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Abstract

In the literature, there are two proofs [MMM] [IY98] that the prepotential of $N = 2$ pure Super-Yang-Mills theory satisfies the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. We show that these two methods are in fact equivalent.
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1 Introduction

In the beginning of the '90s, Witten [Wit91] and Dijkgraaf, E. Verlinde and H. Verlinde [DVV91] defined what are now known as the WDVV equations in the context of 2-dimensional topological field theories. Loosely stated, these equations generate the necessary conditions for a (quasi-homogeneous) function to generate via its third order derivatives the structure constants of an associative algebra. Work on these equations was continued by Dubrovin (see e.g. [Dub96] and references therein) who put the WDVV equations in the more general setting of Frobenius manifolds. Among other things, he classified the polynomial solutions.

In 1994, a breakthrough was achieved in theoretical physics with the work of Seiberg and Witten [SW94a],[SW94b] who gave the exact quantum solution of 4-dimensional $N = 2$ SYM theories. This solution is presented in terms of the so-called prepotential, and it was suggested by Bonelli and Matone [BM96] that the prepotential of this *4-dimensional* theory is a non-polynomial solution to the WDVV equations. This was proven for the classical ($ABCD$) gauge groups by Marshakov, Mironov and Morozov [MMM] and for simply laced (ADE) groups by Ito and Yang [IY98].

The first group of authors uses an associative algebra of differentials to give their proof. The classical groups are associated with hyperelliptic Riemann surfaces and therefore the algebra of differentials is relatively easy to make. Since the exceptional groups are not associated with hyperelliptic Riemann surfaces, such an algebra becomes more difficult.

The second group of authors uses a connection to the 2-dimensional theory via the so-called Landau-Ginzburg framework. In this manner, they derive an associative algebra of functions and they use the Picard-Fuchs equations on the Riemann surface to find the WDVV equations. A crucial aspect of their proof seems to be the ADE classification and the 'flat coordinates' belonging to it. They also give a proof for the B, C type gauge groups by rewriting the Picard-Fuchs equations in such a way that they become WDVV equations **if** some constants $\widetilde{C}_{ij}^k$ are structure constants of an associative algebra.

In this paper, we will show that the methods of these two groups of authors are in fact equivalent. In particular, it is found that the constants $\widetilde{C}_{ij}^k$ are the structure constants of the associative algebra of differentials for the corresponding gauge groups. This gives new hope for a proof of the last remaining[1] case for pure $N = 2$ SYM theory: the gauge group F_4.

The paper is organized as follows: in section 2, the WDVV equations are introduced. In section 3, the prepotential of 4-dimensional $N = 2$ SYM theory is defined. Section 4 contains an introduction to holomorphic q-differentials which are necessary for the first proof. Then in sections 5 and 6 the two proofs for the WDVV equations are given and finally in section 7 it is proven that the two methods are equivalent.

2 The WDVV equations

The WDVV equations were introduced in [Wit91],[DVV91] in the context of 2-dimensional topological N=2 superconformal field theory. It is a system of third order non-linear differential equations given by the following definition:

Definition 1 *A solution to the WDVV equations is given by a holomorphic function $F(a^1, ..., a^r)$ of r complex variables such that*

1. *F is quasihomogeneous, i.e. $F(\lambda^{d_1} a^1, ..., \lambda^{d_r} a^r) = \lambda^{d_F} F(a^1, ..., a^r)$*
2. *The matrix F_r is constant (independent of the a^i) and invertible*
3. *$F_i F_r^{-1} F_j = F_j F_r^{-1} F_i \qquad i, j = 1, ..., r$*

Here, F_i (for any $i = 1, ..., r$) denotes the Hesse matrix of $\frac{\partial F}{\partial a^i}$, so $(F_i)_{jk} = \frac{\partial^3 F}{\partial a^i \partial a^j \partial a^k}$.

Together with the introduction of these equations, a whole class of polynomial solutions to them was given. It turned out however that it is very hard to give non-polynomial solutions. Nevertheless, some years later, it was conjectured [BM96] that the prepotentials of 4-dimensional N=2 SYM theory[2] are (non-polynomial) solutions to these equations in some sense. However, in this physical theory no natural special coordinate a^r exists such that F_r is constant and invertible and therefore the generalized WDVV system was introduced:

Definition 2 *A solution $F(a^1, ..., a^r)$ to the generalized WDVV system is a function characterized by*

$$F_i K^{-1} F_j = F_j K^{-1} F_i \qquad i, j = 1, ..., r \tag{2.1}$$

Here, the matrix K is an invertible linear combination of the F_k.

[1] For G_2 the WDVV equations become trivial.

[2] This is a physical 4-dimensional field theory with gauge group G. We will introduce the prepotential of such a theory in section 3.3.

This is obviously a generalization of the WDVV system, since a solution to the WDVV system is also a solution to the generalized system. In this paper, we will show that the prepotential of pure N=2 SYM theory satisfies the generalized system for any gauge group (with the possible exception of F_4), but the ones with simply laced algebras even satisfy the WDVV system itself. Note that we dropped the requirement of quasi-homogeneity becaus the prepotential does not satisfy this property. Nevertheless, there is still a grading present in the problem and it will be useful in the proof[3].

An interesting observation is that if we define $C_i = F_i K^{-1}$ then (2.1) becomes equivalent to

$$C_i C_j = C_j C_i \tag{2.2}$$

This suggests the following

Proposition 3 *If there exists a function $F(a^1, ..., a^r)$ and an invertible matrix K such that we can write $F_i = C_i K$ then F satisfies (2.1) if and only if the $(C_i)^k_j = C^k_{ij}$ are structure constants of a commutative associative algebra.*

Proof. ($\Leftarrow$) If there exists a commutative algebra A

$$\phi_i * \phi_j = \sum_{k=1}^{r} C^k_{ij} \phi_k \tag{2.3}$$

then A is associative iff for all i, j, k

$$\begin{aligned}
(\phi_i * \phi_j) * \phi_k &= \phi_i * (\phi_j * \phi_k) \\
\left(\sum_{l=1}^{n} C^l_{ij} \phi_l \right) * \phi_k &= \phi_i * \left(\sum_{l=1}^{n} C^l_{jk} \phi_l \right) \\
\sum_{l=1}^{n} C^l_{ij} \sum_{m=1}^{n} C^m_{lk} \phi_m &= \sum_{l=1}^{n} C^l_{jk} \sum_{m=1}^{n} C^m_{il} \phi_m \\
\sum_{l=1}^{n} \sum_{m=1}^{n} C^l_{ij} C^m_{kl} \phi_m &= \sum_{l=1}^{n} \sum_{m=1}^{n} C^l_{kj} C^m_{il} \phi_m \\
C_i C_k &= C_k C_i
\end{aligned} \tag{2.4}$$

where the ϕ_i should be taken linearly independent. From this we can prove that F satisfies the generalized WDVV equations:

$$F_i K^{-1} F_j = C_i F_j = C_i C_j K = C_j C_i K = F_j K^{-1} F_i$$

($\Rightarrow$) If F satisfies the generalized WDVV system then the C_i commute:

$$\begin{aligned}
F_i K^{-1} F_j &= C_i F_j = C_i C_j K = \\
F_j K^{-1} F_i &= C_j C_i K
\end{aligned}$$

[3] We could include the 'physical scale' μ (see section 3.1) as one of the variables a^i to yield a quasihomogeneous function again [MMM96].

By multiplying these equations with K^{-1} we get the desired result. Now we define an (abstract) algebra A which has the $(C_i)_j^k$ as its structure constants. Then A is associative because of (2.4). ■

Therefore, if we are given a set of $(F_i)_{jk}$ then there exists a solution F of the generalized WDVV system if

1. The $(F_i)_{jk}$ are the third order derivatives of some function $F(a^1, ..., a^r)$
2. There exists an invertible matrix K such that we can write $F_i = C_i K$
3. The $(C_i)_j^k$ are structure constants of an associative algebra

This is the approach we will use to prove that the prepotential of pure N=2 SYM theory indeed satisfies the generalized WDVV system.

3 Candidate solutions: the prepotential of pure N=2 SYM

The aim of this section is to introduce a class of possible solutions to the generalized WDVV system. These solutions arise naturally in a physical theory called *Seiberg-Witten theory* or *N=2 Super-Yang-Mills (SYM) theory* [SW94a],[Bil97], [AGH97]. The object of interest in this paper, which is the prepotential, can be introduced in purely mathematical terms, and this is done below. First a family of Riemann surfaces is associated with every gauge group, then a special meromorphic differential on the families is introduced (the Seiberg-Witten differential) and finally the prepotential is given in terms of integrals of this Seiberg-Witten differential over cycles on the Riemann surfaces.

3.1 A family of Riemann surfaces

With every gauge group G we associate a physical theory, classified by the twisted affine Lie algebra $\left(g^{(1)}\right)^\vee$ where g denotes the Lie algebra of G [Kac90]. This Seiberg-Witten theory has a natural setting in terms of a family of Riemann surfaces, which is given by the following data [MW96]:

1. In terms of a Chevalley basis $\{h_i, e_{\alpha_i}, e_{-\alpha_i}\}$ of $\left(g^{(1)}\right)^\vee$ we define

$$A = \sum_{i=1}^{r} \left(b^i h_i + a^i e_{\alpha_i} + e_{-\alpha_i}\right) + z e_{\alpha_0} + z^{-1} e_{-\alpha_0} \tag{3.1}$$

2. In terms of an irreducible representation ρ, we define a family of Riemann surfaces (depending on $\left(g^{(1)}\right)^\vee$ and ρ) through the spectral equation

$$\det\left[\rho\left(A\right) - x \cdot Id\right] = 0 \tag{3.2}$$

which is a polynomial equation in x.

We have defined different families depending on the specific representation. It turns out however, that although the families for various representations are indeed different, the prepotentials depend only on the gauge group. We have used this freedom to give the families for the smallest representation: the results can be found in appendix A. At the end of this section we will come back briefly on this representation independence.

We will illustrate some general properties of the families by discussing the simplest case of gauge group with Lie algebra A_r in the fundamental representation. In this case, the family of surfaces is given by

$$z + \frac{\mu}{z} - \left(x^{r+1} - u^1 x^{r-1} - u^2 x^{r-2} - ... - u^{r-1} x - u^r\right) = 0 \tag{3.3}$$

where the u^i, μ are (for all groups) polynomial expressions in the complex numbers a^i and b^i. If we assign appropriate degrees to them ($[a^i] = 2, [b^i] = 1$) then $[u^i] = d_i$ where d_i is a *degree* of the Lie algebra and $[\mu] = 2h_g^\vee$ ($h_g^\vee$ is the dual Coxeter number of the Lie algebra), see table 1. With these degrees (3.1), (3.2) and (3.3) become graded. The origin of this

	h	$h^\vee$	d_i
A_r	$r+1$	$r+1$	$2,..,r+1$
B_r	$2r$	$2r-1$	$2,4,..,2r$
C_r	$2r$	$r+1$	$2,4,..,2r$
D_r	$2r-2$	$2r-2$	$2,4,..,2r-2,r$
E_6	12	12	$2,5,6,8,9,12$
E_7	18	18	$2,6,8,10,12,14,18$
E_8	30	30	$2,8,12,14,18,20,24,30$
F_4	12	9	$2,6,8,12$
G_2	6	4	$2,6$

Table 1: Data for the Lie algebras

grading lies in a well-known grading [Kac90] of the Lie algebra $(g^{(1)})^\vee$. For completeness, we list the degrees of all the variables:

$$[x]=1 \quad [z]=h_g^\vee \quad [\mu]=2h_g^\vee \quad \left[u^i\right]=d_i \quad \left[a^i\right]=2 \quad \left[b^i\right]=1$$

The u^i are moduli parameters of the family of Riemann surfaces since fixing the values $\{u^i,\mu\}$ fixes a Riemann surface in the family. Throughout this paper, we will keep the value of μ fixed, so the families we will consider depends on r moduli where r denotes the rank of the gauge group. This view will become important when discussing the Picard-Fuchs equations in section 6.1.2: these equations express how certain integrals over cycles on the Riemann surface depend on the moduli parameters.

Before moving on to the Seiberg-Witten differential, which contains the rest of the data needed to define the prepotential, we will analyze the curves closer. The surfaces can all be rewritten in the convenient form

$$\mathcal{C}_z \quad : \quad z+\frac{\mu}{z}=W(x,u^1,...,u^r) \tag{3.4}$$

which is trivial in the case of A_r but can yield a nonpolynomial (and non-rational) W for other gauge groups (see appendix A). The surfaces of the classical groups can even be taken hyperelliptic:

Proposition 4 *For the A,B,C,D Lie algebras, the prepotential can also be described by a family of hyperelliptic Riemann surfaces. This family is defined by the transformation*

$$\begin{aligned} & & y &= x^m\left(z-\frac{\mu}{z}\right) \\ \mathcal{C}_y &: & y^2 &= P(x) \end{aligned} \tag{3.5}$$

for certain m depending on the gauge group.

Proof. First we give the list of transformations to the (nonsingular) hyperelliptic surfaces:

$$\begin{array}{ll} A_r: & y = z - \frac{\mu}{z} \\ & y^2 = \left(z + \frac{\mu}{z}\right)^2 - 4\mu = W^2 - 4\mu = P(x) \\ & g = r \end{array} \tag{3.6}$$

$$\begin{array}{ll} B_r: & y = x\left(z - \frac{\mu}{z}\right) \\ & y^2 = W_{BC}^2 - 4\mu x^2 = P(x) \\ & g = 2r - 1 \end{array} \tag{3.7}$$

$$\begin{array}{ll} C_r: & y = x^{-1}\left(z - \frac{\mu}{z}\right) \\ & y^2 = x^{-2}(\left(z + \frac{\mu}{z}\right)^2 - 4\mu) = W_{BC}(x^2 W_{BC} + 4\sqrt{\mu}) = P(x) \\ & g = 2r \end{array} \tag{3.8}$$

$$\begin{array}{ll} D_r: & y = x^2\left(z - \frac{\mu}{z}\right) \\ & y^2 = x^4 W^2 - 4\mu x^4 = P(x) \\ & g = 2r - 1 \end{array} \tag{3.9}$$

For any (z, x) satisfying (3.4) the point

$$(y, x) = \left(x^m\left(z - \frac{\mu}{z}\right), x\right)$$

satisfies (3.5). Conversely, if (y, x) satisfies (3.5) then

$$(z, x) = \left(\frac{1}{2}\left[W(x) + x^{-m}y\right], x\right)$$

satisfies (3.4). ■

It would make computations for the exceptional groups easier if their prepotentials can also be expressed in terms of hyperelliptic surfaces, but it does not seem likely that this is possible [LPG97].

