On the WDVV equations in
five-dimensional gauge theories

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Abstract

It is well-known that the perturbative prepotentials of four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theories satisfy the generalized WDVV equations, regardless of the gauge group. In this paper we study perturbative prepotentials of the five-dimensional theories for some classical gauge groups and determine whether or not they satisfy the WDVV system.

1 Introduction

The original WDVV equations were put forward by Witten [1] and Dijkgraaf, E. Verlinde and H. Verlinde [2] in the context of 2-dimensional topological conformal field theory. They form the following system of third order nonlinear partial differential equations for a function F of N variables

$$F_i F_1^{-1} F_m = F_m F_1^{-1} F_i \qquad\qquad i, m = 1, ..., N \qquad\qquad (1.1)$$

where F_i is the matrix

$$(F_i)_{jk} = \frac{\partial^3 F(a_1, ..., a_N)}{\partial a_i \partial a_j \partial a_k} \qquad\qquad (1.2)$$

Moreover, one requires that F_1 is a constant and invertible matrix.

The generalized WDVV equations however are given by the following system

$$F_i F_k^{-1} F_m = F_m F_k^{-1} F_i \qquad\qquad i, k, m = 1, ..., N \qquad\qquad (1.3)$$

for any k, and there are no further requirements with respect to a special coordinate. It is not difficult to show (see e.g. [3]) that this system can be written equivalently in a form which is more convenient for our purposes

$$F_i B^{-1} F_m = F_m B^{-1} F_i \qquad\qquad i, m = 1, ..., N \qquad\qquad (1.4)$$

for a linear combination B of the matrices F_k, possibly with a_i dependent coefficients. In fact, (1.4) holds for all linear combinations B simultaneously, provided their inverse exists. The original equations (1.1) are therefore indeed a special case of (1.4), which explains the terminology.

The generalized WDVV system was proven to hold for prepotentials of certain four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theories by Marshakov, Mironov and Morozov [4],[3],[5]. Such prepotentials consist of a perturbative and nonperturbative part, and often it can be shown that the perturbative part itself satisfies the WDVV system. In [5],[6],[7] and [8] the authors study perturbative prepotentials of five-dimensional theories in combination with the WDVV equations, and it is our main goal in the present article to provide proofs of some of the statements made there and to deduce new results for perturbative prepotentials of the five-dimensional theory.

For sake of transparency we first give a summary of the results of this paper in section 2, followed by the central part section 3 containing all the proofs.

2 Summary of the results

In general we consider functions of the following type

$$
\begin{aligned}
F(a_1,...,a_N) \;=\;& \sum_{1\leq i<j\leq N}\Big(\alpha_- f(a_i - a_j) + \alpha_+ f(a_i + a_j)\Big) + \eta\sum_{i=1}^{N} f(a_i) \\
+\;& \frac{a}{6}\left(\sum_{i=1}^{N} a_i\right)^3 + \frac{b}{2}\left(\sum_{i=1}^{N} a_i\right)\left(\sum_{j=1}^{N} a_j^2\right) + \frac{c}{6}\sum_{i=1}^{N} a_i^3
\end{aligned}
\tag{2.1}
$$

where we adopt the notation of [5]. The function f is defined by

$$
f(x) = \frac{1}{6}x^3 - \frac{1}{4}Li_3(e^{-2x}) = \frac{1}{6}x^3 - \frac{1}{4}\sum_{k=1}^{\infty}\frac{e^{-2kx}}{k^3}
\tag{2.2}
$$

and therefore

$$
f'''(x) = \coth(x)
\tag{2.3}
$$

The general form (2.1) is motivated by physics, see for instance [9],[10],[11]. In particular, the second line contains cubic terms coming from string theory, serving as corrections to the naive field theoretic perturbative prepotentials. These represent the most general cubic expression which is preserved by permutations of the variables $a_1,...,a_N$.

For various combinations of the parameters we will investigate whether or not F satisfies the WDVV system (1.4). The method used involves making an appropriate choice for the matrix B, although the results are of course independent of this particular choice.

2.1 The simplest case

The simplest set of parameters we consider is $\alpha_+ = \eta = 0$. These values do not correspond to an actual prepotential from physics, but we do find solutions to the WDVV system. Without loss of generality we can chose $\alpha_- = 1$ by scaling a, b, c.

We can prove the following result

Theorem 2.1 *The function (2.1) with $\alpha_- = 1$, $\alpha_+ = 0$ and $\eta = 0$ satisfies the WDVV system (1.4) if and only if the following relation holds*

$$
Nb^3 + 3b^2 c - ac^2 + 3Nb + c + N^2 a = 0
\tag{2.4}
$$

More accurately, this relation is correct in the generic case that both $Nb + c \neq 0$ and $Na + 2b \neq 0$. Special cases will be discussed separately in section 3.1.1.

