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A new PTAS for maximum independent sets in unit disk graphs

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A new PTAS for Maximum Independent Sets in Unit Disk Graphs

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Abstract

A unit disk graph is an intersection graph of unit disks in the euclidean plane. We present a polynomial-time approximation scheme for the maximum independent set problem in unit disk graphs. In contrast to previously known approximation schemes, our approach does not require a geometric representation (specifying the coordinates of the disk centers).

Key words: Independent Set, Geometric Intersection Graph, PTAS
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1 Introduction

A unit disk graph (*UDG*) is the intersection graph of unit disks in the plane. In other words, $G = (V, E)$ is a UDG if there exists a map $f : V \to \mathbb{R}^2$ (a *geometric representation*) satisfying

$$(u, v) \in E \iff \|f(u) - f(v)\| \leq 2, \tag{1}$$

where $\|.\|$ denotes the euclidean norm.

Our goal is to give a *polynomial-time approximation scheme* (PTAS) for the maximum independent set problem in unit disk graphs for the case that a geomtric representation is not given. In other words, we seek for an algorithm which, given as input a UDG $G = (V, E)$ and a parameter $\varepsilon > 0$, computes an independent set $I \subset V$ of size at least $(1+\varepsilon)^{-1}$ times the maximum size of an independent set in G. For an independent set, the induced subgraph contains no edges. The running time of the algorithm is allowed to depend on ε, but should be polynomial in $n = |V|$ for fixed $\varepsilon > 0$.

Most of the work concering approximation schemes in unit disk graphs has been done assuming a geometric representation given. It then becomes possible to perform separation of the graph alongside a grid. This so called *shifting strategy* is presented in [1] and [4]. Combined with a dynamic programming approach, the shifting strategy is used by Erlebach et. al. [3] to give a PTAS for the maximum (weight) independent set in disk graphs of arbitrary diameter. The shifting strategy can also be applied to related problems like minimum vertex cover, and minimum dominating set [5].

The case where no geometric representation f for the UDG $G = (V, E)$ is available is significantly different: Computing a corresponding representation function f for a given UDG $G = (V, E)$ is NP-hard. Indeed, any polynomial time algorithm computing geometric representation functions for unit disk graphs could be used in a straightforward way to solve the UDG recognition problem (determine wether a given graph is a UDG), which is known to be NP-hard [2].

When there is no geometric representation given, a PTAS for the maximum independent set or related problems was not known previously. However, constant factor approximation algorithms can be applied. A simple greedy strategy gives a 5-approximation for the maximum independent set, and a more sophisticated choice of a node to be greedily added to the partial set of independent nodes gives an approximation within a factor of 3 [6]. Both algorithms work without given representation, however, the running time can be improved a lot when the representation is given.

It is therefore interesting to investigate to what extend the geometric representation is really needed for solving optimization problems on unit disk graphs. The problem of finding a maximum independent set, for example, arises in the context of clustering in ad-hoc networks [7]. The nodes, each equipped with a radio transceiver of fixed transmission range, form a UDG representing the communication network. Nodes of a maximum independent set, which also form a dominating set, can be used as controlling instances of the other nodes within range their radio. In the ad-hoc scenario, it might not be possible to determine the exact position of each node and therefore the resulting network can only be represented by the nodes and communication links between them.

The following section introduces the algorithm that gives the PTAS. In Section 3, we identify some other classes of geometric intersection graphs for which our approach also is efficient. The paper ends with a short conclusion.

2 The Approximation Algorithm

Let $\varepsilon > 0$ and let $\rho := 1+\varepsilon$ denote the desired approximation guarantee. Thus, given a unit disk graph $G = (V, E)$, we seek to construct an independent set $I \subset V$ of size at leat ρ^{-1} times $\alpha(G)$, the maximum size of an independent set in G.

The basic idea is simple. We start with an arbitrary node $v \in V$ and consider for $r = 0, 1, 2, \ldots$, the r^{th} neighborhood

$$N^r = N^r(v) := \{w \in V | w \text{ has distance at most } r \text{ from } v\}.$$

Starting with N^0, we compute a maximum independent set $I_r \subset N^r$ for each $r = 0, 1, 2, \ldots$ as long as

$$|I_{r+1}| > \rho |I_r| \tag{2}$$

holds.