3.2 Holomorphic differentials and λ_{SW}

For further use, we will introduce the holomorphic differentials on a Riemann surface and then the Seiberg-Witten differential λ_{SW} associated with each family of Riemann surfaces.

Definition 5 *A holomorphic differential $f dz$ is given by a set of locally holomorphic functions $\{f_k(z_k)\}$ (one for every coordinate chart) such that if U_i, U_j are charts, $z_i \in U_i$, $z_j \in U_j$ and $z_i, z_j \in U_i \cap U_j$, then*

$$f_j(z_j) = \frac{\partial z_i}{\partial z_j} f_i(z_i) \tag{3.10}$$

Proposition 6 *The holomorphic differentials on a Riemann surface of genus g form a linear space Ω^1, the dimension of which equals g.*

Proof. Since $fdx + gdx$ is given by $\{(f_i + g_i)(z_i)\}$ and $\alpha f dx$ is given by $\{\alpha f_i(z_i)\}$, it is easy to see that the differentials form a linear space. The dimension can for example be read off from the Riemann-Roch theorem (see e.g. [FK92]). ■

We will describe an example of a basis for Ω^1

Example 7 *We consider holomorphic differentials on a hyperelliptic Riemann surface of genus g. Let this surface be defined by*

$$y^2 = \prod_{\alpha=1}^{2g+2} (x - x_\alpha) \tag{3.11}$$

with $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, where $x, y \in \mathbb{C}P = \mathbb{C} \cup \{\infty\}$. Local coordinates are given as follows:

1. *If $y \neq 0$ $(x \neq x_\alpha\ \forall\alpha)$ and $x \neq \infty$ then we can take x as a local coordinate.*
2. *Define an $R \in \mathbb{R}$ such that $y \neq 0$ for all (x, y) on the curve and $|x| > R$. This is a neighbourhood of ∞, and a local coordinate is given as $X := 0$ if $x = \infty$ and $X := \frac{1}{x}$ otherwise.*
3. *If $y = 0$ $(x = x_\alpha)$ then $v_\alpha := \sqrt{x - x_\alpha}$ is a local coordinate in a neighbourhood of this point (which does not contain another x_β).*

The holomorphic differentials are given by (see e.g. [FK92])

$$\psi_i = \frac{x^{i-1}dx}{y} \qquad i = 1, ..., g \tag{3.12}$$

Let us check that these are indeed holomorphic by checking holomorphicity in the coordinate charts

- *If $x \neq x_\alpha$ for all α and $x \neq \infty$ then $\frac{x^{i-1}}{y}$ is certainly holomorphic if $i \geq 1$*
- *Around $x = \infty$ we use the local coordinate X which leads to*

$$\psi_i = \frac{X^{-i+1}\left(-X^{-2}\right)dX}{\sqrt{\prod_{i=1}^{2g+2}(\frac{1}{X} - x_\alpha)}} \overset{X\approx 0}{\sim} -X^{g+1-i+1-2}dX = -X^{g-i}dX$$

  which is nonsingular for $i \leq g$.
- *Around the x_α, we use the local coordinate v_α which yields*

$$\psi_i = \frac{\left(v_\alpha^2 + x_\alpha\right)^{i-1}(2v_\alpha)\,dv_\alpha}{v_\alpha \prod_{\beta\neq\alpha}(v_\alpha^2 + x_\alpha - x_\beta)} \overset{v_\alpha\approx 0}{\sim} \frac{x_\alpha^{i-1}dv_\alpha}{\prod_{\beta\neq\alpha}(x_\alpha - x_\beta)}$$

  which is nonsingular.

We now turn to the introduction of the Seiberg-Witten differential. This is not a holomorphic differential, but a meromorphic one (which is meromorphic in each coordinate chart).

Definition 8 *The Seiberg-Witten differential is defined on every Riemann surface from a family by*

$$\lambda_{SW} = x\frac{dz}{z} \stackrel{(3.4)}{=} \frac{x\partial_x W}{\sqrt{W^2-4\mu}}dx = x^{m+1}\partial_x W \frac{dx}{y} \tag{3.13}$$

This defines the Seiberg-Witten differential in the x-coordinate, and the transformation rule (3.10) can be used to determine what it looks like in other coordinates.

Proposition 9 *The Seiberg-Witten differential has the important property that*

$$\frac{\partial \lambda_{SW}}{\partial u^i} \in \Omega^1 \tag{3.14}$$

Proof. We work out the differentiation:

$$\begin{aligned} \frac{\partial \lambda_{SW}}{\partial u^i} &= \frac{\partial}{\partial u^i}\left(\frac{x\partial_x W}{\sqrt{W^2-4\mu}}dx\right) = -\frac{\frac{\partial W}{\partial u^i}}{\sqrt{W^2-4\mu}}dx + d\left(\frac{x\frac{\partial W}{\partial u^i}}{\sqrt{W^2-4\mu}}\right) \\ &\cong -\frac{\partial W}{\partial u^i}\frac{1}{\sqrt{W^2-4\mu}}dx = -\frac{\partial W}{\partial u^i}x^m\frac{dx}{y} \end{aligned} \tag{3.15}$$

where $\cong$ denotes equality modulo exact differential forms. It is easy to verify that the genus of the hyperelliptic curves for A_r, B_r, C_r, D_r are $g = r, 2r-1, 2r, 2r-1$ respectively and that

$$\frac{\partial \lambda_{SW}}{\partial u^i} = p_i(x)\frac{dx}{y} \tag{3.16}$$

with $p_i(x)$ a polynomial of degree smaller or equal to $g-1$ for the classical groups. Therefore $\frac{\partial \lambda_{SW}}{\partial u^i}$ is a linear combination of the ψ_i of example 7.

For the exceptional groups, we refer to [MMM] and references therein for a proof. ■

3.3 The prepotential

Let g denote the genus of the Riemann surfaces (in all cases $g \geq r$), then given g independent holomorphic differentials ω_i there are $2g$ canonical homology cycles A_i, B_i $(i = 1, ..., g)$ such that

$$\oint_{A_i} \omega_j = \delta_{ij} \tag{3.17}$$

$$A_i \circ B_j = \delta_{ij} = -B_j \circ A_i \tag{3.18}$$

where $A_i \circ B_j$ denotes the intersection number of the two cycles. It can be shown that the B_i satisfy

$$\oint_{B_i} \omega_j = \Pi_{ij} \tag{3.19}$$

where Π is called the period matrix, which can be shown to be symmetric, i.e. $\Pi_{ij} = \Pi_{ji}$. We now have enough data to define the prepotential

Definition 10 *Let a family of surfaces be given, associated with the gauge group. In terms of some special canonical homology cycles, the **prepotential** is a function $F(a^i)$ where*

$$a^i = \oint_{A_i} \lambda_{SW} \tag{3.20}$$

and this function is given implicitly by

$$a_D^j = \frac{\partial F}{\partial a^j} = \oint_{B_j} \lambda_{SW} \tag{3.21}$$

In this definition, the cycles A_i, B_i are a subset which contains $2r$ (not $2g$) elements. We will come back to the particular choice of the subset.

To see that the function F really exists (at least locally), note that

$$\oint_{A_j} \frac{\partial \lambda_{SW}}{\partial a^i} = \frac{\partial}{\partial a^i} \oint_{A_j} \lambda_{SW} = \frac{\partial a^j}{\partial a^i} = \delta_i^j \tag{3.22}$$

$$\oint_{B_j} \frac{\partial \lambda_{SW}}{\partial a^i} = \frac{\partial}{\partial a^i} \oint_{B_j} \lambda_{SW} = \frac{\partial^2 F}{\partial a^i \partial a^j} = \Pi_{ij} \tag{3.23}$$

and we now use the fact that the period matrix is symmetric to see that a_D^j is a gradient:

$$\frac{\partial}{\partial a^i}\left(a_D^j\right) = \Pi_{ij} = \Pi_{ji} = \frac{\partial}{\partial a^j}\left(a_D^i\right)$$

Therefore, the function F indeed exists.

Proposition 11 *The differentials*

$$\omega_i = \frac{\partial \lambda_{SW}}{\partial a^i} \tag{3.24}$$

are holomorphic and the homology cycles are canonical with respect to them.

Proof. We already saw that $\left\{\frac{\partial \lambda_{SW}}{\partial u^i}\right\}$ are g independent holomorphic differentials. Due to the choice of the cycles (which we haven't explained yet) we get that

$$\frac{\partial a^i}{\partial u^j} = \oint_{A_i} \frac{\partial \lambda_{SW}}{\partial u^j}$$

is an invertible map, so that $\frac{\partial \lambda_{SW}}{\partial a^i} = \sum_k \frac{\partial u^k}{\partial a^i} \frac{\partial \lambda_{SW}}{\partial u^k}$ which is a linear combination of holomorphic differentials and therefore again holomorphic. Also,

$$\oint_{A_j} \frac{\partial \lambda_{SW}}{\partial a^i} = \frac{\partial}{\partial a^i} \oint_{A_j} \lambda_{SW} = \frac{\partial a^j}{\partial a^i} = \delta_i^j$$

Finally, we must show that $\oint_{B_j} \frac{\partial \lambda_{SW}}{\partial a^i}$ is the period matrix of a Riemann surface. But this is automatically guaranteed once the cycles A_i, B_i are a canonical homology basis. ■

Before moving on to q-differentials on the Riemann surfaces, we want to come back briefly to the dependence of the prepotential on the representation for $\left(g^{(1)}\right)^\vee$. First we note that if we keep the gauge group fixed, the different representations give rise to families with a common property:

Proposition 12 *The distinguished Prym subvariety of the family does not depend on the representation of the gauge group*

Proof. The proof can be found in [MW96] and references therein. ■

The Prym is a subvariety of the Jacobian of a surface (which is an r-dimensional torus). The homology cycles one has to take to define the prepotential are determined purely by this distinguished Prym. Since the Prym is representation independent, so are the cycles and therefore the prepotential.

4 The holomorphic q-differentials

Recall the definition of a holomorphic differential (Definition 5). We can think of a holomorphic differential as being a section of a line bundle Ω. Locally, the line bundle is given by $U_i \times \mathbb{C}$ where $U_i \subset \mathbb{C}$ is a coordinate chart around z. A section is then given locally by pairs $(z^i, f_i(z^i))$ where f_i is a holomorphic function $f_i(z^i) : U_i \to \mathbb{C}$. On the overlaps $U_i \cap U_j$, we must have

$$f_j(z^j) = f_i(z^i)\frac{\partial z^i}{\partial z^j} \tag{4.1}$$

A holomorphic q-differential is then given by the following

Definition 13 *For any $q \in \mathbb{N}$ a holomorphic q-differential is a holomorphic section of the line bundle $L^{\otimes q}$, which has transition functions*

$$f_j(z^j) = f_i(z^i)\left(\frac{\partial z^i}{\partial z^j}\right)^q \tag{4.2}$$

on the overlaps $U_i \cap U_j$. This is the q^{th} tensor product of the cotangent bundle; the space of holomorphic sections of $L^{\otimes q}$ is denoted by Ω^q.

The q-differentials are often denoted by $f(z)\,(dz)^q$ (see the definition of holomorphic differentials). It is clear that the q-differentials form a linear space (copy the proof from Ω^1) and its dimension can be read off from the Riemann-Roch theorem (see e.g. [FK92]). The dimensions are given in table 2.

g	q	dim
0	all	0
1	all	1
>1	0	1
	1	g
	>1	$(2q-1)(g-1)$

Table 2: The dimension of the space of q-differentials of a Riemann surface of genus g

Example 14 *As an example, we will give a basis for Ω^2 on hyperelliptic Riemann surfaces*[4]*. We define a product $p : \Omega^1 \times \Omega^1 \to \Omega^2$ locally by*

$$p(fdz, gdz) = p(\{f_i(z^i)dz^i\}, \{g_i(z^i)dz^i\}) = \left\{f_i(z^i)g_i(z^i)\left(dz^i\right)^2\right\}$$

if ϕ is given by $\{f_i\}$ and ψ is given by $\{g_i\}$. To see that this defines an element of Ω^2 we should check the transformation property:

$$f_j(z^j)g_j(z^j) = \left(f_i(z^i)\frac{\partial z^i}{\partial z^j}\right)\left(g_i(z^i)\frac{\partial z^i}{\partial z^j}\right) = f_i(z^i)g_i(z^i)\left(\frac{\partial z^i}{\partial z^j}\right)^2$$

[4] Elements of Ω^1, Ω^2 and Ω^3 are the only q-differentials used in this paper.

which indeed transforms as an element of Ω^2. Furthermore we should check that the functions in the local coordinates are holomorphic, which they are since they are products of holomorphic functions. Therefore the products

$$\frac{x^{i-1}dx}{y}\frac{x^{j-1}dx}{y} \in \Omega^2 \tag{4.3}$$

generate a subspace of Ω^2, whose dimension is $2g-1$. We can take the basis elements of this subspace to be

$$\frac{x^{i-1}}{y^2}(dx)^2 \qquad i = 1, ..., 2g-1 \tag{4.4}$$

The rest of the basis of Ω^2 is formed by (see e.g. [FK92])

$$\frac{x^{i-1}}{y}(dx)^2 \qquad i = 1, ..., g-2 \tag{4.5}$$

The transformation properties are given by requiring that these are 2-differentials, so we should only check that they are holomorphic. Around the x_α where $y = 0$, there is no problem (see section 3.2) because we get

$$\frac{\left(v_\alpha^2 + x_\alpha\right)^{i-1}(2v_\alpha)^2 (dv_\alpha)^2}{v_\alpha \prod_{\beta\neq\alpha}(v_\alpha^2 + x_\alpha - x_\beta)} \overset{v_\alpha \approx 0}{\sim} \frac{x_\alpha^{i-1} v_\alpha (dv_\alpha)^2}{\prod_{\beta\neq\alpha}(x_\alpha - x_\beta)}$$

which is holomorphic. Around $x = \infty$ we get

$$\frac{X^{-i+1}\left(X^{-4}\right)(dX)^2}{\sqrt{\prod_{i=1}^{2g+2}(\frac{1}{X} - x_\alpha)}} \overset{X\approx 0}{\sim} -X^{g+1-i+1-4}(dX)^2 = -X^{g-2-i}(dX)^2$$

which is holomorphic for $1 \leq i \leq g-2$. Since $\dim\left(\Omega^2\right) = 3g-3$*, we have now found all basis elements.*

Example 15 *As a second example, we will construct a basis of Ω^3. We define a product $q : \Omega^1 \times \Omega^2 \to \Omega^3$ locally by*

$$q(\left\{f_i(z^i)dz^i\right\}, \left\{g_i(z^i)\left(dz^i\right)^2\right\}) = \left\{f_i(z^i)g_i(z^i)\left(dz^i\right)^3\right\}$$

and one can check that the outcome is indeed an element of Ω^3. This allows us to construct a basis for the subspace of Ω^3 consisting of triple products coming from Ω^1: these elements are

$$\frac{x^{i-1}}{y^3}(dx)^3 \qquad i = 1, ..., 3g-3 \tag{4.6}$$

Furthermore, one should consider products of elements of Ω^1 with elements of Ω^2 which are themselves not products:

$$\frac{x^{i-1}}{y^2}(dx)^3 \qquad i = 1, ..., 2g-4 \tag{4.7}$$

Finally, we should add

$$\frac{x^{i-1}}{y}(dx)^3 \qquad i = 1, ..., g-5 \tag{4.8}$$

One can check that indeed these are all holomorphic 3*-differentials, but we have too many of them since* $\dim\left(\Omega^3\right) = 5g-5$. *Therefore some of these elements must be linearly dependent on others. In particular, for all* $3g-3 \geq k \geq 2g+3 = \deg(y^2)+1$ *we have*

$$\frac{x^{k-1}}{y^3}(dx)^3 = \frac{x^{k-1}}{y^2}\frac{(dx)^3}{y} = \left[q(x) + \frac{r(x)}{y^2}\right]\frac{(dx)^3}{y}$$

where $q(x), r(x)$ *are polynomials,* $\deg(q) \leq g-6$ *and* $deg(r) \leq 2g+1$. *This reduces the independent number of elements to* $(2g+4)+(2g-4)+(g-5) = 5g-5$. *Again, one should check that these elements are indeed linearly independent.*

5 Proof of the WDVV equations I: algebra of differentials

The aim of this section is to construct a proof that the prepotential for gauge groups of type A, B, C, D (the classical groups) satisfies the generalized WDVV equations following [MMM]. We already saw that these groups are special in the sense that we can describe their prepotential in terms of hyperelliptic Riemann surfaces, and this property will simplify the proof. In the first section, the proof is outlined by supposing that the prepotential satisfies 4 properties. Under the assumption that these properties hold, a proof that the prepotential satisfies the WDVV equations is presented. In section 5.3 the required properties are proven.

