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C-diagrams, shifts and solidarity values

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Abstract

In the theory of cooperative games so called dividends of a coalition S are considered, which are defined as $\frac{c_S}{|S|}$. The costs c_S form a c-diagram. On these c-diagrams several types of shifts are defined and analysed. Different solution concepts and their properties are related to shifts. We introduce reward games and fine games as components of a cooperative game. Some solution concepts for applications are analysed in terms of c-diagrams, as well as the solidarity concept.
Key words: cooperative games, c-diagrams, shifts, solidarity values
AMS classification: 91A44.

1 Introduction

Let N be a set of n players. From the theory of cooperative games, one knows that the space of all characteristic functions v, corresponding to an n-person coalitional game (N, v) is a linear vector space. Moreover, one of its bases is $\{u_S, S \in 2^N \setminus \emptyset\}$, where u_S is the unanimity game for S, defined as

$$u_S(T) = \begin{cases} 1, & \text{if } S \subset T, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, for each cooperative game (N, v), the map v can be expressed as

$$v(S) = \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} c(u_T), \tag{1}$$

where

$$c(u_S) = \sum_{T \subseteq S, T \neq \emptyset} (-1)^{|S|-|T|} v_c(T).$$

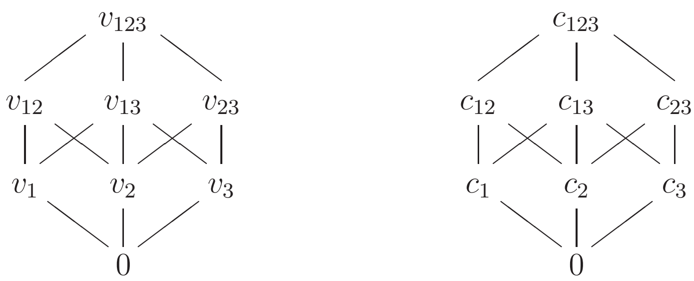


Figure 1: v-diagram and c-diagram of a 3-player cooperative game

In the theory of cooperative games, the quantities $c(u_S), S \subseteq N$ are widely used. Recall that one of the classical proofs for the characterisation of the Shapley value by the *efficiency, anonimity, dummy player and additivity* properties is done by using decomposition (1) (see [S53]). Also, note that the Shapley value associates to each player i, $x_i = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{1}{|S|} c(u_S)$. In another context, $c(u_S)$ are used by Harsanyi [Har59] to define the dividend of set S, which is $\frac{c(u_S}{|S|}$.

The quantities $c(u_S)$ also proved to be essential in establishing the connection between set games and cooperative games (see [BH03]).

As we are essentially considering the partially ordered set of subsets of N, we can make use of the Hasse diagram to indicate the values of the coalitions in the cooperative game as well as the associated numbers $c(u_S)$, see Figure 1.

Remark 1 *Note that we have written v_S for $v_c(S)$, c_S for $c(u_S)$ and that we have added the number 0 in the c-diagram on the right. The diagram on the left is called v-diagram. We will mainly discuss the c-diagram in this paper. Note that the sum of the numbers c_S equals v_{123}. Restriction of the c-diagram to subsets of a coalition S determines a sub-c-diagram with numbers that sum up to v_S. For example, $c_{12} + c_1 + c_2 = v_{12}$.*

In Section 2 the concept of shift is introduced and various shift techniques and their corresponding solution concepts are discussed. In Section 3 we introduce reward games and fine games. In Section 4 we analyse some applications of c-diagrams. In Section 5 two new solidarity concepts are presented.

2 Solution concepts and game shifts

Any solution concept can be written as

$$x_i = \sum_{S \subseteq N} \lambda_{S,i} c_S. \tag{2}$$

This simply expresses that every player i gets a certain share of each c_S.

The positive c_S can be interpreted as a *cooperation bonus or reward* in case

$c_S \geq 0$ or as a *cooperation malus or fine* in case it is negative. Then (2) can be seen as distributing cooperation rewards and fines.

A pure *egalitarian value*, in which every player gets the same, namely $\frac{v(N)}{n}$, is obtained for $\lambda_{S,i} = \frac{1}{n}$, for each S and i. The Shapley value is obtained if $\lambda_{S,i} = \frac{1}{|S|}$ for all $i, i \in S$, and $\lambda_{S,i} = 0$ otherwise. If for player 1 we have $\lambda_{S,i} = 1$ while $\lambda_{S,i} = 0$ for all $i, i \neq 1$, player 1 is allocated $v(N)$ and the other players have allocation zero. That value might be called the *unfair value.* Solution concepts may be therefore be studied or classified by considering the possibilities for $\lambda_{S,i}$. A first distinction might be made by distinguishing such solution concepts where the $\lambda_{S,i}$ do not depend on the function v_c in (N, v_c) from solution concepts where they do. Another distinction might be between the situation where $0 \leq \lambda_{S,i} \leq 1$ and the situation where $\lambda_{S,i}$ may be outside the interval $[0, 1]$. We are inclined to consider only solution concepts in which $\lambda_{S,i}$ is independent of v_c and in the interval $[0, 1]$.

