary* ∂K *obey the error bounds*

$$\|\rho\|^2_{L^2(K)} \leq C(d)\frac{h_K^{2s_K}}{p_K^{2k_K}}\|u\|^2_{H^{k_K}(K)}, \tag{4.17}$$

$$\|\nabla\rho\|^2_{L^2(K)} \leq C(d)\frac{h_K^{2s_K-2}}{p_K^{2k_K-3}}\|u\|^2_{H^{k_K}(K)}, \tag{4.18}$$

$$\|\rho\|^2_{L^2(\partial K)} \leq C(d)\frac{h_K^{2s_K-1}}{p_K^{2k_K-1}}\|u\|^2_{H^{k_K}(K)}, \tag{4.19}$$

with p_K *the local polynomial order on element* K *and* $1 \leq s_K \leq \min(p_K+1, k_K)$*. The constant* $C(d)$ *is a positive value that depends only on the dimension and the shape regularity of the triangulation.*

In addition, the hp estimate for the term $\|\bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{K})}$ for steady state problems has been derived in [14] as follows.

Lemma 7 *Suppose that* K *is the element in* $\mathbb{R}^d$ *of diameter* h_K *and* $u|_K \in H^{k_K}(K), k_K \geq 0$*. Suppose further* P *denotes the orthogonal projector in* L^2 *onto the suitable finite element space. Then the projection error* $\rho = u - Pu$ *on* K *obeys the error bound*

$$\|\bar{r}_S([\![\rho]\!])\|^2_{L^2(K)} \leq C(d)\frac{p_K^2}{h_K}\|[\![\rho]\!]\|^2_{L^2(\partial\mathcal{K})} \leq C(d)\frac{h_K^{2s_K-2}}{p_K^{2k_K-3}}\|u\|^2_{H^{k_K}(K)}, \tag{4.20}$$

with p_K *the local polynomial order on element* K *and* $1 \leq s_K \leq \min(p_K+1, k_K)$*. The constant* $C(d)$ *is a positive value that depends only on the dimension and the shape regularity of the triangulation.*

The hp error estimates in Lemma 6 and Lemma 7 are independent of the dimension of the domain, only the constant $C(d)$ depends on the dimension. Hence, by considering time as an additional dimension, the hp error estimates in those Lemmas can be applied to the hp estimation on space-time element $\mathcal{K}$.

Lemma 8 *Suppose that* $\mathcal{K}$ *is the space-time element in* $\mathbb{R}^{d+1}$ *of diameter* $h_{\mathcal{K}}$ *and* $u|_{\mathcal{K}} \in H^{k_{\mathcal{K}}}(\mathcal{K}), k_{\mathcal{K}} \geq 0$*. Note that the diameter* $h_{\mathcal{K}}$ *now includes the time interval* $\Delta_n t$*. Suppose further* P *denotes the orthogonal projector in* L^2 *onto the finite element space* V_h*. Then the projection error* $\rho = u - Pu$ *on* $\mathcal{K}$ *and its boundary* $\partial\mathcal{K}$ *obey the error bounds*

$$\|\rho\|^2_{L^2(\mathcal{K})} \leq C(d+1)\frac{h_{\mathcal{K}}^{2s_{\mathcal{K}}}}{p_{\mathcal{K}}^{2k_{\mathcal{K}}}}\|u\|^2_{H^{k_{\mathcal{K}}}(\mathcal{K})}, \tag{4.21}$$

$$\|\overline{\nabla}_h\rho\|^2_{L^2(\mathcal{K})} \leq \|\nabla_h\rho\|^2_{L^2(\mathcal{K})} \leq C(d+1)\frac{h_{\mathcal{K}}^{2s_{\mathcal{K}}-2}}{p_{\mathcal{K}}^{2k_{\mathcal{K}}-3}}\|u\|^2_{H^{k_{\mathcal{K}}}(\mathcal{K})}, \tag{4.22}$$

