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Complexity results for restricted instances of a paint shop problem

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Complexity Results for Restricted Instances of a Paint Shop Problem

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Abstract

We study the following problem: an instance is a word with every letter occurring twice. A solution is a 2-coloring of the letters in the word such that the two occurrences of every letter are colored with different colors. The goal is to minimize the number of color changes between adjacent letters. This is a special case of the Paint Shop Problem, which was previously shown to be $\mathcal{NP}$-hard. We show that this special case is also $\mathcal{NP}$-hard and even $\mathcal{APX}$-hard.

Keywords: Paint Shop, APX-hardness, NP-completeness
MSC: 68Q25, 68R15, 90B30

1 Introduction

An *alphabet* is a set of letters. A *word* is an ordered sequence of letters from this alphabet. The same letter can appear at multiple positions in a word. Epping, Hochstättler and Oertel [4] studied the following problem: an instance consists of a word W, a set of colors $S = \{1, \ldots, c\}$ and a color requirement $r_{xi} \in \mathbb{N}$ for every different letter x in the word and every color $i \in S$. If letter x occurs k times in the word, $\sum_{i \in S} r_{xi} = k$. This color requirement states that r_{xi} occurrences of letter x should be colored with color i. A feasible solution for this problem is a coloring of the letters satisfying the color requirement. If two adjacent letters x and y are colored with a different color, we say there is a *color change* between x and y. The goal is to find a feasible solution that minimizes the number of color changes.

This problem is called the *Paint Shop Problem*, and is motivated by an application in car manufacturing: different letters represent different car types. Customers order a combination of a car type and a certain color, so for every car type it is known how many of each color should be produced. The word represents the order in which the uncolored car bodies arrive at the paint shop part of the factory (it is assumed that this order can not be changed). Changing colors is a costly process and should be minimized.

In [4], it is shown that this problem is $\mathcal{NP}$-complete even when restricted to instances where only two different letters occur in the word, and also when restricted to instances where only two different colors are used. In this paper, the problem is shown to be $\mathcal{NP}$-complete for an even smaller subset of the instances: instances with two colors and $r_{xi} = 1$ for every x and i. In addition, the problem is also shown to be $\mathcal{APX}$-hard. This answers one of the problems stated in [4].

2 Preliminaries

Let $G = (V, E)$ denote a simple, undirected graph. The set of neighbors of a vertex $v \in V$ is denoted by $N(v)$. The degree $d(v)$ of a vertex is equal to $|N(v)|$. An *at-most-cubic graph* is a graph in which every vertex has degree at most 3. In the remainder we will use the shorter name *cubic graphs* for these graphs. The edge incident with vertex i and j is denoted by ij or ji. Edges with a direction assigned to them are called *arcs*. An arc from i to j is denoted by

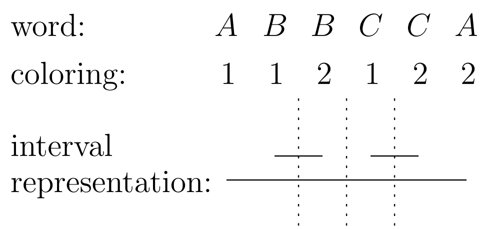

Figure 1: An Interval Representation of a Paint Shop Instance and Solution

(i, j). In this case, i is an *in-neighbor* of j and j is an *out-neighbor* of i. Directed graphs are denoted by $G = (V, A)$. A *vertex cover* of a (directed) graph is a set $S \subset V$ such that every edge (arc) is incident with at least one vertex of S.

An algorithm for a minimization problem is called a *ρ-approximation algorithm* if for every instance with optimal solution value k it gives a solution with value at most ρk. Here, $\rho = 1 + \epsilon$ with $\epsilon > 0$. For maximization problems the definition is similar. A *PTAS* for a minimization problem is a scheme to find polynomial time $(1 + \epsilon)$-approximation algorithms for *every* $\epsilon > 0$. The problem class $\mathcal{APX}$ is the class of problems for which a polynomial time ρ-approximation algorithm exists for some $\rho > 1$ ($0 < \rho < 1$ for maximization problems). A problem is called $\mathcal{APX}$-hard if the existence of a PTAS for this problem would imply that for every problem in $\mathcal{APX}$ a PTAS exists. Moreover, Arora, Lund, Motwani, Sudan and Szegedy [2] have shown that:

Theorem 1 *If there exists a PTAS for some $\mathcal{APX}$-hard problem, then $\mathcal{P}$=$\mathcal{NP}$,*

which is quite unlikely.