Although the proof is based on [MMM], it has to be slightly altered since we take a different curve to describe C_r. This will amount to an algebra of 'even' differentials, whereas in [MMM] only algebras of odd differentials are considered.

5.1 Setting up the proof

The first time the WDVV equations were proven to hold for a certain type of function was in the context of conformal field theory [Wit91],[DVV91]. An essential ingredient in the proof in that context is an associative algebra of polynomials. Here we will present the proof of [MMM], which makes use of an associative algebra of holomorphic differentials. But apart from this difference, the proofs are very similar.

We will need the following properties of the physical theory:

Proposition 16 *For the subset* $\{\omega_i = \frac{\partial \lambda_{SW}}{\partial a^i}\}$ *of* Ω^1 *(see proposition 11), there exist* $G, H_{ij} \in \Omega^1$ *such that the multiplication structure*

$$\omega_i \omega_j = \sum_{k \in I} C_{ij}^k \omega_k G + H_{ij} \frac{dz}{z} \tag{5.1}$$

exists: we can express any product of ω_i *and* ω_j *uniquely in terms of the righthand side.*

Proposition 17 *We can define an algebra* A *by*

$$\omega_i * \omega_j = \sum_{k \in I} C_{ij}^k \omega_k \tag{5.2}$$

with the C_{ij}^k *from (5.1). This algebra is associative, i.e.*

$$(\omega_i * \omega_j) * \omega_k = \omega_i * (\omega_j * \omega_k) \tag{5.3}$$

Proposition 18 *There exists a residue formula for the prepotential*

$$\frac{\partial F}{\partial a^i \partial a^j \partial a^k} = \underset{\partial_x \mathcal{P}=0}{res} \left(\sum_{l,m,n} \sigma_i^l \sigma_j^m \sigma_k^n \frac{\partial \psi_l}{\partial x} \frac{\partial \mathcal{P}}{\partial u^m} \frac{\partial \mathcal{P}}{\partial u^n} \frac{1}{(\partial_x \mathcal{P})^2} \frac{dz}{z} \right) = \underset{\frac{dz}{z}=0}{res} \left(\frac{\omega_i \omega_j \omega_k}{dx \frac{dz}{z}} \right) \tag{5.4}$$

The terms in this formula need some explaining. The Riemann surface is rewritten in terms of a polynomial (see the appendix)

$$\mathcal{P}(x, u^i, z) = 0 \tag{5.5}$$

The middle term in (5.4) is the sum of residues of a meromorphic differential at all points where $\partial_x \mathcal{P} = 0$ and it can be shown that at these points also $\frac{dz}{z} = 0$. Therefore we indeed take the residue of a first order pole in (5.4). We have written

$$\begin{aligned} \sigma_i^j &= \frac{\partial u^j}{\partial a^i} \\ \chi_j &= \frac{\partial \lambda_{SW}}{\partial u^j} = \frac{\partial \psi_j}{\partial x} dx \\ \omega_i &= \frac{\partial \lambda_{SW}}{\partial a^i} = \sum_j \sigma_i^j \chi_j \end{aligned}$$

The right term in (5.4) is a formal way of writing the middle term, and in this form the symmetry under exchange of i, j, k is especially clear.

Proposition 19 *We define matrices F_i by*

$$(F_i)_{jk} = \frac{\partial F}{\partial a^i \partial a^j \partial a^k} \tag{5.6}$$

There is a $G = \sum_{l \in I} \alpha_l \omega_l$ such that $K = \sum_{l \in I} \alpha_l F_l$ is invertible and proposition 16 and 17 hold.

Under these assumptions, we will prove the following important

Theorem 20 *Suppose that the prepotential of an N=2 SYM theory satisfies propositions 16-19. Then this prepotential $F(a^i)$ satisfies the WDVV equations*

$$F_i K^{-1} F_j = F_j K^{-1} F_i \tag{5.7}$$

Proof. First we recall (section 3.3) that although the prepotential was defined implicitly through

$$\begin{aligned} a^i &= \oint_{A_i} \lambda_{SW} \\ \frac{\partial F}{\partial a^j} &= \oint_{B_j} \lambda_{SW} \end{aligned}$$

there does exist a function $F(a^i)$. The next observation is the following

Lemma 21 *We define the matrices C_i by $(C_i)_j^k = C_{ij}^k$. Then*

$$F_i = C_i K \tag{5.8}$$

$$F_{ijk} = \sum_{l=1}^{r} C_{ij}^l K_{kl} \tag{5.9}$$

Note that although C_i and K depend on G, the prepotential itself does not, due to (5.4)..

Proof. Substitute (5.1) into (5.4) and note that the term with $\frac{dz}{z}$ drops out because it cancels the first order pole in (5.4) and the residue of this term then vanishes. ■

We will rewrite the associativity condition of the formal algebra A to obtain a condition for the structure constants C_{ij}^k

$$
\begin{aligned}
(\omega_i * \omega_j) * \omega_k &= \omega_i * (\omega_j * \omega_k) \\
\left(\sum_{l=1}^{n} C_{ij}^l \omega_l\right) * \omega_k &= \omega_i * \left(\sum_{l=1}^{n} C_{jk}^l \omega_l\right) \\
\sum_{l=1}^{n} C_{ij}^l \sum_{m=1}^{n} C_{lk}^m \omega_m &= \sum_{l=1}^{n} C_{jk}^l \sum_{m=1}^{n} C_{il}^m \omega_m \\
\sum_{l=1}^{n} \sum_{m=1}^{n} C_{ij}^l C_{kl}^m \omega_m &= \sum_{l=1}^{n} \sum_{m=1}^{n} C_{kj}^l C_{il}^m \omega_m
\end{aligned}
$$

The lefthand and the righthand side should be equal, leading to the following condition on the structure constants regarded as matrices

$$C_i C_k = C_k C_i \qquad \forall i, k \in I \tag{5.10}$$

if the ω_i are linearly independent. Now we are ready to finish the proof: we observe that K^{-1} exists due to proposition 19, we use the lemma and (5.10) to see that

$$F_i K^{-1} F_j = C_i F_j = C_i C_j K = C_j C_i K = F_j K^{-1} F_i \tag{5.11}$$

■

Note that in the proof, we did not use an algebra of *all* holomorphic differentials, but an algebra for the differentials ω_i which are associated to F. These differentials are given by

$$\omega_i = \frac{\partial \lambda_{SW}}{\partial a^i} \in \Omega^1 \qquad i = 1, ..., r \tag{5.12}$$

5.2 An example: A_r

Before turning to the proofs of propositions 16-19, we will discuss in detail the simplest case of gauge group A_r. In this case, the curve is given by (see section 3.1)

$$y = z - \frac{\mu}{z} \Longrightarrow \begin{cases} z + \frac{\mu}{z} = W = x^{r+1} - u_1 x^{r-1} - u_2 x^{r-2} ... - u_r \\ y^2 = W^2 - 4\mu \\ \lambda_{SW} = x W' \frac{dx}{y} \end{cases} \tag{5.13}$$

The genus of the curve is given by $g = r$ (and therefore the subset of differentials mentioned above are in fact all the elements of Ω^1) and the prepotential is given by

$$
\begin{aligned}
a^i &= \oint_{A_i} \lambda_{SW} \qquad && i \in \{1, ..., g\} \\
\frac{\partial F}{\partial a^j} &= \oint_{B_j} \lambda_{SW} \qquad && j \in \{1, ..., g\}
\end{aligned}
$$

Recall that the differentials

$$\omega_i = \frac{\partial \lambda_{SW}}{\partial a^i} \qquad i \in \{1, ..., g\} \tag{5.14}$$

are canonical with respect to the cycles. We will subsequently check the properties mentioned in the propositions:

Ad prop. 16 Taking $G = \frac{dx}{y}$ and $H_{ij} = \sum_k D_{ij}^k \omega_k$, we want to show that the following multiplication structure exists:

$$\omega_i \omega_j = \sum_{k \in I} \left[C_{ij}^k \omega_k \frac{dx}{y} + D_{ij}^k \omega_k W' \frac{dx}{y} \right] \tag{5.15}$$

In order to see this, we express

$$\omega_i = \sum_j \sigma_i^j \chi_j \tag{5.16}$$

in terms of the basis elements

$$\chi_i = -\frac{\partial W}{\partial u^i} \frac{dx}{y} = \frac{x^{g-i} dx}{y} \qquad i \in \{1, ..., g\} \tag{5.17}$$

so that the multiplication structure becomes

$$\left(\sum_{l=1}^{g} \sigma_i^l \chi_l \right) \left(\sum_{m=1}^{g} \sigma_j^m \chi_m \right) = \tag{5.18}$$

$$\sum_{k=1}^{g} \left[C_{ij}^k \left(\sum_{n=1}^{g} \sigma_k^n \chi_n \right) \frac{dx}{y} + D_{ij}^k \left(\sum_{n=1}^{g} \sigma_k^n \chi_n \right) W' \frac{dx}{y} \right] \tag{5.19}$$

We now construct a multiplication for the χ_i instead of the ω_i: if we can make

$$\chi_l \chi_m = \sum_{n=1}^{g} \left[\widehat{C}_{lm}^n \chi_n \frac{dx}{y} + \widehat{D}_{lm}^n \chi_n W' \frac{dx}{y} \right] \tag{5.20}$$

then this multiplication structure would imply the existence of (5.15) with structure constants

$$(C_i)_j^k = \sigma_i^l \sigma_j^m \widehat{C}_{lm}^n \left(\sigma^{-1}\right)_n^k \tag{5.21}$$

So let us turn to (5.20).

We already observed in section 4 that a basis for the products of elements of Ω^1 is given by

$$\frac{x^{i-1} (dx)^2}{y^2} \qquad i \in \{1, ..., 2g-1\} \tag{5.22}$$

We will need all the $\widehat{C}^n_{ij}$ and all but one of the $\widehat{D}^n_{ij}$: the $\left\{\chi_n \frac{dx}{y}\right\}$ with $n \in \{1, ..., g\}$ form a subset of the basis and $\left\{\chi_n W' \frac{dx}{y}\right\}$ with $n \in \{2, ..., g\}$ precisely form the rest of the basis[5]. So on the right hand side we have a linear combination of basis elements of the space of products and any product on the left hand side can be written as such a combination. This proves the existence of the multiplication structure for the χ_i and hence for the ω_i.

Ad prop. 17 The next step is to show that the factor algebra is associative. We prove this is by using the curve for A_r to see that

$$\omega_i = p_i(x)\frac{dx}{y} \tag{5.23}$$

where $p_i(x)$ are polynomials, because the χ_i are of this form and the transformation σ is independent of x. Next, we note that (5.15) is equivalent to a multiplication of polynomials

$$p_i(x)p_j(x) = \sum_{k=1}^{g}\left[C^k_{ij}p_k(x) + D^k_{ij}p_k(x)W'(x)\right] \tag{5.24}$$

It is well-known that any factor algebra of polynomials over an ideal is associative. Since $W'(x)$ generates an ideal we conclude that $C_iC_j = C_jC_i$.

Ad prop. 18 We will now turn to the residue formula (5.4). We will give a proof following [MMM], which is immediately valid for any gauge group. We start with two observations that we will use in the proof:

$$\frac{\partial F_i}{\partial u^k} = \frac{\partial}{\partial u^k}\int_{B_i}\lambda_{SW} = \sum_j \frac{\partial a^j}{\partial u^k}F_{ij} = \sum_j F_{ij}\int_{A_j}\chi_k \tag{5.25}$$

and

$$\sum_i \left[\int_{B_i}\chi_k\int_{A_i}\chi_l - \int_{B_i}\chi_l\int_{A_i}\chi_k\right] = \sum_{\delta\mathcal{M}} res\left(\psi_l\chi_k\right) \tag{5.26}$$

We use a representation of the Riemann surface by a $4g$-sided polygon $\mathcal{M}$ with identification of the sides. Since $\mathcal{M}$ is simply connected, $\chi_k = d\psi_k$ with ψ_k is a holomorphic function. Using these two equations together with partial integration with respect to u^m we get

$$\sum_{r,s}\frac{\partial F_{rs}}{\partial u^n}\int_{A_r}\chi_l\int_{A_s}\chi_m = \frac{\partial}{\partial u^n}\left(\sum_r\left[\int_{B_r}\chi_m\int_{A_r}\chi_l - \int_{B_r}\chi_m\int_{A_r}\chi_l\right]\right)$$
$$+\sum_r\left[\int_{B_r}\frac{\partial\chi_m}{\partial u^n}\int_{A_r}\chi_l - \int_{B_r}\chi_l\int_{A_r}\frac{\partial\chi_m}{\partial u^n}\right] = 0 + res\left(\psi_l\frac{\partial\chi_m}{\partial u^n}\right)$$

[5] $\chi_n W'\frac{dx}{y} = x^k W'\frac{(dx)^2}{y^2}$ where x^kW' has degree $2g-n$ in x. If we take $n=1$ then we are outside of the basis.