2.2 The type A prepotential

Let us now turn to a prepotential with physical background. We consider the function

$$
\tilde{F}(x_1,...,x_{N+1}) = \sum_{1\leq i<j\leq N+1} f(x_i - x_j) + \frac{N+1}{2}\sum_{1\leq i<j<k\leq N+1} x_i x_j x_k
\tag{2.5}
$$

which is of the form of the previous paragraph with parameters a, b, c given by

$$
a = \frac{N+1}{2} \qquad b = -\frac{N+1}{2} \qquad c = N+1
\tag{2.6}
$$

2

The $SU(N+1)$ perturbative prepotential is obtained from $\tilde{F}$ by the linear [1] change of variables

$$
\begin{aligned}
a_i &= x_i - x_{N+1} & i = 1, .., N \\
a_{N+1} &= x_1 + ... + x_{N+1}
\end{aligned}
\tag{2.7}
$$

and the substitution $a_{N+1} = 0$. Concretely it is given by

$$
\begin{aligned}
F(a_1, ..., a_N) &= \sum_{1 \leq i < j \leq N} f(a_i - a_j) + \sum_{i=1}^{N} f(a_i) \\
&+ \frac{1}{3(N+1)} \left(\sum_{i=1}^{N} a_i \right)^3 - \frac{1}{2} \left(\sum_{i=1}^{N} a_i \right) \left(\sum_{j=1}^{N} a_j^2 \right) + \frac{N+1}{6} \sum_{i=1}^{N} a_i^3
\end{aligned}
\tag{2.8}
$$

This is of the general type (2.1) with parameters

$\alpha_- = 1$	$\alpha_+ = 0$	$\eta = 1$
$a = \frac{2}{N+1}$	$b = -1$	$c = N+1$

It turns out that the sign of the correction term in (2.5) is irrelevant for the WDVV equations.

We can confirm the result in [5] and prove

Theorem 2.2 *The function*

$$
\begin{aligned}
F(a_1, ..., a_N) &= \sum_{1 \leq i < j \leq N} f(a_i - a_j) + \sum_{i=1}^{N} f(a_i) \\
&\pm \frac{1}{3(N+1)} \left(\sum_{i=1}^{N} a_i \right)^3 \mp \frac{1}{2} \left(\sum_{i=1}^{N} a_i \right) \left(\sum_{j=1}^{N} a_j^2 \right) \pm \frac{N+1}{6} \sum_{i=1}^{N} a_i^3
\end{aligned}
\tag{2.9}
$$

satisfies the WDVV system (1.4).

Remark We note that (2.9) is invariant under the Weyl group of A_N. In fact, taking arbitrary values for a, b, c this is still the case. A natural question is therefore whether an F with $\alpha_- = 1$, $\alpha_+ = 0$ and $\eta = 1$ satisfies the WDVV system for any other values of a, b, c. Calculations for ranks up to five suggest that there are no other solutions. This should mean that the string theory corrections are *precisely* the ones needed to satisfy the WDVV equations.

2.3 Other classical Lie algebras

Next we consider a prepotential inspired by the other classical Lie algebras. Without correction terms, the B, D prepotentials are given by $\alpha_- = 1$, $\alpha_+ = 1$ and $\eta = 1, 0$ respectively. Leaving the parameter η unfixed, we can prove the following theorem

Theorem 2.3 *The function*

$$
F(a_1, ..., a_N) = \sum_{1 \leq i < j \leq N} \left(f(a_i - a_j) + f(a_i + a_j) \right) + \eta \sum_{i=1}^{N} f(a_i)
\tag{2.10}
$$

satisfies the WDVV equations (1.4) if and only if $\eta = -2(N-2)$.

---

[1] The WDVV equations are invariant under linear coordinate changes

Remark This solution seems to have little to do with the B, D Lie algebras, since η is different from 1 and 0. One can think about adding correction terms to restore the Lie algebraic interpretation, but the Lie algebras under consideration do not possess any third order Weyl invariant polynomials. This forces one to consider fourth (or higher) order correction terms and therefore nonconstant additions to the third order derivatives of F, which is beyond the scope of this article. For the sake of completion we mention that calculations for low ranks indicate that adding third order correction terms with parameters a, b, c indeed doesn't help: the WDVV equations always force $a = b = c = 0$.

2.3.1 Adding an extra variable

We can add a variable a_{N+1} to the prepotential (2.10) and obtain the following result

Theorem 2.4 *The function*

$$F(a_1, ..., a_{N+1}) = \sum_{1 \le i < j \le N} \left(f(a_i - a_j) + f(a_i + a_j) \right) + \eta \sum_{i=1}^{N} f(a_i) + \frac{\gamma}{6}(a_{N+1}^3 + 3a_{N+1} \sum_{i=1}^{N} a_i^2) \tag{2.11}$$

satisfies the WDVV system (1.4) if and only if

$$\eta = -2(N - 2) - \frac{\gamma^2}{2} \tag{2.12}$$

With respect to the variable a_{N+1} the matrix of third order derivatives is

$$(F_{N+1})_{ij} = \frac{\partial^3 F}{\partial a_{N+1} \partial a_i \partial a_j} \tag{2.13}$$

and therefore becomes a multiple of the identity. So the function (2.11) can even be regarded as a solution to the original WDVV equations (1.1). To the best of our knowledge such functions, both as a solution to the generalized system and to the original one, have not been considered before.

2.4 An additional result

In this section we mention a result which was obtained in the process of proving the main results of this article. The five-dimensional prepotentials are built from a basic function f with $f'''(x) = \coth(x)$. The four-dimensional theories have a basic function with $f'''(x) = \frac{1}{x}$. We can prove the following result, which should be compared with theorem 2.1

Theorem 2.5 *The function (2.1) with $\alpha_- = 0$, $\alpha_+ = 1$, $\eta = 0$ and $f'''(x) = \frac{1}{x}$ satisfies the WDVV equations (1.4) if and only if*

$$Nb^3 + 3b^2c - ac^2 = 0 \tag{2.14}$$

In the next sections, we will describe the proofs of theorems 2.1-2.5.