3 Results

In this section, the $\mathcal{APX}$-hardness of the following problem is proved:

Problem: 1-Regular Two Color Paint Shop
Instance: A word with every letter occurring exactly twice.
Solution: An assignment of the colors 1 and 2 to the letters such that every letter is colored with color 1 once and colored with color 2 once.
Goal: Minimize the number of color changes.

This proof is an example of an L-reduction as introduced in [5].

There is an insightful way to represent instances and solutions to this problem [3] which is shown in Figure 1. Every letter pair is represented by a horizontal line segment (an interval). Vertical line segments represent the color changes. If we know the positions of the color changes, a corresponding color assignment is easily constructed. Observe that this represents a feasible solution iff every interval crosses an odd number of vertical line segments (between every letter pair there is an odd number of color changes). The 1-Regular Two Color Paint Shop problem is proved to be $\mathcal{APX}$-hard using the following problem:

Problem: Cubic Graph Vertex Cover.
Instance: A cubic graph $G = (V, E)$.
Solution: A vertex cover S.
Goal: minimize $|S|$.

In [1] this problem is shown to be $\mathcal{APX}$-hard (actually it is even shown for real cubic graphs, so for graphs in which every vertex has degree 3). First we show that vertex cover is $\mathcal{APX}$-hard when restricted to instances with a special property.

Property 1 *Let $G = (V, A)$ be a directed graph with $V = \{1, \ldots, n\}$. We say that G satisfies Property 1 if every vertex $v \in V$ has at most one in-neighbor u with $u < v$, and at most one in-neighbor w with $w > v$.*

Problem: Directed Graph Vertex Cover.
Instance: A directed graph $G = (V, A)$ with $V = \{1, \ldots, n\}$ satisfying Property 1.
Solution: A vertex cover S.
Goal: Minimize $|S|$.

Lemma 2 *Directed Graph Vertex Cover is $\mathcal{APX}$-hard.*

Proof: Let a graph G be an instance of Cubic Graph Vertex Cover. Subdivide every edge $uv \in E(G)$ with two vertices. These are called the *new uv-vertices*. This gives a graph G'.

Now the edges of G' have to be directed and the vertices of G' have to be numbered such that G' satisfies Property 1. First we number the vertices from $1, \ldots, n$ ($n = |V(G')|$) such that no vertex has 3 neighbors with a lower number or 3 neighbors with a higher number. All the new vertices in G' have degree 2, so for those vertices this property is trivially satisfied. For all $v \in V(G)$ with $d(v) = 3$, label at least one neighbor in G' with 'L' and at least one with 'R'. Only degree 2 vertices will be labeled. Now number the vertices as follows: first number the vertices labeled with 'L' arbitrarily with $1, \ldots, t_1$, then number all unlabeled vertices with $t_1 + 1, \ldots, t_2$, and finally number the vertices labeled with 'R' with $t_2 + 1, \ldots, n$. Clearly, every vertex u with degree 3 has at least one neighbor v with $v < u$ and at least one neighbor w with $w > u$.

Next, we direct the edges one by one. We say the undirected edge uv is *forced at u* if there is an arc (w, u) directed towards u and either $u < v$ and $u < w$ or $v < u$ and $w < u$. If an edge uv is not forced at u or v, it can be directed both ways without violating Property 1. Start with directing an arbitrary edge uv in G' towards u. If this introduces a forced edge at u, direct this edge away from u. Otherwise choose another arbitrary edge to direct. Since every vertex has at most two neighbors with a lower (higher) number, at every step there is at most one forced edge (and it is forced at only one end vertex). Therefore this algorithm will direct all edges in G' such that Property 1 is satisfied.