So by changing $\lambda_{S,i}$ we can obtain a different solution concept, a different value. However, there is another way of looking at the c-diagram. Let the $\lambda_{S,i}$ be fixed and, more particularly, let us choose $\lambda_{S,i}$ as the Shapley value, *i.e.* $\frac{1}{|S|}$ or zero.

We now consider the possibility to distribute the sumtotal of the c_S's over the $2^n - 1$ places of the diagram in another way. We say that such a re-distribution leads to a a *shifted c-diagram.* The Shapley value now calculates the allocation with c_S's that have changed. Naturally, a shift of the c-diagram will result in a shifted $v - diagram$, and hence in a *shifted game.*

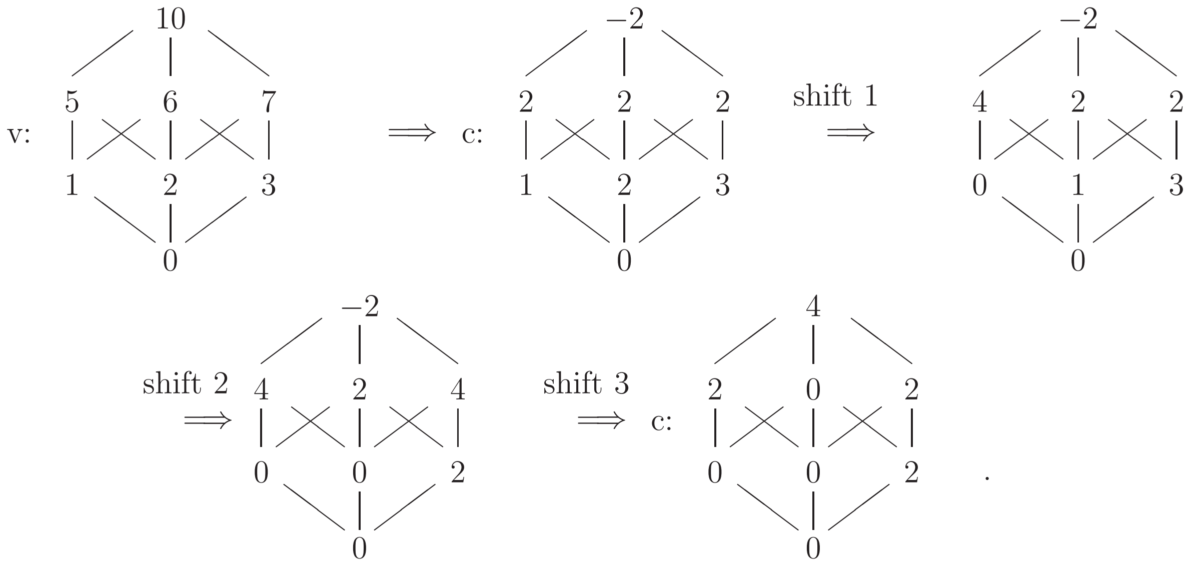
We are interested, among other, in ways of *shifting* the c-diagram such that the solution, in our case the Shapley value, remains the same. For example, consider the c^*-diagram in which $c_i^* = x_i$ according to the Shapley value , applied to the c-diagram of (N, v), and in which $c_S^* = 0$, for all S with $|S| \geq 2$. Applying the Shapley value to this shifted c^*-diagram gives the same allocations x_i as before, but constructing the v^*-diagram corresponding to the c^* diagram we find $v^*(S) = \sum_{i \in S} x_i$, for all $S \subseteq N$, in general differing from $v(S)$ in (N, v). So we have a game shift although the solution stays the same.

Remark 2 *One can prove that by applying the following shifting technique iteratively, the Shapley value remains unchanged, although the game is shifted.*

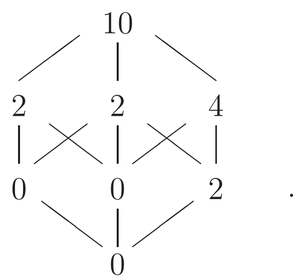
Technique *Either shift* $\frac{c_S}{|S|}$ *to the* c_T *with* $T \subseteq S$ *and* $|T| = |S| - 1$ *or shift* $|S|$ *equal parts from these* c_T *to* c_S*.*

The proof is straightforward and is based only on the definition of the Shapley value.

Example 1 Consider the following 3-persons game and three shifts.



The v-diagram corresponding to the last c-diagram is



From the last c-diagram we can calculate the Shapley value of the shifted game,

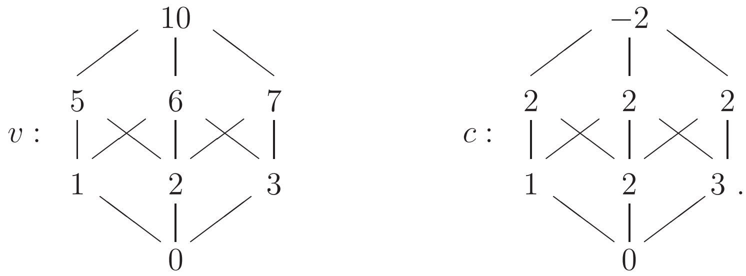
$$x_1 = \frac{2}{2} + \frac{4}{3} = 2\frac{1}{3}$$
$$x_2 = \frac{2}{2} + \frac{2}{2} + \frac{4}{3} = 3\frac{1}{3}$$
$$x_3 = 2 + \frac{2}{2} + \frac{4}{3} = 4\frac{1}{3},$$

which is equal to the Shapley value of the initial game. Hence, although the game changed, the Shapley value remained the same.