3 The proofs

The general idea underlying all the proofs of this section is to find an appropriate linear combination B such that its inverse, appearing in (1.4), becomes manageable. In the four-dimensional theory it is possible to chose B to equal the Killing form of the Lie algebra under consideration [12]. In our five-dimensional setting this is no longer the case. This makes the Lie algebraic structure less transparent than in the four-dimensional case.

We start by proving theorems 2.1 and 2.5 in section 3.1, making use of a constant nondiagonal matrix B. Theorems 2.2, 2.3 and 2.4 on the other hand are proven by taking a diagonal (not always constant) B, which allows for a common general strategy for these cases, given in section 3.2. The individual theorems are subsequently proven in sections 3.3,3.4 and 3.4.1 respectively.

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3.1 The simplest case

We will prove theorem 2.1. Therefore we consider a function of the form

$$F(a_1, ..., a_N) = \sum_{1 \leq i < j \leq N} f(a_i - a_j) + \frac{a}{6}\left(\sum_{i=1}^{N} a_i\right)^3 + \frac{b}{2}\left(\sum_{i=1}^{N} a_i\right)\left(\sum_{j=1}^{N} a_j^2\right) + \frac{c}{6}\sum_{i=1}^{N} a_i^3 \quad (3.1)$$

where

$$f'''(-x) = -f'''(x) \quad (3.2)$$

Writing $\beta_{ij} = f'''(a_i - a_j)$, the third order derivatives of F are

$$
\begin{aligned}
F_{klm} &= a + \delta_{kl}\delta_{lm}\left(\sum_{q \neq k} \beta_{kq} + 3b + c\right) + \delta_{kl}(1 - \delta_{km})(\beta_{mk} + b) \\
&+ \delta_{km}(1 - \delta_{kl})(\beta_{lk} + b) + \delta_{lm}(1 - \delta_{kl})(\beta_{kl} + b) = aU_{lm} + (V_k)_{lm}
\end{aligned} \quad (3.3)
$$

We take a specific linear combination $B = \sum_{j=1}^{N} F_j$ and using (3.2) we find

$$B = (Na + 2b)U + (Nb + c)I \quad (3.4)$$

Special situations occur when $Na + 2b = 0$ and / or $Nb + c = 0$. The first results in B being a multiple of the identity and the second causes B to become singular. For the moment we will work with generic B and we will come back to the special cases later. The inverse of B equals up to a factor

$$B_{kl}^{-1} = 1 + \delta_{kl}\left(-\frac{Nb + c}{Na + 2b} - N\right) \quad (3.5)$$

For the WDVV equations to hold, we should have

$$\left(F_i B^{-1} F_m\right)_{jn} - \left(F_m B^{-1} F_i\right)_{jn} = 0 \quad (3.6)$$

or equivalently

$$B_{ij}B_{mn} - B_{mj}B_{in} - \frac{3Nb + c + N^2 a}{Na + 2b}[F_i, F_m]_{jn} = 0 \quad (3.7)$$

We will first calculate the commutator $[F_i, F_m] = [aU + V_i, aU + V_m]$. We find

$$(UV_m)_{kl} = 2b + \delta_{lm}(1 - \delta_{kl})(Nb + c) \quad (3.8)$$

and since $U^T = U$ and $V_m^T = V_m$ we also know $V_m U = (UV_m)^T$. Furthermore, if we use the identity

$$\beta_{ij}\beta_{ik} + \beta_{ij}\beta_{kj} + \beta_{ik}\beta_{jk} = 1 \quad (3.9)$$

we find

$$
\begin{aligned}
[V_i, V_m]_{jn} &= \delta_{ij}(1 - \delta_{mn})(1 - \delta_{in})(b^2 - 1) + \delta_{mn}(1 - \delta_{jm})(1 - \delta_{ij})(b^2 - 1) \\
&- \delta_{jm}(1 - \delta_{mn})(1 - \delta_{in})(b^2 - 1) - \delta_{in}(1 - \delta_{jm})(1 - \delta_{ij})(b^2 - 1) \\
&+ \delta_{ij}\delta_{mn}(\beta + 2(b^2 - 1)) - \delta_{jm}\delta_{in}(\beta + 2(b^2 - 1))
\end{aligned} \quad (3.10)
$$

and therefore

$$
\begin{aligned}
[F_i, F_m]_{jn} &= \delta_{ij}(1 - \delta_{mn})(1 - \delta_{in})\alpha + \delta_{mn}(1 - \delta_{jm})(1 - \delta_{ij})\alpha - \delta_{jm}(1 - \delta_{mn})(1 - \delta_{in})\alpha \\
&\quad - \delta_{in}(1 - \delta_{jm})(1 - \delta_{ij})\alpha + \delta_{ij}\delta_{mn}(\beta - 2\alpha) - \delta_{jm}\delta_{in}(\beta - 2\alpha)
\end{aligned}
\tag{3.11}
$$

where

$$
\begin{aligned}
\alpha &= b^2 - 1 - ac - Nab \\
\beta &= N + Nb^2 + 2bc
\end{aligned}
\tag{3.12}
$$

On the other hand, we have

$$
B_{ij}B_{mn} - B_{mj}B_{in} = (Na + 2b)^2 \Big[\delta_{ij}(1 - \delta_{mn})(1 - \delta_{in})\gamma + \delta_{mn}(1 - \delta_{jm})(1 - \delta_{ij})\gamma
$$

$$
- \delta_{jm}(1 - \delta_{mn})(1 - \delta_{in})\gamma - \delta_{in}(1 - \delta_{jm})(1 - \delta_{ij})\gamma + \delta_{ij}\delta_{mn}(\delta - 2\gamma) - \delta_{jm}\delta_{in}(\delta - 2\gamma) \Big]
\tag{3.13}
$$

where

$$
\begin{aligned}
\gamma &= \frac{Nb + c}{Na + 2b} \\
\delta &= \gamma^2
\end{aligned}
\tag{3.14}
$$

