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THE ALGEBRAIC STRUCTURE OF LAX EQUATIONS FOR INFINITE MATRICES

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Abstract. In this paper we discuss the algebraic structure of the tower of differential difference equations that one can associate with any commutative subalgebra of $M_k(\mathbb{C})$. These equations can be formulated conveniently in so-called Lax equations for infinite upper- resp. lowertriangular matrices and they are shown in a purely algebraic way to be equivalent with zero curvature equations for a collection of finite band matrices. The uppertriangular and lowertriangular systems corresponding to the same algebra are shown to be compatible. Finally the linearizations of the aforementioned systems are treated, which form the basis of the construction of solutions of these hierarchies. As such this work is an extension of that of Ueno and Takasaki and furnishes a complete algebraic context for it.

1. Introduction

Recall, see [Toda], that the infinite Toda-chain consists of an infinite number of particles on a straight line that are labeled by $\mathbb{Z}$ and whose equations of motion in dimensionless form are described by

$$\frac{dq_n}{dt} = p_n \text{ and } \frac{dp_n}{dt} = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1}-q_n)}, \quad n \in \mathbb{Z}. \tag{1}$$

Here q_n is the displacement of the n-th particle. These equations can be rewritten as an equality between infinite matrices by putting

$$a_n := \frac{1}{2} e^{-(q_n - q_{n-1})} \text{ and } b_n := \frac{1}{2} p_n.$$

The equations (1) get then the following form

$$\frac{da_n}{dt} = a_n(b_n - b_{n-1}) \text{ and } \frac{db_n}{dt} = 2(a_{n-1}^2 - a_n^2), \quad n \in \mathbb{Z}. \tag{2}$$

If we introduce the $\mathbb{Z} \times \mathbb{Z}$-matrices L and B by

$$L = \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & b_{n-1} & a_n & 0 & \ddots \\ \ddots & a_n & b_n & a_{n+1} & \ddots \\ & 0 & a_{n+1} & b_{n+1} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix} \text{ and } B = \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & 0 & -a_n & 0 & \ddots \\ \ddots & a_n & 0 & -a_{n+1} & \ddots \\ & 0 & a_{n+1} & 0 & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix},$$

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then a direct computation shows that the equations (2) amount to the matrix equation

$$\frac{dL}{dt} = BL - LB = [B, L]. \tag{3}$$

This is an example of a so-called *Lax equation*, because it suggests that the matrix L is obtained by conjugating a matrix that does not depend of t with a t-dependent one, its analogue in the KdV-setting being found by P.Lax, see [Lax]. Several variations on the above situation have been considered, see e.g. [Flaschka] , the most general setting being that of [UT]. These systems of equations play a role in various parts of mathematics, like random matrices and orthogonal polynomials, see [AM] and [HH], but also in theoretical physics, see [GMMMO] and [PM]. In this paper we want to stress the algebraic character of this type of Lax equations and we will illustrate this at the hand of a more general class of $\mathbb{Z}\times\mathbb{Z}$-matrices L than the one considered by [UT]. The first section is devoted to algebraic prerequisites of $\mathbb{Z}\times\mathbb{Z}$-matrices , like decompositions. Next we discuss various Lax equations for lower- and uppertriangular matrices and it will be shown that it is natural to group certain equations together, since they correspond to commuting flows, and this procedure leads to infinite towers of nonlinear equations, the so-called Toda-type hierarchies. These equations can equivalently be formulated in the so-called zero curvature form. This is treated in the third section. Finally we discuss the linearizations associated with these systems. They show you the way how to construct solutions of these hierarchies. This is the topic of the last section.

2. The space $M_{\mathbb{Z}}(R)$

Since Lax equations for $\mathbb{Z}\times\mathbb{Z}$-matrices are the central topic of this paper, we will first discuss the necessary ingredients from that space. Let R be a commutative ring. Then we write $M_{\mathbb{Z}}(R)$ for the R-module of $\mathbb{Z}\times\mathbb{Z}$-matrices with coefficients from R. We use the ordering of columns and rows that is compatible with the finite dimensional case, i.e. any matrix $A = (\alpha_{ij})$ is denoted by

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \alpha_{n-1\,n-1} & \alpha_{n-1\,n} & \alpha_{n-1\,n+1} & \ddots \\ \ddots & \alpha_{n\,n-1} & \alpha_{n\,n} & \alpha_{n\,n+1} & \ddots \\ \ddots & \alpha_{n+1\,n-1} & \alpha_{n+1\,n} & \alpha_{n+1\,n+1} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

There are a number of special elements in $M_{\mathbb{Z}}(R)$ that we will frequently use. First of all, there is the basic matrix $E_{(i,j)}$, i and $j \in \mathbb{Z}$, given by

$$(E_{(i,j)})_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}. \tag{4}$$

Thus one can describe every $A = (A_{ij}) \in M_{\mathbb{Z}}(R)$ as a formal linear combination of the basic matrices

$$A = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} A_{ij}E_{(i,j)}. \tag{5}$$

If confusion might occur, we also denote, without further mentioning, the matrixcoefficient A_{ij} of A as $A_{(i,j)}$. The basic matrices also serve to introduce the partial multiplication on $M_{\mathbb{Z}}(R)$. In general, if $A = (A_{ij)})$ and $B = (B_{ij})$ belong to $M_{\mathbb{Z}}(R)$, then the product AB, defined by

$$(AB)_{ik} = \sum_{j \in \mathbb{Z}} A_{ij} B_{jk}, \tag{6}$$

is not well-defined. However, it is if A or B belongs to one of the classes of matrices that we introduce in a moment. Any map $\Delta; R \to R$ extends in a natural way to a map $\Delta : M_{\mathbb{Z}}(R) \to M_{\mathbb{Z}}(R)$ by putting $\Delta(A)_{ij} = \Delta(\alpha_{ij})$, if $A = (\alpha_{ij})$. We say that Δ has the *derivation property* if it is linear and it satisfies

$$\Delta(AB) = \Delta(A)B + A\Delta(B), \tag{7}$$

whenever all the products in this expression are well-defined.

A dominant role is played by the shift matrix Λ given by $\Lambda = \sum_{i \in \mathbb{Z}} E_{(i-1,i)}$. Recall that the KP-hierarchy is an infinite tower of nonlinear equations between operators in $\partial := \frac{d}{dx}$. In the present context where a mixture of differential and difference equations is discussed, this role will be taken over by the operator Λ. With every collection $\{d(ks) | s \in \mathbb{Z}\}$ of matrices in $M_k(R)$ we associate the diagonal of k-blocks $\text{diag}(d(ks))$ in $M_{\mathbb{Z}}(R)$ given by

$$\text{diag}(d(ks)) := \sum_{s \in \mathbb{Z}} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} d(ks)_{\alpha\beta} E_{(s+\alpha-1, s+\beta-1)}. \tag{8}$$

Its matrix looks as follows

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & d(kn-k) & 0 & 0 & \ddots \\ \ddots & 0 & d(kn) & 0 & \ddots \\ \ddots & 0 & 0 & d(kn+k) & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{9}$$

For each $k \geq 1$ we denote the ring of k-block diagonal matrices in $M_{\mathbb{Z}}(R)$ by

$$\mathcal{D}_k(R) = \{d = \text{diag}(d(ks)) | d(ks) \in M_k(R) \text{ for all } s \in \mathbb{Z}\}.$$

We have a ringhomomorphism i_k from $M_k(R)$ into $\mathcal{D}_k(R)$ by taking for an $A \in M_k(R)$ all diagonal blocks of $i_k(A)$ equal to A. In particular every $\mathcal{D}_k(R)$ becomes a $M_k(R)$-algebra in this way.

The elements Λ^{km}, $m \in \mathbb{Z}$ act on $\mathcal{D}_k(R)$ according to

$$\Lambda^{km} diag(d(ks)) \Lambda^{-km} = diag(d(ks + km)). \tag{10}$$

Therefore the image of i_k consists of all matrices in $\mathcal{D}_k(R)$ that commute with Λ^k. Each matrix in $M_{\mathbb{Z}}(R)$ can be divided into so-called k-block diagonals. For, if $A = (A_{ij}) \in M_{\mathbb{Z}}(R)$, then we put

Definition 1. The *j-th k-block diagonal* of any matrix A, $j \in \mathbb{Z}$, is the matrix

$$\sum_{i \in \mathbb{Z}} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} A_{(ki-kj+\alpha-1, ki+\beta-1)} E_{(ki-kj+\alpha-1, ki+\beta-1}.$$

From equation 10 it is clear that the j-th k-block diagonal of a $\mathbb{Z} \times \mathbb{Z}$-matrix A can uniquely be written in the form $diag(d(ks))\Lambda^{kj}$ or $\Lambda^{kj} diag(c(ks))$ with $diag(d(ks))$ and $diag(c(ks)) \in \mathcal{D}_k(R)$. Thus each $A = (A_{(i,j)}) \in M_{\mathbb{Z}}(R)$ can uniquely be written as

$$A = \sum_{j \in \mathbb{Z}} d_j \Lambda^{kj} \text{ or } A = \sum_{j \in \mathbb{Z}} \Lambda^{kj} c_j, \tag{11}$$

with d_j and c_j in $\mathcal{D}_k(R)$. In particular any matrix that commutes with Λ^k has the form (11) with d_j and c_j in the image of i_k.

To the first decomposition in (11) we link two notations: if $A = \sum_{j \in \mathbb{Z}} d_j \Lambda^j$ as in (11) then we write

$$A_+(k) = \sum_{j \geq 0} d_j \Lambda^{kj} \text{ and } A_-(k) = \sum_{j < 0} d_j \Lambda^{kj}. \tag{12}$$

Inside $M_{\mathbb{Z}}(R)$ we consider two subspaces that form a ring w.r.t. the product (6).

Definition 2. An element A in $M_{\mathbb{Z}}(R)$ is called *upper k-block triangular of level m*, if it can be written as

$$A = \sum_{j \geq m} d_j \Lambda^{kj}, \text{ with } d_j \in \mathcal{D}_k(R).$$

We call m the *order* of A in Λ^k, if d_m is nonzero. The collection of all these elements we denote by UT_m, $UT_m(R)$ or $UT_m^{(k)}(R)$, depending, if we have to stress where the coefficients come from, what the size of the blocks along the diagonal is or both. Likewise we use the notations

$$UT(R) := \bigcup_{k \in \mathbb{Z}} UT_k =: UT$$

for the set of all uppertriangular matrices.

One verifies directly that UT with the product (6) forms an R-algebra. All commutative R-subalgebras of UT that contain the element Λ^k have the following form: choose any commutative R-subalgebra C of $M_k(R)$, then the required algebra consists of the

$$U(C) := \{\sum_{i \geq N} i_k(c_i)\Lambda^{ki} \mid \text{with } c_i \in C \text{ for all } i\}. \tag{13}$$

Likewise one introduces the opposite class of matrices

Definition 3. An element A in $M_{\mathbb{Z}}(R)$is called *lower k-block triangular of level m*, if it can be written as

$$A = \sum_{j \leq m} d_j \Lambda^{kj}, \text{ with } d_j \in \mathcal{D}(R).$$

Like for UT we call m the *order* of A in Λ^k, if d_m is nonzero. The collection of all these elements we denote again by LT_m, $LT_m(R)$ or $LT_m^{(k)}(R)$, depending of the dependence that has to be stressed. Similarly we use the notations

$$LT(R) := \bigcup_{k \in \mathbb{Z}} LT_k =: LT$$

for the set of all lowertriangular matrices.

Again one verifies easily that LT with the product (6) forms an algebra over R. The commutative R-subalgebras of UT containing the element Λ^k can be described as follows: let C as above be any commutative R-subalgebra of $M_k(R)$, then

$$L(C) := \{\sum_{i\leq N} i_k(c_i)\Lambda^{ki} \mid \text{with } c_i \in C \text{ for all } i\} \tag{14}$$

is the required algebra. If C is maximal commutative inside $M_k(R)$, then the same holds for $L(C)$.

Note that, if $U \in UT$ and $V \in LT$ have the form respectively

$$U = \sum_{i\geq 0} u_i\Lambda^{ik} \text{ and } V = \sum_{i\leq 0} v_i\Lambda^{ik},$$

with u_0 and v_0 invertible in $\mathcal{D}_k(R)$, then the elements U and V are invertible and the diagonal k-block components of their inverses can be computed recursively.

For each $k \geq 1$ there is a ring isomorphism between UT and LT that enables you to translate properties of the one directly to the other. Consider namely the element $w_k = \sum_{\alpha=1}^{k}\sum_{j\in\mathbb{Z}} E_{(-jk+\alpha-1, jk+\alpha-1)}$ in $M_{\mathbb{Z}}(R)$. Then this element of order two acts as follows

$$w_k \sum_{j\in\mathbb{Z}} diag(d(kj))w_k = \sum_{j\in\mathbb{Z}} diag(d(-kj)) \text{ and } w_k\Lambda^k w_k = \Lambda^{-k}. \tag{15}$$

Hence conjugating with w_k gives a ring isomorphism between UT and LT that transforms the k-block decompositions in UT to those in LT and vice versa. Its usefulness can be seen from the proof of the following proposition, which states that inside LT or UT every operator whose leading k-block diagonal part is invertible, can be obtained by "dressing" a suitable power of Λ . As it is well-known this does not have to hold in the ring of pseudodifferential operators in ∂ with coefficients from $M_n(R)$.