Let r_1 denote the smallest $r \geq 0$ for which (2) is violated. Such an $r_1 \geq 0$ indeed exists and it is bounded by a constant (depending on ρ):

Lemma 1 *There exists a constant $c = c(\rho)$ such that $r_1 \leq c$.*

PROOF. From (1), we conclude that any $w \in N^r$ satisfies

$$\|f(v) - f(w)\| \leq 2r.$$

So, the unit disks corresponding to nodes in I_r are pairwise disjoint and are all contained in a disk of radius $R = 2r + 1$ around $f(v)$. This implies

$$|I_r| \leq \pi R^2/\pi = O(r^2). \tag{3}$$

On the other hand, by definition of r_1, we have for $r < r_1$

$$|I_r| > \rho|I_{r-1}| > \ldots > \rho^r|I_0| = \rho^r. \tag{4}$$

Comparing (3) and (4), the claim follows. $\square$

Due to (3), we may compute I_r by complete enumeration in time $O(n^C)$, where $C = O(r^2) = O(c^2)$ for $r \leq r_1$.

Now consider $G' := G \setminus N^{r_1+1}$ and assume inductively that we can efficiently compute a ρ-approximate independent set $I' \subset V \setminus N^{r_1+1}$ for G'. We claim that $I := I_{r_1} \cup I'$ is a ρ-approximate independent set for G.

Since each $v \in I' \subset V \setminus N^{r_1+1}$ has no neighbor in N^{r_1}, and thus not in $I_{r_1} \subset N^{r_1}$, I is an independent set.

Furthermore, by definition of r_1, we have

$$|I_{r_1+1}| \leq \rho|I_{r_1}|.$$

In other words, the subgraph $G[N^{r_1+1}]$ induced by N^{r_1+1} has a maximum independent set size bounded by

$$\alpha(G[N^{r_1+1}]) \leq \rho|I_{r_1}|.$$

Further, by assumption, I' is approximately optimal for $G' = G[V \setminus N^{r_1+1}]$. Thus,

$$\alpha(G[V \setminus N^{r_1+1}]) \leq \rho|I'|.$$

Adding the two inequalities, we obtain

$$\alpha(G) \leq \alpha(G[N^{r_1+1}]) + \alpha(V \setminus G[N^{r_1+1}]) \leq \rho|I|,$$

as claimed.

3 Extensions

It is straightforward to verify that our arguments apply equally well to intersection graphs of some other geometrical objects related to unit disks. For example, the unit disks may be replaced by disks with fixed lower and upper bound on the radius (*bounded disk graphs*). Similarly, an extension to (fixed) dimension $d \geq 2$ is possible. Indeed, all that is needed in the proof is a polynomial bound on the maximum geometric diameter divided by the minimum volume of the objects under consideration.

The algorithm can also be applied to λ-precision disk graphs that have no explicit bound on the radius of the disks and thus no bound on the minimum volume of the disks. A λ-precision disk graph is an intersection disk graph where two vertices are at least λ apart in a geometric representation that has been scaled so that the disks have a maximum radius of 1 [5]. In this case, the size of the r^{th} neighborhood, $|N^r|$, is already polynomially bounded in r. Note that this is a different condition as the one given above: for example, in a UDG, two vertices may be arbitrarily close as the graph can contain arbitrarily large cliques.

The independent set created by the PTAS may not be maximal. However, a simple greedy strategy on the nodes in $N^{r_1+1} \setminus N^{r_1}$ that are not connected by an edge to a node from the independent set can resolve this problem. The obtained independent set then also forms a dominating set in the graph.

4 Conclusion

In this paper, we presented a new PTAS for the maximum independent set problem in UDGs that does not depend on a geometric representation of the vertices. Some extensions to different, related families of geometric intersection graphs where presented as well.

Future work focuses on application of the same ideas to related problems like minimum vertex cover or dominating sets in UDGs without given geometric representation, and extensions towards weighted versions of these problems.

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