This shows how a Directed Graph Vertex Cover instance G' can be constructed from a Cubic Graph Vertex Cover instance G. We proceed to show that a vertex cover of G' with cardinality $k + |E(G)|$ can be used to construct a vertex cover of cardinality k of G and vice versa.

Let S be a vertex cover of G'. Consider an edge $uv \in E(G)$. If $u \notin S$ and $v \notin S$, then both of the new uv-vertices must be in S. Add u to S, and remove the new uv-vertex closest to u from S. This gives a new vertex cover. Repeating this for every edge in G gives a vertex cover S' of G' such that every edge in G is incident with at least one vertex in S', with $|S'| = |S|$. Furthermore, for every edge $uv \in E(G)$, at least one of the new uv-vertices is present in S'. So $S'' = S' \cap V(G)$ is a vertex cover of G with cardinality $|S''| = |S| - |E(G)|$.

Next, let S be a vertex cover of G. We extend this to a vertex cover S' of G'. Start with $S' = S$. For every edge $uv \in E(G)$, if only one of the vertices u and v is in S, then add the uv-vertex with distance 2 to this vertex to S'. If both are in S, arbitrarily choose one of the uv-vertices and add it to S'. This gives a vertex cover S' of G' with $|S'| = |S| + |E(G)|$.

We use this reduction to prove the $\mathcal{APX}$-hardness of Directed Graph Vertex Cover. Suppose a polynomial approximation algorithm for Directed Graph Vertex Cover exists with performance guarantee $1 + \epsilon$. For every instance $G = (V, E)$ of Cubic Graph Vertex Cover with minimum vertex cover S, we construct a Directed Graph Vertex Cover instance G', with minimum vertex cover of cardinality $|S'| = |S| + |E|$. The approximation algorithm for this instance gives a solution T', with $|T'| \leq (1 + \epsilon)|S'|$. Use T' to find a solution T for instance G with cardinality $|T'| - |E|$. Observe that since G is cubic, $3|S| \geq |E|$, so $4|S| \geq |E| + |S|$. Now:

$$\frac{|T| - |S|}{|S|} = \frac{|T'| - |E| - (|S'| - |E|)}{|S|} \leq 4\frac{|T'| - |S'|}{|E| + |S|} = 4\frac{|T'| - |S'|}{|S'|} \leq 4\epsilon$$

So a PTAS for Directed Graph Vertex Cover would give a PTAS for Cubic Graph Vertex Cover. Since Cubic Graph Vertex Cover is $\mathcal{APX}$-hard, it follows that Directed Graph Vertex Cover is also $\mathcal{APX}$-hard. □