It would be very interesting to completely characterize the shifting techniques that preserve certain values. Clearly, as any solution ϕ is equal to the Shapley value of the game v_ϕ, defined as $v_\phi(\{i\}) = \phi(i), i \in N$ and $v_\phi(S) = 0$, for each $S \subseteq N, |S| \geq 2$, any game shift of v_ϕ that preserves the Shapley value would also preserve ϕ. Of course, there are many other shifts that preserve solution concepts.

We will give an example for the egalitarian solution proposed by Dutta and Ray [DR89]. Two principles are used: Maximization of average profit by coalition members and equal sharing.

Example 2 Consider the same 3-player game as in Example 1. The v-diagram and c-diagram are:

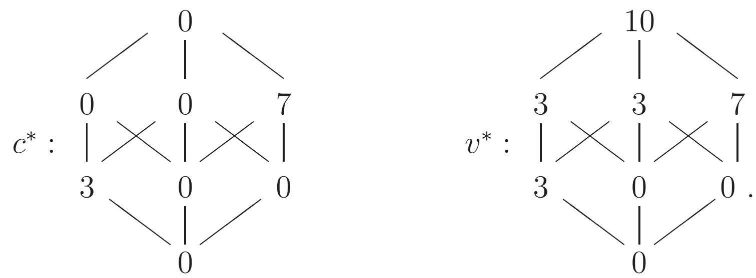


We have seen that the Shapley value is

$$x_1 = 2\frac{1}{3}, x_2 = 3\frac{1}{3}, x_3 = 4\frac{1}{3}.$$

The egalitarian solution concept determines the S with highest average. This is in our case $S = \{2, 3\}$, with average $3\frac{1}{2}$. Both players 2 and 3 get $3\frac{1}{2}$ and player 1 gets the rest, namely $10 - 7 = 3$. So $x_1^* = 3$, $x_2^* = 3\frac{1}{2}$, $x_3^* = 3\frac{1}{2}$.

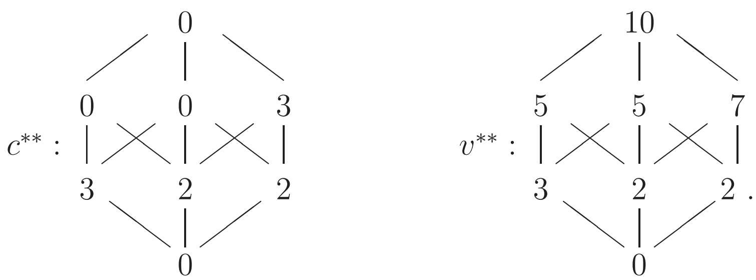
We now shift the c-diagram as follows. c_2 and c_3 are shifted to c_{23} and c_{123}, c_{12} and c_{13} are shifted to c_1. This yields a shift to the following c^*-diagram with corresponding v^*-diagram:



We have a game shift $v_c \mapsto v_c^*$ and applying the Shapley value to the c^*-diagram we obtain the allocation

$$\begin{aligned} x_1^* &= 3 + \frac{0}{2} + \frac{0}{2} + \frac{0}{3} = 3 \\ x_2^* &= 0 + \frac{0}{2} + \frac{7}{2} - \frac{0}{3} = 3\frac{1}{2} \\ x_3^* &= 0 + \frac{0}{2} + \frac{7}{2} - \frac{0}{3} = 3\frac{1}{2}. \end{aligned}$$

Note that we might also have shifted to maintain $c_2^{**} = 2$, $c_3^{**} = 2$ and $c_{23}^{**} = 3$. The c_{123}, c_{12}, c_{13} should, however, be shifted to c_1 as from them no contributions to player 2 or player 3 should be coming. We would have the diagrams:



We observe a c-diagram shift from c to c^{**}, with corresponding v-diagram shift, or game shift, from v to v^{**}, but without solution shift.

In the example above we have seen that shifting techniques might be used in the study of coalition formation. There is an essential difference between the situation in which an allocation is calculated from the cooperation rewards and fines by a value designed by a person from the outside, say a judge, and the situation in which the players are considered to determine the allocation themselves by *forming coalitions* on the basis of the cooperation rewards and fines. To give a very simple example we recall the egalitarian method, where, in Example 2, players 2 and 3 formed a coalition yielding them a share of $3\frac{1}{2}$ each, leaving 3 for player 1.