We will now rewrite this residue in the desired form. Since the curve is described by

$$\mathcal{P}(x, z, u^i) = 0 \tag{5.27}$$

we can use the implicit function theorem to define $x(u^i, \log(z))$ so that u^i and $\log(z)$ become the independent variables and

$$\chi_m = \frac{\partial}{\partial u^m}\lambda_{SW} = \frac{\partial}{\partial u^m}\left(x\frac{dz}{z}\right) = \left(\frac{\partial}{\partial u^m}x\right)\frac{dz}{z} = -\frac{1}{\partial_x \mathcal{P}}\frac{\partial \mathcal{P}}{\partial u^m}\frac{dz}{z} \tag{5.28}$$

and therefore

$$\begin{aligned}\frac{\partial \chi_m}{\partial u^n} &= -\frac{\partial^2 \mathcal{P}}{\partial u^n \partial u^m}\frac{1}{\partial_x \mathcal{P}}\frac{dz}{z} + \partial_x\left(\frac{\partial \mathcal{P}}{\partial u^m}\right)\frac{\partial \mathcal{P}}{\partial u^n}\frac{1}{\left(\partial_x \mathcal{P}\right)^2}\frac{dz}{z} + \\ & \frac{\partial \mathcal{P}}{\partial u^m}\frac{\partial\left(\partial_x \mathcal{P}\right)}{\partial u^n}\frac{1}{\left(\partial_x \mathcal{P}\right)^2}\frac{dz}{z} - \frac{\partial \mathcal{P}}{\partial u^m}\frac{\partial \mathcal{P}}{\partial u^n}\frac{\partial^2 \mathcal{P}}{\partial x^2}\frac{1}{\left(\partial_x \mathcal{P}\right)^3}\frac{dz}{z}\end{aligned} \tag{5.29}$$

Note that due to the derivative with respect to u^n the holomorphic differential χ_m becomes meromorphic: the extra poles are caused by the zeroes of $\partial_x \mathcal{P}$. We will now show that where $\partial_x \mathcal{P}$ is zero also $\frac{dz}{z}$ is zero: from equation (5.27) we see that for fixed u^i

$$\begin{aligned}0 = \frac{d\mathcal{P}}{d\log(z)}d\log(z) = \left(\frac{\partial \mathcal{P}}{\partial x}\frac{dx}{d\log(z)} + \frac{\partial \mathcal{P}}{\partial \log(z)}\right)d\log(z) \Longrightarrow \\ \partial_x \mathcal{P} dx + \frac{\partial \mathcal{P}}{\partial \log(z)}\frac{dz}{z} = 0\end{aligned} \tag{5.30}$$

One of the conditions for the implicit function theorem for fixed u^i is that

$$grad(\mathcal{P}) = \left[\frac{\partial \mathcal{P}}{\partial x}, \frac{\partial \mathcal{P}}{\partial \log(z)}\right] \neq [0, 0] \tag{5.31}$$

Therefore $\frac{dz}{z}$ must be zero if $\partial_x \mathcal{P}$ is zero.

From this reasoning, we can see that if the $\frac{\partial \mathcal{P}}{\partial u^i}$ do not cause singularities (which should be checked), the first term in (5.29) can be ignored since it is not singular and the last term can be ignored since it is too singular. The residue now becomes

$$\begin{aligned}\sum res\left(\psi_l \frac{\partial \chi_m}{\partial u^n}\right) &= \sum res\left(\psi_l \partial_x\left(\frac{\partial \mathcal{P}}{\partial u^m}\frac{\partial \mathcal{P}}{\partial u^n}\frac{1}{\left(\partial_x \mathcal{P}\right)^2}\frac{dz}{z}\right)\right) \\ &= -\sum \underset{\partial_x \mathcal{P}=0}{res}\left(\frac{\partial \psi_l}{\partial x}\frac{\partial \mathcal{P}}{\partial u^m}\frac{\partial \mathcal{P}}{\partial u^n}\frac{1}{\left(\partial_x \mathcal{P}\right)^2}\frac{dz}{z}\right)\end{aligned} \tag{5.32}$$

The final step towards the residue formula (5.4) is to add the Jacobians σ (5.16)

$$\begin{aligned}\sum_{r,s,l,m,n}\sigma_i^l \sigma_j^m \sigma_k^n \frac{\partial F_{rs}}{\partial u^n}\int_{A_r}\chi_l \int_{A_s}\chi_m = \sum_{r,s,n}\sigma_k^n \frac{\partial F_{rs}}{\partial u^n}\int_{A_r}\omega_i \int_{A_s}\omega_j = \\ \sum_n \sigma_k^n \frac{\partial F_{ij}}{\partial u^n} = \sum_{m,n}\sigma_k^n \frac{\partial F_{ij}}{\partial a^m}\frac{\partial a^m}{\partial u^n} = \sum_{m,n}\sigma_k^n \frac{\partial F_{ij}}{\partial a^m}\int_{A_m}\chi_n = \frac{\partial^3 F}{\partial a^i \partial a^j \partial a^k}\end{aligned}$$

On the other hand, we have

$$\frac{\partial^3 F}{\partial a^i \partial a^j \partial a^k} = \sum_{r,s,l,m,n} \sigma_i^l \sigma_j^m \sigma_k^n \frac{\partial F_{rs}}{\partial u^n} \int_{A_r} \chi_l \int_{A_s} \chi_m =$$

$$- \sum_{l,m,n} \sigma_i^l \sigma_j^m \sigma_k^n \underset{\partial_x \mathcal{P}=0}{res} \left(\frac{\partial \psi_l}{\partial x} \frac{\partial \mathcal{P}}{\partial u^m} \frac{\partial \mathcal{P}}{\partial u^n} \frac{1}{(\partial_x \mathcal{P})^2} \frac{dz}{z} \right) =$$

$$- \sum_{l,m,n} \sigma_i^l \sigma_j^m \sigma_k^n \underset{\partial_x \mathcal{P}=0}{res} \left(\frac{\chi_l \chi_m \chi_n}{dx \frac{dz}{z}} \right) = - \underset{\partial_x \mathcal{P}=0}{res} \left(\frac{\omega_i \omega_j \omega_k}{dx \frac{dz}{z}} \right)$$

which is the desired formula. Note that this derivation is valid for all gauge groups.

Ad prop. 19 Finally, we want to show that there exists an invertible matrix $\sum_{l=1}^g \alpha_l F_l$ where $\sum_{l=1}^g \alpha_l \omega_l = G = \frac{dx}{y}$. Since $\frac{dx}{y} = \chi_g$, we find that $\alpha_l = \left(\sigma^{-1}\right)_g^l$ so that

$$\sum_{l=1}^g \alpha_l \omega_l = \sum_{k,l=1}^g \left(\sigma^{-1}\right)_g^l \sigma_l^k \chi_k = \sum_k \delta_g^k \chi_k = \chi_g \tag{5.33}$$

and we need to show that

$$K := \sum_{l=1}^g \left(\sigma^{-1}\right)_g^l F_l = \sum_{l=1}^g \frac{\partial a^l}{\partial u^g} \frac{\partial F}{\partial a^l} = \frac{\partial F}{\partial u^g} \tag{5.34}$$

with entries

$$K_{ij} = \frac{\partial^3 F}{\partial u^g \partial a^i \partial a^j} \tag{5.35}$$

is invertible. For this we refer to [IY98] and references therein. In these articles, so-called 'flat coordinates' t^i (see also section 6) are constructed on the moduli space in such a way that

$$\eta_{ij} = \frac{\partial^3 F}{\partial t^g \partial t^i \partial t^j} \tag{5.36}$$

is invertible. It can be checked that $\frac{\partial F}{\partial u^g} = \frac{\partial F}{\partial t^g}$ and by adding the correct Jacobians from the $t's$ to the $a's$ (these are invertible) we get the desired result.

5.3 Proofs of the propositions

5.3.1 Proposition 16

Proof. The statement to prove is that for the classical groups A, B, C, D there exists a multiplication structure

$$\omega_i \omega_j = \sum_{k \in I} C_{ij}^k \omega_k G + H_{ij} \frac{dz}{z} \tag{5.37}$$

Our first observation is that for all these gauge groups, we can take hyperelliptic Riemann surfaces. We need to identify the differentials of which we want to make an algebra. In the case of A_r we need all the differentials, but in the rest of the cases we need only about half of them (the rank of these groups is about half the genus). It is easy to see[6] that the Riemann surfaces are invariant under the involution $\sigma : x \to -x$. The g holomorphic differentials will be split into even and odd differentials, the number of these depends on whether g is an even or odd number. This immediately implies that

$$\omega_i = \frac{\partial \lambda_{SW}}{\partial a^i} = -\sigma_i^j \frac{\partial W}{\partial u^j} x^m \frac{dx}{y} \tag{5.38}$$

is odd under this symmetry for groups B_r, D_r but even for C_r. Therefore we should restrict ourselves to products of the odd (B_r, D_r) or even (C_r) holomorphic differentials and these products will be even under σ. This will also affect the choice of G and H_{ij} since these must be chosen even or odd as well. The multiplication structure (5.37) expresses the possibility to expand these products in the linear combinations on the right hand side. We will give the proof of this case by case:

1. A_r: We already saw in the previous section that the right hand side is a linear combination of all basis elements if we take $G = \frac{dx}{y}, H_{ij} = \sum_k D_{ij}^k \omega_k$.

2. B_r: We recall that

$$z + \frac{\mu}{z} = W = \frac{W_{BC}}{x} = \frac{x^{2r} - \sum_{i=1}^r u^i x^{2r-2i}}{x} \tag{5.39}$$

$$y = x\left(z - \frac{\mu}{z}\right) \tag{5.40}$$

$$y^2 = W_{BC}^2 - 4\mu x^2 \tag{5.41}$$

and therefore

$$\frac{dz}{z} = x\frac{dW}{y} = (W_{BC}' - \frac{W_{BC}}{x})\frac{dx}{y} \tag{5.42}$$

$$\frac{\partial \lambda_{SW}}{\partial a^i} = -\frac{\partial W_{BC}}{\partial a^i}\frac{dx}{y} \tag{5.43}$$

(see (3.15)) and therefore the ω_i are odd. Since $g = 2r - 1$, there are $\frac{g-1}{2}$ even differentials, and $\frac{g+1}{2}$ odd holomorphic differentials, a basis of these is given by

$$x^{2i+1}\frac{dx}{y} \qquad i = 0, ..., \frac{g-3}{2} \tag{5.44}$$

$$\frac{x^{2i}dx}{y} \qquad i = 0, ..., \frac{g-1}{2} \tag{5.45}$$

and therefore the products of the odd differentials form a subspace of the quadratic differentials of dimension g, which is spanned by

$$\frac{x^{2i}\,(dx)^2}{y^2} \qquad i = 0, ..., g-1 \tag{5.46}$$

[6]The expressions for the curves can be found in the appendix.

We can take $G = \frac{dx}{y}$, since this is an odd differential and multiplied by odd holomorphic differentials it yields even ones. Therefore, the first term on the right hand side of (5.37) is a linear combination of

$$\frac{x^{2i}\,(dx)^2}{y^2} \qquad i = 0, ..., \frac{g-1}{2} \tag{5.47}$$

This takes care of the first $\frac{g+1}{2}$ basis elements in which the products must be expanded. Since $\frac{dz}{z}$ is even under σ, it need only be multiplied by even differentials, and therefore we take $H_{ij} = \sum_k D_{ij}^k \zeta_k^e$ where ζ_k^e form a basis of even differentials. Then the second term of (5.37) adds the following $\frac{g-1}{2}$ linearly independent elements to (5.47)

$$\frac{x^{2i}\,(dx)^2}{y^2} \qquad i = \frac{g+1}{2}, ..., g-1 \tag{5.48}$$

What happens is that the second term generates differentials from which we subtract linear combinations of differentials in (5.47) to get these new basis elements. The total number of basis elements that we constructed is now g, which is exactly enough to expand the products of odd holomorphic differentials in. Therefore we see that the multiplication structure exists.

3. C_r: The curve is given by

$$\begin{aligned}\left(z + \frac{\mu}{z}\right) &= x^2 W_{BC} + 2\sqrt{\mu} \\ y &= x^{-1}\left(z - \frac{\mu}{z}\right) \\ y^2 &= W_{BC}\left(x^2 W_{BC} + 4\sqrt{\mu}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{dz}{z} &= \frac{d\left(x^2 W_{BC} + 2\sqrt{\mu}\right)}{xy} = \left(2W_{BC} + xW'_{BC}\right)\frac{dx}{y} \\ \frac{\partial \lambda_{SW}}{\partial a^i} &= -x\frac{\partial W_{BC}}{\partial a^i}\frac{dx}{y}\end{aligned}$$

Since the genus is in this case even ($g = 2r$), we have $\frac{g}{2}$ even and $\frac{g}{2}$ odd differerentials with bases

$$x^{2i+1}\frac{dx}{y} \qquad i = 0, ..., \frac{g-2}{2} \tag{5.49}$$

$$x^{2i}\frac{dx}{y} \qquad i = 0, ..., \frac{g-2}{2} \tag{5.50}$$

respectively. We must take G even, and if we take $G = x\frac{dx}{y}$ then the first term on the right hand side of (5.37) is a linear combination of

$$\frac{x^{2i}\,(dx)^2}{y^2} \qquad i \in \{1, ..., \frac{g}{2}\} \tag{5.51}$$

Since $\frac{dz}{z}$ is odd, it must be multiplied by odd differentials and therefore we take $H_{ij} = \sum_k D_{ij}^k \zeta_k^o$ where the ζ_k^o denote a basis for the odd differentials. After subtracting linear combinations of the elements in (5.51) from the second term, it is seen that it consists of a linear combination of

$$\frac{x^{2i}\,(dx)^2}{y^2} \qquad i = \{\frac{g}{2}+1, ..., g-1\} \tag{5.52}$$

This means that we have a total of $\frac{g}{2}+\frac{g-2}{2} = g-1$ independent terms on the right hand side, which is exactly enough[7] to expand the products of odd differentials in. Note that although the right hand side does *not* form a complete basis for the products of even quadratic differentials (since this space has dimension g and not $g-1$), we still have enough elements since $\frac{dx^2}{y^2}$ is never formed as a product of even differentials.