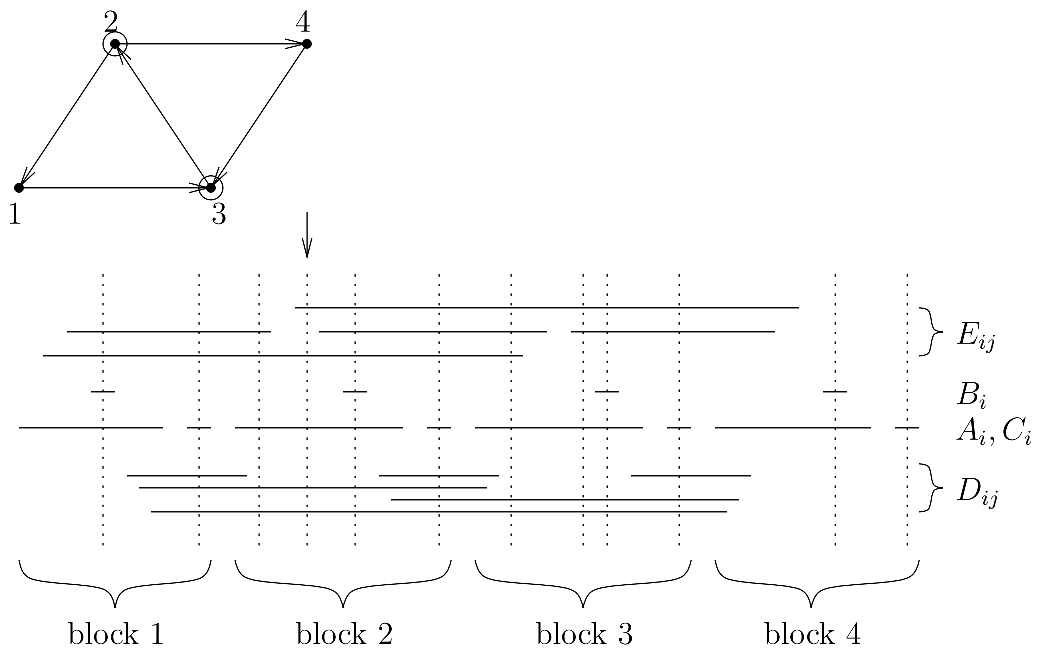

Figure 2: Graph G with Cover S and the Corresponding Paint Shop Instance and Solution

Theorem 3 *1-Regular Two Color Paint Shop is $\mathcal{APX}$-hard.*

Proof: Let $G = (V, A)$ with $V = \{1, \ldots, n\}$ be an instance of Directed Graph Vertex Cover. We use G to construct an instance for 1-Regular Two Color Paint Shop. This instance will consist of n blocks of letters, one block for every vertex. For every $i \in V$ we will introduce letters A_i, B_i and C_i that only appear in block i. For every arc $a \in A$ we will introduce letter E_a that appears once in both of the blocks corresponding to the end vertices. For every vertex pair $i, j \in V$ with $i < j$ we will introduce letter D_{ij}, that appears once in block i and once in block j.

Observe that because G satisfies Property 1, for every vertex v we can find an order $\prec$ on the arcs incident with v such that:

- if $a = (u, v)$ and $u < v$ then $a \prec b$ for every other arc b.
- if $a = (u, v)$ and $v < u$ then $b \prec a$ for every other arc b.
- If $a \in A$ is incident with u and v and $b \in A$ is incident with v and w and $u < v < w$ then $a \prec b$.

If vertex i is incident with arcs a, b and c and the order $a \prec b \prec c$ satisfies the above properties, then we introduce the following block for v:

$$A_i D_{1i} D_{2i} \ldots D_{(i-1)i} E_a E_b E_c B_i B_i D_{i(i+1)} D_{i(i+2)} \ldots D_{in} A_i C_i C_i$$

If a vertex i has fewer than three neighbors, the block is similar. Now order the blocks left to right from 1 to n. Taken together this gives the word W for the paint shop problem. See Figure 2 for an example.

Claim 1 *If the 1-Regular Two Color Paint Shop instance W has a solution with at most $2|V| + 2k$ color changes, then G has a vertex cover using at most k vertices.*

To prove this, we make the following observations that are true for every feasible solution of instance W:

1. Because of the place of the B_i pair and the C_i pair in block i, in block i there are at least two color changes, for every i.

2. If there is an arc $a \in A$ between i and j, then in any optimal solution of the paint shop instance, either in block i or in block j there are at least four color changes.

We prove this last observation by contradiction. Suppose a is an arc between i and j, $i < j$ and in block i and in block j there are at most 3 color changes. Since one of the color changes in block i must be between C_iC_i, there are at most two color changes between the A_i pair. Since the number of color changes between a letter pair must be odd, there is only one color change between the A_i pair. The same holds for the A_j pair. This color change between the A_i (A_j) pair must be between the B_i (B_j) pair. In block i, E_a appears before the B_i pair, whereas D_{ij} appears after it. In block j, E_a and D_{ij} both appear before the B_j pair. This means that there is an odd number of color changes between the E_a pair iff there is an even number of color changes between the D_{ij} pair, a contradiction.

From the two observations above it follows that if there is a solution with at most $2|V| + 2k$ color changes, then G has a vertex cover with at most k vertices.