3 Reward games and fine games

In normal cooperative game theory margins play an important role. The quantity

$$m_i(S, v_c) = v_c(S \cup i) - v_c(S)$$

gives the contribution of player i to the value of the coalition S, on joining that coalition. Marginalistic values are solution concepts expressed in terms of margins. As $v_c(S)$ can be expressed in the costs c_T of the elements u_T of the standard set game $< N, v_s >$ associated with (N, v_c), we can express margins in terms of the c_T's. As $v_c(S) = \sum_{T \subseteq S} c_T$, we immediately have

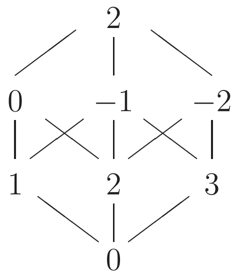
$$m_i(S, v_c) = \sum_{T \subseteq S} c_{T \cup \{i\}}$$

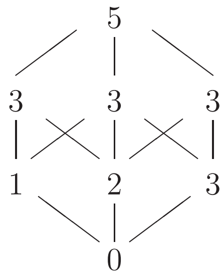
So the margin of player i with respect to the coalition S is the sum of all cooperation rewards and fines of the subcoalitions of which player i is a member. For a 3-player game we have e.g.

$$\begin{aligned} m_3(\{1,2\}, v_c) &= v_c(\{1,2,3\}) - v_c(\{1,2\}) \\ &= c_{123} + c_{12} + c_{13} + c_{23} + c_1 + c_2 + c_3 - c_{12} - c_1 - c_2 \\ &= c_{123} + c_{13} + c_{23} + c_3. \end{aligned}$$

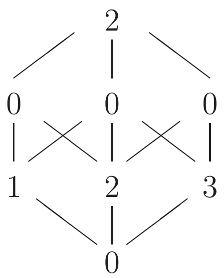
A solution concept expressed in terms of the c_S's has a finer structure than the same solution concept in terms of the margins. The c-diagram of the standard set game allows an interesting remark on solution concepts, also due to the distinction of cooperation rewards and fines. Before designing a solution concept we can separate the c-diagram into two c-diagrams, one having the positive c_S's and zero on those places in the diagram where the c_S's are negative and another having the negative c_S's and zeroes on those places in the diagram where the c_S's are positive. As a c-diagram determines a v-diagram, *i.e.* a game, we have split the game into two games now. As we have separated the cooperation rewards from the cooperation fines we call these two games *the reward game* and *the fine game* of the original game.

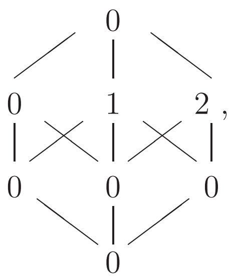
Example 3 Let the c-diagram for a 3-player game $(\{1, 2, 3\}, v)$ be

c :


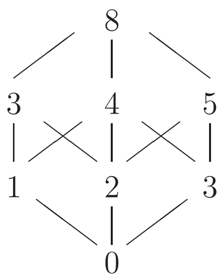
, so v :


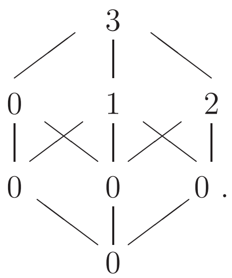
The c-diagram splits into

c_r :


and c_f :


where the minus signs have been omitted. The c_r-diagram and c_f-diagram define two games, with v_r-diagram and v_f-diagram.

v_r :


v_f :


Note that in both games the c_S's are nonnegative. It is known that both games are therefore convex and have non-empty core.

Designing a solution concept may be seen as deciding on how the cooperation rewards and cooperation fines should be allocated. Let x_r be a solution for the reward game and let x_f be a solution for the fine game. One can then allocate for the original game $x = x_r - x_f$. Note that we did not say anything about the solution concepts used. In fact for allocating the fines one may use another solution concept than for the rewards. The fairest way to split a cooperation reward, for some coalition, seems to split the reward into equal parts and allocate them to each of the members of the coalition. This would mean using the Shapley value for the reward game. In Example 3 we would get $x_r = (1\frac{2}{3}, 2\frac{2}{3}, 3\frac{2}{3})$. A cooperation fine, for some coalition, might be dealt with in the same way, and we would get $x_f = (\frac{1}{2}, 1, 1\frac{1}{2})$, so $x = (1\frac{1}{6}, 1\frac{2}{3}, 2\frac{1}{6})$, the Shapley value for the original game. However, as $c_f(\{1,3\}) = 1$ and $c_f(\{2,3\}) = 2$ are the only non-zero elements of the c_f -diagram, one might argue that player 3 is to blame for causing fines, so that he should be allocated the full fine of 3. We would then have $x_f = (0, 0, 3)$ and $x = x_r - x_f = (1\frac{2}{3}, 2\frac{2}{3}, \frac{2}{3})$. If this would be considered too hard a punishment for the non-cooperation of player 3, an alternative solution concept for the fine game would be to count the number of times a player is part of a coalition with cooperation fine. In Example 3 this would give 1, 1, and 2 as weights that are to be used in distributing the cooperation fines. Then $c_f(\{1,3\}) = 1$ would be split into $\frac{1}{3}$ for player 1 and $\frac{2}{3}$ for player 3 and $c_f(\{2,3\}) = 2$ would be split into $\frac{2}{3}$ for player 2 and $\frac{4}{3}$ for player 3. This would yield $x_f = (\frac{1}{3}, \frac{2}{3}, 2)$ and thus $x = (1\frac{2}{3}, 2\frac{2}{3}, 3\frac{2}{3}) - (\frac{1}{3}, \frac{2}{3}, 2) = (1\frac{1}{3}, 2, 1\frac{2}{3})$. As in this way the player who is involved most times in a cooperation fine is fined most, we consider this a fair procedure for allocating the fines. We summarize our considerations in the following solution method, which we call the *Reward and Fine Method*.