Proposition 1. *a) For any nonzero $r \in \mathbb{Z}$, let $\mathcal{L}$ in LT_k have the k-block form $\mathcal{L} = \sum_{j\leq r}\alpha_j\Lambda^{jk}$ with α_r in $\mathcal{D}_k(R)$ invertible. Then there exists an element U in LT of the k-block form $U = \sum_{i\leq 0}\beta_i\Lambda^{ki}$ with β_0 in $\mathcal{D}_k(R)$ invertible such that*

$$\mathcal{L} = U\Lambda^{rk}U^{-1}.$$

b) For any nonzero $s \in \mathbb{Z}$, let $\mathcal{M}$ in UT_k have the k-block form $\mathcal{M} = \sum_{j\geq s}\gamma_j\Lambda^{kj}$ with γ_s invertible. Then there exists an element V in UT of the k-block form $V = \sum_{i\geq 0}\delta_i\Lambda^{ik}$, where δ_0 in $\mathcal{D}_k(R)$ is invertible, satisfying

$$\mathcal{M} = V\Lambda^{sk}V^{-1}.$$

Proof. Thanks to the isomorphism between the rings LT and UT described in (15), it is sufficient to prove part a) of the proposition. From the fact that the leading coefficient α_r is invertible, one deduces that $\mathcal{L}$ is invertible and that $\mathcal{L}^{-1}$ belongs to LT_{-r} with $\Lambda^{-r}\alpha_r^{-1}\Lambda^r$ as its leading coefficient. Therefore we can restrict ourselves to the case $r > 0$. The matrix U we are looking for, should satisfy the equation $\mathcal{L}U = U\Lambda^{kr}$. This amounts to the following equations for the coefficients β_j of U

$$\beta_t = \sum_{i=t}^{0}\alpha_{t+r-i}\Lambda^{k(t+r-i)}\beta_i\Lambda^{-k(t+r-i)} \tag{16}$$

for all $t \leq 0$. If each $\beta_i = diag(b_i(kj))$ and $\alpha_i = diag(a_i(kj))$, then this equation for $t = 0$ says for all $j \in \mathbb{Z}$

$$diag(b_0(kj)) = diag(a_r(kj))diag(b_0(k(j+r))). \tag{17}$$

Thus one sees that in order to solve this equation, one can pick arbitrary invertible elements $b_0(ks), 0 \leq s < r$, and then the equation (17) determines the element $\beta_0 \in \mathcal{D}_k(R)$ uniquely. Now that we have shown how to find a solution for β_0, we may assume that we have found all the β_l, with $l > s$. The next coefficient β_s of the operator U is then again fully determined by equation (16), once one has chosen the $b_s(kv), 0 \leq v < r$. In this way one finds an operator U that satisfies the equations. All other solutions have the form UU_0, with $U_0 = \sum_{i\geq 0} u_i\Lambda^{ki}$, with all u_i in the image of the map i_k and u_0 invertible. This completes the proof of the proposition. □

Remark 1. Following the terminology used in the case of differential operators, we say that the operators $\mathcal{L}$ and $\mathcal{M}$ from the proposition are obtained by dressing the operators Λ^{rk} resp. Λ^{sk}.

Remark 2. The example $\alpha_0 = Id$ and $\mathcal{L} = \sum_{j\leq 0} \alpha_j\Lambda^j$ with $\mathcal{L} - Id$ nonzero, shows that proposition (1) does not have to hold for the case $r = 0$.

Remark 3. The foregoing proposition reduces the structure of $Z_{LT}(\mathcal{L})$, the centralizer of $\mathcal{L}$ in LT, resp. $Z_{UT}(\mathcal{M})$, the centralizer of $\mathcal{M}$ in UT to that of Λ^{rk} in LT resp. UT. In particular all commutative subalgebras of LT that contain an element with an invertible leading coefficient in it, are conjugated with the ones described in (14) and similarly those in UT can be obtained by dressing the subalgebras from (13).

Remark 4. Note that, if L and M are as in the proposition, then L has also a $|rk|$-block decomposition whose leading term is invertible in $\mathcal{D}_{|kr|}(R)$ and likewise M has a $|ks|$-block decomposition with an invertible leading term in $\mathcal{D}_{|ks}(R)$. Therefore, it is enough to consider the cases $r = \pm 1$ and $s = \pm 1$.

3. Lax equations in LT and UT

We want to discuss here the algebraic structure that lies at the basis of equations like (3). From that equation one can see already that something special is going on. For, the order of $\frac{dL}{dt}$ in Λ will be less or equal to that of L and the order of $[B, L]$ in Λ is less or equal to order(L)+order(B).

Thus one arrives naturally at the question, given a matrix $\mathcal{A} = \sum_{j\leq l} \alpha_j\Lambda^{kj}$ in LT, with $l > 0$, how to find matrices in LT like B of positive order in Λ^k, such that its commutator with $\mathcal{A}$ has order in Λ^k smaller or equal to that of $\mathcal{A}$. Similarly one can consider a matrix $\mathcal{B} = \sum_{j\geq -l} \beta_j\Lambda^j$ in UT, with $l > 0$, and ask if there is a systematic way to come up with operators C in UT of order ≤ 0 in Λ^k such that $[C, \mathcal{B}]$ has order in Λ^k bigger or equal to that of $\mathcal{B}$. For such operators it makes sense to consider Lax equations for $\mathcal{A}$ and $\mathcal{B}$, analogous to (3).

Let P be an element of $Z_{LT}(\mathcal{A})$ of positive order in Λ^k. Since $[P_+ + P_-, \mathcal{A}] = 0$, we see that

$$[P_+, \mathcal{A}] = -[P_-, \mathcal{A}].$$

Hence $[P_+, \mathcal{A}]$ has order $\leq$ order($\mathcal{A}$)-1. Likewise, if Q is an element in UT that commutes with $\mathcal{B}$ and has negative order in Λ^k, then we have that

$$[Q_-, \mathcal{B}] = -[Q_+, \mathcal{B}]$$

and the right hand side clearly has order in $\Lambda^k \geq$ order($\mathcal{B}$). For such matrices P and Q, it makes sense to look for suitable derivations ∂_P and ∂_Q of R such that the equations

$$\partial_P(\mathcal{A}) = [P_+, \mathcal{A}] \text{ and} \tag{18}$$

$$\partial_Q(\mathcal{B}) = [Q_-, \mathcal{B}] \tag{19}$$

hold. Here the action of ∂_P and ∂_Q on elements of $M_{\mathbb{Z}}(R)$ is defined coefficientwise. If a matrix $\mathcal{A}$ satisfies the equation (18), then this implies for its leading coefficient $\alpha_l = diag(a_l(ks))$ that all its matrixentries are constant for ∂_P, i.e. for all α and β and all $s \in \mathbb{Z}$, $\partial_P((a_k(ks))_{\alpha\beta}) = 0$.

Remark 5. By conjugating with the element w_k from the foregoing subsection one can translate equations in UT to LT and vice versa. If you do so for the equation (19), then it corresponds to splitting an element $P \in Z_{LT}(\mathcal{A})$ of positive order in Λ^k not as $P = P_+ + P_-$, but as $P = P_{>0} + P_{\leq 0}$, where $P_{>0}$ contains the terms in P that are strictly positive powers in Λ^k. Because of a coupling that will be laid later on, we prefer to keep one system in LT and the other in UT.

The equations (18) can be considered for various choices of $\mathcal{A}$ and P. First of all we take for $\mathcal{A}$ a number of generators of a commutative algebra in LT. More precisely, we consider in LT the elements

$$\mathcal{L} := \sum_{i \leq 1} l_i \Lambda^{ki} \text{ with } l_1 \text{ invertible and } U_\alpha = \sum_{i \leq 0} u_{i,\alpha} \Lambda^{ki}, \tag{20}$$

where the U_α are a basis over R of the space of elements of order zero in this algebra modulo the space of elements of negative order in this algebra. It is convenient to write U_0 for the identity element in $LT(R)$. Since we assume the algebra to be commutative, there holds

$$[\mathcal{L}, U_\alpha] = 0 \text{ and } [U_\alpha, U_\beta] = 0 \text{ for all } \alpha \text{ and } \beta. \tag{21}$$

From proposition (6) we know that there is an invertible element $w_0 \in \mathcal{D}_k(R)$ such that $l_1 = w_0 \Lambda^k w_0^{-1} \Lambda^{-k}$. Thus we see that $u_{0,\alpha} = w_0 i_k(E_\alpha) w_0^{-1}$, where the E_α are a basis over R of a commutative R-subalgebra of $M_k(R)$. For the matrices P we choose the $P_{i\alpha} := \mathcal{L}^i U_\alpha$ for all α and all $i \geq 0$. The nonlinear differential equations we want $\mathcal{L}$ and the U_β to satisfy are

$$\partial_{P_{i\alpha}}(\mathcal{L}) = [(P_{i\alpha})_+, \mathcal{L}] \text{ and } \partial_{P_{i\alpha}}(U_\beta) = [(P_{i\alpha})_+, U_\beta] \tag{22}$$

Recall that these relations imply that the leading coefficients l_1 and $u_{0,\alpha}$ have entries that are constants for all the derivations $\partial_{P_{i\alpha}}$. The element w_0 such that $l_1 = w_0 \Lambda^k w_0^{-1} \Lambda^{-k}$ can then also be chosen constant. By conjugating with this w_0^{-1} one can reduce the solution of the system (22) to the case $l_1 = Id$ and $u_{0,\alpha} = i_k(E_\alpha)$. This we will do from now on.

We make similar choices for the matrices $\mathcal{B}$ and Q in the equations (19). Now we take for $\mathcal{B}$ a number of generators of a commutative algebra in UT. More precisely, we consider in LT the elements

$$\mathcal{M} := \sum_{i\geq -1} m_i\Lambda^{ki} \text{ with } m_{-1} \text{ invertible and } V_\alpha = \sum_{i\geq 0} v_{i,\alpha}\Lambda^{ki}, \tag{23}$$

where the V_α are a basis over R of the space of elements of order zero in this algebra modulo the space of elements of positive order in this algebra. The identity element in $UT(R)$ we denote again by V_0. Since the algebra they generate is commutative, there has to hold

$$[\mathcal{M}, V_\alpha] = 0 \text{ and } [V_\alpha, V_\beta] = 0 \text{ for all } \alpha \text{ and } \beta. \tag{24}$$

From proposition (6) we know that one can gauge $\mathcal{M}$ to $\mathcal{M} = V\Lambda^{-k}V^{-1}$. Hence the first equation in (24) implies that each $V_\alpha = V(i_k(F_\alpha) + \sum_{s=1}^{\infty} i_k(F_\alpha(s))\Lambda^{sk})V^{-1}$ and the second one gives that the F_α are a basis over R of a commutative R-subalgebra of $M_k(R)$. For the matrices Q that commute with $\mathcal{M}$ and the V_β we choose the $Q_{j\alpha} := \mathcal{M}^j V_\alpha = \sum_{s=-j}^{\infty} q_{j\alpha}(s)\Lambda^{ks}$, $j \geq 1$. The nonlinear differential equations we want $\mathcal{M}$ and the V_β to satisfy are

$$\partial_{Q_{j\alpha}}(\mathcal{M}) = [(Q_{j\alpha})_-, \mathcal{M}] \text{ and } \partial_{Q_{j\alpha}}(V_\beta) = [(Q_{j\alpha})_-, V_\beta] \tag{25}$$

This means for example for the leading coefficients of $\mathcal{M}$ and V_α that they have to satisfy the equations

$$\partial_{Q_{j\alpha}}(m_{-1}) = m_{-1}\Lambda^{-k}q_{j\alpha}(0)\Lambda^k - q_{j\alpha}(0)m_{-1} \text{ and } \partial_{Q_{j\alpha}}(v_{0,\alpha}) = -[q_{j\alpha}(0), v_{0,\alpha}]. \tag{26}$$

Next we give an algebraic interpretation of both the relations (22) and (25). Now there are three situations that we want to consider, namely each set of equations seperately and the combination of both sets. There is a minimal model in all three cases, where these sets of equations hold by definition. In this formal set-up one starts for the equations in LT with a commutative algebra h_L of $M_k(\mathbb{C})$ with basis $\{E_\alpha\}$ and for those in UT with a commutative algebra h_U of $M_k(\mathbb{C})$ with basis $\{F_\alpha\}$. The choice made in [UT] is taking the diagonal matrices for h_L and h_U. Inside h_L resp. h_U one has the relations

$$E_\alpha E_\beta = \sum_\gamma C^\gamma_{\alpha\beta}E_\gamma \text{ resp. } F_\alpha F_\beta = \sum_\gamma D^\gamma_{\alpha\beta}F_\gamma. \tag{27}$$

By taking h_L and h_U maximal commutative we may further assume that both h_L and h_U contain the identity matrix and thus we can write

$$Id = \sum_\alpha c_\alpha E_\alpha = \sum_\alpha d_\alpha F_\alpha, \text{ with } c_\alpha \text{ and } d_\alpha \in \mathbb{C}. \tag{28}$$

For the algebra h_L, one chooses the ring R equal to

$$R_L := \mathbb{C}[\tilde{l}_j(ks)_{\gamma\delta}, \tilde{u}_{i,\alpha}(kt)_{\rho\sigma}],$$

with all $\{\gamma,\ \delta,\ \rho,\ \sigma\}$ in $\{1,\ldots,k\}$; both $s,$ and $t \in \mathbb{Z}; j < 1, i < 0$. These indeterminates are the matrixcoefficients of the matrices occurring in the following operators in $M_{\mathbb{Z}}(R_L)$.

$$\tilde{\mathcal{L}} := Id + \sum_{j\leq 0} \tilde{l}_j\Lambda^{kj},\ \ \tilde{U}_\alpha := E_\alpha + \sum_{i<0} \tilde{u}_{i,\alpha}\Lambda^{ki}$$

For the equations in UT we take into account that the leading term of the operator $\mathcal{M}$ has to be invertible and choose the ring R equal to the localization

$$R_U := S^{-1}\mathbb{C}[\tilde{m}_n(kp)_{\epsilon\eta}, \tilde{v}_{m,\beta}(kt)_{\mu\nu}],$$

of the ring $\mathbb{C}[\tilde{m}_n(kp)_{\epsilon\eta}, \tilde{v}_{m,\beta}(kt)_{\mu\nu}]$ w.r.t. the multiplicative subset S generated by the determinants of the matrices $\tilde{m}_{-1}(kp), p \in \mathbb{Z}$. Here all the indices $\{\epsilon,\ \eta,\ \mu,\ \nu\}$ belong to $\{1, \dots, k\}$; both p and $q \in \mathbb{Z}; m, n \geq -1$. The indeterminates in the ring R_U are again the matrixcoefficients of the matrices from the following operators in $M_{\mathbb{Z}}(R_U)$

$$\tilde{\mathcal{M}} := \sum_{n \geq -1} \tilde{m}_i \Lambda^{ki} \text{ and } \tilde{V}_\alpha := w_0 F_\alpha w_0^{-1} + \sum_{i>0} \tilde{v}_{i,\alpha} \Lambda^{ki}.$$

Here $w_0 = diag(w_0(ks))$ is the gauge given by $w_0(0) = Id$ and

$$w_0(t) = m_{-1}(t) \dots m_{-1}(1) \text{ for } t \geq 1 \text{ and } w_0(t) = m_{-1}(t+1)^{-1} \dots m_{-1}(0)^{-1} \text{ for } t < 0.$$

In the situation that we consider that we consider both the equations in UT and LT we take for R the ring

$$R_{LU} := \mathbb{C}[\tilde{l}_j(ks)_{\gamma\delta}, \tilde{u}_{i,\alpha}(kt)_{\rho\sigma}, \tilde{m}_n(kp)_{\epsilon\eta}, \tilde{v}_{m,\beta}(kt)_{\mu\nu}],$$

where S is the same multiplicative subset as above and all the indices $\{\gamma,\ \delta,\ \rho,\ \sigma,\ \epsilon,\ \eta,\ \mu,\ \nu\}$ belong to $\{1, \dots, k\}$; all the $\{s, t, p, q\}$ belong to $\mathbb{Z}; j < 1, i < 0, m > 0, n \geq -1$. The operators $\tilde{\mathcal{L}}$, $\tilde{U}_\alpha$, $\tilde{\mathcal{M}}$ and $\tilde{V}_\beta$ have then their matrixcoefficients in the ring R_{LU}.