$$\|\rho\|^2_{L^2(\partial\mathcal{K})} \leq C(d+1)\frac{h_{\mathcal{K}}^{2s_{\mathcal{K}}-1}}{p_{\mathcal{K}}^{2k_{\mathcal{K}}-1}}\|u\|^2_{H^{k_{\mathcal{K}}}(\mathcal{K})}, \tag{4.23}$$

$$\|\bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{K})} \leq C(d+1)\frac{h_{\mathcal{K}}^{2s_{\mathcal{K}}-2}}{p_{\mathcal{K}}^{2k_{\mathcal{K}}-3}}\|u\|^2_{H^{k_{\mathcal{K}}}(\mathcal{K})}, \tag{4.24}$$

with $p_{\mathcal{K}}$ the local polynomial order on element $\mathcal{K}$ and $1 \leq s_{\mathcal{K}} \leq \min(p_{\mathcal{K}}+1, k_{\mathcal{K}})$. The constant $C(d+1)$ is a positive value that depends only on the dimension and the shape regularity of $\mathcal{T}_h$.

Using Lemma 8, we substitute the estimates for ρ into the right-hand side of (4.15). The resulting error bound is formulated in the next theorem.

Theorem 2 *Let $\mathcal{E} \subset \mathbb{R}^{d+1}$ be a bounded polyhedral space-time domain and let $\mathcal{T}_h = \{\mathcal{K}\}$ be a shape-regular subdivision into space-time element $\mathcal{K}$ in $\mathbb{R}^{d+1}$ of diameter $h_{\mathcal{K}}$. Note that the diameter $h_{\mathcal{K}}$ now includes the time interval $\Delta_n t$. Let $u_h \in V_h$ be the discontinuous Galerkin approximation to u defined by (2.19) and $u|_{\mathcal{K}} \in H^{k_{\mathcal{K}}}(\mathcal{K}), k_{\mathcal{K}} \geq 0$. Then, the following error bound holds:*

$$|||u - u_h|||_{DG}^2 \leq \mathcal{C} \sum_{\mathcal{K} \in \mathcal{T}_h} \Big(\bar{D}_{\mathcal{K}} \frac{h_{\mathcal{K}}^{2s_{\mathcal{K}}-2}}{p_{\mathcal{K}}^{2k_{\mathcal{K}}-3}} + (\bar{\alpha}_{\mathcal{K}} + \bar{\beta}_{\mathcal{K}}) \frac{h_{\mathcal{K}}^{2s_{\mathcal{K}}-1}}{p_{\mathcal{K}}^{2k_{\mathcal{K}}-1}} + \gamma_{\mathcal{K}} \frac{h_{\mathcal{K}}^{2s_{\mathcal{K}}}}{p_{\mathcal{K}}^{2k_{\mathcal{K}}}} \Big) \|u\|_{H^{k_{\mathcal{K}}}(\mathcal{K})}^2 \tag{4.25}$$

for any integer $s_{\mathcal{K}}, 1 \leq s_{\mathcal{K}} \leq \min(p_{\mathcal{K}}+1, k_{\mathcal{K}})$, and $p_{\mathcal{K}} \geq 0$ the local polynomail order on element $\mathcal{K}$. Here

$$\gamma_{\mathcal{K}} = 1 + \|D^{-1/2} b\|_{L^\infty(\mathcal{K})}^2 + \|c\|_{L^\infty(\mathcal{K})}^2, \qquad \bar{\alpha}_{\mathcal{K}} = \|\alpha\|_{L^\infty(\mathcal{K})}, \qquad \bar{\beta}_{\mathcal{K}} = \|B\|_{L^\infty(\mathcal{K})}, \tag{4.26}$$

and $\bar{D} = |\sqrt{D}|_2^2$, where $|\cdot|_2$ denotes the matrix norm subordinate to the ℓ^2 vector norm on $\mathbb{R}^d$ and $\bar{D}_{\mathcal{K}} = \bar{D}|_{\mathcal{K}}$. The constant $\mathcal{C}$ is a positive value that depends on the dimension $d+1$, parameters η_0 and β_c (defined in Lemma 5), and the shape regularity of $\mathcal{T}_h$.