4. D_r: The construction is very similar to the B_r case. The genus $g = 2r-1$ Riemann surfaces are given by

$$z + \frac{\mu}{z} = W = \frac{W_D}{x^2} = \frac{x^{2r} - u_1 x^{2r-2} - u_2 x^{2r-4} - ...u_{r-2}x^4 - u_r x^2 - u_{r-1}^2}{x^2} \tag{5.53}$$

$$y = x^2(z - \frac{\mu}{z}) \tag{5.54}$$

$$y^2 = W_D^2 - 4\mu x^4 \tag{5.55}$$

and

$$\frac{dz}{z} = \frac{dW}{x^{-2}y} = (W_D' - 2\frac{W_D}{x})\frac{dx}{y} \tag{5.56}$$

$$\frac{\partial \lambda_{SW}}{\partial a^i} = -\frac{\partial W_D}{\partial a^i}\frac{dx}{y} \tag{5.57}$$

As in the B_r case, the genus $g = 2r-1$ is odd, so there are $\frac{g-1}{2}$ even differentials, and $\frac{g+1}{2}$ odd holomorphic differentials, a basis of these is given by (5.44),(5.45) and therefore the products of the odd differentials form a subspace of the quadratic differentials of dimension g, which is spanned by

$$\frac{x^{2i}\,(dx)^2}{y^2} \qquad i = 0, ..., g-1 \tag{5.58}$$

If we take $G = \frac{dx}{y}$, then this odd differential need only be multiplied by odd holomorphic differentials to yield even ones. Therefore, the first term on the right hand side of (5.37) is a linear combination of

$$\frac{x^{2i}\,(dx)^2}{y^2} \qquad i = 0, ..., \frac{g-1}{2} \tag{5.59}$$

[7] see equation (5.46).

This takes care of the first $\frac{g+1}{2}$ basis elements in which the products must be expanded. Since $\frac{dz}{z}$ is even, we take $H_{ij} = \sum_k D^k_{ij} \zeta^e_k$ where the ζ^e_k denote a basis for the even differentials. Then the second term adds the following $\frac{g-1}{2}$ linearly independent elements to (5.59)

$$\frac{x^{2i}\,(dx)^2}{y^2} \qquad i = \frac{g+1}{2}, ..., g-1 \tag{5.60}$$

Note that the term with highest degree $\frac{x^{2g}(dx)^2}{y^2}$ drops out (its coefficient is zero) since it is outside of Ω^2. There are g independent elements to express the products of odd differentials in, which is exactly enough.

For future purposes, we will also describe a different method of constructing the D_r algebra of forms: take $G = \frac{x^2 dx}{y}$ and $H_{ij} = \sum_k D^k_{ij} \zeta^e_k$. Then the first term on the right hand side of the algebra is a linear combination of

$$\frac{x^{2i}\,(dx)^2}{y^2} \qquad i = 2, ..., \frac{g+1}{2} \tag{5.61}$$

and it would seem at first sight that we loose the basis element $\frac{(dx)^2}{y^2}$. However, subtracting these basis elements properly from the second term, we see that this second term becomes a linear combination of

$$\frac{x^{2i}\,(dx)^2}{y^2} \qquad i = 0, \frac{g+3}{2}, \frac{g+5}{2}, ..., g-1 \tag{5.62}$$

In particular, the term

$$\left(x\frac{dx}{y}\right)\frac{dz}{z} = x^3 W' \left(\frac{dx}{y}\right)^2 = (xW_D' - 2W_D)\left(\frac{dx}{y}\right)^2 \tag{5.63}$$

becomes equal to

$$\left(\frac{dx}{y}\right)^2 \tag{5.64}$$

if we subtract proper linear combinations of (5.61). So what we have actually done by taking $G = \frac{x^2 dx}{y}$ is to move the basis element $\frac{(dx)^2}{y^2}$ from the first term to the second term on the right hand side. Then we have again g basis elements of the space of products of differentials and therefore the left hand side can again be expressed in terms of them.

Since all classical groups are dealt with, this ends the proof.

■

5.3.2 Proposition 17

Proof. Again we handle the algebras case by case. The general strategy is to construct an isomorphism from the algebra of differentials to a polynomial algebra, which is associative if we factor it over an ideal. We can use the isomorphism to see that the corresponding algebra of differentials is also associative.

1. A_r: The algebra is already isomorphic to a polynomial algebra if we remove all the $\left(\frac{dx}{y}\right)^2$. The polynomial $\frac{\partial W}{\partial x}$ generates an ideal so the factor algebra

$$p_i p_j = \sum_{k=1}^{r} C_{ij}^k p_k \mod \left(\frac{\partial W}{\partial x}\right) \tag{5.65}$$

over this polynomial is associative. Now just plug in the $\left(\frac{dx}{y}\right)^2$ again and we find that

$$\omega_i \omega_j = \sum_{k=1}^{r} C_{ij}^k \omega_k \mod \left(\frac{\partial W}{\partial x}\frac{dx}{y}\right) \tag{5.66}$$

is associative.

2. B_r: Again we remove all the $\left(\frac{dx}{y}\right)^2$ and we get the algebra

$$p_i p_j = \sum_{k=1}^{r} C_{ij}^k p_k + \sum_{k=1}^{r} D_{ij}^k q_k \left(xW_{BC}' - W_{BC}\right) \tag{5.67}$$

This algebra is also a polynomial algebra. The reasoning we applied for A_r now also applies here.

3. C_r: If we remove the $\left(\frac{dx}{y}\right)^2$, the algebra becomes a polynomial algebra straight away.

4. D_r: If we remove the $\left(\frac{dx}{y}\right)^2$, the algebra becomes a polynomial algebra straight away.

■

5.3.3 Propositions 18&19

Proof. For the proof of these propositions, we refer to the example A_r discussed in section 5.2. ■

6 Proof of the WDVV equations II

The aim of this section is to construct a second proof that the prepotential satisfies the WDVV equations, following the lines of [IY98]. This proof is (at least superficially) somewhat different from the proof using the q-differentials, and it can be applied directly for the A, D, E type gauge groups (instead of the classical groups A, B, C, D). The authors also managed to use their method to prove in some particular cases (B_3, C_3, B_4, C_4) that the WDVV equations hold there also, which was expected at the time since the proof using the q-differentials was already known.

6.1 The A, D, E groups

6.1.1 Existence of an associative algebra

For these groups the Riemann surfaces look like

$$z + \frac{\mu}{z} = W(x, u^i) \tag{6.1}$$

where $W(x, u^i)$ is a function known from singularity theory and Landau-Ginzburg models, see for instance [DVV91]. In fact, mutiple variable descriptions of the singularities are known for a long time[AGZV85], but the single-variable version of E_6 was inspired by Seiberg-Witten theory[EY97]. We will now cite some results that were found in the context of Landau-Ginzburg theory. One can define so-called flat coordinates t^i on the moduli space:

Definition 22 *Flat coordinates on the moduli space are given by*

$$t^i = c_i \oint W^{\frac{e_i}{h^\vee}} dx \tag{6.2}$$

where the residue should be taken[8] *at* $x = \infty$ *and*

$$e_i = \deg(u^i) - 1 \tag{6.3}$$

where the degree of u^i *(as well as the* e_i *and* $h^\vee$*) was defined in section 3.1. The unknown constants can be chosen in such a way that the 'metric'*

$$\eta_{ij} = \frac{\partial^2}{\partial t^i \partial t^j} \oint W^{1+\frac{1}{h^\vee}} dx \tag{6.4}$$

becomes proportional to $\delta_{e_i+e_j,h^\vee}$ *(so that it is flat, hence the name flat coordinates). Equivalently, these coordinates ensure that*

$$\frac{\partial^2 W}{\partial t^i \partial t^j} = \frac{\partial Q_{ij}}{\partial x} \tag{6.5}$$

with Q_{ij} *defined as in (6.7).*

[8] Except for t_r in the case of group D_r, which should be taken around $x = 0$.

It can be checked in all cases that the transformation from u^i to t^i is invertible: from (6.2) it can be deduced that the t^i are polynomial in the u^i. Furthermore we find from the grading that the u with highest index that occurs in t^k is u^k. From explicit calculation one finds that its coefficient is not zero, and therefore the transformation is invertible (the Jacobian is upper triangular with nonzero diagonal elements).

If we define

$$\phi_i := -\frac{\partial W}{\partial t^i} \qquad i = 1, ..., r = rank(G) \tag{6.6}$$

then one of the results of Landau-Ginzburg theory says that

Proposition 23 *There exists a multiplication structure*

$$\phi_i \phi_j = \sum_{k=1}^{r} C_{ij}^k \phi_k + Q_{ij} \frac{\partial W}{\partial x} \tag{6.7}$$

which is equivalent to some algebra of polynomials (there exists a polynomial algebra with the same structure constants). Since a polynomial algebra factorized over an ideal is associative, we get as a result that the algebra

$$\phi_i \cdot \phi_j = \sum_{k=1}^{r} C_{ij}^k \phi_k \quad \mathrm{mod} \left[\frac{\partial W}{\partial x} \right] \tag{6.8}$$

is associative

Proof. Originally, it was proven [Wit91],[DVV91] that there exists a polynomial multiplication structure for the multi-variable version of the Landau-Ginzburg superpotential $\widetilde{W}$ for which all of this is true. However, the W that we have presented for the D, E type gauge groups is a one-variable nonpolynomial version. The fact that they satisfy the same algebra as the polynomial $\widetilde{W}$ associated with these groups was proven in [EY97]. Therefore we can make an isomorphism between these algebras. Since the polynomial algebra is associative, we can factor out $\frac{\partial \widetilde{W}}{\partial x}$ (which is now itself polynomial) and use the isomorphism to see that the algebra

$$\phi_i \cdot \phi_j = \sum_{k=1}^{r} C_{ij}^k \phi_k \quad \mathrm{mod} \left[\frac{\partial W}{\partial x} \right] \tag{6.9}$$

is associative itself. ■

6.1.2 Picard-Fuchs equations

The Picard-Fuchs equations are a set of first order differential equations associated with each family of Riemann surfaces, which expresses how the integrals of holomorphic forms over the homology cycles depend on the moduli space parameters. In this section we will derive a subset of the Picard-Fuchs equations by using the algebra we just found.

Proposition 24 *The differential equations*

$$\frac{\partial}{\partial t^i}\left(\oint_\Gamma \frac{\partial \lambda_{SW}}{\partial t^j}\right) = \sum_{k=1}^{r} C_{ij}^k \frac{\partial}{\partial t^k}\left(\oint_\Gamma \frac{\partial \lambda_{SW}}{\partial t^r}\right) \tag{6.10}$$

hold and they are a subset of the Picard-Fuchs equations. Here Γ *is any homology cycle.*

Proof. First recall (section 3.2) that the derivatives of λ_{SW} with respect to the moduli parameters are holomorphic differentials. Therefore equation (6.10) is a differential equation of period integrals of holomorphic forms over a homology cycle and are therefore by definition a subset of the Picard-Fuchs equations if they hold.

The Seiberg-Witten differential was defined by

$$\lambda_{SW} = x\frac{dz}{z} = \frac{xW'}{\sqrt{W^2-4\mu}}dx \tag{6.11}$$

and therefore its derivative with respect to the moduli parameters is given by

$$\frac{\partial \lambda_{SW}}{\partial t^i} \cong \frac{-1}{\sqrt{W^2-4\mu}}\frac{\partial W}{\partial t^i}dx = \frac{\phi_i}{\sqrt{W^2-4\mu}}dx \tag{6.12}$$

where we already anticipated taking an integral over these differentials, since we only calculate modulo exact forms. Taking another derivative leads to

$$\frac{\partial^2 \lambda_{SW}}{\partial t^i \partial t^j} = \frac{-1}{\sqrt{W^2-4\mu}}\frac{\partial^2 W}{\partial t^j \partial t^i}dx + \frac{\phi_i\phi_j W}{(W^2-4\mu)^{\frac{3}{2}}}dx \tag{6.13}$$

Now we use (6.7) and (6.5) to see that

$$\begin{aligned}\frac{\partial^2 \lambda_{SW}}{\partial t^j \partial t^i} &= \frac{-1}{\sqrt{W^2-4\mu}}\frac{\partial Q_{ij}}{\partial x}dx + \frac{\left(\sum_{k=1}^r C_{ij}^k\phi_k + Q_{ij}\frac{\partial W}{\partial x}\right)W}{(W^2-4\mu)^{\frac{3}{2}}}dx \\ &\cong \sum_{k=1}^{r} C_{ij}^k \frac{\partial^2 \lambda_{SW}}{\partial t^k \partial t^r} \end{aligned} \tag{6.14}$$

Taking an integral over any homology cycle yields the desired result. ∎

6.1.3 The proof

Theorem 25 *The prepotentials of the* A, D, E *gauge groups satisfy the WDVV equations.*

Proof. Since we want to give a proof that the prepotential $F(a^i)$ satisfies the WDVV equations, we will have to convert (6.10) to the a^i coordinates (we will sometimes write capital indices for a variables in order to distinguish $F_i = \sum \frac{\partial F}{\partial a^I}\frac{\partial a^I}{\partial t^i}$ from $F_I = \frac{\partial F}{\partial a^I}$). The change of coordinates leads to

$$\left(\frac{\partial a^I}{\partial t^i}\frac{\partial a^J}{\partial t^j} - \sum_{l=1}^{r} C_{ij}^l \frac{\partial a^I}{\partial t^l}\frac{\partial a^J}{\partial t^r}\right)\frac{\partial^2}{\partial a^I \partial a^J}\oint_\Gamma \lambda_{SW} + \left(\frac{\partial^2 a^I}{\partial t^i \partial t^j} - \sum_{k=1}^{r} C_{ij}^k \frac{\partial^2 a^I}{\partial t^k \partial t^r}\right)\oint_\Gamma \frac{\partial \lambda_{SW}}{\partial a^I} = 0 \tag{6.15}$$

Since

$$a^I = \oint_{A_I} \lambda_{SW}$$

is itself a period integral, we can use (6.10) to see that the second term in (6.15) vanishes. If we now take

$$\oint_{\Gamma} \lambda_{SW} = \oint_{B_K} \lambda_{SW} = \frac{\partial F}{\partial a^K}$$

then we get

$$\frac{\partial a^I}{\partial t^i}\frac{\partial a^J}{\partial t^j}\frac{\partial^3 F}{\partial a^I \partial a^J \partial a^K} = \sum_{l=1}^{r} C_{ij}^l \frac{\partial a^I}{\partial t^l}\frac{\partial a^J}{\partial t^r}\frac{\partial^3 F}{\partial a^I \partial a^J \partial a^K}$$

Multiplying both sides with $\frac{\partial a^K}{\partial t^k}$ (for any fixed k), summing over κ and putting

$$F_{ijk} = \frac{\partial a^I}{\partial t^i}\frac{\partial a^J}{\partial t^j}\frac{\partial a^K}{\partial t^k}\frac{\partial^3 F}{\partial a^I \partial a^J \partial a^K} \tag{6.16}$$

we get

$$F_{ijk} = \sum_{l=1}^{r} C_{ij}^l F_{lrk} \tag{6.17}$$

Note the similarity between this expression and (5.8) if we take $K_{kl} = F_{rlk}$. From now on, it will be assumed that F_r is an invertible matrix [IY98]. We can use exactly the same reasoning that led to equation (5.11) to see that

$$F_i F_r^{-1} F_j = C_i F_j = C_i C_j F_r = C_j C_i F_r = F_j F_r^{-1} F_i$$

Since the Jacobians on the left- and righthand side are the same and since they are invertible, this result is equivalent to

$$F_I F_R^{-1} F_J = F_J F_R^{-1} F_I \qquad \forall I, J$$

and therefore the prepotential for the ADE groups satisfies the WDVV system. ■

6.2 The B type groups

In this section we will give a proof that the prepotential for gauge groups B_3 also satisfies the WDVV equations. The method we use follows the authors of [IY98], who first construct the Riemann surfaces for B_r from A_{2r-1} and then use the polynomial algebra of the corresponding A_{2r-1} group to derive the Picard-Fuchs equations. From these Picard-Fuchs equations, it is again shown that the prepotential satisfies the WDVV equations.