A priori these matrices do not commute so the first relation we sanction upon them is

$$[\tilde{\mathcal{M}}, \tilde{V}_\alpha] = [\tilde{\mathcal{L}}, \tilde{U}_\alpha] = 0 \text{ and } [\tilde{U}_\alpha, \tilde{U}_\beta] = [\tilde{V}_\alpha, \tilde{V}_\beta] = 0, \text{ for all } \alpha \text{ and } \beta. \tag{29}$$

One can also require, like it is done in [UT], that the deformations $\{\tilde{U}_\alpha\}$ and $\{\tilde{V}_\alpha\}$ of respectively the $\{E_\alpha\}$ and the $\{F_\alpha\}$ preserve the relations (27) and (28) inside the algebras h_L resp.h_U, i.e.

$$\tilde{U}_\alpha \tilde{U}_\beta = \sum_\gamma C^\gamma_{\alpha\beta} \tilde{U}_\gamma \text{ resp. } \tilde{V}_\alpha \tilde{V}_\beta = \sum_\gamma D^\gamma_{\alpha\beta} \tilde{V}_\gamma. \tag{30}$$

$$Id = \sum_\alpha c_\alpha \tilde{U}_\alpha = \sum_\alpha d_\alpha \tilde{V}_\alpha. \tag{31}$$

We do not include them in the considerations at this point since they are only an additional burden here. We come back to them after the discussion of the linearizations of the nonlinear equations, which are the central issue.

Note that every $\mathbb{C}$-linear derivation $\Delta \in \mathrm{Der}_{\mathbb{C}}(R_L)$ is completely determined by describing freely all the $\{\Delta(\tilde{l}_j(ks)_{\gamma\delta}),\ \Delta(\tilde{u}_{i,\alpha}(kt)_{\rho\sigma})\}$. Therefore we can define for each P in $LT(R_L)$ of degree ≥ 0 in Λ^k that commutes with $\tilde{\mathcal{L}}$ and the $\tilde{U}_\alpha$ a unique derivation $\tilde{\partial_P} :\ R_L \to R_L$ by the matrix equalities

$$\tilde{\partial_P}(\tilde{\mathcal{L}}) := [P_+, \tilde{\mathcal{L}}] \text{ and } \tilde{\partial_P}(\tilde{U}_\alpha) := [P_+, \tilde{U}_\alpha]. \tag{32}$$

Since $\mathrm{ad}(P_+)$ and $\mathrm{ad}(P)$ have the derivation property, the derivation $\tilde{\partial_P}$ factorizes over the relations (29). Thus we have built a context in which the equations of the form (18) hold by definition. What we want now is a realization of the equations (32), i.e. we look for a $\mathbb{C}$-algebra R equiped with a number of $\mathbb{C}$-linear derivations $\partial_P : R \to R$, with P as above, and a $\mathbb{C}$-algebra morphism $\lambda : R_L \to R$ such that for all relevant P there holds

$$\partial_P \circ \lambda = \lambda \circ \tilde{\partial_P}, \tag{33}$$

The morphism λ determines a $\mathbb{C}$-linear map from $M_{\mathbb{Z}}(R_L)$ to $M_{\mathbb{Z}}(R)$ and we write $\mathcal{L} := \lambda(\tilde{\mathcal{L}})$ and $U_\alpha := \lambda(\tilde{U}_\alpha)$. First of all we want that λ factorizes over the relevant relations from (29), i.e. they have to satisfy (21). Clearly, the relation (33) implies relations for the matrices $\mathcal{L}$ and U_α in $M_{\mathbb{Z}}(R)$. Namely they have to satisfy respectively

$$\partial_P(\mathcal{L}) = [\lambda(P_+), \mathcal{L}], \tag{34}$$

$$\partial_P(U_\alpha) = [\lambda(P_+), U_\alpha]. \tag{35}$$

The system of equations (34), (35) for all the $P = P_{i,\alpha}$ are called the *Lax equations* for the hierarchy associated with the algebra h_L or shortly the h_L-hierarchy. If the matrices $\mathcal{L}$ and U_α in $M_{\mathbb{Z}}(R)$ satisfy, besides these equations, also those in (21), then they are called a *solution* of this hierarchy. Note that there is at least one solution of these equations, namely $\mathcal{L} = \Lambda^k$ and $U_\alpha = E_\alpha$. It is called the *trivial solution* of the h_L-hierarchy.

For the hierarchy related to h_U one follows the same line. Every $\mathbb{C}$-linear derivation $\Delta \in \mathrm{Der}_{\mathbb{C}}(R_U)$ is completely determined by describing freely all the $\{\Delta(\tilde{m}_n(kp)_{\epsilon\eta}),\ \Delta(\tilde{v}_{m,\beta}(kt)_{\mu\nu})\}$. Therefore we can define for each Q in UT of degree < 0 in Λ^k that commutes with $\tilde{\mathcal{M}}$ and the $\tilde{V}_\alpha$ a unique derivation $\tilde{\partial_Q} : R_U \to R_U$ by the matrix equalities

$$\tilde{\partial_Q}(\tilde{\mathcal{M}}) := [Q_-, \tilde{\mathcal{M}}] \text{ and } \tilde{\partial_Q}(\tilde{V}_\alpha) := [Q_-, \tilde{V}_\alpha]. \tag{36}$$

For the same reason as above, also the derivation $\tilde{\partial_Q}$ factorizes over the relations (29). Thus we have built a context in which the equations of the form (19) hold by definition. What we want now is a realization of the equations (36), i.e. we look for a $\mathbb{C}$-algebra R equiped with a number of $\mathbb{C}$-linear derivations $\partial_Q : R \to R$, with Q as above, and a $\mathbb{C}$-algebra morphism $\mu : R_U \to R$ such that for all relevant Q respectively

$$\partial_Q \circ \mu = \mu \circ \tilde{\partial_Q}. \tag{37}$$

Again the morphism μ determines a $\mathbb{C}$-linear map from $M_{\mathbb{Z}}(R_U)$ to $M_{\mathbb{Z}}(R)$ and we write $\mathcal{M} := \mu(\tilde{\mathcal{M}})$ and $V_\alpha := \mu(\tilde{V}_\alpha)$. First of all we want that μ factorizes over the relevant relations in (29), i.e. they have to satisfy (24). Clearly, the properties (45) translates into relations for the matrices $\mathcal{M}$ and V_α in $M_{\mathbb{Z}}(R)$. Namely they have to satisfy respectively

$$\partial_Q(\mathcal{M}) = [\mu(Q_-), \mathcal{M}], \tag{38}$$

$$\partial_Q(V_\alpha) = [\lambda(Q_-), V_\alpha]. \tag{39}$$

The system of equations (38) and (39) for all the $Q = Q_{j,\beta}$ are called the *Lax equations* for the hierarchy associated with the algebra h_U or shortly the h_U-hierarchy. If the matrices $\mathcal{M}$ and V_α in $M_{\mathbb{Z}}(R)$ satisfy, besides these equations, also those in (21) and (24), then they are called a *solution* of this hierarchy. Also here we have a *trivial solution* of the h_U-hierarchy, namely $\mathcal{M} = \Lambda^{-k}$ and $V_\alpha = F_\alpha$.

Next we make a combination of the two foregoing systems. As before every $\mathbb{C}$-linear derivation $\Delta \in \mathrm{Der}_{\mathbb{C}}(R_{LU})$ is completely determined by prescribing freely all the elements $\{\Delta(\tilde{l}_j(ks)_{\gamma\delta}),\ \Delta(\tilde{u}_{i,\alpha}(kt)_{\rho\sigma}),\ \Delta(\tilde{m}_n(kp)_{\epsilon\eta}),\ \Delta(\tilde{v}_{m,\beta}(kt)_{\mu\nu})\}$. Therefore we can define for each P in LT of degree ≥ 0 in Λ^k that commutes with $\tilde{\mathcal{L}}$ and the $\tilde{U}_\alpha$ a unique derivations $\tilde{\partial_P} : R_{LU} \to R_{LU}$ by the matrix equalities

$$\tilde{\partial_P}(\tilde{\mathcal{L}}) := [P_+, \tilde{\mathcal{L}}] \text{ and } \tilde{\partial_P}(\tilde{U}_\alpha) := [P_+, \tilde{U}_\alpha], \text{ resp.} \tag{40}$$

$$\tilde{\partial_P}(\tilde{\mathcal{M}}) := [P_+, \tilde{\mathcal{M}}] \text{ and } \tilde{\partial_P}(\tilde{V}_\alpha) := [P_+, \tilde{V}_\alpha]. \tag{41}$$

The derivation $\tilde{\partial_P}$ factorizes again over the relations (29). Also for each Q in UT of degree < 0 in Λ^k that commutes with $\tilde{\mathcal{M}}$ and the $\tilde{V}_\alpha$, there is a unique derivation $\tilde{\partial_Q}$: $R_{LU} \to R_{LU}$ satisfying

$$\tilde{\partial_Q}(\tilde{\mathcal{M}}) := [Q_-, \tilde{\mathcal{M}}] \text{ and } \tilde{\partial_Q}(\tilde{V}_\alpha) := [Q_-, \tilde{V}_\alpha], \text{ resp.} \tag{42}$$

$$\tilde{\partial_Q}(\tilde{\mathcal{L}}) := [Q_-, \tilde{\mathcal{L}}] \text{ and } \tilde{\partial_Q}(\tilde{U}_\alpha) := [Q_-, \tilde{U}_\alpha]. \tag{43}$$

For the same reason as above, also the derivation $\tilde{\partial_Q}$ factorizes over the relations (29).

Now that we have built a context in which the equations of the form (18) and (19) hold by definition, the next step is a realization of the equations (40) and (42), i.e. we look for a $\mathbb{C}$-algebra R equiped with a number of $\mathbb{C}$-linear derivations $\partial_P : R \to R$ and $\partial_Q : R \to R$, with P and Q as above, and a $\mathbb{C}$-algebra morphism $\nu : R_{LU} \to R$ such that for all relevant P and Q respectively

$$\partial_P \circ \nu = \nu \circ \tilde{\partial_P}, \tag{44}$$

$$\partial_Q \circ \nu = \nu \circ \tilde{\partial_Q}. \tag{45}$$

The morphism ν determines a $\mathbb{C}$-linear map from $M_{\mathbb{Z}}(R_{LU})$ to $M_{\mathbb{Z}}(R)$ and we write $\mathcal{L} := \nu(\tilde{\mathcal{L}})$, $U_\alpha := \nu(\tilde{U}_\alpha)$, $\mathcal{M} := \nu(\tilde{\mathcal{M}})$ and $V_\alpha := \nu(\tilde{V}_\alpha)$. First of all we want that ν factorizes over the relation (29), i.e. they have to satisfy (21) and (24). Clearly, the properties (44) and (45) translates into relations for the matrices $\mathcal{L}$, U_α, $\mathcal{M}$ and V_α in $M_{\mathbb{Z}}(R)$. Namely they have to satisfy respectively

$$\partial_P(\mathcal{L}) = [\nu(P_+), \mathcal{L}],\ \partial_P(U_\alpha) = [\nu(P_+), U_\alpha], \tag{46}$$

$$\partial_P(\mathcal{M}) = [\nu(P_+), \mathcal{M}] \text{ and } \partial_P(V_\alpha) = [\nu(P_+), V_\alpha], \tag{47}$$

$$\partial_Q(\mathcal{M}) = [\nu(Q_-), \mathcal{M}],\ \partial_Q(V_\alpha) = [\nu(Q_-), V_\alpha], \tag{48}$$

$$\partial_Q(\mathcal{L}) = [\nu(Q_-), \mathcal{L}] \text{ and } \partial_Q(U_\alpha) = [\nu(Q_-), U_\alpha]. \tag{49}$$

The system of equations (46), (47), (48) and (48) for all the $P = P_{i,\alpha}$ and all the $Q = Q_{j,\beta}$ are called the *Lax equations* for the hierarchy associated with the algebras h_L and h_U, shortly the (h_L, h_U)-hierarchy. If the matrices $\mathcal{L}$, U_α, $\mathcal{M}$ and V_α in $M_{\mathbb{Z}}(R)$ satisfy, besides these equations, also those in (21) and (24), then they are called a *solution* of this hierarchy. Recall that the h_L-hierarchy had the trivial solution $\mathcal{L} = \Lambda^k$ and $U_\alpha = E_\alpha$ and the h_U-hierarchy $\mathcal{M} = \Lambda^{-k}$ and $V_\alpha = F_\alpha$. The equations (46) and (48) clearly hold for these operators, but the equations (47) and (49) hold if and only if all the matrices E_α and F_β commute. Therefore, in order that the union of the trivial solutions of the h_L-hierarchy and the one of the h_U is a solution of the (h_L, h_U)-hierarchy, we assume from now on that the algebras h_L and h_U commute.