The equation (3.7) therefore reduces to two algebraic relations among the parameters a, b, c. These relations are

$$
-\frac{3Nb + c + N^2 a}{Na + 2b}\alpha + (Na + 2b)^2\gamma = 0
\tag{3.15}
$$

and

$$
-\frac{3Nb + c + N^2 a}{Na + 2b}(2\alpha - \beta) + (Na + 2b)^2(2\gamma - \delta) = 0
\tag{3.16}
$$

which combine into only one relation

$$
Nb^3 + 3b^2 c - ac^2 + 3Nb + c + N^2 a = 0
\tag{3.17}
$$

This finishes the proof of theorem 2.1. In the next section we will look at the nongeneric values of the parameters a, b, c.

3.1.1 Special values for a, b, c.

Recall from (3.4) that there are special situations for either $Na + 2b = 0$ or $Nb + c = 0$ or both. If $Na + 2b = 0$ and $Nb + c \neq 0$ then we find that the WDVV equations hold if and only if

$$
[F_i, F_m] = 0
\tag{3.18}
$$

and therefore if and only if

$$
\alpha = 1 + \left(\frac{Na}{2}\right)^2 - ac = 0
\tag{3.19}
$$

and

$$2\alpha - \beta = -(N-2)\left(1 + \left(\frac{Na}{2}\right)^2 - ac\right) = 0 \tag{3.20}$$

Note that just substituting $b = -\frac{Na}{2}$ in (3.17) gives

$$(N^2a - 2c)\left(1 + \left(\frac{Na}{2}\right)^2 - ac\right) = 0 \tag{3.21}$$

which is only partially correct since $N^2a - 2c = 0$ does not yield a solution.

Furthermore, if $Nb + c = 0$ and $Na + 2b \neq 0$, then (3.4) shows that B becomes singular. Experience tells us that for $N \neq 3$ there exist no solutions to the WDVV equations without the extra requirement $b = \pm 1$. For $N = 3$ there is no such condition on b and the WDVV equations are satisfied. We will now consider $b = 1$ and $b = -1$ separately. If $b = 1$ then we chose a new nonsingular B equal to

$$B = \sum_{j=1}^{N} h_j F_j = \sum_{j=1}^{N} \left(-(2 + a(N-1))e^{2a_j} + a\sum_{i\neq j} e^{2a_i}\right) F_j \tag{3.22}$$

and working this out we find that B equals up to a factor

$$\left(\frac{Na}{2} + b\right) I \tag{3.23}$$

which is a nonzero multiple of the identity since $Na + 2b \neq 0$. If $b = -1$ on the other hand, we take

$$B = \sum_{j=1}^{N} h_j F_j = \sum_{j=1}^{N} \left(\prod_{k\neq j}(2 - a(N-1))e^{2a_k} + a\sum_{k\neq j}\prod_{i\neq k} e^{2a_i}\right) F_j \tag{3.24}$$

which also leads to B being a multiple of the identity. So in both cases we must solve (3.18) again, which leads to

$$Na + 2b = 0 \tag{3.25}$$

which is precisely what we excluded before.

Finally, if we take both $Na + 2b = 0$ and $Nb + c = 0$ then all linear combinations of the F_j become singular and the WDVV equations are meaningless.

Summarizing, we conclude that if $Na + 2b = 0$ and $Nb + c \neq 0$ there are solutions if and only if

$$1 + \left(\frac{Na}{2}\right)^2 - ac = 0 \tag{3.26}$$

and if $Nb + c = 0$ and $Na + 2b \neq 0$ there are solutions if and only if $N = 3$ and finally if both $Na + 2b = 0$ and $Nb + c = 0$ then there are no solutions at all.

This finishes the discussion of theorem 2.1.

3.1.2 Taking $f'''(x) = \frac{1}{x}$ instead of $\coth(x)$

In the previous section, we took $f'''(x) = \coth(x)$ which leads to the relation

$$\beta_{ij}\beta_{ik} + \beta_{ij}\beta_{kj} + \beta_{ik}\beta_{jk} = 1 \tag{3.27}$$

and under the condition $f'''(-x) = -f'''(x)$ this is the only differentiable solution to this relation. We could also consider instead the equation

$$\beta_{ij}\beta_{ik} + \beta_{ij}\beta_{kj} + \beta_{ik}\beta_{jk} = 0 \tag{3.28}$$

and assuming again $f'''(-x) = -f'''(x)$ we find that the only solution is $f'''(x) = \frac{1}{x}$. This is the basic function for the four-dimensional theory. Adding correction terms, we find that $[F_i, F_m]$ is of the same form as (3.11) but with

$$\begin{aligned} \alpha &= -b^2 - ac - Nab \\ \beta &= Nb^2 + 2bc \end{aligned} \tag{3.29}$$

and we find precisely the same $B_{ij}B_{mn} - B_{mj}B_{in}$ as in (3.13). For $N \neq 2$ this again leads to a single relation, namely

$$Nb^3 + 3b^2c - ac^2 = 0 \tag{3.30}$$

which is to be compared with (3.17). This finishes the proof of theorem 2.5.