5 Dual problems

We define the following dual problem:

Find a $z_h \in V_h$ such that for all $w \in V_h$, the following relation is satisfied:

$$a(w, z_h) := a_a(w, z_h) + a_d(w, z_h) = \ell^\star(w), \tag{5.1}$$

with

$$\ell^\star(w) = \sum_{\mathcal{K} \in \mathcal{T}_h^{N_t-1}} \int_{K(t_{N_t})} \phi w^- \, \mathrm{d}K = \sum_{\mathcal{K} \in \mathcal{T}_h^{N_t-1}} (\phi, w^-)_{K(t_{N_t})}, \tag{5.2}$$

where ϕ is the solution at time t_{N_t} and the bilinear forms are defined in (2.18a) and (2.18b). Replacing t by $t_{N_t} + t_0 - t$, the definitions of the inflow-outflow boundaries, boundary norm and DG norm remain the same. In addition, the dual problem has a unique solution and other results obtained for the original problem can be translated to this case, such as the orthogonality relation.

Using the functions ρ and θ as in (4.3) and the orthogonality relation (4.2), we obtain

$$a(\theta, v) = -a(\rho, v), \ \forall v \in V_h. \tag{5.3}$$

Take now $w = \theta$ in the dual problem (5.1) and use (5.3) to obtain

$$\sum_{\mathcal{K} \in \mathcal{T}_h^{N_t-1}} (\phi, \theta_-)_{K(t_{N_t})} = a_a(\theta, z_h) + a_d(\theta, z_h) \leq |a_a(\rho, z_h)| + |a_d(\rho, z_h)|. \tag{5.4}$$

We try to estimate each term separately. First, we compute the estimate for the bilinear form $a_a(\rho, zh)$. Using (3.14), we can write a_a as

$$\begin{aligned} a_a(\rho, z_h) = &- \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \rho \frac{\partial z_h}{\partial t} \, \mathrm{d}\mathcal{K} - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} b\rho \cdot \overline{\nabla}_h z_h \, \mathrm{d}\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} c\rho z_h \, \mathrm{d}\mathcal{K} \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} B\cdot n\rho^- z_h^- \, \mathrm{d}\partial\mathcal{K} \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} B\cdot n\rho^+ (z_h^- - z_h^+) \, \mathrm{d}\partial\mathcal{K}. \end{aligned} \tag{5.5}$$

Since $\frac{\partial z_h}{\partial t} \in V_h$, we can use the orthogonality relation to drop the term $\int_{\mathcal{K}} \rho \frac{\partial z_h}{\partial t} \, \mathrm{d}\mathcal{K}$ from formulation. Then, using Schwarz' and arithmetic-geometric mean inequalities, we have the estimate for a_a as follows

$$\begin{aligned} |a_a(\rho, z_h)| \leq &\sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{-1/2}b\|_{L^\infty(\mathcal{K})} \|\rho\|_{L^2(\mathcal{K})} \|D^{1/2}\overline{\nabla}_h z_h\|_{L^2(\mathcal{K})} \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h} \|c\|_{L^\infty(\mathcal{K})} \|\rho\|_{L^2(\mathcal{K})} \|z_h\|_{L^2(\mathcal{K})} \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h} \|\rho^-\|_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} \|z_h^-\|_{\partial_+\mathcal{K}\cap\Gamma_{+,D,M}} \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h} \|\rho^+\|_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} \|z_h^- - z_h^+\|_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} \\ \leq\ & b_D |||\rho|||_{DG} |||z_h|||_{DG} + \bar{c} |||\rho|||_{DG} |||z_h|||_{DG} \\ &+ 2|||\rho|||_{DG} |||z_h|||_{DG} + \sqrt{2} \Big(\sum_{\mathcal{K}\in\mathcal{T}_h} \|\rho^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}} \Big)^{1/2} |||z_h|||_{DG}, \\ \leq\ & \Big(C_a |||\rho|||_{DG} + \sqrt{2} \|\rho\|_\star \Big) |||z_h|||_{DG}, \end{aligned} \tag{5.6}$$