4. The zero curvature form of the hierarchies

The Lax equations of the hierarchies introduced above involve in principle for each equation an infinite number of k-block diagonals. Like their differential operator analogues, they can be written in a different way, the so-called *zero curvature form*, in which for each equation only a finite number of k-block diagonals plays a role. Before presenting it, we fix some notations. For all $i \geq 0, n \geq 1$, α, $0 \leq \alpha \leq m_L = dim(h_L)$ and all γ,$0 \leq \gamma \leq m_U = dim(h_U)$, we write $\tilde{B}_{i\alpha}$ for $(\tilde{\mathcal{L}}^i\tilde{U}_\alpha)_+$ and $\tilde{C}_{n\gamma}$ for $(\tilde{\mathcal{M}}^n\tilde{V}_\gamma)_-$. These operators satisfy the following so-called *zero curvature equations*

Proposition 2. *In the lowertriangular setting the operators $\{\tilde{B}_{n\alpha}\}$ in $M_{\mathbb{Z}}(R_L)$ satisfy for all $m \geq 0$ and all β,$0 \leq \beta \leq m_U = dim(h_L)$,*

$$\partial_{\tilde{P}_{n\alpha}}(\tilde{B}_{m\beta}) - \partial_{\tilde{P}_{m\beta}}(\tilde{B}_{n\alpha}) - [\tilde{B}_{n\alpha}, \tilde{B}_{m\beta}] = 0. \tag{50}$$

In the uppertriangular case the operators $\{\tilde{C}_{m\gamma}\}$ in $M_{\mathbb{Z}}(R_U)$ satisfy similarly for all $n \in \mathbb{N}$ and all α, $0 \leq \alpha \leq m_U$,

$$\partial_{\tilde{Q}_{n\alpha}}(\tilde{C}_{m\gamma}) - \partial_{\tilde{Q}_{m\gamma}}(\tilde{C}_{n\alpha}) - [\tilde{C}_{n\alpha}, \tilde{C}_{m\gamma}] = 0. \tag{51}$$

In the mixed case the operators $\{\tilde{B}_{i\alpha}\}$ and $\{\tilde{C}_{n\gamma}\}$ in $M_{\mathbb{Z}}(R_{LU})$ satisfy , besides the equations (50) and (51), moreover

$$\partial_{\tilde{Q}_{i\alpha}}(\tilde{B}_{j\beta}) - \partial_{\tilde{P}_{j\beta}}(\tilde{C}_{i\alpha}) - [\tilde{C}_{i\alpha}, \tilde{B}_{j\beta}] = 0. \tag{52}$$

Proof. In all three cases we will show that the left hand side belongs both to UT_0 and to LT_{-1} and thus has to be zero. For the first equation, the left hand side of equation (50) clearly belongs to UT_0. To get the other property, we note that for all n and $m \geq 0$ and all α and $\beta \in \{0, \ldots, m_L\}$ there holds

$$\partial_{\tilde{P}_{n\alpha}}(\tilde{\mathcal{L}}^m\tilde{U}_\beta) = [\tilde{B}_{n\alpha}, \tilde{\mathcal{L}}^m\tilde{U}_\alpha] = -[(\tilde{\mathcal{L}}^n\tilde{U}_\alpha)_-, \tilde{\mathcal{L}}^m\tilde{U}_\beta]. \tag{53}$$

This is a direct consequence of the fact that both $\partial_{\tilde{P}_{n\alpha}}$ and $[\tilde{B}_{n\alpha}, -]$ have the derivation property (7) and that their action on $\tilde{\mathcal{L}}$ and $\tilde{U}_\beta$ is equal. Now we substitute $\tilde{B}_{n\alpha} = \tilde{\mathcal{L}}^n\tilde{U}_\alpha - (\tilde{\mathcal{L}}^n\tilde{U}_\alpha)_-$ in the left hand side of (50) and we use equation (53) to obtain

$$\begin{aligned}
&\partial_{\tilde{P}_{n\alpha}}(\tilde{\mathcal{L}}^m\tilde{U}_\beta) - \partial_{\tilde{P}_{n\alpha}}(\tilde{\mathcal{L}}^m\tilde{U}_\beta)_- - \partial_{\tilde{P}_{m\beta}}(\tilde{\mathcal{L}}^n\tilde{U}_\alpha) + \partial_{\tilde{P}_{m\beta}}(\tilde{\mathcal{L}}^n\tilde{U}_\alpha)_- - \\
&\qquad [\tilde{\mathcal{L}}^n\tilde{U}_\alpha - (\tilde{\mathcal{L}}^n\tilde{U}_\alpha)_-, \tilde{\mathcal{L}}^m\tilde{U}_\beta - (\tilde{\mathcal{L}}^m\tilde{U}_\beta)_-] \\
&= -\partial_{\tilde{P}_{n\alpha}}(\tilde{\mathcal{L}}^m\tilde{U}_\beta)_- + \partial_{\tilde{P}_{m\beta}}(\tilde{\mathcal{L}}^n\tilde{U}_\alpha)_- - [(\tilde{\mathcal{L}}^n\tilde{U}_\alpha)_-, (\tilde{\mathcal{L}}^m\tilde{U}_\beta)_-].
\end{aligned}$$

This last expression belongs clearly to LT_{-1}, which proves the first statement.

To obtain the second equation, we note first that, since $\tilde{C}_{m\gamma}$ belongs to LT_{-1} for all $m \geq 1$, it is clear that the left hand side of the zero curvature equations for the $\{\tilde{C}_{m\gamma}\}$ belongs to LT_{-1}. Next we use again the Lax equations for $\tilde{\mathcal{M}}$ to get the relations: for all n and $m \in \mathbb{N}$ and all α and $\gamma \in \{0, \ldots, m_U\}$

$$\partial_{\tilde{Q}_n}(\tilde{\mathcal{M}}^m\tilde{V}_\alpha) = [\tilde{C}_{n\gamma}, \tilde{\mathcal{M}}^m\tilde{V}_\alpha] = -[(\tilde{\mathcal{M}}^n\tilde{V}_\alpha)_+, \tilde{\mathcal{M}}^m\tilde{V}_\alpha]. \tag{54}$$

If we substitute in the zero curvature equation $\tilde{C}_{m\gamma} = \tilde{\mathcal{M}}^m\tilde{V}_\alpha - (\tilde{\mathcal{M}}^m\tilde{V}_\alpha)_+$ and use property (54) , then this leads to

$$\begin{aligned}
&\partial_{\tilde{Q}_{n\alpha}}(\tilde{\mathcal{M}}^m\tilde{V}_\gamma) - \partial_{\tilde{Q}_{n\alpha}}(\tilde{\mathcal{M}}^m\tilde{V}_\gamma)_+ - \partial_{\tilde{Q}_{m\gamma}}(\tilde{\mathcal{M}}^n\tilde{V}_\alpha) + \partial_{\tilde{Q}_{m\gamma}}(\tilde{\mathcal{M}}^n\tilde{V}_\alpha)_+ - \\
&\qquad [\tilde{\mathcal{M}}^n\tilde{V}_\alpha - (\tilde{\mathcal{M}}^n\tilde{V}_\alpha)_+, \tilde{\mathcal{M}}^m\tilde{V}_\gamma - (\tilde{\mathcal{M}}^m\tilde{V}_\gamma)_+] \\
&= -\partial_{\tilde{Q}_{n\alpha}}(\tilde{\mathcal{M}}^m\tilde{V}_\gamma)_+ + \partial_{\tilde{Q}_{m\gamma}}(\tilde{\mathcal{M}}^n\tilde{V}_\alpha)_+ - [(\tilde{\mathcal{M}}^n\tilde{V}_\alpha)_+, (\tilde{\mathcal{M}}^m\tilde{V}_\gamma)_+].
\end{aligned}$$

The right hand side of this expression belongs to UT_0 and we have shown the desired property.

Finally we consider the combined case. On one hand the left hand side of equation (52) can be written as

$$\partial_{\tilde{Q}_{n\alpha}}(\tilde{\mathcal{L}}^m\tilde{U}_\beta) - \partial_{\tilde{Q}_{n\alpha}}((\tilde{\mathcal{L}}^m\tilde{U}_\beta)_-) - \partial_{\tilde{P}_{m\beta}}(\tilde{C}_{n\alpha}) - [\tilde{C}_{n\alpha}, \tilde{\mathcal{L}}^m\tilde{U}_\beta - (\tilde{\mathcal{L}}^m\tilde{U}_\beta)_-] =$$
$$-\partial_{\tilde{Q}_{n\alpha}}((\tilde{\mathcal{L}}^m\tilde{U}_\beta)_-) - \partial_{\tilde{Q}_{m\beta}}(\tilde{C}_{n\alpha}) + [\tilde{C}_{n\alpha}, (\tilde{\mathcal{L}}^m\tilde{U}_\beta)_-]$$

and this is clearly an element of LT_{-1}. On the other hand, it also equals

$$\partial_{\tilde{Q}_{n\alpha}}(\tilde{B}_{m\beta}) - \partial_{\tilde{P}_{m\beta}}(\tilde{\mathcal{M}}^n\tilde{V}_\alpha) + \partial_{\tilde{P}_{m\beta}}((\tilde{\mathcal{M}}^n\tilde{V}_\alpha)_+) - [\tilde{\mathcal{M}}^n\tilde{V}_\alpha - (\tilde{\mathcal{M}}^n\tilde{V}_\alpha)_+, \tilde{B}_{m\beta}] =$$
$$\partial_{\tilde{Q}_{n\alpha}}(\tilde{B}_{m\beta}) + \partial_{\tilde{P}_{m\beta}}((\tilde{\mathcal{M}}^n\tilde{V}_\alpha)_+) + [(\tilde{\mathcal{M}}^n\tilde{V}_\alpha)_+, \tilde{B}_{m\beta}],$$

which is an element of UT_0. This completes the proof of the proposition. □

This proposition permits you to show the following property that unites the equations that belong to the same hierarchy

Corollary 1. *The derivations $\{\partial_{\tilde{P}_{n\alpha}} | n \geq 0, 0 \leq \alpha \leq m_L\}$ of the algebra R_L all commute. The same holds for the set of derivations $\{\partial_{\tilde{Q}_{m\beta}} | m \geq 1, 0 \leq \beta \leq m_L U\}$ of the algebra R_U and on the algebra R_{LU} the derivations $\{\partial_{\tilde{P}_{n\alpha}} | n \geq 0, 1 \leq \alpha \leq m_L\}$ also commute with all the $\{\partial_{\tilde{Q}_{m\beta}} | m \geq 1, 0 \leq \beta \leq m_U\}$.*

Proof. In the lowertriangular case the coefficients of $\tilde{\mathcal{L}}$ and $\tilde{U}_\beta$ generate the algebra R_L so that we merely have to show respectively that

$$(55) \qquad (\partial_{\tilde{P}_{k\gamma}} \circ \partial_{\tilde{P}_{n\alpha}} - \partial_{\tilde{P}_{n\alpha}} \circ \partial_{\tilde{P}_{k\gamma}})(\tilde{\mathcal{L}}) = 0 \text{ and } (\partial_{\tilde{P}_{k\gamma}} \circ \partial_{\tilde{P}_{n\alpha}} - \partial_{\tilde{P}_{n\alpha}} \circ \partial_{\tilde{P}_{k\gamma}})(\tilde{U}_\beta) = 0.$$

Likewise the coefficients of $\tilde{\mathcal{M}}$ and $\tilde{V}_\alpha$ generate the algebra R_U. Therefore it suffices to prove

$$(56) \qquad (\partial_{\tilde{Q}_{m\beta}} \circ \partial_{\tilde{Q}_{r\delta}} - \partial_{\tilde{Q}_{r\delta}} \circ \partial_{\tilde{Q}_{m\beta}})(\tilde{\mathcal{M}}) = 0 \text{ and } (\partial_{\tilde{Q}_{m\beta}} \circ \partial_{\tilde{Q}_{r\delta}} - \partial_{\tilde{Q}_{r\delta}} \circ \partial_{\tilde{Q}_{m\beta}})(\tilde{V}_\alpha) = 0.$$

The algebra R_LU is generated by the coefficients of the 4 operators $\tilde{\mathcal{L}}$, $\tilde{U}_\beta$, $\tilde{\mathcal{M}}$ and $\tilde{V}_\alpha$. Hence one only needs to prove the identities

$$(57) \qquad \begin{aligned} (\partial_{\tilde{P}_{k\gamma}} \circ \partial_{\tilde{P}_{n\alpha}} - \partial_{\tilde{P}_{n\alpha}} \circ \partial_{\tilde{P}_{k\gamma}})(A) &= 0, \\ (\partial_{\tilde{Q}_{m\beta}} \circ \partial_{\tilde{Q}_{r\delta}} - \partial_{\tilde{Q}_{r\delta}} \circ \partial_{\tilde{Q}_{m\beta}})(A) &= 0, \\ (\partial_{\tilde{P}_{n\alpha}} \circ \partial_{\tilde{Q}_{m\beta}} - \partial_{\tilde{Q}_{m\beta}} \circ \partial_{\tilde{P}_{n\alpha}})(A) &= 0, \end{aligned}$$

where A is either one of the four operators. In all three cases one gets the desired identities by applying the following property: let ∂_1 and ∂_2 be derivations of a ring $\mathcal{R}$ and let X be a matrix in $M_{\mathbb{Z}}(\mathcal{R})$ such that for $i = 1, 2$ $\partial_i(X) = [D_i, X]$ for some $D_i \in M_{\mathbb{Z}}(\mathcal{R})$. Then a straightforward computation shows that

$$(58) \qquad (\partial_1 \circ \partial_2 - \partial_2 \circ \partial_1)(X) = [\partial_1(D_2) - \partial_2(D_1) - [D_1, D_2], X]$$

By inserting the zero curvature relations from proposition (2) in this identity for the operators and derivations listed above, one obtains the statements in the corollary. □

We list a number of other consequences of proposition (2). Let $\mathcal{L} := \lambda(\tilde{\mathcal{L}})$ and $U_\alpha := \lambda(\tilde{U}_\alpha)$ be solutions of the h_L-hierarchy. For each $n \geq 0$ and all α, $0 \leq \alpha \leq m_L$ we write $P_{n\alpha} := \mathcal{L}^n U_\alpha$ and $B_{n\alpha} := (\mathcal{L}^n U_\alpha)_+$. Then there holds

$$(59) \qquad \partial_{P_{n\alpha}}(B_{m\beta}) - \partial_{P_{m\beta}}(B_{n\alpha}) - [B_{n\alpha}, B_{m\beta}] = 0$$

Similarly, let $\mathcal{M} := \mu(\tilde{\mathcal{M}})$ and $V_\alpha := \mu(\tilde{V}_\alpha)$ be a solution of the h_U-hierarchy. For each $n \geq 1$ and all α, $0 \leq \alpha \leq m_U$ we write $Q_{n\alpha} := \mathcal{M}^n V_\alpha$ and $C_{n\alpha} := (\mathcal{M}^n V_\alpha)_-$. Then there holds

$$\partial_{Q_{n\alpha}}(C_{m\gamma}) - \partial_{Q_{m\gamma}}(C_{n\alpha}) - [C_{n\alpha}, C_{m\gamma}] = 0. \tag{60}$$

Finally, let $\mathcal{L} := \nu(\tilde{\mathcal{L}})$, $U_\alpha := \nu(\tilde{U}_\alpha)$, $\mathcal{M} := \nu(\tilde{\mathcal{M}})$ and $V_\alpha := \nu(\tilde{V}_\alpha)$ be a solution of the (h_L,h_U)-hierarchy. With the same notations as above, the operators $B_{i\alpha}$ and $C_{j\beta}$ satisfy besides the equations (59) and (60) also

$$\partial_{Q_{j\beta}}(B_{i\alpha}) - \partial_{P_{i\alpha}}(C_{j\beta}) - [C_{j\beta}, B_{i\alpha}] = 0. \tag{61}$$

Next we will show that, reversely, the equations (59), (60) and (61) also imply the Lax equations for the hierarchy under consideration. This gives the result announced at the beginning of this subsection.