Claim 2 *If G has a vertex cover of cardinality k, there is a solution for the paint shop instance W with $2|V| + 2k$ color changes.*

Let S be a vertex cover of G. If $i \notin S$, we apply only two color changes in block i: one between B_iB_i and one between C_iC_i. If $i \in S$, we use four color changes in block i: between B_iB_i and C_iC_i but also between the consecutive letters $D_{(i-1)i}E_a$ and between the consecutive letters E_cB_i. If there is an arc $a \in A$ with end vertices $i \in S$ and $j \in S$ ($i < j$), we know that either E_aB_i or $D_{(j-1)j}E_a$ is a consecutive letter combination in W, since a is directed towards i or towards j. In this case we move the color change one position to the left (between E_bE_a for some b) respectively one position to the right (between E_aE_b for some b). This ensures that for every a between i and j with $i < j$, there is either one color change in block i between E_a and B_i and no color change in block j between $D_{(i-1)i}$ and E_a or there is no color change in block i between E_a and B_i and one color change in block j between $D_{(i-1)i}$ and E_a. In Figure 2 an example of such a paint shop solution corresponding to a vertex cover is shown. The color change marked with an arrow is the only color change that was moved (to the left) in this last step. We prove that this method gives a feasible solution:

1. In block i, there is one color change to the right of D_{ij}: this is the color change between C_iC_i. In block j, there is no color change to the left of D_{ij}. In every block between i and j there is an even number of color changes. There are no color changes between blocks. So there is an odd number of color changes between the two D_{ij} letters for every $i < j$.

2. Let a be an arc between i and j, $i < j$. We know that either there is a color change in block i between E_a and the first B_i, or there is a color change in block j between D_{ij} and E_a, but not both. In addition, there is exactly one color change in block i between the first B_i and D_{ij} (this is the color change between B_iB_i). So there is an odd number of color changes between the E_a pair if there is an odd number of color changes between the D_{ij} pair. This was shown to be true, so for every a there is an odd number of color changes between the two E_a letters.

3. Between every A_i pair, there are either one or three color changes.

4. Between B_iB_i and between C_iC_i there is a color change.

We conclude that this is a feasible solution with exactly $2|V| + 2|S|$ color changes.

Now suppose a $(1 + \epsilon)$-approximation algorithm for 1-Regular Two Color Paint Shop exists. We use this to construct a $(1 + 5\epsilon)$-approximation algorithm for Directed Graph Vertex Cover. Let $G = (V, A)$ be a Directed Graph Vertex Cover instance with minimum vertex cover S. We use the above transformation to construct a paint shop instance W, with minimum number of color changes $m = 2|S| + 2|V|$. Use the approximation algorithm to find a solution with $k \leq (1 + \epsilon)m$ color changes. This gives a vertex cover of cardinality $k' \leq k/2 - |V|$. Observe that since G is cubic, $|V| = |\cup_{v \in S} (N(v) \cup \{v\})| \leq 4|S|$, so $5|S| \geq |V| + |S|$. So:

$$\frac{k' - |S|}{|S|} \leq \frac{k/2 - |V| - (m/2 - |V|)}{|S|} \leq 5\frac{k/2 - m/2}{|V| + |S|} = 5\frac{k - m}{m} \leq 5\epsilon.$$

So a PTAS for 1-Regular Two Color Paint Shop would give a PTAS for Directed Graph Vertex Cover. Therefore 1-Regular Two Color Paint Shop is also $\mathcal{APX}$-hard. □

Corollary 4 *The decision version of the 1-Regular Two Color Paint Shop Problem is $\mathcal{NP}$-complete.*

Proof: If a polynomial algorithm for the decision problem exists, fewer than $|W|$ calls ($|W|$ denotes the number of letters in W) have to be used to solve the optimization version of the problem. So, using Theorem 1, $\mathcal{P}$=$\mathcal{NP}$, and $\mathcal{NP}$-hardness follows. Since the decision problem is in $\mathcal{NP}$, the problem is $\mathcal{NP}$-complete. □

4 Conclusion

We have answered one open problem stated in [4]. Unfortunately it turns out that even with so much restrictions on the instances of the paint shop problem, the problem is still hard, even relatively hard to approximate. The next open question from [4] is whether there is a constant factor approximation algorithm for these instances, or even for more generalized instances: instances with $r_{xi} = k$ for every x and i, with possibly more than 2 colors.

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