with $b_D = \max_{\mathcal{K}\in\mathcal{T}_h} \|D^{-1/2}b\|_{L^\infty(\mathcal{K})}$, $\bar{c} = \max_{\mathcal{K}\in\mathcal{T}_h} \|c\|_{L^\infty(\mathcal{K})}$, and $C_a = b_D + \bar{c} + 2$. Next, we compute the estimate for the bilinear form a_d. Using the local lifting operator r_S and the symmetric properties of matrix D, the bilinear form a_d can be written as

$$\begin{aligned} a_d(\rho, z_h) = &\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} D^{1/2}\overline{\nabla}_h \rho \cdot D^{1/2}\overline{\nabla}_h z_h v \, \mathrm{d}\mathcal{K} \\ &+ \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} D^{1/2}\bar{r}_S([\![\rho]\!]) \cdot D^{1/2}\overline{\nabla}_h z_h \, \mathrm{d}\mathcal{K} \\ &+ \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} D^{1/2}\overline{\nabla}_h \rho \cdot D^{1/2}\bar{r}_S([\![z_h]\!]) \, \mathrm{d}\mathcal{K} + \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_M^n} \int_S \alpha\rho^- z_h^- \, \mathrm{d}S \\ &+ \sum_{n=0}^{N_t-1} \sum_{S\in\mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_0 \int_{\mathcal{K}} D^{1/2}\bar{r}_S([\![\rho]\!]) \cdot D^{1/2}\bar{r}_S([\![z_h]\!]) \, \mathrm{d}\mathcal{K}. \end{aligned} \tag{5.7}$$

The estimate for bilinear form a_d is as follows

$$\begin{aligned}|a_d(\rho, z_h)| \leq & \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\overline{\nabla}_h\rho\|_{L^2(\mathcal{K})}\|D^{1/2}\overline{\nabla}_h z_h\|_{L^2(\mathcal{K})} \\ & + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_{ID}^n}\sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\bar{r}_S([\![\rho]\!])\|_{L^2(\mathcal{K})}\|D^{1/2}\overline{\nabla}_h z_h\|_{L^2(\mathcal{K})} \\ & + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_{ID}^n}\sum_{\mathcal{K}\in\mathcal{T}_h} \|D^{1/2}\overline{\nabla}_h\rho\|_{L^2(\mathcal{K})}\|D^{1/2}\bar{r}_S([\![z_h]\!])\|_{L^2(\mathcal{K})} \\ & + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_M^n} \|\sqrt{\alpha}\rho^-\|_{L^2(S)}\|\sqrt{\alpha}z_h^-\|_{L^2(S)} \\ & + \sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_{ID}^n}\sum_{\mathcal{K}\in\mathcal{T}_h} \eta_0\|D^{1/2}\bar{r}_S([\![\rho]\!])\|_{L^2(\mathcal{K})}\|D^{1/2}\bar{r}_S([\![z_h]\!])\|_{L^2(\mathcal{K})} \\ \leq & C_d|||\rho|||_{DG}|||z_h|||_{DG}, \end{aligned} \tag{5.8}$$

with $C_d = 4 + \max_{\mathcal{K}\in\mathcal{T}_h}\eta_{\mathcal{K}}$. Collecting all the terms together, then we have

$$\sum_{\mathcal{K}\in\mathcal{T}_h^{N_t-1}} (\phi, \theta^-)_{K(t_{N_t})} \leq C_e|||\rho|||_{DG}|||z_h|||_{DG} + \sqrt{2}\|\rho\|_\star|||z_h|||_{DG}, \tag{5.9}$$

with $C_e = C_a + C_d$.