6.2.1 Construction from A groups

The construction of the Rieman surfaces [MW96] of the B_r groups is done from the A_{2r-1} groups as follows: take $W_{A_{2r-1}}(\widetilde{u}^i)$ and set

$$\begin{aligned} \widetilde{u}^{2k} &= 0 \\ u^k &= \widetilde{u}^{2k-1} \qquad k = 1, ..., r \end{aligned} \tag{6.18}$$

The W for B_r is obtained by dividing the restricted W for A_{2r-1} by x

$$W_{B_r} = \frac{W_{A_{2r-1}}\big|_{\widetilde{u}^{2k}=0}}{x} = \frac{W_{BC}}{x} \tag{6.19}$$

The flat coordinates are obtained from those of A_{2r-1} by setting

$$\begin{aligned} t^k &= \widetilde{t}^{2k-1}\Big|_{\widetilde{u}^{2k}=0} \\ \phi_k &= \frac{-\partial W_{BC}}{\partial t^k} \end{aligned} \tag{6.20}$$

Note that from the multiplication structure

$$\widetilde{\phi}_i \widetilde{\phi}_j = \sum_{k=1}^{2r-1} C_{ij}^k \widetilde{\phi}_k + \widetilde{Q}_{ij} \frac{\partial W_{A_{2r-1}}}{\partial x} \tag{6.21}$$

belonging to A_{2r-1} ($\widetilde{\phi}_i = -\frac{\partial W_{A_{2r-1}}}{\partial \widetilde{t}_i}$), we can make a substructure:

Proposition 26 *The multiplication structure*

$$\phi_i \phi_j = \sum_{k=1}^{r} C_{ij}^k \phi_k + Q_{ij} \frac{\partial W_{BC}}{\partial x} \tag{6.22}$$

exists (note that we renumbered the indices, so the C_{ij}^k in (6.21) and (6.22) are not the same). Also the coordinates are still flat in the sense that

$$\frac{\partial Q_{ij}}{\partial x} = \frac{\partial^2 W_{BC}}{\partial t^i \partial t^j} \tag{6.23}$$

Proof. From the definition of $W_{A_{2r-1}}$, we observe that $\widetilde{\phi}_{2i-1}\Big|_{\widetilde{t}^{2k}=0}$ contain only polynomials which are even in x, and $\widetilde{\phi}_{2i}$ contain polynomials which are odd. Therefore after setting $\widetilde{t}^{2k} = 0$ (which results in a subalgebra) we can split the odd and even parts of the algebra (6.21) and these are again subalgebras. Let us focus on the even part: this amounts to taking products of two odd or two even polynomials on the left hand side, and keeping only the even $\widetilde{\phi}_{2k-1}$ and restricting to $W_{A_{2r-1}}\big|_{\widetilde{t}^{2k}=0} = W_{BC}$ and Q_{ij} on the right hand side, where Q_{ij} is the odd part of $\widetilde{Q}_{ij}$ with $\widetilde{t}^{2k} = 0$. If we consider only products of even polynomials, we get (6.22), which is therefore a subalgebra. Equation (6.23) clearly holds since it is simply the even part of (6.5). ■

We will use this subalgebra of A_{2r-1} to derive the Picard-Fuchs equations.

6.2.2 The Picard-Fuchs equations

Following [IY98], we will now write the Picard-Fuchs equations in a specific form. We will start with

$$\frac{\partial^2\lambda_{SW}}{\partial t^i\partial t^j} = \frac{-1}{\sqrt{W^2-4\mu}}\frac{\partial^2 W}{\partial t^i\partial t^j}dx + \frac{\frac{\partial W}{\partial t^i}\frac{\partial W}{\partial t^j}W}{(W^2-4\mu)^{\frac{3}{2}}}dx \tag{6.24}$$

keeping in mind that $-\frac{\partial W}{\partial t^i} = \frac{\phi_i}{x}$. We can use the same steps as before to find that the Picard-Fuchs equations become

$$\begin{aligned}\frac{\partial^2\lambda_{SW}}{\partial t^i\partial t^j} &\cong \sum_{k=1}^{r} C_{ij}^k\frac{\partial^2\lambda_{SW}}{\partial t^k\partial t^r} + \frac{4\mu x^{-2}Q_{ij}}{\left(W_{B_r}^2-4\mu\right)^{\frac{3}{2}}}dx \\ &= \sum_{k=1}^{r} C_{ij}^k\frac{\partial^2\lambda_{SW}}{\partial t^k\partial t^r} + 2\mu\frac{\partial}{\partial\mu}\left(\frac{x^{-2}Q_{ij}}{\sqrt{W_{B_r}^2-4\mu}}\right)dx\end{aligned} \tag{6.25}$$

Since Q_{ij} is odd and has maximal degree $2r-3$ in x, we can divide it by x to get

$$Q_{ij} = x\sum_{k=1}^{r} D_{ij}^k\widehat{\phi}_k \tag{6.26}$$

where D_{ij}^k are constants defined through this relation. Now we can prove the following

Proposition 27 *The Picard-Fuchs equations for B_r are given by*

$$\frac{\partial^2\lambda_{SW}}{\partial t^i\partial t^j} = \sum_{l=1}^{r}\left[C_{ij}^l\frac{\partial^2\lambda_{SW}}{\partial t^l\partial t^r} + \sum_{n=1}^{r}\frac{d_n t^n}{h^\vee}D_{ij}^l\frac{\partial^2\lambda_{SW}}{\partial t^l\partial t^n}\right] \tag{6.27}$$

where $h^\vee = 2r-1$ and $d_n = 2n$ for B_r.

Proof. Starting from (6.25), we make the following remarks. We will use the fact that W_{B_r} is graded to see that

$$x\frac{\partial W_{B_r}}{\partial x} + \sum_{i=1}^{r} d_n t^n\frac{\partial W_{B_r}}{\partial t^n} = h^\vee W_{B_r} \tag{6.28}$$

Using this, we find that[9]

$$\begin{aligned}\lambda_{SW} - \sum_{n=1}^{r} d_n t^n\frac{\partial}{\partial t^n}\lambda_{SW} &= \frac{h^\vee W_{B_r}}{\sqrt{W_{B_r}^2-4\mu}}dx \\ h^\vee\mu\frac{\partial}{\partial\mu}\lambda_{SW} = -\frac{h^\vee}{2}d\left[\frac{x\partial_x W_{B_r}}{\sqrt{W_{B_r}^2-4\mu}}\right] &+ \frac{1}{2}\frac{h^\vee W_{B_r}}{\sqrt{W_{B_r}^2-4\mu}}dx \cong \frac{1}{2}\frac{h^\vee W_{B_r}}{\sqrt{W_{B_r}^2-4\mu}}dx\end{aligned} \tag{6.29}$$

[9] Our conventions differ from [IY98], since they use $\sqrt{W_{B_r}^2-4\mu^2}$ instead of $\sqrt{W_{B_r}^2-4\mu}$.

Combining these equations and using (6.26), we see that the Picard-Fuchs equations can be rewritten as

$$\frac{\partial^2\lambda_{SW}}{\partial t^i\partial t^j}-\sum_{k=1}^{r}C_{ij}^k\frac{\partial^2\lambda_{SW}}{\partial t^k\partial t^r}-\sum_{k=1}^{r}\sum_{n=1}^{r}\frac{d_nt^n}{h^\vee}D_{ij}^k\frac{\partial^2\lambda_{SW}}{\partial t^k\partial t^n}+\sum_{l=1}^{r}D_{ij}^k\frac{1}{h^\vee}\left(1-d_k\right)\frac{\partial\lambda_{SW}}{\partial t^k}=0 \quad (6.30)$$

We now switch from the flat coordinates to the a^I and find

$$\begin{aligned}&\left[\partial_{t^i}a^I\partial_{t^j}a^J-C_{ij}^k\partial_{t^k}a^I\partial_{t^r}a^J-D_{ij}^l\frac{d_nt^n}{h^\vee}\partial_{t^n}a^I\partial_{t^l}a^I\right]\partial_{a^I}\partial_{a^J}\oint_\Gamma\lambda_{SW}+\\&\left[\partial_{t^i}\partial_{t^j}a^I-C_{ij}^k\partial_{t^k}\partial_{t^r}a^I+D_{ij}^l\frac{1}{h^\vee}\left(1-d_l\right)\partial_{t^l}a^I-\frac{1}{h^\vee}D_{ij}^ld_nt^n\partial_{t^l}\partial_{t^n}a^I\right]\partial_{a^I}\oint_\Gamma\lambda_{SW}\overset{(6.30)}{=}\\&\qquad\left[\partial_{t^i}a^I\partial_{t^j}a^J-C_{ij}^k\partial_{t^k}a^I\partial_{t^r}a^J-D_{ij}^l\frac{d_nt^n}{h^\vee}\partial_{t^n}a^I\partial_{t^l}a^I\right]\partial_{a^I}\partial_{a^J}\oint_\Gamma\lambda_{SW}+0=0\end{aligned}$$

where we used again that a^I is of the form $\oint_\Gamma\lambda_{SW}$ and therefore satisfies the Picard-Fuchs equations itself. ■

6.2.3 The proof

Now that we have the Picard-Fuchs equations, we can give a proof that the prepotential satisfies the WDVV system.

Theorem 28 *The prepotential of the B_3 gauge group satisfies the WDVV equations.*

Proof. We use the same coordinate transformation to the a^I and the same reasoning as before. This leads directly to the equation

$$\begin{aligned}\frac{\partial a^I}{\partial t^i}\frac{\partial a^J}{\partial t^j}\frac{\partial^3F}{\partial a^I\partial a^J\partial a^K}&=\sum_{l=1}^{r}\left[C_{ij}^l\frac{\partial a^I}{\partial t^l}\frac{\partial a^J}{\partial t^r}\frac{\partial^3F}{\partial a^I\partial a^J\partial a^K}+\sum_{n=1}^{r}\frac{2nt^n}{h^\vee}D_{ij}^l\frac{\partial a^I}{\partial t^l}\frac{\partial a^J}{\partial t^n}\frac{\partial^3F}{\partial a^I\partial a^J\partial a^K}\right]\\F_{ijk}&=\sum_{l=1}^{r}\left[C_{ij}^lF_{rlk}+\sum_{n=1}^{r}\frac{2nt^n}{h^\vee}D_{ij}^lF_{lkn}\right]\end{aligned} \quad (6.31)$$

If we set in the matrix notation

$$\begin{aligned}\left[K\right]_{ij}&=\left[F_r\right]_{ij}\\F_iK^{-1}&=\widetilde{C}_i\end{aligned}$$

then it is clear from (6.31) that

$$\begin{aligned}\widetilde{C}_iK&=C_iK+D_i\sum_{n=1}^{r}\frac{2nt^n}{h^\vee}\widetilde{C}_nK\\\widetilde{C}_i&=C_i+D_i\sum_{n=1}^{r}\frac{2nt^n}{h^\vee}\widetilde{C}_n\end{aligned} \quad (6.32)$$

If the $\widetilde{C}_{ij}^k$ turn out to be the structure constants of an associative algebra, we can use the exact same argument as before to prove that the prepotential satisfies the WDVV equations. To check that this is indeed the case, we will solve $\widetilde{C}_i$ from (6.32) in the case of B_3 and check explicitly that these matrices commute among each other:

$$\tilde{C}_1 := \begin{bmatrix} 0 & 0 & 1 \\ \frac{6}{5}\,t_3 + 2\,t_1\,t_2 & -\frac{1}{5}\,t_2 + {t_1}^2 & \frac{2}{5}\,t_1 \\ {t_1}^4 + \frac{42}{25}\,t_3\,t_1 + \frac{1}{5}\,{t_2}^2 + \frac{8}{5}\,t_2\,{t_1}^2 & \frac{2}{5}\,{t_1}^3 + \frac{6}{5}\,t_3 + \frac{48}{25}\,t_1\,t_2 & \frac{14}{25}\,{t_1}^2 + \frac{4}{5}\,t_2 \end{bmatrix}$$

$$\tilde{C}_2 := \begin{bmatrix} 0 & 1 & 0 \\ -t_2 + {t_1}^2 & -t_1 & 1 \\ \frac{6}{5}\,t_3 + 2\,t_1\,t_2 & -\frac{1}{5}\,t_2 + {t_1}^2 & \frac{2}{5}\,t_1 \end{bmatrix}$$

$$\tilde{C}_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Later in section 7 we will prove that indeed there exists a natural interpretation of these constants as structure constants of the algebra of differentials belonging to B_3. ■

6.3 The C type groups

In this section we will give a proof that the prepotential for gauge group C_3 also satisfies the WDVV equations. The method we use again follows the authors of [IY98], who first construct the Riemann surfaces for C_r from D_{r+1} and then use the algebra of the corresponding D_{r+1} group to derive the Picard-Fuchs equations. From these Picard-Fuchs equations, it is again shown that the prepotential satisfies the WDVV equations.

6.3.1 Construction from D groups

The construction of the Rieman surfaces [MW96] of the C_r groups is done from the D_{r+1} groups as follows: take $W_{D_{r+1}}(\widetilde{u}^i)$
and set

$$\widetilde{u}^{r-1} = 0$$

The curve for C_r is obtained by

$$z + \frac{\mu}{z} = x^4\, W_{D_{r+1}}\big|_{u^{r-1}=0} + 2\sqrt{\mu} = x^2 W_{BC} + 2\sqrt{\mu} \qquad (6.33)$$

The algebra we'll be using in this case is again the algebra

$$\begin{aligned} t^i &= \widetilde{t}^i\big|_{\widetilde{t}^{r-1}=0} \\ \phi_i &= -\frac{\partial W_{BC}}{\partial t^i} \\ \phi_i\phi_j &= \sum_{k=1}^{r} C_{ij}^k \phi_k + Q_{ij}\frac{\partial W_{BC}}{\partial x} \end{aligned} \tag{6.34}$$

the existence of which we already proved. Note that the flat coordinates are also the same as those in (6.20).