Theorem 1. *a) Let $\mathcal{R}_1$ be a $\mathbb{C}$-algebra equiped with a number of $\mathbb{C}$-linear derivations $\partial_{P_{i\alpha}} : \mathcal{R}_1 \to \mathcal{R}_1, i \geq 0, 0 \leq \alpha \leq m_L.$, and let $\mathcal{L} := \lambda(\tilde{\mathcal{L}})$ and $U_\alpha := \lambda(\tilde{U}_\alpha) \in M_{\mathbb{Z}}(\mathcal{R}_1)$ satisfy the relations (21). Then the Lax equations for $\mathcal{L}$ and U_α are equivalent to the zero curvature relations (59) for the operators $\{B_{n\alpha} := (\mathcal{L}^n U_\alpha)_+\}$.*

b) Similarly, let $\mathcal{R}_2$ be a $\mathbb{C}$-algebra equiped with a number of $\mathbb{C}$-linear derivations $\partial_{Q_{j\beta}}$: $\mathcal{R}_2 \to \mathcal{R}_2, j \geq 1, 0 \leq \beta \leq m_U.$, and let $\mathcal{M} := \mu(\tilde{\mathcal{M}})$ and $V_\alpha := \mu(\tilde{V}_\alpha) \in M_{\mathbb{Z}}(\mathcal{R}_2)$ satisfy the relations (24). Then the Lax equations for $\mathcal{M}$ and V_α are equivalent to the zero curvature relations (60) for the operators $\{C_{k\gamma} := (\mathcal{M}^k V_\gamma)_-\}$.

c) Finally, assume we have a $\mathbb{C}$-algebra $\mathcal{R}_3$ equiped with both a set of $\mathbb{C}$-linear derivations $\partial_{P_{i\alpha}} : \mathcal{R}_1 \to \mathcal{R}_1, i \geq 0, 0 \leq \alpha \leq m_L$, as well as $\partial_{Q_{j\beta}} : \mathcal{R}_3 \to \mathcal{R}_3, j \geq 1, 0 \leq \beta \leq m_U.,$. Now let $\mathcal{L} := \nu(\tilde{\mathcal{L}})$, $U_\alpha := \nu(\tilde{U}_\alpha)$, $\mathcal{M} := \nu()$ and $V_\alpha := \nu(\tilde{V}_\alpha) \in M_{\mathbb{Z}}(\mathcal{R}_3)$ satisfy the relations (21) and (24). Then these matrices are a solution of the (h_L, h_U)-hierarchy if and only if the zero curvature relations (59),(60) and (61) hold for the operators $\{B_{n\alpha} := (\mathcal{L}^n U_\alpha)_+\}$ and $\{C_{k\gamma} := (\mathcal{M}^k V_\gamma)_-\}$.

Proof. In all three cases we merely have to prove the sufficiency still. To get the Lax equations for $\mathcal{L}$ and U_β we consider for all $m \geq 1$ the operator $\partial_{P_{i\alpha}}(\mathcal{L}^m U_\beta) - [B_{i\alpha}, \mathcal{L}^m U_\beta]$. By substituting in them $\mathcal{L}^m U_\beta = B_{m\beta} + (\mathcal{L}^m U_\beta)_-$ and by using the zero curvature equations, we get the equality

$$\partial_{P_{i\alpha}}(\mathcal{L}^m U_\beta) - [B_{i\alpha}, \mathcal{L}^m U_\beta] = \partial_{P_{i\alpha}}(\mathcal{L}^m U_\beta)_- + \partial_{P_{m\beta}}(B_{i\alpha}) - [B_{i\alpha}, (\mathcal{L}^m U_\beta)_-] \in LT^{(k)}_{i-1}$$

From the first equality we see that the order in Λ^k in all the $\partial_{P_{i\alpha}}(\mathcal{L}^m U_\beta) - [B_{i\alpha}, \mathcal{L}^m U_\beta]$ is uniformly bounded above by $i - 1$. We first consider the operator $\mathcal{L}$. Assume now that

$$\partial_{P_{i\alpha}}(\mathcal{L}) - [B_{i\alpha}, \mathcal{L}] = \alpha \Lambda^{rk} + \text{ lower order in } \Lambda^k, \tag{62}$$

with $\alpha \in \mathcal{D}_k(R)$ nonzero. Since both $\partial_{P_{i\alpha}}$ and $[B_{i\alpha}, -]$ have the derivation property (7), there follows with induction that

$$\partial_{P_{i\alpha}}(\mathcal{L}^m) - [B_{i\alpha}, \mathcal{L}^m] = \sum_{i=0}^{m-1} \mathcal{L}^i \{\partial_{P_{i\alpha}}(\mathcal{L}) - [B_{i\alpha}, \mathcal{L}]\} \mathcal{L}^{m-i-1}.$$

Now we focuss on the leading term of this equality and we get

$$\partial_{P_{i\alpha}}(\mathcal{L}) - [B_{i\alpha}, \mathcal{L}] = (\sum_{i=0}^{m-1} \Lambda^{ik} \alpha \Lambda^{-ik}) \Lambda^{(r+m-1)k} + \text{ lower order in } \Lambda^k.$$

If we let m tend to infinity this contradicts the fact that the left hand side belongs to $LT^{(k)}_{i-1}$, unless we have for all sufficiently large m that $\sum_{i=0}^{m-1} \Lambda^{ik}\alpha\Lambda^{-ik}$ is zero. This last fact would, however, immediately imply that for sufficiently large i, $\Lambda^{ik}\alpha\Lambda^{-ik} = 0$ and hence $\alpha = 0$. This contradicts the assumption we made, hence there holds for all $i \geq 0$, and all $\alpha, 0 \leq \alpha \leq m_L$, $\partial_{P_{i\alpha}}(\mathcal{L}) - [B_n, \mathcal{L}] = 0$. So the Lax equation holds for $\mathcal{L}$. Next we consider the operator $\mathcal{L}^m U_\beta$ and apply the Lax equations for $\mathcal{L}^m$ to get

(63)
$$\begin{aligned}\partial_{P_{i\alpha}}(\mathcal{L}^m U_\beta) - [B_{i\alpha}, \mathcal{L}^m U_\beta] &= (\partial_{P_{i\alpha}}(\mathcal{L}^m) - [B_{i\alpha}, (\mathcal{L}^m) U_\beta] + \mathcal{L}^m(\partial_{P_{i\alpha}}(U_\beta) - [B_{i\alpha}, U_\beta])\\ &= 0 + \mathcal{L}^m(\partial_{P_{i\alpha}}(U_\beta) - [B_{i\alpha}, U_\beta])\end{aligned}$$

If the operator U_β does not satisfy the Lax equation, then we would have

$$\partial_{P_{i\alpha}}(U_\beta) - [B_{i\alpha}, U_\beta] = \beta\Lambda^{sk} + \text{ lower order in } \Lambda^k, \tag{64}$$

with $\beta \in \mathcal{D}_k(R)$ nonzero. Then the right hand side of (63) has a nonzero leading coefficient in Λ^k of order $m+s$. This contradicts the fact that the order in Λ^k of the left hand side is bounded above. Therefore the Lax equation for U_β has to hold.

To get the Lax equations in the uppertriangular case for $\mathcal{M}$ and the $\{V_\alpha\}$, we proceed similarly and we consider for all $m \geq 1$ the operator $\partial_{Q_{n\beta}}(\mathcal{M}^m V_\alpha) - [C_{n\beta}, \mathcal{M}^m V_\alpha]$. By substituting in them $\mathcal{M}^m V_\alpha) = C_{m\alpha} + (\mathcal{M}^m V_\alpha))_+$ and by using the zero curvature equations, we get the equality

$$\partial_{Q_{n\beta}}(\mathcal{M}^m V_\alpha)) - [C_{n\beta}, \mathcal{M}^m V_\alpha)] = \partial_{Q_{n\beta}}(\mathcal{M}^m V_\alpha))_+ + \partial_{Q_{m\alpha}}(C_{n\beta}) - [C_{n\beta}, (\mathcal{M}^m V_\alpha))_+] \in UT^{(k)}_{-n}.$$

From this equality we see that the order in Λ^k in all the $\partial_{Q_{n\beta}}(\mathcal{M}^m V_\alpha)) - [C_{n\beta}, \mathcal{M}^m V_\alpha)]$ is bounded below by $-n$. We first consider the operator $\mathcal{M}$. Assume that we have

$$\partial_{Q_{n\beta}}(\mathcal{M}) - [C_{n\beta}, \mathcal{M}] = \beta\Lambda^{lk} + \text{ higher order in } \Lambda^k, \tag{65}$$

with $\beta \in \mathcal{D}_k(R)$ nonzero. Now the same formula as above holds, so we have

$$\partial_{Q_{n\beta}}(\mathcal{M}^m) - [C_{n\beta}, \mathcal{M}^m] = \sum_{i=0}^{m-1} \mathcal{M}^i\{\partial_{Q_{n\beta}}(\mathcal{M}) - [C_{n\beta}, \mathcal{M}]\}\mathcal{M}^{m-i-1}. \tag{66}$$

Hence the leading term in Λ^k of the right hand side is

$$\sum_{i=0}^{m-1} (m_{-1}\Lambda^{-k})^i \beta\Lambda^{lk}(m_{-1}\Lambda^{-k})^{-i}(m_{-1}\Lambda^{-k})^{m-1}, \tag{67}$$

which is of order $l+1-m$ in Λ^k. If m tends to infinity this contradicts again that the left hand side belongs to UT_{-n}, unless for all sufficiently large m

$$\sum_{i=0}^{m-1} (m_{-1}\Lambda^{-k})^i \beta\Lambda^{lk}(m_{-1}\Lambda^{-1})^{-i} = 0.$$

This implies again that $\beta\Lambda^{lk} = 0$ and hence that $\beta = 0$. So, our assumption was wrong and we know that the Lax equation has to hold for $\mathcal{M}$ for all $n \geq 1$ and all $\alpha, 0 \leq \alpha \leq m_U$. Again we consider the operator $\mathcal{M}^m V_\alpha$ and apply the Lax equations for $\mathcal{M}^m$ to get

$$\partial_{Q_{n\beta}}(\mathcal{M}^m V_\alpha)) - [C_{n\beta}, \mathcal{M}^m V_\alpha)] = 0 + \mathcal{M}^m(\partial_{Q_{n\beta}}(V_\alpha) - [C_{n\beta}, V_\alpha)]). \tag{68}$$

If the operator V_α would not satisfy the Lax equation, then the left hand side of (68) would be of order $-m$ in Λ^k and this contradicts the fact that it was bounded below. Therefore the Lax equation holds also for V_α.