Lemma 9 *The DG norm* (3.1) *is bounded by known data*

$$\alpha_s|||z_h|||_{DG}^2 \leq \sum_{\mathcal{K}\in\mathcal{T}_h^{N_t-1}} (\phi, \phi)_{K(t_{N_t})},$$

with $0 < \alpha_s = \min(\frac{1}{2}, \bar{c}_0, 1-\epsilon, \eta_0 - \frac{N_f}{\epsilon})$, *for* $\epsilon \in (\frac{N_f}{\eta_0}, 1)$, *and* $\bar{c}_0 = \inf_{x\in\mathcal{E}} c_0(x) > 0$. *If* $\bar{c}_0 \leq 0$, *then we use the Garding inequality.*

Proof. Substituting z_h for v in (5.1), we obtain:

$$\begin{aligned} a^\star(z_h, z_h) = \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial_+\mathcal{K}\cap\Gamma_+} B\cdot n\phi z_h^- \,\mathrm{d}\partial\mathcal{K} &\leq \sum_{\mathcal{K}\in\mathcal{T}_h}\|\phi\|_{\partial_+\mathcal{K}\cap\Gamma_+}\|z_h^-\|_{\partial_+\mathcal{K}\cap\Gamma_+} \\ &\leq \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{1}{2\alpha_1}\|\phi\|^2_{\partial_+\mathcal{K}\cap\Gamma_+} + \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{1}{2}\alpha_1\|z_h^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D}}. \end{aligned} \tag{5.10}$$

Since Lemma 2 also applies for the backward problems, we can use (3.22) as follows:

$$\begin{aligned} a^\star(z_h, z_h) \geq & \left(1-\epsilon\right)\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\overline{\nabla}_h z_h\|^2_{L^2(\mathcal{K})} + \left(\eta_0 - \frac{N_f}{\epsilon}\right)\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}_{ID}^n}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\bar{r}_S([\![z_h]\!])\|^2_{L^2(\mathcal{K})} \\ & + \bar{c}_0\sum_{\mathcal{K}\in\mathcal{T}_h}\|z_h\|^2_{L^2(\mathcal{K})} + \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{1}{2}\|z_h^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D}} + \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{1}{2}\|z_h^-\|^2_{\partial_-\mathcal{K}\cap\Gamma_{-,D}} \\ & + \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{1}{2}\|z_h^- - z_h^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}}. \end{aligned} \tag{5.11}$$

Combining (5.10) and (5.11) and choosing $\alpha_1 = \frac{1}{2}$, we then obtain:

$$\begin{aligned}
&\big(1-\epsilon\big)\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\overline{\nabla}_h z_h\|^2_{L^2(\mathcal{K})} + \big(\eta_0 - \frac{N_f}{\epsilon}\big)\sum_{n=0}^{N_t-1}\sum_{S\in\mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^{1/2}\bar{r}_S([\![z_h]\!])\|^2_{L^2(\mathcal{K})}\\
&+\bar{c}_0\sum_{\mathcal{K}\in\mathcal{T}_h}\|z_h\|^2_{L^2(\mathcal{K})} + \frac{1}{4}\sum_{\mathcal{K}\in\mathcal{T}_h}\|z_h^-\|^2_{\partial_+\mathcal{K}\cap\Gamma_{+,D}} + \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\|z_h^-\|^2_{\partial_-\mathcal{K}\cap\Gamma_{-,D}} + \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\|z_h^- - z_h^+\|^2_{\partial_-\mathcal{K}\setminus\Gamma_{-,D}}\\
&\le \sum_{\mathcal{K}\in\mathcal{T}_h}\|\phi\|^2_{\partial_-\mathcal{K}\cap\Gamma_-} = \sum_{\mathcal{K}\in\mathcal{T}_h^{N_t-1}}(\phi,\phi)_{K(t_{N_t})}. \qquad (5.12)
\end{aligned}$$