1. Consider a cooperative game (N, v).
2. Construct the c-diagram for its associated standard set game
3. Split the c-diagram into a reward diagram and and a fine digram
4. Consider the reward game (N, v_r) corresponding to the c_r-diagram and determine the Shapley solution x_r.
5. Consider the fine diagram and determine the vector w, with components w_i, giving the number of times player i is member of a coalition with a cooperation fine.
6. Split a cooperation fine c_S into parts proportional to the components w_i corresponding to the players that belong to S. Determine the allocation vector x_f of the fine game (N, v_f) by summing these parts for all n players.
7. Calculate $x = x_r - x_f$.

There are many solution concepts proposed in the literature of cooperative games. For games modeling a situation in which some *gain* is to be allocated to players, the proposed method seems fair, in particular the use of the Shapley value for the reward game. For the fine game easily other splittings of the cooperation fines might be proposed, that seem fair too. For a game in which a *cost* is to be shared the c-diagram can be split into two diagrams again, but the interpretation is then changing the picture. The positive c_S's are now costs, resulting from joint activities, whereas the negative c_S's can be seen as savings on the costs, due to cooperation. One is inclined to speak about a *Cost and Saving* method and to exchange the ways the two games, the *cost game* and the *saving game* are given a solution. Now the Shapley value seems fair for the saving game, whereas the method used for the fine game might be chosen for the cost game. After all, rewards and savings are typically cooperation bonuses, whereas fines and costs are cooperation maluses.

4 Analysis of some solution concepts for applications

In this section we include only a few of the many problems considered in the literature on cooperative game theory.

4.1 Bankruptcy problems

Bankruptcy problems belong to the oldest problems considered, see Moulin [M01], as already 2000 years ago in the Babylonian Talmud rules were given, solution concepts, how to allocate a given total T to two claimants with demands d_1 and d_2, $d_1 \leq d_2$, the so called Contested Garment problem. Let us consider this problem against the background of c-diagrams.

The value of $v(S)$ has to be defined. One way to do this is to simply take $v(\{1\})$ and $v(\{2\})$ as $mind_1, T$ and $mind_2, T$ and to take $v(\{1,2\}) = T$ as their joint claim. We then have the following v-diagram and c-diagram:

$$v : \begin{array}{ccc} & min\{d_1 + d_2, T\} & \\ min\{d_1, T\} & & min\{d_2, T\} \\ & 0 & \end{array} \qquad c_v : \begin{array}{ccc} & T - min\{d_1, T\} - min\{d_2, T\} & \\ min\{d_1, T\} & & min\{d_2, T\} \\ & 0 & \end{array} .$$

Now let us define $w(S)$ in another way, namely $w(S) = (T - d_{N \setminus S})_+$, where d_S is the sum of the claims of the players of S and $(a)_+ = max\{a, 0\}$. We obtain the following diagrams:

$$w \quad \begin{array}{ccc} & T & \\ (T - d_1)_+ & & (T - d_2)_+ \\ & 0 & \end{array} \qquad c_w : \begin{array}{ccc} & T - (T - d_1)_+ - (T - d_2)_+ & \\ (T - d_2)_+ & & (T - d_1)_+ \\ & 0 & \end{array} .$$

Denote by $D = T - \min\{d_1, T\} - \min\{d_2, T\}$. Applying the reward and fine method to the game v, we obtain

$$x_r = (\min\{d_1, T\} + \max\{\frac{D}{2}, 0\}, \min\{d_2, T\} + \max\{\frac{D}{2}, 0\})$$
$$x_f = (\min\{\frac{D}{2}, 0\}, \min\{\frac{D}{2}, 0\}).$$

. Hence, $x = (\min\{d_1, T\} + \frac{D}{2}, \min\{d_2, T\} + \frac{D}{2})$.

Now, applying the reward and fine method to w, we obtain:

$$x_r = ((T - d_2)_+ + \max\{\frac{\bar{D}}{2}, 0\}, (T - d_1)_+ + \max\{\frac{\bar{D}}{2}, 0\})$$
$$x_f = (\min\{\frac{\bar{D}}{2}, 0\}, \min\{\frac{\bar{D}}{2}, 0\}),$$

where $\bar{D} = T - (T - d_1)_+ - (T - d_2)_+)$. We obtain the allocation $x = ((T - d_2)_+ \frac{\bar{D}}{2}, (T - d_1)_+ \frac{\bar{D}}{2})$.

Note that in both cases we have obtained the, same, Contested Garment solution (see [M01]).