From part a) of this theorem follows that the equations (46) hold and from part b) that those of (48) are satisfied, so that we only have to show the remaining two: (47) and (49). In order to get these Lax equations for $\mathcal{L}$, U_δ, V_γ and $\mathcal{M}$, we consider for fixed indices the operators $\partial_{Q_{j\beta}}(\mathcal{L}^m U_\delta) - [C_{j\beta}, \mathcal{L}^m U_\delta]$ and $\partial_{P_{i\alpha}}(\mathcal{M}^m V_\gamma) - [B_{i\alpha}, \mathcal{M}^m V_\gamma]$ for all $m \geq 1$. By substituting in them $\mathcal{L}^m U_\delta = B_{m\delta} + (\mathcal{L}^m U_\delta)_-$ and $\mathcal{M}^m V_\gamma = C_{m\gamma} + (\mathcal{M}^m V_\gamma)_+$ and by using the third zero curvature equation, we get the equalities

$$\partial_{Q_{j\beta}}(\mathcal{L}^m U_\delta) - [C_{j\beta}, \mathcal{L}^m U_\delta] = \partial_{Q_{j\beta}}((\mathcal{L}^m U_\delta)_-) + \partial_{P_{m\delta}}(C_{j\beta}) - [C_{j\beta}, (\mathcal{L}^m U_\delta)_-] \in LT_{-1}^{(k)}$$

$$\partial_{P_{i\alpha}}(\mathcal{M}^m V_\gamma) - [B_{i\alpha}, \mathcal{M}^m V_\gamma] = \partial_{P_{i\alpha}}((\mathcal{M}^m V_\gamma)_+) + \partial_{Q_{m\gamma}}(B_{i\alpha}) - [B_{i\alpha}, (\mathcal{M}^m V_\gamma)_+] \in UT_0^{(k)}.$$

From the first equality we see that the order in Λ^k of all the $\partial_{Q_{j\beta}}(\mathcal{L}^m U_\delta) - [C_{j\beta}, \mathcal{L}^m U_\delta]$ is uniformly bounded above by -1 and from the second one, we conclude that the order in Λ^k of all the operators $\partial_{P_{i\alpha}}(\mathcal{M}^m V_\gamma) - [B_{i\alpha}, \mathcal{M}^m V_\gamma]$ is bounded below by 0. Again we look first at the operator $\mathcal{L}$. Assume now that

$$\partial_{Q_n}(\mathcal{L}) - [C_n, \mathcal{L}] = \rho\Lambda^{ks} + \text{ lower order in } \Lambda, \tag{69}$$

with $\rho \in \mathcal{D}_k(R)$ nonzero. Since both $\partial_{Q_{j\beta}}$ and $[C_{j\beta}, -]$ possess the derivation property (7), there follows with induction that

$$\partial_{Q_{j\beta}}(\mathcal{L}^m) - [C_{j\beta}, \mathcal{L}^m] = \sum_{i=0}^{m-1} \mathcal{L}^i \{\partial_{Q_{j\beta}}(\mathcal{L}) - [C_{j\beta}, \mathcal{L}]\} \mathcal{L}^{m-i-1}$$

If we look at the leading term of this equality, we get

$$\partial_{Q_{j\beta}}(\mathcal{L}^m) - [C_{j\beta}, \mathcal{L}^m] = (\sum_{i=0}^{m-1} \Lambda^{ik}\rho\Lambda^{-ik})\Lambda^{(s+m-1)k} + \text{ lower order in } \Lambda.$$

If we let m tend to infinity this contradicts the fact that the left hand side belongs to LT_{-1}, unless we have for all sufficiently large m that $\sum_{i=0}^{m-1} \Lambda^{ik}\rho\Lambda^{-ik}$ is zero. This last fact would, however, immediately imply that for sufficiently large i $\Lambda^{ik}\rho\Lambda^{-ik} = 0$ and hence $\rho = 0$. This contradicts the assumption we made, hence there holds $\partial_{Q_{j\beta}}(\mathcal{L}) - [C_{j\beta}, \mathcal{L}] = 0$. Then the Lax equations also hold for all the positive powers of $\mathcal{L}^m$ and thus we get

$$\partial_{Q_{j\beta}}(\mathcal{L}^m U_\delta) - [C_{j\beta}, \mathcal{L}^m U_\delta] = 0 + \mathcal{L}^m(\partial_{Q_{j\beta}}(U_\delta) - [C_{j\beta}, U_\delta]). \tag{70}$$

If the Lax equations would not hold for U_δ, then the right hand side of (70) shows that for sufficiently large m the operator $\partial_{Q_{j\beta}}(\mathcal{L}^m U_\delta) - [C_{j\beta}, \mathcal{L}^m U_\delta]$ has a positive order in Λ^k, contrary to what we have shown before.

To get the Lax equations for $\mathcal{M}$ and V_γ, we proceed similarly and first consider $\mathcal{M}$. Assume that we have

$$\partial_{P_{i\alpha}}(\mathcal{M}) - [B_{i\alpha}, \mathcal{M}] = \sigma\Lambda^{lk} + \text{ higher order in } \Lambda, \tag{71}$$

with $\sigma \in \mathcal{D}_k(R)$ nonzero. Applying the same formula as above, we obtain

(72)

$$\partial_{P_{i\alpha}}(\mathcal{M}^m) - [B_{i\alpha}, \mathcal{M}^m] = (\sum_{i=0}^{m-1} (m_{-1}\Lambda^{-k})^i \sigma\Lambda^{lk}(m_{-1}\Lambda^{-k})^{-i})(m_{-1}\Lambda^{-k})^{m-1} + \text{ higher order in } \Lambda^k.$$

If m tends to infinity this contradicts again that the left hand side belongs to UT_0, unless for all sufficiently large m

$$\sum_{i=0}^{m-1}(m_{-1}\Lambda^{-k})^i\sigma\Lambda^{lk}(m_{-1}\Lambda^{-k})^{-i}=0.$$

This implies again that $\sigma=0$. So, our assumption was wrong and we know that the Lax equation holds for $\mathcal{M}$ and all its positive powers. Using this we see that

$$\partial_{P_{i\alpha}}(\mathcal{M}^mV_\gamma)-[B_{i\alpha},\mathcal{M}^mV_\gamma]=O+\mathcal{M}^m(\partial_{P_{i\alpha}}(V_\gamma)-[B_{i\alpha},V_\gamma]). \tag{73}$$

If $\partial_{P_{i\alpha}}(V_\gamma)-[B_{i\alpha},V_\gamma]$ would not be zero, then the right hand side of (73) shows that a sufficiently large m would produce negative powers of Λ^k, contrary to what we have shown before. Therefore also V_γ satisfies the Lax equations and this completes the proof of the theorem.

□

5. Wavematrices

In the foregoing subsection we saw that the Lax equations of the hierarchies are equivalent to a set of zero curvature relations. This indicates already that there could be a linear system of which they form the compatibility conditions. We will discuss here these linearizations that render the Lax equations of all the three hierarchies. For the h_L-hierarchy we start with a $\mathbb{C}$-algebra R equiped with a collection of $\mathbb{C}$-linear commuting derivations $\{\partial_{P_{i\alpha}},i\geq 0,1\geq\alpha\geq m_L\}$. Further we have the corresponding potential solutions, namely operators $\mathcal{L}$ and U_β in $LT(R)$ of the form (21), with $l_1=Id$ and $u_{o\alpha}=E_\alpha$. First we present the *linearization of the h_L-hierarchy* and show how you get the Lax equations from them. Later we discuss precisely the objects that occur in them. They read as follows:

$$\mathcal{L}\psi=\psi\Lambda^k, U_\beta\psi=\psi E_\beta \text{ and } \partial_{P_{i\alpha}}(\psi)=B_{i\alpha}\psi, \tag{74}$$

where $i\geq 0$, $1\leq\alpha\leq m_L$. To get the Lax equations for $\mathcal{L}$ one applies the derivation $\partial_{P_{i\alpha}}$ to the first equation in (74) and sustitutes the last one. This leads to the following manipulations

$$\partial_{P_{i\alpha}}(\mathcal{L}\psi-\psi\Lambda^k)=\partial_{P_{i\alpha}}(\mathcal{L})\psi+\mathcal{L}(\partial_{P_{i\alpha}}(\psi))-(\partial_{P_{i\alpha}}(\psi))\Lambda^k= \tag{75}$$

$$\partial_{P_{i\alpha}}(\mathcal{L})\psi+\mathcal{L}B_{i\alpha}\psi-B_{i\alpha}\psi\Lambda^k=\{\partial_{P_{i\alpha}}(\mathcal{L})-[B_{i\alpha},\mathcal{L}\,]\}\psi=0. \tag{76}$$

Hence, if we may scratch the function ψ from the foregoing equation, we obtain the Lax equations for $\mathcal{L}$. For the operator U_β one applies $\partial_{P_{i\alpha}}$ to the second equation in (74) and sustitutes the last one. Thus one gets

$$\partial_{P_{i\alpha}}(U_\beta\psi-\psi E_\alpha)=\partial_{P_{i\alpha}}(U_\beta)\psi+U_\beta(\partial_{P_{i\alpha}}(\psi))-(\partial_{P_{i\alpha}}(\psi))E_\beta= \tag{77}$$

$$\partial_{P_{i\alpha}}(U_\beta)\psi+U_\beta B_{i\alpha}\psi-B_{i\alpha}\psi E_\beta=\{\partial_{P_{i\alpha}}(U_\beta)-[B_{i\alpha},U_\beta\,]\}\psi=0. \tag{78}$$

and if we can leave out ψ again we get the Lax equations for U_β.

Next we build a context in which these manipulations make sense. In the case of $\mathcal{L}$ one needs an action of operators like $\mathcal{L}$, the U_β and all the $B_{i\alpha}$ from the left and from matrices like Λ^k and E_α from the right. So we will build at least a left $LT(R)$-module and the actual form of the elements in the module is guided by the trivial solution $\mathcal{L} = \Lambda^k$, $U_\beta = E_\beta$ of the hierarchy. In that case the equations (74) become

$$\Lambda^k\psi = \psi\Lambda^k, E_\beta\psi = \psi E_\beta, \text{ and } \partial_{P_{i\alpha}}(\psi) = \Lambda^{ki}E_\alpha\psi. \tag{79}$$

Hence ψ commutes with Λ^k and all the E_β and if we think of $\partial_{P_{i\alpha}}$ as taking the derivative $\partial_{t_{i\alpha}}$ w.r.t. a parameter $t_{i\alpha}$, then the function

$$\psi_0 := exp(\sum_{i=0}^{\infty} t_{i\alpha}E_\alpha(\Lambda^{ki}) \tag{80}$$

is a solution of these equations. Therefore we use from now on the suggestive notation $\partial_{t_{i\alpha}}$ instead of $\partial_{P_{i\alpha}}$. The module that we will consider consists of perturbations in $LT(R)$ of this trivial solution. We consider namely the collection $M^{(\infty)}$ consisting of formal products

$$\{\sum_{j=-\infty}^{N} d_j\Lambda^j\}exp(\sum_{n=1}^{\infty} t_{i\alpha}E_\alpha(\Lambda^{ik})), \text{ where } d_j \in \mathcal{D}_k(R). \tag{81}$$

Following the terminology used in the case of pseudodifferential operators we call the elements of $M^{(\infty)}$ *oscillating matrices at infinity.* In general these formal products do not give a well-defined element of $M_{\mathbb{Z}}(R)$. Nevertheless there is a well-defined left action of $LT(R)$ on it. For all p_1 and $p_2 \in LT(R)$ we put namely

$$p_1\{p_2\}exp(\sum_{i=0}^{\infty} t_{i\alpha}E_\alpha\Lambda^{ik}) = \{p_1p_2\}exp(\sum_{i=0}^{\infty} t_{i\alpha}E_\alpha\Lambda^{ik}). \tag{82}$$

Also the right multiplication with Λ^k and E_α is well-defined on elements of $M^{(\infty)}$. We can define an action of the derivations $\{\partial_{t_{i\alpha}}\}$ on M^∞ as follows

$$\partial_{t_{i\alpha}}\{\sum_{j=-\infty}^{N} d_j\Lambda^j\}\psi_0 = \{\sum_{j=-\infty}^{N} \partial_{t_{i\alpha}}(d_j)\Lambda^j + \sum_{j=-\infty}^{N} d_j\Lambda^j\Lambda^{ik}E_\alpha\}\psi_0. \tag{83}$$

All the actions occurring in the linearization have been introduced now. Note that $M^{(\infty)}$ is a free $LT(R)$-module with generator ψ_0. Hence the scratching of ψ from the equation (89) is permitted as soon as we know that $\psi = \hat{\psi}\psi_0$ with $\hat{\psi} \in LT(R)$ invertible. In this last case the equation $\mathcal{L}\psi = \psi\Lambda^k$ implies then that $\mathcal{L} = \hat{\psi}\Lambda^k\hat{\psi}^{-1}$ and the equation $U_\beta\psi = \psi E_\beta$ renders that $U_\beta = \hat{\psi}E_\beta\hat{\psi}^{-1}$. Note that these operators satisfy the commutativity relations (24) trivially. In view of the fact that the leading coefficient in Λ^k of $\mathcal{L}$ is the identity and that of U_β is E_β, we gauge the leading coefficient of $\hat{\psi}$. An oscillating matrix at infinity $\psi = \hat{\psi}\psi_0$, with $\hat{\psi} - Id \in LT_{-1}$ is called *a wavematrix at infinity* for the operators $\mathcal{L} := \hat{\psi}\Lambda^k\hat{\psi}^{-1}$ and $U_\beta = \hat{\psi}E_\beta\hat{\psi}^{-1}$, if it satisfies the equations (74). Since the manipulations to get the Lax equations are well-defined on such a ψ, the corresponding operators $\mathcal{L}$ and U_β are a solution of the hierarchy. If one wants to prove the equations (74) for an oscillating matrix at infinity ψ of the right form, it suffices to prove a weaker result, for there holds

Proposition 3. *Let $\psi = \hat{\psi}\psi_0$, with $\hat{\psi} - Id \in LT_{-1}$, be an oscillating matrix at infinity. If it satisfies for all $i \geq 0$ and all $\alpha, 1 \leq \alpha \leq m_L$,*

$$\partial_{t_{i\alpha}}(\psi) = F_{i\alpha}\psi, \text{ with } F_{i\alpha} \in LT(R) \cap UT_0(R),$$

then $F_{i\alpha} = (\mathcal{L}^i U_\alpha)_+$, *where* $\mathcal{L} := \hat{\psi}\Lambda^k\hat{\psi}^{-1}$ *and* $U_\beta = \hat{\psi}E_\beta\hat{\psi}^{-1}$. *In particular the* $\mathcal{L}$ *and* U_β *form a solution to the* h_L*-hierarchy.*

Proof. From the definition of the action of $\partial_{t_{i\alpha}}$ on $M^{(\infty)}$ and the fact that $M^{(\infty)}$ is a free $LT(R)$-module with generator ψ_0, we get the operator equation

$$\partial_{t_{i\alpha}}(\hat{\psi}) + \hat{\psi}(\Lambda^k)^i E_\alpha = F_{i\alpha}\hat{\psi}. \tag{84}$$

Since $\partial_{t_{i\alpha}}(\hat{\psi})\hat{\psi}^{-1} \in LT_{-1}$, multiplying this equation from the right with $\hat{\psi}^{-1}$ and taking the uppertriangular part gives the desired result. □

Different wavematrices at infinity may lead to the same solution of the h_L-hierarchy. Assume $\mathcal{L} = \hat{\psi_1}\Lambda^k\hat{\psi_1}^{-1} = \hat{\psi_2}\Lambda^k\hat{\psi_2}^{-1}$ and $U_\beta = \hat{\psi_1}E_\beta\hat{\psi_1}^{-1} = \hat{\psi_2}E_\beta\hat{\psi_2}^{-1}$, where both ψ_1 and ψ_2 are wavematrices at infinity. Then we have first of all that

$$\hat{\psi_1}^{-1}\hat{\psi_2} = \sum_{s\leq 0} u_s(\Lambda^k)^s, \text{ where } u_s \in i_k(M_n(R)) \text{ commutes with } h_L.$$