3.2 General strategy

In the previous section proofs were given of theorems 2.1 and 2.5. To discuss the proofs of the remaining theorems we will use the following general strategy. We take the general function of (2.1) and return to the case $f'''(x) = \coth(x)$. This F has third order derivatives equal to

$$F_{klm} = a + \delta_{kl}\delta_{lm}K_k + \delta_{kl}\beta_{mk} + \delta_{km}\beta_{lk} + \delta_{lm}\beta_{kl} \tag{3.31}$$

where

$$\begin{aligned} K_k &= \sum_{q \neq k} \beta_{kq} + \beta_k \\ \beta_k &= \eta \coth(a_k) + (4 - N)b + c \\ \beta_{ij} &= \begin{cases} 0 & \text{if } i = j \\ \alpha_- \coth(a_i - a_j) + \alpha_+ \coth(a_i + a_j) + b & \text{if } i \neq j \end{cases} \end{aligned} \tag{3.32}$$

Consider a linear combination B of the following form

$$B_{kl} = \delta_{kl}\frac{1}{A_k} \tag{3.33}$$

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where A_k depends on the specific prepotential under consideration and will be specified later. We find

$$
\begin{aligned}
\left(F_i B^{-1} F_m\right)_{jl} &= \sum_{k=1}^{N} F_{ijk} A_k F_{klm} \\
&= \delta_{im}\left(A_i \beta_{ji}\beta_{lm} + \delta_{ij}\beta_{lm}A_i K_i + \delta_{il}\beta_{ji}A_i K_i + \delta_{ij}\delta_{lm}A_i K_i K_m \right) \\
&+ \delta_{jl}(1-\delta_{lm})A_j\beta_{ij}\beta_{mj} \\
&+ \delta_{jl}\delta_{lm}A_m K_l \beta_{im} \\
&+ \delta_{ij}\delta_{il}A_i K_l \beta_{mi} \\
&+ \delta_{il}(1-\delta_{jm})A_i\beta_{ji}\beta_{mi} \\
&+ \delta_{lm}(1-\delta_{ij})\left(A_i\beta_{ji}\beta_{im} + A_j\beta_{ij}\beta_{jm} + aA_m K_m + a\sum_k A_k\beta_{km}\right) \\
&+ \delta_{jm}(1-\delta_{il})A_j\beta_{ij}\beta_{lj} \\
&+ \delta_{ij}(1-\delta_{lm})\left(A_l\beta_{li}\beta_{ml} + A_m\beta_{mi}\beta_{lm}aA_i K_i + a\sum_k A_k\beta_{ki}\right) \\
&+ \delta_{ij}\delta_{lm}\left(A_i K_i \beta_{im} + A_m K_m \beta_{mi} + \sum_{k\neq i,m} A_k\beta_{km}\beta_{ki} + aA_m K_m \right. \\
&\qquad\qquad \left. +a\sum_k A_k\beta_{km} + aA_i K_i + a\sum_k A_k\beta_{ki}\right) \\
&+ \delta_{jm}\delta_{il}\left(A_m\beta_{im}^2 + A_i\beta_{mi}^2\right) \\
&+ a^2\sum_k A_k + a(A_j\beta_{ij} + A_i\beta_{ji}) + a(A_m\beta_{lm} + A_l\beta_{ml}) \qquad (3.34)
\end{aligned}
$$

Here it should be noted that the last line contributes to all the previous ones. For example, if $i=l, i\neq m, i\neq j, j\neq m$ then $\left(F_i B^{-1} F_m\right)_{jl}$ is not

$$
A_j\beta_{ij}\beta_{mj} \qquad (3.35)
$$

but

$$
A_j\beta_{ij}\beta_{mj} + a^2\sum_k A_k + a(A_j\beta_{ij} + A_i\beta_{ji}) + a(A_m\beta_{im} + A_i\beta_{mi}) \qquad (3.36)
$$

In order to satisfy the WDVV system we should check whether or not (3.34) is symmetric in i and m. For example, the first two lines automatically are preserved under the interchange of i and m. The third and fourth lines on the other hand are mutually exchanged. The rest of condition (1.4) is nontrivial and depends on the details of the function F.

3.3 The type A prepotential

As mentioned in section 2.2, after taking $\alpha_- = 1$, $\alpha_+ = 0$ and $\eta = 1$ the only solutions to the WDVV equations exist for the parameters

$$
a = \pm\frac{2}{N+1} \quad , \quad b = \mp 1 \quad , \quad c = \pm(N+1) \qquad (3.37)
$$

These two cases will be treated separately in the following sections.

3.3.1 The parameters $a = -\frac{2}{N+1}$, $b = 1$, $c = -N - 1$

The third order derivatives of F are given by

$$F_{klm} = -\frac{2}{N+1} + \delta_{kl}\delta_{lm}K_k + \delta_{kl}\beta_{mk} + \delta_{km}\beta_{lk} + \delta_{lm}\beta_{kl} \tag{3.38}$$

with

$$
\begin{aligned}
K_k &= \sum_{q \neq k} \beta_{kq} + \beta_k \\
\beta_k &= \frac{2}{1 - e^{-2a_k}} - 2(N-1) \\
\beta_{ij} &= \begin{cases} 0 & \text{if } i = j \\ 2\frac{e^{2a_i}}{e^{2a_i} - e^{2a_j}} & \text{if } i \neq j \end{cases}
\end{aligned}
\tag{3.39}
$$

We take a specific linear combination $B = \sum_j h_j F_j$ where

$$h_j = e^{2a_j} + \sum_{i=1}^{N} e^{2a_i} \tag{3.40}$$

and we find up to a factor

$$B_{kl} = \delta_{kl}\left(\frac{1}{1 - e^{-2a_k}}\right) = \delta_{kl}\frac{1}{A_k} \tag{3.41}$$