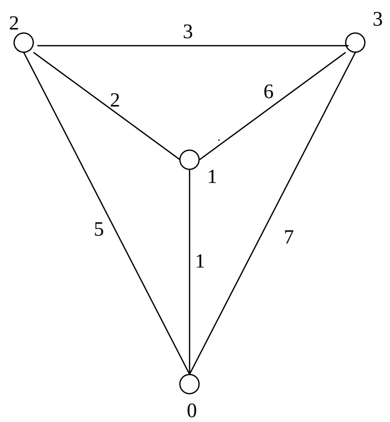
4.2 Minimum cost spanning tree problems

A well known type of cooperative game was introduced by Bird [B76], see also Aarts [A94]. Some source is supplying some facilities, say gas, to n players, say houses, by a network of transportation means, say gas pipes. Given are the costs for connecting the players to the source between each other. We can describe the situation by a complete graph on $n+1$ vertices, with edges carrying labels representing these costs. A minimum cost spanning tree is constructed. The problem is to allocate the cost of the constructed network to the players.

The modeling as a game, a so called *mcst-game*, is by defining the worth of a coalition S as the cost of the minimum cost spanning subtree of the graph on the vertices of S and the source. We will number the source by 0 and the n players by $1, ..., n$.

We will again choose a very simple example, with $n = 3$, to discuss the way set game theory ideas can be used to deal with this problem.

The graph is chosen to be

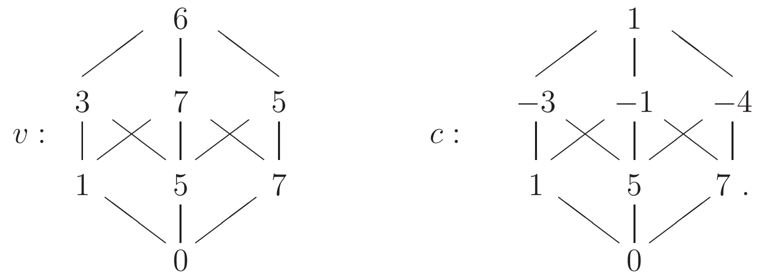


The labels represent the costs of the edges. The minimum cost spanning trees and subtrees are easily calculated. We find

$$
\begin{aligned}
v(\emptyset) &= 0 \qquad & v(\{1,2\}) &= 3 \\
v(\{1\}) &= 1 & v(\{1,3\}) &= 7 \\
v(\{2\}) &= 5 & v(\{2,3\}) &= 8 \\
v(\{3\}) &= 7 & v(\{1,2,3\}) &= 6
\end{aligned}
$$

Let us first consider the v-diagram and the c-diagram:

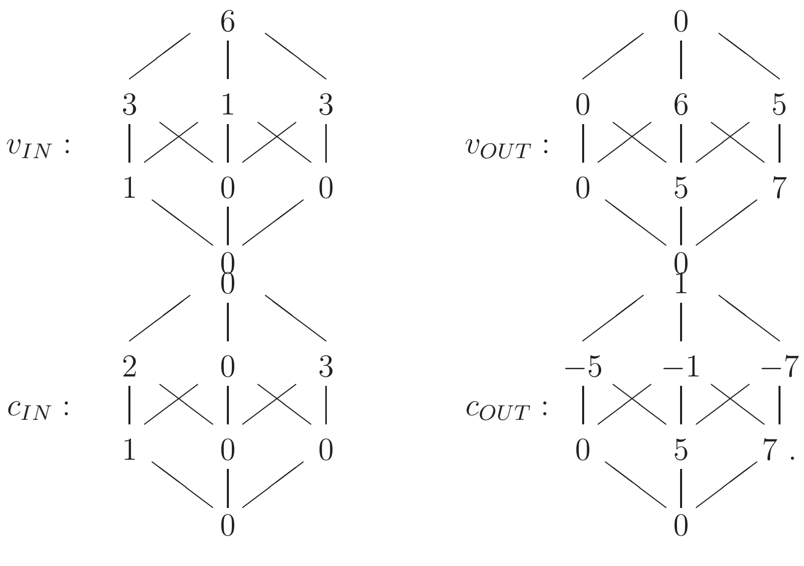
The Shapley value gives $x = (-\frac{2}{3}, 1\frac{5}{6}, 4\frac{5}{6})$. It turns out that player 1, closest to



the source 0, even gets paid. The reason can be said to be that this house is used as transit point for the other players.

One might, however, argue that, once the network is constructed, having in this case the form of a path,from 0 to 1 to 2 to 3, the egalitarian method of Dutta and Ray may be applied. This solution gives $x = (1, 2\frac{1}{2}, 2\frac{1}{2})$.

Yet another solution can be obtained on applying the idea of reward game and fine game. We make the distinction between elements that are IN the minimum cost spanning tree and elements that are OUTside. We can then split the v-diagram into two diagrams, v_{IN}-diagram and v_{OUT}-diagram, where the first describes the elements for which costs are actually made, and the second describes elements for which costs are avoided. The v_{IN}-diagram, v_{OUT}-diagram and the corresponding c-diagrams are:



We now can deal with the two game settings separately. The Shapley value gives $x_{IN} = (2, 2\frac{1}{2}, 1\frac{1}{2})$, the solution we met before, and $x_{OUT} = (-2\frac{2}{3}, -\frac{2}{3}, 3\frac{1}{3})$, so that $x = x_{IN} + x_{OUT} = (-\frac{2}{3}, 1\frac{5}{6}, 4\frac{5}{6})$ is regained. However, we might argue that there are avoided costs involved, profits that should be taken into account. They occur for the coalitions $\{2\}$, $\{3\}$, $\{1,3\}$ and $\{2,3\}$. The *profit frequencies* are given by $w = (1, 2, 3)$. Using this weight vector for the c_{OUT}-diagram we obtain

$$x_{OUT} = (-1\frac{3}{4}, -\frac{4}{5}, \frac{51}{20})$$

and combining this outcome with x_{IN}, formed by the Shapley value from the c_{IN}-diagram, we obtain

$$x = (\frac{1}{4}, 1\frac{7}{10}, 4\frac{1}{20}).$$

This is the reward and fine solution for the mcst-game.