We have seen in the proof of proposition (3) that for all $i \geq 0$, and $j = 1, 2$,

$$\partial_{t_{i\alpha}}(\hat{\psi_j}) = (\mathcal{L}^i U_\alpha)_+\hat{\psi_j} - \hat{\psi_j}(\Lambda^k)^i E_\alpha.$$

Hence, if we apply $\partial_{t_{i\alpha}}$ to the equality $\hat{\psi_2} = \hat{\psi_1}\sum_s u_s(\Lambda^k)^s$, then we obtain

$$\partial_{t_{i\alpha}}(\hat{\psi_2}) = \partial_{t_{i\alpha}}(\hat{\psi_1})\sum_{s\leq 0} u_s(\Lambda^k)^s + \hat{\psi_1}\sum_{s\leq 0}\partial_{t_{i\alpha}}(u_s)(\Lambda^k)^s = \tag{85}$$

$$(\mathcal{L}^i U_\alpha)_+\hat{\psi_1} - \hat{\psi_1}(\Lambda^k)^i E_\alpha)\sum_{s\leq 0} u_s(\Lambda^k)^s + \hat{\psi_1}\sum_{s\leq 0}\partial_{t_{i\alpha}}(u_s)(\Lambda^k)^s = \tag{86}$$

$$(\mathcal{L}^i E_\alpha)_+\hat{\psi_2} - \hat{\psi_2}(\Lambda^k)^i E_\alpha + \hat{\psi_1}\sum_{s\leq 0}\partial_{t_{i\alpha}}(u_s)(\Lambda^k)^s. \tag{87}$$

Hence we must have for all $i \geq 0$ and all $\alpha, 1 \leq \alpha \leq m_U$ that $\partial_{t_{i\alpha}}(u_s) = 0$. For completeness sake, we resume this result in a corollary

Corollary 2. *If* ψ_1 *and* ψ_2 *are wavematrices at infinity w.r.t. the same operator* $\mathcal{L}$, *then we have*

$$\psi_2 = \psi_1\sum_{i\leq 0} u_s(\Lambda^k)^s,$$

where all the $u_s \in i_k(M_k(R))$ *commute with* h_L *and are constant for the derivations* $\partial_{t_{i\alpha}}$, *i.e.* $\partial_{t_{i\alpha}}(u_s) = 0$.

The *linearization of the* h_U*-hierarchy* follows the same lines. One starts out with a $\mathbb{C}$-algebra R equiped with a collection of $\mathbb{C}$-linear commuting derivations $\{\partial_{Q_{j\beta}}, j \geq 1, 1 \leq \beta \leq m_U\}$. Further we have the corresponding potential solutions, namely operators $\mathcal{M}$ and V_α in $UT(R)$ of the form (21), with m_{-1} invertible and $v_{o\alpha} = F_\alpha$. The *linearization of the* h_U*-hierarchy* consists of the following equations for $\mathcal{M}$ and V_α

$$\mathcal{M}\phi = \phi\Lambda^{-k}, V_\alpha\phi = \phi F_\alpha \text{ and } \partial_{Q_{j\beta}}(\phi) = C_{j\beta}\phi. \tag{88}$$

To get the Lax equations for $\mathcal{M}$ one applies the derivation $\partial_{Q_{j\beta}}$ to the first equation in (88) and subtitutes the last one. This leads to the following manipulations

$$\partial_{Q_{j\beta}}(\mathcal{M}\phi - \phi\Lambda^{-k}) = \partial_{Q_{j\beta}}(\mathcal{M})\phi + \mathcal{M}(\partial_{Q_{j\beta}}(\phi)) - (\partial_{Q_{j\beta}}(\phi))\Lambda^{-k} = \tag{89}$$

$$\partial_{Q_{j\beta}}(\mathcal{M})\phi + \mathcal{M}C_{j\beta}\phi - C_{j\beta}\phi\Lambda^{-k} = \{\partial_{Q_{j\beta}}(\mathcal{M}) - [C_{j\beta}, \mathcal{M}\,]\}\phi = 0. \tag{90}$$

Hence, if we may scratch the function ϕ from the foregoing equation, we obtain the Lax equations for $\mathcal{M}$. For the operator V_α one applies $\partial_{Q_{j\beta}}$ to the second equation in (88) and substitutes the last one. Thus one gets

$$\partial_{Q_{j\beta}}(V_\alpha\phi - \psi F_\alpha) = \partial_{Q_{j\beta}}(V_\alpha)\psi + V_\alpha(\partial_{Q_{j\beta}}(\phi)) - (\partial_{Q_{j\beta}}(\phi))F_\alpha = \tag{91}$$

$$\partial_{Q_{j\beta}}(V_\alpha)\phi + V_\alpha C_{j\beta}\phi - C_{j\beta}\phi F_\alpha = \{\partial_{Q_{j\beta}}(V_\alpha) - [C_{j\beta}, V_\alpha\,]\}\phi = 0. \tag{92}$$

and if we can leave out ϕ again we get the Lax equations for V_α.

For the linearization (88) we need a left action of $\mathcal{M}$ and all the $C_{j\beta}$ on the functions ϕ and from matrices like Λ^{-k} from the right. So we will build a left $UT(R)$-module. Again the actual form of the elements in the module is guided by the trivial solution $\mathcal{M} = \Lambda^{-k}$ and $V_\alpha = F_\alpha$ of the hierarchy. In that case the equations (88) become

$$\Lambda^{-k}\phi = \phi\Lambda^{-k}, F_\alpha\phi = \phi F_\alpha \text{ and } \partial_{Q_{j\beta}}(\phi) = (\Lambda)^{-jk}F_\beta\phi. \tag{93}$$

These equations of the derivations $\partial_{Q_{j\beta}}$ can be integrated simultaneously and the corresponding ϕ commutes with Λ^{-k}. For, if we think of $\partial_{Q_{j\beta}}$ as taking the partial derivative $\partial_{s_{j\beta}}$ w.r.t. a parameter $s_{j\beta}$, then the function

$$\phi_0 := exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{m_U} s_{j\beta}F_\alpha\Lambda^{-jk}) \tag{94}$$

is a solution of these equations. The module for this linearization will consist of perturbations in $UT(R)$ of this trivial solution. Consider namely the collection $M^{(0)}$ consisting of formal products

$$\{\sum_{j=N}^{\infty} d_j\Lambda^{kj}\}exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{m_U} s_{j\beta}F_\beta\Lambda^{-kj}), \text{ where } d_j \in \mathcal{D}_k(R). \tag{95}$$

The elements of $M^{(0)}$ are called *oscillating matrices at zero*. In general these formal products do not give a well-defined element of $M_{\mathbb{Z}}(R)$. Nevertheless there is a well-defined left action of $UT(R)$ on it. For all u_1 and $u_2 \in UT(R)$ we put namely

$$u_1\{u_2\}exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{m_U} s_{j\beta}F_\beta\Lambda^{-kj}) = \{u_1u_2\}exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{m_U} s_{j\beta}F_\beta\Lambda^{-kj}). \tag{96}$$

Also the right multiplication with Λ^{-k} and F_β is well-defined on elements of $M^{(0)}$. From now on we will think of the derivations $\partial_{Q_{j\beta}}$ as the derivatives w.r.t. the flowparameters $\{s_{j\beta}\}$ and we will write $\partial_{s_{j\beta}}$ instead of $\partial_{Q_{j\beta}}$. An action of these derivations on $M^{(0)}$ can be defined as follows

$$\partial_{s_{j\beta}}\{\sum_{j=N}^{\infty} d_j\Lambda^j\}\phi_0 = \{\sum_{j=N}^{\infty}\partial_{s_{j\beta}}(d_j)\Lambda^j + \sum_{j=N}^{\infty} d_j\Lambda^j(\Lambda)^{-kj}F_\beta\}\phi_0. \tag{97}$$

All the actions occurring in the linearization have been introduced now. Note that $M^{(0)}$ is a free $UT(R)$-module with generator ϕ_0. Hence the scratching of ϕ from the equation

(??) is permitted as soon as we know that $\phi = \hat{\phi}\phi_0$ with $\hat{\phi} \in UT(R)$ invertible. In this last case the equation $\mathcal{M}\phi = \phi\Lambda^{-1}$ implies then that $\mathcal{M} = \hat{\phi}\Lambda^{-1}\hat{\phi}^{-1}$. An oscillating matrix at zero $\phi = \hat{\phi}\phi_0$, with $\hat{\phi} = \sum_{i=0}^{\infty} d_i\Lambda^{ki}$, with d_0 invertible, is called *a wavematrix at zero* for the operator $\mathcal{M} = \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1}$, if it satisfies the equations (88). Since the manipulations to get the Lax equations are well-defined on such a ϕ, the corresponding operator $\mathcal{M}$ is a solution of the hierarchy. If one wants to prove the equations (88) for an oscillating matrix at zero ϕ of the right form, it suffices to prove a weaker result, for there holds

Proposition 4. *Let $\phi = \hat{\phi}\phi_0$, with $\hat{\phi} - d_0 \in UT_1$ and $d_0 \in D_k(R)$ invertible, be an oscillating matrix at zero. If it satisfies for all $j \geq 1$ and all $\beta, 1 \leq \beta \leq m_U$*

$$\partial_{s_{j\beta}}(\phi) = G_{j\beta}\psi, \text{ with } G_{j\beta} \in UT(R) \cap LT_0(R),$$

then $G_{j\beta} = (\mathcal{M}^j V_\beta)_-$, where $\mathcal{M} := \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1}$ and $V_\beta = \hat{\phi}F_\beta\hat{\phi}^{-1}$. In particular $\mathcal{M}$ and the V_β form a solution to the h_U-hierarchy

Proof. From the definition of the action of $\partial_{s_{j\beta}}$ on $M^{(0)}$ and the fact that $M^{(0)}$ is a free $UT(R)$-module with generator ϕ_0, we get the operator equation

$$\partial_{s_{j\beta}}(\hat{\phi}) + \hat{\phi}(\Lambda)^{-kj}F_\beta = G_{j\beta}\hat{\phi}. \tag{98}$$

Multiplying this equation from the right with $\hat{\psi}^{-1}$ and taking the lowertriangular part gives the desired result. □

Different wavematrices at zero may lead to the same solution of the h_U-hierarchy. Assume $\mathcal{M} = \hat{\phi_1}\Lambda^{-k}\hat{\phi_1}^{-1} = \hat{\phi_2}\Lambda^{-k}\hat{\phi_2}^{-1}$ and $V_\beta = \hat{\phi_1}F_\beta\hat{\phi_1}^{-1} = \hat{\phi_2}F_\beta\hat{\phi_2}^{-1}$, where both ψ_1 and ψ_2 are wavematrices at zero. Then we have first of all that

$$\hat{\phi_1}^{-1}\hat{\phi_2} = \sum_{i\geq 0} v_i(\Lambda)^{ki},$$

where $v_i \in i_k(M_k(R))$ commutes with Λ^{ki} and the F_β. We have seen in the proof of proposition (4) that for all $j \geq 1$ and $i = 1, 2$,

$$\partial_{s_{j\beta}}(\hat{\phi_i}) = (\mathcal{M}^j V_\beta)_-\hat{\phi_i} - \hat{\phi_i}(\Lambda)^{-kj}F_\beta.$$

Hence, if we apply the operator $\partial_{s_{j\beta}}$ to the equality $\hat{\phi_2} = \hat{\psi_1}\sum_i v_i(\Lambda)^i$, then we obtain

$$\partial_{s_{j\beta}}(\hat{\phi_2}) = \partial_{s_{j\beta}}(\hat{\phi_1})\sum_{i\geq 0} u_i(\Lambda)^{ki} + \hat{\psi_1}\sum_{i\geq 0}\partial_{s_{j\beta}}(v_i)(\Lambda)^{ki} = \tag{99}$$

$$((\mathcal{M}^j V_\beta)_-\hat{\phi_1} - \hat{\phi_1}(\Lambda)^{-kj}F_\beta)\sum_{i\geq 0} v_i(\Lambda)^{ki} + \hat{\phi_1}\sum_{i\geq 0}\partial_{s_{j\beta}}(v_i)(\Lambda)^{ki} = \tag{100}$$

$$(\mathcal{M}^j V_\beta)_-\hat{\phi_2} - \hat{\phi_2}(\Lambda)^{-kj}F_\beta + \hat{\phi_1}\sum_{i\geq 0}\partial_{s_n}(v_i)(\Lambda)^{ki}. \tag{101}$$

Hence we must have for all $i \geq 0$ and all $j \geq 1$ and all $\beta, 1 \leq \beta \leq m_U$ that $\partial_{s_{j\beta}}(v_i) = 0$. For completeness sake, we resume this result in a corollary

Corollary 3. *If ϕ_1 and ϕ_2 are wavematrices at zero w.r.t. the same operators $\mathcal{M}$ and V_β, then we have*

$$\phi_2 = \phi_1\sum_{i\geq 0} v_i(\Lambda)^{ki},$$

where all the v_i are constant for the derivations $\partial_{s_{j\beta}}$, i.e. $\partial_{s_{j\beta}}(v_i) = 0$.

Remark 6. The actual construction of wavematrices in $M^{(\infty)}$ resp. $M^{(0)}$ that satisfy the differential equations in the propositions 3 resp. 4 can be done by considering the commuting flows generated by the commutative subalgebras on suitable infinite dimensional flag manifolds. This is carried out in [GFH].