Using this information we can derive the following identities

$$
\begin{aligned}
A_j\beta_{ij} + A_i\beta_{ji} &= 2 - \frac{2}{e^{2a_i}} - \frac{2}{e^{2a_j}} \tag{3.42} \\
A_i\beta_{ij} + A_j\beta_{ji} &= 2 \tag{3.43} \\
A_i\beta_{ji}\beta_{im} + A_j\beta_{ij}\beta_{jm} - A_m\beta_{jm}\beta_{im} &= \frac{4}{e^{2a_m}} \tag{3.44}
\end{aligned}
$$

Turning to the WDVV condition we find that the first two lines of (3.34) are preserved under the interchange of i and m and that the third and fourth lines become mutually exchanged. We will now study the fifth and sixth lines. Keeping in mind that the last line of (3.34) contributes to both of these, we find that the fifth line becomes

$$\delta_{il}(1 - \delta_{lm})\left(A_j\beta_{ij}\beta_{mj} + a^2\sum_k A_k + a\left(A_j\beta_{ij} + A_i\beta_{ji}\right) + a\left(A_m\beta_{lm} + A_l\beta_{ml}\right)\right)$$

and the sixth becomes

$$\delta_{lm}(1 - \delta_{ij})\left(A_i\beta_{ji}\beta_{im} + A_j\beta_{ij}\beta_{jm} + a^2\sum_k A_k + a\left(A_j\beta_{ij} + A_i\beta_{ji}\right) + a\sum_k A_k\beta_{km} + aA_mK_m\right)$$

Using the definition of K_m and the relations (3.42), (3.43) and (3.44) we see that these are indeed exchanged under the interchange of i and m. The seventh and eighth lines of (3.34) are mutually exchanged for the same reasons, which leaves us with the ninth and tenth lines. The complete ninth line becomes

$$
\begin{aligned}
\delta_{ij}\delta_{lm}\Big(& A_iK_i\beta_{im} + A_mK_m\beta_{mi} + \sum_{k \neq i,m} A_k\beta_{km}\beta_{ki} + aA_mK_m + a\sum_k A_k\beta_{km} \\
& +aA_iK_i + a\sum_k A_k\beta_{ki} + a^2\sum_k A_k\Big)
\end{aligned}
\tag{3.45}
$$

and the tenth line is

$$\delta_{jm}\delta_{il}\left(A_m\beta_{im}^2 + A_i\beta_{mi}^2 + 2a\left(A_m\beta_{im} + A_i\beta_{mi}\right) + a^2\sum_k A_k\right) \tag{3.46}$$

Using the definition of K_m and working out (3.45) we find

$$\delta_{ij}\delta_{lm}\left(\sum_{k\neq i,m}\left(A_k\beta_{ik}\beta_{im} + A_m\beta_{mk}\beta_{mi} + A_k\beta_{km}\beta_{ki}\right) + A_i\beta_{im}^2 + A_m\beta_{mi}^2 + A_i\beta_i\beta_{im} + A_m\beta_m\beta_{mi}\right.$$
$$\left. + a\sum_{k\neq m}\left(A_m\beta_{mk} + A_k\beta_{km}\right) + aA_m\beta_m + a\sum_{k\neq i}\left(A_i\beta_{ik}A_k\beta_{ki}\right) + aA_i\beta_i + a^2\sum_k A_k\right) \tag{3.47}$$

We will make use of (3.43) and the following relations

$$A_i\beta_{ik}\beta_{im} + A_m\beta_{mk}\beta_{mi} + A_k\beta_{km}\beta_{ki} = 4 \tag{3.48}$$

$$A_i\beta_i\beta_{im} + A_m\beta_m\beta_{mi} = 8 - 4N \tag{3.49}$$

and we find that the ninth line becomes

$$\delta_{ij}\delta_{lm}\left(A_i\beta_{im}^2 + A_m\beta_{mi}^2 + 4a(N-1) + a\left(A_m\beta_m + A_i\beta_i\right) + a^2\sum_k A_k\right) \tag{3.50}$$

Using (3.43) again we find that the tenth line becomes

$$\delta_{jm}\delta_{il}\left(A_m\beta_{im}^2 + A_i\beta_{mi}^2 + 4a - \frac{4a}{e^{2a_i}} - \frac{4a}{e^{2a_m}} + a^2\sum_k A_k\right) \tag{3.51}$$

and using the relations

$$A_i\beta_{im}^2 + A_m\beta_{mi}^2 - A_m\beta_{im}^2 - A_i\beta_{mi}^2 = \frac{4}{e^{2a_i}} + \frac{4}{e^{2a_m}} \tag{3.52}$$

$$A_m\beta_m + A_i\beta_i = 8 - 4N + \frac{2N-2}{e^{2a_i}} + \frac{2N-2}{e^{2a_m}} \tag{3.53}$$

we find that the ninth and tenth lines are indeed exchanged under the interchange of i and m. Therefore the prepotential (2.9) satisfies the WDVV equations and we have proven half of theorem 2.2.