5 A class of solidarity values

Next we will analyse some solidarity values from the perspective of c-diagrams. Nowak and Radzik [NR94] consider the marginal contribution

$$m_i(S) = v(S) - v(S - \{i\})$$

as it occurs in the Shapley value and replace it by

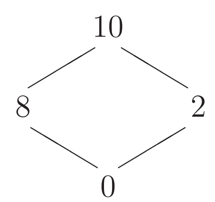
$$A^v(S) = \frac{1}{|S|} \sum_{j \in S} m_j(S),$$

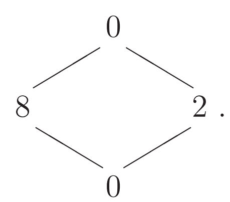
so by the average contribution for the players belonging to the coalition S. This way players with a high marginal contribution share it to some extent with the players with a low marginal contribution. For this reason

$$f_i = \sum_{S \ni i} \frac{(n - |S|)!(|S| - 1)!}{n!} A^v(S)$$

is called a *solidarity value*.

We will consider the idea of sharing contributions in the context of c-diagrams and shifting of c-diagrams. We start with a simple two player example. Let the v-diagram and the c-diagram of the two-player game be

v : 

c : 

The Shapley value $f_i = \sum\limits_{S \ni i} \frac{1}{|S|} c(S)$ gives the allocation $a = (8, 2)$. We introduce a variable σ, for solidarity, describing what is shifted from $c(\{1\})$ to $c(\{2\})$ and from $c(\{2\})$ to $c(\{1\})$ in the following way. The c^{SOL}-diagram becomes:

$$\begin{array}{ccc} & 0 & \\ \frac{1}{2}\sigma 8 + (1 - \frac{1}{2}\sigma)2 & & (1 - \frac{1}{2}\sigma)8 + \frac{1}{2}\sigma 2 \\ & 0 & \end{array}.$$

So a fraction of the c's is given to the other player. If $0 \leq \sigma \leq 1$, then at most half of the c-contribution is given to the other player. $f_i(\sigma) = \sum\limits_{S \ni i} \frac{1}{|S|} c^{SOL}(S)$, with $c^{SOL}(\{1\}) = 8 - 3\sigma$ and $c^{SOL}(\{2\}) = 2 + 3\sigma$, gives an allocation $a = (8 - 3\sigma, 2 + 3\sigma)$ for the simple example. Note that for $\sigma = 0$ no solidarity is involved and that for $\sigma = 1$ we obtain complete solidarity as then, calculating the Shapley value applied to the shifted c^{SOL}-diagram, $a = (5, 5)$. In fact, we have an infinite class of solidarity values.

We now generalize this procedure as follows. Given a c-diagram, we consider all coalitions with the same cardinality and let these coalitions share the same part of their c_S with the other $\begin{pmatrix} n \\ |S| \end{pmatrix} - 1$ coalitions. Note that this differs from the procedure chosen by Novak and Radzik. We define

$$c_S^{SOL}(\sigma) = \left[1 - \frac{\begin{pmatrix} n \\ |S| \end{pmatrix} - 1}{\begin{pmatrix} n \\ |S| \end{pmatrix}} \sigma \right] c_S + \frac{\begin{pmatrix} n \\ |S| \end{pmatrix} - 1}{\begin{pmatrix} n \\ |S| \end{pmatrix}} \sigma \left[\frac{1}{\begin{pmatrix} n \\ |S| \end{pmatrix} - 1} \sum_{\substack{S^* \neq S \\ |S^*| = |S|}} c_{S^*} \right].$$

The class of values is given by

$$f_i(\sigma) = \sum_{S \ni i} \frac{1}{|S|} c_S^{SOL}(\sigma), \; 0 \leq \sigma \leq 1.$$

So we consider the Shapley value for the c^{SOL}-diagram. For $\sigma = 0$ we regain the Shapley value, whereas for $\sigma = 1$ we obtain

$$c_S^{SOL}(1) = \frac{1}{\begin{pmatrix} n \\ |S| \end{pmatrix}} c_S + \frac{1}{\begin{pmatrix} n \\ |S| \end{pmatrix}} \sum_{\substack{S^* \neq S \\ |S^*| = |S|}} c_{S^*} = \frac{1}{\begin{pmatrix} n \\ |S| \end{pmatrix}} \sum_{|T| = |S|} c_T.$$

So, for $\sigma = 1$ layer-wise the values of the c_S's are equal and therefore the Shapley value is the same for all players and we have complete solidarity. The c-diagram makes the sharing process quite transparent.