6.3.2 The Picard-Fuchs equations

Following [IY98], we will now write the Picard-Fuchs equations in a specific form. We will start with (6.24), keeping in mind that $-\frac{\partial W_{C_r}}{\partial t^i} = x^2\phi_i$. We can use the same steps as before to find that the Picard-Fuchs equations become

$$\frac{\partial^2 \lambda_{SW}}{\partial t^j \partial t^i} = \sum_{l=1}^{r}\left[C_{ij}^l \frac{\partial^2 \lambda_{SW}}{\partial t^l \partial t^r} - \sum_{n=1}^{r} \frac{2nt^n}{h^\vee} D_{ij}^l \frac{\partial^2 \lambda_{SW}}{\partial t^l \partial t^n}\right] \tag{6.35}$$

where we should note that $h^\vee = r+1$ for C_r.

6.3.3 The proof

Since the Picard-Fuchs equations for B_r and C_r are almost alike (there is an extra minus sign for C_r), the proof that the prepotential of pure C_r SYM theory satisfies the WDVV equations is virtually identical to the proof for B_r: just place the minus sign in all the formulas. In particular, to find the $\widetilde{C}_{ij}^k$ one has to replace $\frac{2nt^n}{h^\vee_{B_r}}$ by $-\frac{2nt^n}{h^\vee_{C_r}}$.

7 The two methods are identical

In sections 5 and 6 we have given two different proofs that the prepotentials of pure SYM theory with classical gauge group satisfy the WDVV equations. In the first proof, an associative algebra of differential forms is constructed. However, since they make explicit use of the fact that the Riemann surfaces are hyperelliptic for classical gauge groups, the authors do not know how to extend their proof to the exceptional groups. In the second proof, an associative algebra of polynomials with structure constants C_{ij}^k and the Picard-Fuchs equations are used. It is then shown that if some constants $\widetilde{C}_{ij}^k$ are structure constants of an associative algebra, the WDVV equations hold. In this section, we want to show that in the cases of A_r, B_r, C_r and D_r they indeed behave as structure constants since they are precisely the structure constants of the algebra of the differential forms

$$\psi_i = \frac{\partial \lambda_{SW}}{\partial t^i}. \tag{7.1}$$

Theorem 29 *The proofs of sections 5 and 6 are in fact identical. The Picard-Fuchs equations express exactly the same content as the residue formula together with the associative algebra of holomorphic differentials:*

$$F_{ijk} = \sum_{l=1}^{r} C_{ij}^l K_{kl} \tag{7.2}$$

Proof. The general idea is that the algebra of differentials ω_i can be obtained from an algebra of $\psi_i = \frac{\partial \lambda_{SW}}{\partial t^i}$ (instead of the $\chi_i = \frac{p_i dx}{y}$ that were considered in section 5). The transformation from this algebra to the algebra of $\omega_i = \frac{\partial \lambda_{SW}}{\partial a^i}$ is exactly given by multiplying with the Jacobians $\frac{\partial a^i}{\partial t^j}$. We will start with the simplest case

1. A_r : In section 5 the algebra of differentials was obtained from $\chi_i = \frac{\partial \lambda_{SW}}{\partial u^i}$. Since (see section 6.1.1)

$$t^i = u^i + g(u^1, ..., u^{i-1}) \tag{7.3}$$

we see that

$$\psi_i := \frac{\partial \lambda_{SW}}{\partial t^i} = \sum_{j=1}^{r} \frac{\partial u^j}{\partial t^i} \frac{\partial \lambda_{SW}}{\partial u^j} = \sum_{j=1}^{r} \frac{\partial u^j}{\partial t^i} \chi_j \tag{7.4}$$

and therefore $\{\psi_i\}$ and $\{\chi_i\}$ span the same set of differentials. We now see that the algebra of holomorphic differentials is isomorphic to the algebra of polynomials:

$$\begin{aligned} \psi_i \psi_j &= \sum_{k=1}^{r} C_{ij}^k \psi_k + \left(Q_{ij} \frac{dx}{y} \right) \frac{dz}{z} \quad \Longrightarrow \\ \phi_i \phi_j \left(\frac{dx}{y} \right)^2 &= \sum_{k=1}^{r} C_{ij}^k \phi_k \left(\frac{dx}{y} \right)^2 + Q_{ij} \frac{\partial W}{\partial x} \left(\frac{dx}{y} \right)^2 \end{aligned} \tag{7.5}$$

We simply drop the $\left(\frac{dx}{y}\right)^2$ (which means that we forget about the transformation properties) and get

$$\phi_i \phi_j = \sum_{k=1}^{r} C_{ij}^k \phi_k + Q_{ij} \frac{\partial W}{\partial x} \tag{7.6}$$

Therefore the two algebras get the same structure constants. Also, the Picard-Fuchs equations together with the algebra give

$$F_{ijk} = \sum_{l=1}^{r} C_{ij}^l K_{kl} \tag{7.7}$$

with $K_{kl} = F_{rkl}$ and this formula was also obtained in section 5 by using the residue formula together with the algebra.

2. D_r : Recall that for the algebra of differentials, we needed odd differentials $x^{2i}\frac{dx}{y}$ which we now take as

$$\psi_i = \frac{\partial \lambda_{SW}}{\partial t^i} = -\frac{\partial W_D}{\partial t^i}\frac{dx}{y} = x^2 \phi_i \frac{dx}{y} \tag{7.8}$$

Again, these span the same set of differentials. Instead of using the involution $\sigma : x \to -x$ to determine which differentials to keep, as was done in section 5, we take $\frac{\partial \lambda_{SW}}{\partial t^i}$ which automatically gives the correct differentials. Also, we will use $G = \frac{x^2 dx}{y}$ so that the algebra of differentials becomes

$$\begin{aligned} x^2\phi_i x^2 \phi_j \left(\frac{dx}{y}\right)^2 &= \sum_{k=1}^{r} C_{ij}^k x^2 \phi_k x^2 \left(\frac{dx}{y}\right)^2 + \sum_{k=1}^{r} D_{ij}^k x \phi_k x^2 \frac{\partial W}{\partial x}\left(\frac{dx}{y}\right)^2 \\ \phi_i \phi_j x^4 \left(\frac{dx}{y}\right)^2 &= \left[\sum_{k=1}^{r} C_{ij}^k \phi_k + \sum_{k=1}^{r} \frac{D_{ij}^k \phi_k}{x} \partial_x W\right] x^4 \left(\frac{dx}{y}\right)^2 \end{aligned} \tag{7.9}$$

and by dropping the $x^4\left(\frac{dx}{y}\right)^2$ we find the algebra (6.7). Again, the Picard-Fuchs equations in combination with the algebra gives the same result as the residue formula in combination with the algebra:

$$F_{ijk} = \sum_{l=1}^{r} C_{ij}^l K_{kl} \tag{7.10}$$

3. B_r : We take as differentials

$$\psi_i = \frac{\partial \lambda_{SW}}{\partial t^i} = \frac{\phi_i dx}{y} = -\frac{\partial W_{BC}}{\partial t^i}\frac{dx}{y} \tag{7.11}$$

where the t^i, ϕ_i are introduced for B_r in section 6. Furthermore, we take G and H_{ij} as in section 5.3.1. The content of the Picard-Fuchs equations together with the algebra $\phi_i \cdot \phi_j = C_{ij}^k \phi_k$ is expressed as

$$F_{ijk} = \sum_{l=1}^{r} \widetilde{C}_{ij}^{l} K_{kl} \tag{7.12}$$

The content of the residue formula together with the algebra of differentials $\psi_i \psi_j = \gamma_{ij}^k \psi_k$ is expressed as

$$F_{ijk} = \sum_{l=1}^{r} \gamma_{ij}^{l} K_{kl}. \tag{7.13}$$

We will prove the following

Proposition 30 *The γ_{ij}^k and the $\widetilde{C}_{ij}^k$ are identical.*

Proof. We start with the algebra

$$\phi_i \phi_j = \sum_{k=1}^{r} C_{ij}^k \phi_k + \sum_{k=1}^{r} D_{ij}^k \phi_k x \partial_x W_{BC} \tag{7.14}$$

and we will rewrite it in such a way that it becomes of the form

$$\phi_i \phi_j = \sum_{k=1}^{r} \widetilde{C}_{ij}^k \phi_k + P_{ij} \left[x \partial_x W_{BC} - W_{BC} \right] \tag{7.15}$$

To this obtain this, we will use (6.32):

$$\begin{aligned} \phi_i \phi_j &= \left[C_i \cdot \phi + D_i \cdot \phi x \partial_x W_{BC} \right]_j \\ &= \left[\left(\widetilde{C}_i - D_i \cdot \sum_{n=1}^{r} \frac{d_n t^n}{h^\vee} \widetilde{C}_n \right) \cdot \phi + D_i \cdot \phi x \partial_x W_{BC} \right]_j \\ &= \left[\widetilde{C}_i \cdot \phi - D_i \cdot \sum_{n=1}^{r} \frac{d_n t^n}{h^\vee} \widetilde{C}_n \cdot \phi + D_i \cdot \phi x \partial_x W_{BC} \right]_j \\ &= \left[\widetilde{C}_i \cdot \phi \right]_j + \left[A_i \cdot \phi \right]_j \end{aligned} \tag{7.16}$$

The notation $\widehat{\phi}$ stands for the vector with components ϕ_k. We introduced the matrix a^i here because it will come back later in the proof. The strategy will be to get rid of the second term of (7.16) since we want an algebra in which the first term gives the structure constants, and we will do this by splitting up the third term and rewriting part of it using (6.28):

$$\begin{aligned}
\phi_i\phi_j &= \left[\widetilde{C}_i\cdot\phi - D_i\cdot\sum_{n=1}^{r}\frac{d_n t^n}{h^\vee}\widetilde{C}_n\cdot\phi + D_i\cdot\phi x\partial_x W_{BC}\right]_j \\
&= \left[\widetilde{C}_i\cdot\phi - D_i\cdot\sum_{n=1}^{r}\frac{d_n t^n}{h^\vee}\widetilde{C}_n\cdot\phi + \alpha D_i\cdot\phi x\partial_x W_{BC} + (1-\alpha)\,D_i\cdot\phi x\partial_x W_{BC}\right]_j \\
&= \left[\widetilde{C}_i\cdot\phi - D_i\cdot\sum_{n=1}^{r}\frac{d_n t^n}{h^\vee}\widetilde{C}_n\cdot\phi + \alpha D_i\cdot\phi\left(\left(1+h^\vee\right)W_{BC} - \sum_{n=1}^{r} d_n t^n\phi_n\right)\right]_j \\
&\quad + \left[(1-\alpha)\,D_i\cdot\phi x\partial_x W_{BC}\right]_j
\end{aligned} \tag{7.17}$$

Now we use the algebra to work out the product $\phi_k\phi_n$:

$$\phi_i\phi_j = \left[\widetilde{C}_i\cdot\phi - D_i\cdot\sum_{n=1}^{r}\frac{d_n t^n}{h^\vee}\widetilde{C}_n\cdot\phi + \alpha D_i\cdot\phi\left(1+h^\vee\right)W_{BC}\right]_j \tag{7.18}$$

$$-\left[\alpha D_i\cdot\sum_{n=1}^{r} d_n t^n C_n\cdot\phi + \alpha D_i\cdot\sum_{n=1}^{r} d_n t^n D_n\cdot\phi x\partial_x W_{BC} + (1-\alpha)\,D_i\cdot\phi x\partial_x W_{BC}\right]_j$$

We now use (6.32) again to rewrite the first term in the second line in order to cancel the second term in the first line:

$$\begin{aligned}
\phi_i\phi_j &= \left[\widetilde{C}_i\cdot\phi - D_i\cdot\sum_{n=1}^{r}\frac{d_n t^n}{h^\vee}\widetilde{C}_n\cdot\phi + \alpha D_i\cdot\phi\left(1+h^\vee\right)W_{BC}\right]_j \\
&\quad - \left[\alpha D_i\cdot\sum_{n=1}^{r} d_n t^n\widetilde{C}_n\cdot\phi - \alpha D_i\cdot\sum_{n=1}^{r} d_n t^n D_n\cdot\sum_{m=1}^{r}\frac{q_m t^m}{h^\vee}\widetilde{C}_m\cdot\phi\right]_j \\
&\quad - \left[\alpha D_i\cdot\sum_{n=1}^{r} d_n t^n D_n\cdot\phi x\partial_x W_{BC} + (1-\alpha)\,D_i\cdot\phi x\partial_x W_{BC}\right]_j
\end{aligned} \tag{7.19}$$

We now see that we must take $\alpha = -\frac{1}{h^\vee}$ to cancel. Note that by cancelling this one term, we automatically calculate modulo $x\partial_x W_{BC} - W_{BC}$:

$$\begin{aligned}
\phi_i\phi_j &= \left[\widetilde{C}_i\cdot\phi + \left(1+\frac{1}{h^\vee}\right)D_i\cdot\phi\left(x\partial_x W_{BC} - W_{BC}\right)\right]_j \\
&\quad + \left[D_i\cdot\sum_{n=1}^{r}\frac{d_n t^n}{h^\vee}\left(-D_n\cdot\sum_{m=1}^{r}\frac{q_m t^m}{h^\vee}\widetilde{C}_m\cdot\phi + D_n\cdot\phi x\partial_x W_{BC}\right)\right]_j \\
&= \left[\widetilde{C}_i\cdot\phi + \left(1+\frac{1}{h^\vee}\right)D_i\cdot\phi\left(x\partial_x W_{BC} - W_{BC}\right)\right]_j \\
&\quad + \left[D_i\cdot\sum_{n=1}^{r}\frac{d_n t^n}{h^\vee}A_n\cdot\phi\right]_j
\end{aligned} \tag{7.20}$$

The second line seems to spoil our achievement but it doesn't: until now we rewrote

$$[A_i \cdot \phi]_j \quad \Longrightarrow \quad \left[D_i \cdot \left[\left(1+\frac{1}{h^\vee}\right)\phi\left(x\partial_x W_{BC} - W_{BC}\right) + \sum_{n=1}^{r} \frac{d_n t^n}{h^\vee} A_n \cdot \phi\right]\right]_j$$

and we can now in turn rewrite

$$[A_n \cdot \phi]_j \quad \Longrightarrow \quad \left[D_n \cdot \left[\left(1+\frac{1}{h^\vee}\right)\phi\left(x\partial_x W_{BC} - W_{BC}\right) + \sum_{n=1}^{r} \frac{q_m t^m}{h^\vee} A_m \cdot \phi\right]\right]_j$$