3.3.2 The parameters $a = \frac{2}{N+1}$, $b = -1$, $c = N+1$

In this section we prove the other half of theorem 2.2. The third order derivatives of F are given by

$$F_{klm} = \frac{2}{N+1} + \delta_{kl}\delta_{lm}K_k + \delta_{kl}\beta_{mk} + \delta_{km}\beta_{lk} + \delta_{lm}\beta_{kl} \tag{3.54}$$

with

$$K_k = \sum_{q\neq k}\beta_{kq} + \beta_k$$

$$\beta_k = \frac{-2}{1 - e^{2a_k}} + 2(N-1)$$

$$\beta_{ij} = \begin{cases} 0 & \text{if } i = j \\ 2\frac{e^{2a_j}}{e^{2a_i} - e^{2a_j}} & \text{if } i \neq j \end{cases} \tag{3.55}$$

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Taking

$$h_j = e^{2a_j} + \sum_{i=1}^{N} e^{2a_i} \tag{3.56}$$

we find that up to a factor $B = \sum_j h_j F_j$ equals

$$\delta_{kl} \left(\frac{1}{1 - e^{2a_k}} \right) = \delta_{kl} \frac{1}{A_k} \tag{3.57}$$

So with respect to the previous paragraph there are modifications in the definitions of β_k, β_{ij} and A_k. This causes the relations (3.42), (3.43), (3.44), (3.48), (3.49), (3.52) and (3.53) to be changed to the following ones

$$
\begin{align}
A_j\beta_{ij} + A_i\beta_{ji} &= -2 + 2e^{2a_i} + 2e^{2a_j} \tag{3.58} \\
A_i\beta_{ij} + A_j\beta_{ji} &= -2 \tag{3.59} \\
A_i\beta_{ji}\beta_{im} + A_j\beta_{ij}\beta_{jm} - A_m\beta_{jm}\beta_{im} &= 4e^{2a_m} \tag{3.60} \\
A_i\beta_{ik}\beta_{im} + A_m\beta_{mk}\beta_{mi} + A_k\beta_{km}\beta_{ki} &= 4 \tag{3.61} \\
A_i\beta_i\beta_{im} + A_m\beta_m\beta_{mi} &= 4N - 8 \tag{3.62} \\
A_i\beta_{im}^2 + A_m\beta_{mi}^2 - A_m\beta_{im}^2 - A_i\beta_{mi}^2 &= 4e^{2a_i} + 4e^{2a_m} \tag{3.63} \\
A_m\beta_m + A_i\beta_i &= 4N - 8 + -(2N - 2)e^{2a_i} - (2N - 2)e^{2a_m} \tag{3.64}
\end{align}
$$

and using these relations we find that the WDVV equations are again satisfied. This proves theorem 2.2.

3.4 Other classical Lie algebras

In this section theorem 2.3 will be proven. Therefore we take a prepotential of the form

$$F(a_1, ..., a_N) = \sum_{1 \le i < j \le N} \left(f(a_i - a_j) + f(a_i + a_j) \right) + \eta \sum_{i=1}^{N} f(a_i) \tag{3.65}$$

where again

$$f'''(x) = \coth(x) \tag{3.66}$$

The third order derivatives are given by (3.31) with

$$
\begin{align}
K_k &= \sum_{q \ne k} \beta_{kq} + \beta_k \\
\beta_k &= \eta \coth(a_i - a_j) \\
\beta_{ij} &= \begin{cases} 0 & \text{if} \quad i = j \\ \coth(a_i - a_j) + \coth(a_i + a_j) & \text{if} \quad i \ne j \end{cases} \tag{3.67} \\
a &= b = c = 0 \tag{3.68}
\end{align}
$$

One can derive the following relations

$$
\begin{align}
\beta_{ji}\beta_{im} + \beta_{ij}\beta_{jm} - \beta_{jm}\beta_{im} &= 0 \tag{3.69} \\
\beta_{ik}\beta_{im} + \beta_{mk}\beta_{mi} + \beta_{km}\beta_{ki} &= 4 \tag{3.70} \\
\beta_i\beta_{im} + \beta_m\beta_{mi} &= 2 \tag{3.71}
\end{align}
$$

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which are identities that we will need later. Furthermore, we take

$$B_{kl} = \sum_{j=1}^{N} \sinh(2a_j) F_{jkl} \tag{3.72}$$

and using (3.69) we find

$$B_{kl} = \delta_{kl} \left(1 - N + \sum_{j=1}^{N} \cosh^2(a_j) + \frac{1}{2} \left(2(N-2) + \eta \right) \cosh^2(a_k) \right) \tag{3.73}$$

This becomes independent of k and l precisely for $\eta = -2(N-2)$. So for this value of η we can regard B as a multiple of the identity. First let us consider all other values of η, so that B is equal to (3.33) with

$$A_k = \frac{1}{1 - N + \sum_{j=1}^{N} \cosh^2(a_j) + \frac{1}{2} \left(2(N-2) + \eta \right) \cosh^2(a_k)} = \frac{1}{X + Y_k} \tag{3.74}$$

In order to satisfy the WDVV equations, the expression (3.34) should be symmetric in i and m. Just as in the previous section, the first nontrivial condition is that the fifth and sixth lines of (3.34) are exchanged under the interchange of i and m. This condition translates into

$$A_i \beta_{ji} \beta_{im} + A_j \beta_{ij} \beta_{jm} - A_m \beta_{jm} \beta_{im} = 0 \tag{3.75}$$

and therefore

$$(X + Y_j)(X + Y_m)\beta_{ji}\beta_{im} + (X + Y_i)(X + Y_m)\beta_{ij}\beta_{jm} - (X + Y_i)(X + Y_j)\beta_{jm}\beta_{im} = 0 \tag{3.76}$$

Working this out further we find

$$-\frac{1}{16} \frac{(e^{4a_i} - 1)(e^{4a_j} - 1)(2(N-2) + \eta)^2}{e^{2(a_i + a_j)}} = 0 \tag{3.77}$$