In order to discuss the concept of solidarity value somewhat further, against the background of c-diagrams, we consider the axiomatization of values. Let F be a value which satisfies the additivity axiom (ADD), the efficiency axiom (EFF) and the equal treatment property (ETP). Consider the decomposition of a cooperative game into unanimity games. The additivity axiom leads to the consideration of one the $c_S's$ separately for each subset S. The efficiency axiom forces F to distribute c_S completely over the players. The equal treatment property leads to breaking up c_S into $s = |S|$ equal parts, *if no part of c_S is distributed over the players outside S.* If we only have the axioms ADD, EFF and ETP, we are left with the decision of breaking up the dividend/cost c_S.

Let us now look how such a decision is taken. The dummy player property can be expressed as "no dummy players should get a part of c_S". This leads to the Shapley value. However, in the context of solidarity values, we might also say "no part of c_S is to be allocated to any player outside S". Let us now look at the solidarity value of Nowak and Radzik. Replacing the marginal contribution $m_i(S)$ by the average contribution $A^v(S)$, is a rather complicated transformation in terms of costs. In terms of c_S's we have

$$m_i(S) = \sum_{\substack{T \subseteq S \\ T \ni i}} c_T \text{ is replaced by } A^v(S) = \frac{1}{|S|} \sum_{j \in S} \sum_{\substack{T \subseteq S \\ T \ni j}} c_T.$$

For the characterization of their value, Novak and Radzik had to replace the dummy player axiom by the socalled A-dummy player axiom, thus allowing that a normal normal dummy player can have a share in the marginal contributions of other players of a coalition. Only when the $A^v(S)$'s are zero for those S to which a player belongs, the player should have allocation zero. However, the sharing procedure is restricted to the members of S, and many more sharing procedures are thinkable.

At this point, the whole axiomatization is put in a different light. ADD, EFF and ETP are axioms that reduce the design of a value to the choice of a sharing rule for c_S. Any sharing rule determines the value, i.e. determines the $\lambda_{S,i}$ in the general expression of a solution. Instead of formulating some extra axiom and proving that now some new value has been uniquely determined, one might focus on sharing rules added to the axioms mentioned. In particular for solidarity values this seems a natural approach.

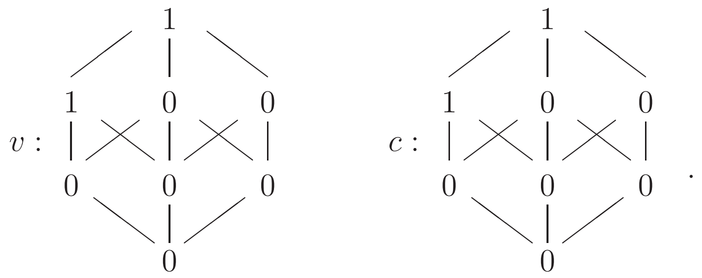
Of the many possibilities to define a solidarity value, let us consider the following sharing rule. Part of c_S is distributed over the $n - |S|$ players outside S, "out of solidarity". The completely solidary way would be to distribute c_S equally over all n players. Then $\frac{|S|}{n} c_S$ is going to the members of S and $\frac{n-|S|}{n} c_S$ is going to the players outside S. Note that we have taken into account axiom ETP.

A player not in S might receive $\sigma \frac{c_S}{n}$, $0 \leq \sigma \leq 1$, leaving $c_s - \sigma \frac{(n-|S|)}{n} c_S$ for the members of S, who get allocated $\frac{1}{|S|}\left(1 - \sigma \frac{(n-|S|)}{n}\right) c_S$ each. The allocation to player i now becomes

$$f_i(\sigma) = \sum_{S \ni i} \frac{1}{|S|} \left(1 - \sigma \frac{(n - |S|)}{n}\right) c_S + \sum_{Sni \notin S} \frac{\sigma}{n} c_S, 0 \leq \sigma \leq 1,$$

which is a class of rather natural allocations, maybe preferable to the one given before. For each player something drops off the costs c_S of coalition S to which he does not belong. Again for $\sigma = 0$ we regain the Shapley value and for $\sigma = 1$ we have $f_i(1) = \frac{1}{n}v(N)$, the completely solidary allocation.

We conclude with an example of Novak and Radzik, called "the three brothers". The v-diagram and c-diagram are



The Shapley value gives $f = (\frac{1}{2}, \frac{1}{2}, 0)$, so the helpless brother, player 3, gets nothing. The solidarity value of Novak and Radzik gives $f = (\frac{7}{18}, \frac{7}{18}, \frac{4}{18})$, and brother 3 is allocated $\frac{4}{18}$ out of solidarity. Our second class of solidarity values gives $f(\sigma) = (\frac{1}{2}(1 - \frac{\sigma}{3}), \frac{1}{2}(1 - \frac{\sigma}{3}), \frac{\sigma}{3})$. Indeed, we have $f(0) = (\frac{1}{2}, \frac{1}{2}, 0)$ and $f(1) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The Novak and Radzik value shows solidarity $\sigma = \frac{2}{3}$.

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