Choosing $\alpha_s = \min(\frac{1}{2}, \bar{c}_0, 1-\epsilon, \eta_0 - \frac{N_f}{\epsilon})$ for $\epsilon \in (\frac{N_f}{\eta_0}, 1)$ completes the proof. □

Using Lemma 9, the estimate (5.9) can be written further as

$$\sum_{\mathcal{K}\in\mathcal{T}_h^{N_t-1}}(\phi,\theta^-)_{K(t_{N_t})} \le \Big(\frac{C_e}{\alpha_s}|||\rho|||_{DG} + \frac{2}{\alpha_s}\|\rho\|_\star\Big)\Big(\sum_{\mathcal{K}\in\mathcal{T}_h^{N_t-1}}(\phi,\phi)_{K(t_{N_t})}\Big)^{1/2}. \qquad (5.13)$$

Defining

$$\|\theta^-\|_{K(t_{N_t})} = \frac{\sup_{0\neq\phi\in L^2(\mathcal{E})}\sum_{\mathcal{K}\in\mathcal{T}_h^{N_t-1}}(\phi,\theta^-)_{K(t_{N_t})}}{\Big(\sum_{\mathcal{K}\in\mathcal{T}_h^{N_t-1}}(\phi,\phi)_{K(t_{N_t})}\Big)^{1/2}}, \qquad (5.14)$$

we then have

$$\|\theta^-\|_{K(t_{N_t})} \le \frac{C_e}{\alpha_s}|||\rho|||_{DG} + \frac{2}{\alpha_s}\|\rho\|_\star. \qquad (5.15)$$

Using the *hp* estimates for ρ as in Lemma 8, we obtain the following bound.

Theorem 3 *Let $\mathcal{E}\subset\mathbb{R}^{d+1}$ be a bounded polyhedral space-time domain and let $\mathcal{T}_h = \{\mathcal{K}\}$ be a shape-regular subdivision into space-time element $\mathcal{K}$ in $\mathbb{R}^{d+1}$ of diameter $h_\mathcal{K}$. Note that the diameter $h_\mathcal{K}$ now includes the time interval $\Delta_n t$. Further let $u|_\mathcal{K} \in H^{k_\mathcal{K}}(\mathcal{K}), k_\mathcal{K} \ge 0$. Using* (5.15) *and Lemma 8, then the following error bound holds:*

$$\|\theta^-\|_{K(t_{N_t})} \le \mathcal{C}\sum_{\mathcal{K}\in\mathcal{T}_h}\Big(\bar{D}_\mathcal{K}\frac{h_\mathcal{K}^{2s_\mathcal{K}-2}}{p_\mathcal{K}^{2k_\mathcal{K}-3}} + (\bar{\alpha}_\mathcal{K}+\bar{\beta}_\mathcal{K})\frac{h_\mathcal{K}^{2s_\mathcal{K}-1}}{p_\mathcal{K}^{2k_\mathcal{K}-1}} + \frac{h_\mathcal{K}^{2s_\mathcal{K}}}{p_\mathcal{K}^{2k_\mathcal{K}}}\Big)\|u\|^2_{H^{k_\mathcal{K}}(\mathcal{K})} \qquad (5.16)$$

for any integer $s_\mathcal{K}, 1 \le s_\mathcal{K} \le \min(p_\mathcal{K}+1, k_\mathcal{K})$, and $p_\mathcal{K} \ge 0$ the local polynomail order on element $\mathcal{K}$, with $\bar{\alpha}_\mathcal{K}, \bar{\beta}_\mathcal{K}$, and $\bar{D}_\mathcal{K}$ already defined in Theorem 2. The constant $\mathcal{C}$ is a positive value that depends on the dimension $d+1$, parameter α_s (defined in Lemma 9) and the shape regularity of $\mathcal{T}_h$.

6 Concluding Remarks

In this report we analyzed the new space-time DG method for the advection- diffusion-reaction equation. We proved that the method is consistent, coercive, stable, and gives a unique solution. We also proved the error estimate and *hp* convergence of the method.

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