Therefore we find that for $\eta \neq -2(N-2)$ the WDVV equations are not satisfied. We will now determine what happens for the value $\eta = -2(N-2)$, for which B becomes a multiple of the identity. Then (3.34) becomes

$$\begin{aligned}
\sum_{k=1}^{N} F_{ijk} F_{klm} &= \delta_{im} \left(\beta_{ji}\beta_{lm} + \delta_{ij}\beta_{lm}K_i + \delta_{il}\beta_{ji}K_i + \delta_{ij}\delta_{lm}K_iK_m \right) \\
&+ \delta_{jl}(1 - \delta_{lm})\beta_{ij}\beta_{mj} \\
&+ \delta_{jl}\delta_{lm}K_l\beta_{im} \\
&+ \delta_{ij}\delta_{il}K_l\beta_{mi} \\
&+ \delta_{il}(1 - \delta_{jm})\beta_{ji}\beta_{mi} \\
&+ \delta_{lm}(1 - \delta_{ij})\left(\beta_{ji}\beta_{im} + \beta_{ij}\beta_{jm} \right) \\
&+ \delta_{jm}(1 - \delta_{il})\beta_{ij}\beta_{lj} \\
&+ \delta_{ij}(1 - \delta_{lm})\left(\beta_{li}\beta_{ml} + \beta_{mi}\beta_{lm} \right) \\
&+ \delta_{ij}\delta_{lm}\left(K_i\beta_{im} + K_m\beta_{mi} + \sum_{k \neq i,m} \beta_{km}\beta_{ki} \right) \\
&+ \delta_{jm}\delta_{il}\left(\beta_{im}^2 + \beta_{mi}^2 \right) \tag{3.78}
\end{aligned}$$

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The seventh and eighth lines are exchanged under the interchange of i and m for the same reasons as the fifth and sixth lines. Therefore it remains to check that the ninth and tenth lines are exchanged. To do this, we use (3.70) and (3.71) and find

$$
\begin{aligned}
K_i \beta_{im} + K_m \beta_{mi} \quad + \quad & \sum_{k \neq i,m} \beta_{km} \beta_{ki} \\
= \quad & \sum_{k \neq i,m} (\beta_{ik}\beta_{im} + \beta_{mk}\beta_{mi} + \beta_{km}\beta_{ki}) + \beta_{im}^2 + \beta_{mi}^2 + \eta(\beta_i \beta_{im} + \beta_m \beta_{mi}) \\
= \quad & \sum_{k \neq i,m} 4 + \beta_{im}^2 + \beta_{mi}^2 + 2\eta = \beta_{im}^2 + \beta_{mi}^2 + 2\left(2(N-2) + \eta\right) \quad (3.79)
\end{aligned}
$$

So for the special value $\eta = -2(N-2)$ we can conclude that F satisfies the generalized WDVV system. This finishes the proof of theorem 2.3.

3.4.1 Adding a new variable

We add a new variable to the prepotential of BCD type and consider the function

$$
\begin{aligned}
F(a_1, ..., a_{N+1}) \quad = \quad & \sum_{1 \leq i < j \leq N} \left(f(a_i - a_j) + f(a_i + a_j) \right) + \eta \sum_{i=1}^{N} f(a_i) \\
+ \quad & \gamma \left(\frac{1}{6} a_{N+1}^3 + \frac{1}{2} a_{N+1} \sum_{k=1}^{N} a_K^2 \right) \quad (3.80)
\end{aligned}
$$

The B that we will use is

$$
B_{kl} = (F_{N+1})_{kl} = \gamma \delta_{kl} \quad (3.81)
$$

and the WDVV condition becomes

$$
\sum_{k=1}^{N+1} F_{ijk} F_{klm} = \sum_{k=1}^{N+1} F_{mjk} F_{kli} \qquad i, j, l, m = 1, ..., N+1 \quad (3.82)
$$

If $i = N+1$ or $m = N+1$ then this condition is automatically fulfilled. Restricting ourselves to $i, m \leq N$ we can rewrite the left hand side of (3.82) in the form

$$
\sum_{k=1}^{N} F_{ijk} F_{klm} + F_{ij,N+1} F_{N+1,lm} = \sum_{k=1}^{N} F_{ijk} F_{klm} + \gamma^2 \delta_{ij} \gamma_{lm} \quad (3.83)
$$

Therefore we can repeat the analysis of the previous paragraph, starting with the expression (3.78). This expression is now changed by adding γ^2 to the $\delta_{ij}\delta_{lm}$ term in the penultimate line. Since this is the only change, the condition $\eta = -2(N-2)$ is modified to

$$
\eta = -2(N-2) - \frac{\gamma^2}{2} \quad (3.84)
$$

This proves theorem 2.4.

Remark The condition (3.84) allows us to use the values $\eta = 1, 0$ which are associated with the B, D Lie algebras. By adding the extra variable we can regard the corresponding prepotentials of the B, D theories as solutions not only to the generalized WDVV equations, but even to the

original system (1.1). Since η is negative for real γ, the condition (3.84) requires γ to be imaginary, and the prepotentials no longer satisfy the property that real variables a_k lead to a real value of F. To restore this property one can change the variables a_k to ia_k and change f to

$$f(x) = \frac{1}{6}(ix)^3 - \frac{1}{4}Li_3(e^{-2ix}) = -\frac{i}{6}x^3 - \frac{1}{4}\sum_{k=1}^{\infty}\frac{e^{-2ikx}}{k^3} \qquad (3.85)$$

This f is sometimes used in the literature, see e.g. [7].

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