In the case of the (h_L, h_U)-hierarchy, we assume that we have a $\mathbb{C}$-algebra R equiped with two collections of $\mathbb{C}$-linear commuting derivations namely the $\{\partial_{P_{i\alpha}}, i \geq 0, 1 \geq \alpha \geq m_L\}$ and the $\{\partial_{Q_{j\beta}}, j \geq 1, 1 \leq \beta \leq m_U\}$. Further we have the corresponding potential solutions, namely the operators $\mathcal{L}$ and U_α in $LT(R)$ of the form (20), with $l_1 = Id$ and $u_{0\alpha} = E_\alpha$, and the operators $\mathcal{M}$ and V_α in $UT(R)$ of the form (23), with m_{-1} invertible and $v_{0\alpha} = F_\alpha$. The *linearization of the* (h_L, h_U)*-hierarchy* consists of a number of equations for all these operators There should hold

$$\mathcal{L}\psi = \psi\Lambda^k, U_\gamma\psi = \psi E_\gamma,\ \partial_{Q_{j\beta}}(\psi) = C_{j\beta}(\psi),\ \text{and } \partial_{P_{i\alpha}}(\psi) = B_{i\alpha}\psi, \tag{102}$$

$$\mathcal{M}\phi = \phi\Lambda^{-k}, V_\sigma\phi = \phi F_\sigma, \partial_{P_{i\alpha}}(\phi) = B_{i\alpha}\phi,\ \text{and } \partial_{Q_{j\beta}}(\phi) = C_{j\beta}\phi. \tag{103}$$

By applying again both sets of derivations to the first two equations of (102) and scratching the function ψ one obtains the Lax equations for $\mathcal{L}$ and the U_α. The same procedure on the first two equations of (103) renders those for $\mathcal{M}$ and the V_α.

Since we have assumed that h_L and h_U commute, the union of the trivial solution to the h_L-hierarchy and that of the h_U-hierarchy gives a solution of the coupled hierarchy and a solution of the corresponding linearization in this case is the pair (ψ_0, ϕ_0). Therefore we look in the (h_L, h_U)-linearization for solutions $\psi \in M^{(\infty)}$ and $\phi \in M^{(0)}$ and we write $\partial_{s_{j\beta}}$ instead of $\partial_{Q_{j\beta}}$ and $\partial_{t_{i\alpha}}$ instead of $\partial_{P_{i\alpha}}$. The action of $\partial_{t_{i\alpha}}$ on $M^{(\infty)}$ and of $\partial_{s_{j\beta}}$ on $M^{(0)}$ is the same as above, so that we merely have to define still that of $\partial_{s_{j\beta}}$ on $M^{(\infty)}$ and the one of $\partial_{t_{i\alpha}}$ on $M^{(0)}$ to have the various actions in (102) and (102) well-defined. They are given respectively by

$$\partial_{s_{j\beta}}(\{\sum_{r=-\infty}^{N} d_r\Lambda^{kr}\}\psi_0) = \{\sum_{r=-\infty}^{N} \partial_{s_{j\beta}}(d_r)\Lambda^{kr}\}\psi_0 \tag{104}$$

$$\partial_{t_{i\alpha}}(\{\sum_{s=N}^{\infty} d_s\Lambda^{ks}\}\phi_0) = \{\sum_{s=N}^{\infty} \partial_{t_{i\alpha}}(d_s)\Lambda^{ks}\}\phi_0. \tag{105}$$

Again we consider the oscillating function ψ at infinity to be of the form $\psi = \hat{\psi}\psi_0$, with $\hat{\psi} - Id \in LT_{-1}$. Likewise we want the oscillating function ϕ at zero to have the form $\phi = \hat{\phi}\phi_0$, with the leading term of $\hat{\phi}$ invertible in $D_k(R)$. The elements (ψ, ϕ) in $M^{(\infty)} \times M^{(0)}$ of this form are called *wavematrices of the coupled hierarchy* if they satisfy the equations in (102) and (103) for the operators $\mathcal{L} := \hat{\psi}\Lambda^k\hat{\psi}^{-1}$, $U_\alpha := \hat{\psi}E_\alpha\hat{\psi}^{-1}$, $\mathcal{M} := \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1}$ and $V_\alpha := \hat{\phi}E_\alpha\hat{\phi}^{-1}$. This collection operators forms then a solution of the Lax equations of the (h_L, h_U)-hierarchy. Also in the coupled case, it suffices that an apparently weaker version of the last two equations of (102) resp. (103) holds for a candidate pair (ψ, ϕ). By combining the propositions (3) and (4) we get namely

Proposition 5. *Let* (ψ, ϕ) *in* $M^{(\infty)} \times M^{(0)}$ *have the form* $\psi = \hat{\psi}\psi_0$*, with* $\hat{\psi} - Id \in LT_{-1}$*, and* $\phi = \hat{\phi}\phi_0$*, with* $\hat{\phi} - Id \in UT_1$*. If they satisfy the equations*

$$\partial_{t_{i\alpha}}(\psi) = F_{i\alpha}\psi \text{ and } \partial_{t_{i\alpha}}(\phi) = F_{i\alpha}\phi, \text{ with } F_{i\alpha} \in LT(R) \cap UT_0(R),$$

$$\partial_{s_{j\beta}}(\psi) = G_{j\beta}\psi \text{ and } \partial_{s_{j\beta}}(\phi) = G_n\phi, \text{ with } G_{j\beta} \in UT(R) \cap LT_0(R),$$

then $F_{i\alpha} = (\mathcal{L}^i U_\alpha)_+$*, where* $\mathcal{L} := \hat{\psi}\Lambda^k\hat{\psi}^{-1}$ *and* $U_\alpha = \hat{\psi}E_\alpha\hat{\psi}^{-1}$*, and* $G_{j\beta} = (\mathcal{M}^j V_\beta)_-$*, with* $\mathcal{M} := \hat{\phi}\Lambda^{-1}\hat{\phi}^{-1}$ *and* $V_\beta = \hat{\phi}F_\beta\hat{\phi}^{-1}$*. In particular the set* $(\mathcal{L}, U_\alpha, \mathcal{M}, V_\beta)$ *is a solution of the* (h_L, h_U)*-hierarchy.*

Also in this coupled situation various sets of wavefunctions may determine the same solution of the coupled hierarchy. For, let (ψ_1, ϕ_1) and (ψ_2, ϕ_2) be wavefunctions that both lead to the solution $(\mathcal{L}, U_\alpha, \mathcal{M}, V_\beta)$. Then we have shown in (2) and (3) that

$$\psi_2 = \psi_1 \sum_{i \le 0} u_i(\Lambda^k)^i \text{ and } \phi_2 = \phi_1 \sum_{i \ge 0} v_i\Lambda^{ki}, \tag{106}$$

where the $\{u_i\}$ are constant w.r.t. the $\partial_{t_{i\alpha}}$ and the $\{v_i\}$ are constant w.r.t. the $\partial_{s_{j\beta}}$. We will show now that they are also constant w.r.t. to the other derivations. By the definition of the action of $\partial_{t_{i\alpha}}$ on $M^{(0)}$ and that of $\partial_{s_{j\beta}}$ on $M^{(\infty)}$ there holds

$$\partial_{t_{i\alpha}}(\phi_2) = \partial_{t_{i\alpha}}(\phi_1) \sum_{i \ge 0} v_i\Lambda^{ki} + \phi_1 \sum_{i \ge 0} \partial_{t_{i\alpha}}(v_i)(\Lambda)^{ki} = \tag{107}$$

$$(\mathcal{L}^i U_\alpha)_+\phi_1 \sum_{i \ge 0} v_i(\Lambda)^{ki} + \phi_1 \sum_{i \ge 0} \partial_{t_{i\alpha}}(v_i)(\Lambda)^{ki} = (\mathcal{L}^i U_\alpha)_+\phi_2, \text{ resp.}, \tag{108}$$

$$\partial_{s_{j\beta}}(\psi_2) = \partial_{s_{j\beta}}(\psi_1) \sum_{i \le 0} u_i(\Lambda)^{ki} + \psi_1 \sum_{i \le 0} \partial_{s_{j\beta}}(u_i)(\Lambda)^{ki} = \tag{109}$$

$$(\mathcal{M}^j V_\beta)_-\psi_1 \sum_{i \le 0} u_i(\Lambda)^{ki} + \psi_1 \sum_{i \le 0} \partial_{s_{j\beta}}(u_i)(\Lambda)^{ki} = (\mathcal{M}^j V_\beta)_-\psi_2. \tag{110}$$

This gives you the remaining equalities $\partial_{s_{j\beta}}(u_i) = 0$ and $\partial_{t_{i\alpha}}(v_i) = 0$. We summarize this result as follows

Corollary 4. *If* (ψ_1, ϕ_1) *and* (ψ_2, ϕ_2) *are wavefunctions for the solution* $(\mathcal{L}, U_\alpha, \mathcal{M}, V_\beta)$ *of the* (h_L, h_U)*-hierarchy, then we have that*

$$\psi_2 = \psi_1 \sum_{i \le 0} u_i\Lambda^{ki} \text{ and } \phi_2 = \phi_1 \sum_{i \ge 0} v_i\Lambda^{ki}, \tag{111}$$

where both the $\{u_i\}$ *as well as the* $\{v_i\}$ *are constant w.r.t. the derivations* $\partial_{t_{i\alpha}}$ *and* $\partial_{s_{j\beta}}$*.*

Remark 7. Note that, if (ψ, ϕ) in $M^{(\infty)} \times M^{(0)}$ are wavematrices, then $A := \phi^{-1}\psi$ is the formal product of four invertible matrices

$$exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{m_U} -s_{j\beta}F_\beta(\Lambda)^{-kj})\hat{\phi}^{-1}\hat{\psi}exp(\sum_{i=0}^{\infty}\sum_{\alpha=1}^{m_L} t_{i\alpha}E_\alpha\Lambda^{ki}).$$

If we apply $\partial_{t_{i\alpha}}$ and $\partial_{s_{j\beta}}$ to the equality $\phi A = \psi$, then we get respectively

$$\partial_{t_{i\alpha}}(\phi A) = \partial_{t_{i\alpha}}(\phi)A + \phi\partial_{t_{i\alpha}}(A) = (\mathcal{L}^i U_\alpha)_+\phi A + \phi\partial_{t_{i\alpha}}(A) = (\mathcal{L}^i U_\alpha)_+\psi, \tag{112}$$

$$\partial_{s_{j\beta}}(\phi A) = \partial_{s_{j\beta}}(\phi)A + \phi\partial_{s_{j\beta}}(A) = (\mathcal{M}^j V_\beta)_-\phi A + \phi\partial_{s_{j\beta}}(A) = (\mathcal{M}^j V_\beta)_-\psi. \tag{113}$$

In other words A does not depend on the variables $\{t_{i\alpha}\}$ and $\{s_{j\beta}\}$. This gives you a way to construct solutions of the hierarchy. Starting with a matrix A that is constant for the derivations $\partial_{t_{i\alpha}}$ and $\partial_{s_{j\beta}}$, one forms the formal product

$$A(t,s) = exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{m_U} s_{j\beta}F_\beta(\Lambda)^{-kj})Aexp(\sum_{i=0}^{\infty}\sum_{\alpha=1}^{m_L} -t_{i\alpha}E_\alpha\Lambda^{ki}).$$

If you can find a decomposition $A(t,s) = \hat{\phi}^{-1}\hat{\psi}$ where $\hat{\phi} \in UT_0(R)$ has an invertible leading coefficient in $D_k(R)$ and $\hat{\psi} - Id \in LT_{-1}(R)$, then $\psi := \hat{\psi}\psi_0$ and $\phi := \hat{\phi}\phi_0$ are wavefunctions of the coupled hierarchy. They satisfy namely the conditions in corollary (4). For ψ and ϕ there holds then $\phi A = \psi$. Applying $\partial_{t_{i\alpha}}$ to this equality gives on one hand

$$\partial_{t_{i\alpha}}(\psi) = (\partial_{t_{i\alpha}}(\hat{\psi})\hat{\psi}^{-1} + \hat{\psi}\Lambda^{ki}E_\alpha\hat{\psi}^{-1})\psi, \tag{114}$$

with $\partial_{t_{i\alpha}}(\hat{\psi})\hat{\psi}^{-1} + \hat{\psi}\Lambda^k\hat{\psi}^{-1} \in LT_i^{(k)}$. On the other hand it equals

$$\partial_{t_{i\alpha}}(\phi A) = (\partial_{t_{i\alpha}}(\hat{\phi})\hat{\phi}^{-1})\phi A = (\partial_{t_{i\alpha}}(\hat{\phi})\hat{\phi}^{-1})\psi, \tag{115}$$

but the factor $\partial_{t_{i\alpha}}(\hat{\phi})\hat{\phi}^{-1}$ now belongs to $UT_0^{(k)}$. Thus we have that $\partial_{t_{i\alpha}}(\psi)\psi^{-1}$ belongs to $UT_0 \cap LT_i$. By corollary (4) we may conclude that $\partial_{t_{i\alpha}}(\psi) = (\mathcal{L}^iU_\alpha)_+\psi$, where $\mathcal{L} = \hat{\psi}\Lambda^{ki}\hat{\psi}^{-1}$ and $U_\alpha = \hat{\psi}E_\alpha\hat{\psi}^{-1}$. Next we apply $\partial_{s_{j\beta}}$ to the equality $\phi A = \psi$. This renders for the right hand side

$$\partial_{s_{j\beta}}(\psi) = (\partial_{s_{j\beta}}(\hat{\psi})\hat{\psi}^{-1})\psi, \tag{116}$$

with the operator $\partial_{s_{j\beta}}(\hat{\psi})\hat{\psi}^{-1}$ in $LT_{-1}^{(k)}$. For the left hand side we get

$$\partial_{s_{j\beta}}(\phi A) = (\partial_{s_{j\beta}}(\hat{\phi})\hat{\phi}^{-1} + \hat{\phi}\Lambda^{-kj}F_\beta\hat{\phi}^{-1})\phi A, \tag{117}$$

with $\partial_{s_{j\beta}}(\hat{\phi})\hat{\phi}^{-1} + \hat{\phi}\Lambda^{-kj}F_\beta\hat{\phi}^{-1} \in UT_{-j}^{(k)}$. Therefore the factor $\partial_{s_{j\beta}}(\hat{\psi})\hat{\psi}^{-1}$ has to equal $(\mathcal{M}^jV_\beta)_-$, with $\mathcal{M} = \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1}$ and $V_\beta = \hat{\phi}F_\beta\hat{\phi}^{-1}$, and we have $\partial_{s_{j\beta}}(\psi) = (\mathcal{M}^jV_\beta)_-\psi$. This proves the equations for ψ, those of ϕ are a direct consequence of the relation $\phi = \psi A^{-1}$.

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