This is a recursive process. If it stops at some point, then we get a multiplication structure

$$\phi_i \phi_j = \sum_{k=1}^{r} \widetilde{C}_{ij}^k \phi_k + P_{ij}\left(x\partial_x W_{BC} - W_{BC}\right) \tag{7.21}$$

for some polynomial P_{ij}. To see that the process indeed stops, we note that the matrices D_i are nilpotent:

Lemma 31 *The D_i are nilpotent matrices.*

Proof. One finds [IY97] that the degree of Q_{ij} is

$$\deg\left(Q_{ij}\right) = 2r + 1 - 2\left(i + j\right)$$

Dividing this by x, we get something of degree $2r - 2\left(i+j\right)$. The D_{ij}^k are defined through

$$Q_{ij} = x \sum_{k=1}^{r} D_{ij}^k \phi_k$$

and if we can show that for $j \geq k$ we can't divide $\frac{Q_{ij}}{x}$ by ϕ_k, we have shown that D_i is nilpotent since it is strictly upper triangular. Since

$$\deg\left(\phi_k\right) = 2r - 2k$$

we find that indeed for $j \geq k$ the degree of ϕ_k is bigger than the degree of $\frac{Q_{ij}}{x}$ and we can't divide the two. This finishes the proof of the lemma. ■

Since the D_i are nilpotent, the increasing products of the D_i that we get by the recursion become zero at some point. Having found now that the recursive process stops due to the nilpotency, we get the algebra $\phi_i \cdot \phi_j = \sum_{k=1}^{r} \widetilde{C}_{ij}^k \phi_k$. We will now show that $\widetilde{C}_{ij}^k = \gamma_{ij}^k$ by noting that the ϕ_i are linearly independent and that none of them is divisable by $x\partial_x W_{BC} - W_{BC}$. From the two algebras we find that if we calculate modulo $x\partial_x W_{BC} - W_{BC}$ we get

$$\phi_i * \phi_j = \sum_{k=1}^{r} \widetilde{C}_{ij}^k \phi_k = \sum_{k=1}^{r} \gamma_{ij}^k \phi_k \tag{7.22}$$

Since the ϕ_i are linearly independent (even modulo $x\partial_x W_{BC} - W_{BC}$) we see that $\widetilde{C}_{ij}^k = \gamma_{ij}^k$. ■

As an example, we calculated these structure constants for the algebra of these differentials for B_3 and they are

$$\tilde{C}_1 := \begin{bmatrix} 0 & 0 & 1 \\ \frac{6}{5}\, t_3 + 2\, t_1\, t_2 & -\frac{1}{5}\, t_2 + {t_1}^2 & \frac{2}{5}\, t_1 \\ {t_1}^4 + \frac{42}{25}\, t_3\, t_1 + \frac{1}{5}\, {t_2}^2 + \frac{8}{5}\, t_2\, {t_1}^2 & \frac{2}{5}\, {t_1}^3 + \frac{6}{5}\, t_3 + \frac{48}{25}\, t_1\, t_2 & \frac{14}{25}\, {t_1}^2 + \frac{4}{5}\, t_2 \end{bmatrix}$$

$$\tilde{C}_2 := \begin{bmatrix} 0 & 1 & 0 \\ -t_2 + {t_1}^2 & -t_1 & 1 \\ \frac{6}{5}\, t_3 + 2\, t_1\, t_2 & -\frac{1}{5}\, t_2 + {t_1}^2 & \frac{2}{5}\, t_1 \end{bmatrix}$$

$$\tilde{C}_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is identical to the constants found in section 6.2.3.

4. C_r : We use the differentials

$$\psi_i = \frac{\partial \lambda_{SW}}{\partial t^i} = \frac{x\phi_i dx}{y} \tag{7.23}$$

which span the same subset of differentials as the even differentials $x^{2i+1}\frac{dx}{y}$ used in section 5. We get a multiplication structure and an algebra:

$$\psi_i\psi_j = \sum_{k=1}^{r} \gamma_{ij}^k \psi_k + P_{ij}\left[2W_{BC} + x\frac{\partial W_{BC}}{\partial x}\right] x^2 \left(\frac{dx}{y}\right)^2 \tag{7.24}$$

$$\psi_i * \psi_j = \sum_{k=1}^{r} \gamma_{ij}^k \psi_k \tag{7.25}$$

$$\phi_i\phi_j = \sum_{k=1}^{r} \gamma_{ij}^k \phi_k \tag{7.26}$$

where the last line can be obtained by dropping $x^2\left(\frac{dx}{y}\right)^2$. The Picard-Fuchs equations for this gauge group were expressed in terms of $\widetilde{C}_{ij}^k$ which can be found from

$$\widetilde{C}_i = C_i - D_i \sum_n \frac{d_n t^n}{h^\vee} \widetilde{C}_n \tag{7.27}$$

Proposition 32 *We have* $\widetilde{C}_{ij}^k = \gamma_{ij}^k$.

Proof. We should first prove that

$$\phi_i \phi_j = \sum_{k=1}^{r} \widetilde{C}_{ij}^k \phi_k + P_{ij} \left[x \partial_x W_{BC} + 2 W_{BC} \right] \tag{7.28}$$

The proof is very similar to that of the B_r case, and therefore we will not present it here. Note that due to our conventions, equation (6.28) becomes

$$x \partial_x W_{C_r} + \sum_n d_n t^n \frac{\partial W_{C_r}}{\partial t^n} = 2 h_{C_r}^\vee W_{C_r} \tag{7.29}$$

Then we use the same argument as for B_r to see that since

$$\sum_{k=1}^{r} \widetilde{C}_{ij}^k \phi_k = \sum_{k=1}^{r} \gamma_{ij}^k \phi_k \tag{7.30}$$

we must have $\widetilde{C}_{ij}^k = \gamma_{ij}^k$. ■

■

8 Conclusions and outlook

In this paper, we have considered two different proofs [MMM],[IY98] that the prepotential for pure Seiberg-Witten theory with classical gauge group satisfies the WDVV equations. The first of these proofs [MMM] deals with holomorphic differential forms on hyperelliptic Riemann surfaces. This proof works naturally for the classical groups, but the problem for the exceptional groups is that the Riemann surfaces are no longer hyperelliptic. The second proof [IY98] makes use of an algebra which is known for A, D, E groups in the context of Landau-Ginzburg theories. The problem with this method is that it is only natural for these particular groups and the derivation the authors give for B_r, C_r is somewhat ad hoc. What we have shown in this paper is that the two seemingly different methods are in fact equivalent. For the classical groups, the flat coordinates naturally determine which differentials to take to build the algebra. Whether this is also true for the exceptional groups is an open question that requires further research. We hope that indeed this can be tested for E_6, E_7, E_8 (for which a proof is given in [IY98]) and then used to give a proof for the F_4 prepotential, which is the only case of pure Seiberg-Witten theory which is unsolved in the literature.

A The Riemann surfaces

The prescription to find the Riemann surfaces for pure N=2 SYM theory with gauge group G was given in section 3. Apart from E_7 and E_8 (which are not of particular interest to us) using the minimal representation for the gauge group the surfaces become [MW96], [Ito]:

- A_r

$$z + \frac{\mu}{z} = W = x^{r+1} - u_1 x^{r-1} - ... - u_r \tag{A.1}$$

- B_r

$$x(z + \frac{\mu}{z}) = W_{BC} = x^{2r} - u_1 x^{2r-2} - u_2 x^{2r-4} ... - u_r \tag{A.2}$$

In standard form this becomes

$$z + \frac{\mu}{z} = W = \frac{W_{BC}}{x} \tag{A.3}$$

- C_r

$$\left(z - \frac{\mu}{z}\right)^2 = x^2 W_{BC} \tag{A.4}$$

In standard form this becomes

$$\widetilde{z} + \frac{\widetilde{\mu}}{\widetilde{z}} = x^2 W_{BC} + 2\sqrt{\widetilde{\mu}} \tag{A.5}$$

A short remark on the C_r curve: there is a large number curves describing N=2 SYM theory with gauge group C_r in the literature and a choice has to be made. We adopted the curve given in [Tak], which can be obtained from (A.4) by setting $\widetilde{z} = z^2$, $\widetilde{\mu} = \mu^2$. We will use this curve (and omit the tildes) throughout this article. The grading from the Lie algebra is respected. Other curves that are used in the literature are for example those of [IY98] and [MMM]. In the first of these articles, the authors work with

$$\left(z - \frac{\mu}{z}\right)^2 = x^2 W_{BC}^2 \tag{A.6}$$

$$z + \frac{\mu}{z} = \sqrt{x^2 W_{BC}^2 + 4\mu} \tag{A.7}$$

and in the second with

$$\left(z - \frac{\mu}{z}\right)^2 = x^4 W_{BC}^2 \tag{A.8}$$

$$\widetilde{z} + \frac{\widetilde{\mu} x^{-4}}{\widetilde{z}} = W_{BC} \tag{A.9}$$

where $\widetilde{z} = \frac{z}{x^2}$ and $\widetilde{\mu} = -\mu$. In all cases, the grading is respected. However, taking a different curve from [MMM] means that their proof of the WDVV equations must be slightly altered. This is reflected in the fact that we make an algebra of even differentials instead of odd ones in section 5.3.1. Also, the derivation of the WDVV equations in [IY98] becomes different: it is slightly harder to calculate the Picard-Fuchs equations. The reason why we choose to work with (A.5) is that it is a natural choice with regard to the distinguished Prym variety [Don98] (see section 3.3).

- D_r

$$x^2(z+\frac{\mu}{z}) = x^{2r} - u_1 x^{2r-2} - ... - u_{r-2}x^4 - u_r x^2 - u_{r-1}^2 \tag{A.10}$$

$$(z+\frac{\mu}{z}) = x^{2r-2} - u_1 x^{2r-4} - ... - u_{r-2}x^2 - u_r - \frac{u_{r-1}^2}{x^2} \tag{A.11}$$

- E_6

$$\frac{1}{2}x^3(z+\frac{\mu}{z}+u_6)^2 - q_1(x)(z+\frac{\mu}{z}+u_6) + q_2(x) = 0 \tag{A.12}$$

where

$$\begin{aligned} q_1 &= 270x^{15} + 342u_1x^{13} + 162u_1^2x^{11} - 252u_2x^{10} + (26u_1^3 + 18u_3)x^9 \\ &\quad -162u_1u_2x^8 + (6u_1u_3 - 27u_4)x^7 - (30u_1^2u_2 - 36u_5)x^6 + (27u_2^2 - 9u_1u_4)x^5 \\ &\quad -(3u_2u_3 - 6u_1u_5)x^4 - 3u_1u_2^2x^3 - 3u_2u_5x - u_2^3, \\ q_2 &= \frac{1}{2x^3}(q_1^2 - p_1^2p_2), \\ p_1 &= 78x^{10} + 60u_1x^8 + 14u_1^2x^6 - 33u_2x^5 + 2u_3x^4 - 5u_1u_2x^3 - u_4x^2 - u_5x - u_2^2, \\ p_2 &= 12x^{10} + 12u_1x^8 + 4u_1^2x^6 - 12u_2x^5 + u_3x^4 - 4u_1u_2x^3 - 2u_4x^2 + 4u_5x + u_2^2. \end{aligned} \tag{A.13}$$

and finally we get the simple representation

$$\left(z+\frac{\mu}{z}\right) = \frac{1}{x^3}\left(q_1 \pm p_1\sqrt{p_2}\right) - u_6 \tag{A.14}$$

- F_4

$$-8\left(z+\frac{\mu^2}{z}\right)^3 + a_1(x)\left(z+\frac{\mu^2}{z}\right)^2 + a_2(x)\left(z+\frac{\mu^2}{z}\right) + a_3(x) = 0 \tag{A.15}$$

where

$$
\begin{aligned}
a_1(x) &= -636x^9 - 300u_1x^7 - 48{u_1}^2x^5 - 5u_3x^3 + 2u_4x, \\
a_2(x) &= -168x^{18} - 348u_1x^{16} - 276{u_1}^2x^{14} + (-116{u_1}^3 + 14u_3)x^{12} \\
&\quad +(-92u_4 - 20{u_1}^4 - 8u_1u_3)x^{10} + (-42u_1u_4 - 6{u_1}^2u_3)x^8 \\
&\quad +(-4u_6 - \frac{10}{3}{u_1}^2u_4 - \frac{2}{3}{u_3}^2)x^6 + (\frac{1}{3}u_3u_4 - \frac{2}{3}u_6u_1)x^4, \\
a_3(x) &= x^{27} + 6u_1x^{25} + 15{u_1}^2x^{23} + (20{u_1}^3 + u_3)x^{21} + (5u_4 + 4u_1u_3 + 15{u_1}^4)x^{19} \\
&\quad +(6{u_1}^2u_3 + 12u_1u_4 + 6{u_1}^5)x^{17} + (\frac{1}{3}{u_3}^2 + 5u_6 + 4{u_1}^3u_3 + \frac{26}{3}{u_1}^2u_4 + {u_1}^6)x^{15} \\
&\quad +(\frac{4}{3}{u_1}^3u_4 + \frac{19}{3}u_6u_1 + {u_1}^4u_3 + \frac{4}{3}u_3u_4 + \frac{2}{3}{u_3}^2u_1)x^{13} \\
&\quad +(\frac{1}{3}{u_1}^2{u_3}^2 - \frac{1}{3}{u_1}^4u_4 - \frac{15}{4}{u_4}^2 + 3u_6{u_1}^2)x^{11} \\
&\quad +(\frac{1}{3}u_6u_3 - \frac{4}{9}{u_1}^2u_3u_4 + \frac{1}{27}{u_3}^3 - \frac{13}{6}{u_4}^2u_1 + \frac{13}{27}u_6{u_1}^3)x^9 \\
&\quad +(-\frac{1}{9}{u_3}^2u_4 - \frac{1}{2}u_6u_4 + \frac{1}{9}u_6u_1u_3 - \frac{7}{36}{u_1}^2{u_4}^2)x^7 + (\frac{1}{12}{u_4}^2u_3 - \frac{1}{6}u_6u_1u_4)x^5 \\
&\quad +(-\frac{1}{54}{u_4}^3 - \frac{1}{108}{u_6}^2)x^3. \qquad \text{(A.16)}
\end{aligned}
$$

and the simple representation is

$$
\left(z + \frac{\mu}{z}\right) = \frac{a_1(x)}{24} - \frac{1}{2}\left\{\left(-q + \sqrt{q^2 + 4p^3}\right)^{1/3} + \left(-q - \sqrt{q^2 + 4p^3}\right)^{1/3}\right\} \qquad \text{(A.17)}
$$

where

$$
\begin{aligned}
p(x) &= -\frac{a_2}{6} - \frac{a_1^2}{144}, \\
q(x) &= \frac{1}{27}\left(\frac{a_1^3}{32} + \frac{9}{8}a_1a_2 + 27a_3\right) \qquad \text{(A.18)}
\end{aligned}
$$

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