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Abstract

In this paper we describe how one can construct the dual wavefunction of the n-component KP-hierarchy in the analytic setting. This geometric description will be useful at the construction of Bäcklund-Darboux transformations for these nonlinear matrix equations. The construction implies moreover that the bilinear relations have a convergent interpretation in this context.

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1 Introduction

Recall that the KP-hierarchy is a tower of non linear differential equations in infinitely many variables $\{t_n | n \geq 1\}$. A convenient condensed formulation of these equations is as a set of Lax equations for a scalar pseudodifferential operator L. The coefficients of L belong to a ring R of functions in the $\{t_n\}$ that is stable w.r.t. all the derivations $\{\partial_n := \frac{\partial}{\partial t_n}, n \geq 1\}$. If $\partial = \frac{\partial}{\partial t_1}$, then the operator L has the form

$$L = \partial + \sum_{j<0} l_j \partial^j \quad , \qquad \text{with } \; l_j \in R. \tag{1}$$

The equations of the KP-hierarchy are then

$$\partial_n(L) = [(L^n)_+, L], \tag{2}$$

where P_+ denotes the differential operator part of a pseudodifferential operator P. The n-component KP-hierarchy is a matrix version of the KP-hierarchy that made its first appearance in the work of the Sato school, see [DKJM] and [UT]. Recently, see ([LM]), it was found that this integrable system is an important mean to construct solutions of a non linear system from two-dimensional topological field theory, the so-called $WDVV$-equations. We start with a description of the algebraic framework of this system of equations and how they result from their linearization. Next we treat the form of the eigenfunctions occurring in this linear system. They determine the solutions of the non linear equations completely and are called wavefunctions. Associated with them is an eigenfuction of another linear system, the so-called dual wavefunction, which forms the subject of the next section. In particular we present there a useful characterization of these dual wavefunctions and we treat the duality between both systems. A wide class of wavefunctions can be constructed from planes in the Grassmann manifold corresponding to the Hilbertspace

$H = L^2(S^1, \mathbb{C}^n)$ with decomposition $H = H_+ \oplus H_-$, where

$$H_+ = \{\sum_{n\geq 0} a_n z^n \in H\} \text{ and } H_- = \{\sum_{n<0} a_n z^n \in H\}.$$

This is the subject of the subsequent section. We end with a pure geometric construction of the dual wavefuctions without making use of the τ-function.

2 The algebraic set-up

The compact form in which the equations of the n-component KP-hierarchy are presented in [UT], is the so-called Lax form. To give some insight in this form and to formulate precisely what we want, we describe here shortly the underlying algebraic structure in the style of [W]. Consider the following two collections of commuting unknowns $\{(u^{(i)}_{\alpha,j})_{\rho\sigma}\}$ and $\{(l^{(i)}_j)_{\rho\sigma}\}$, where $j \geq 1$, $i \geq 0$ and the indices α, ρ and $\sigma \in \{1, \ldots, n\}$. Now let A be the ring generated by them, i.e.

$$A = \mathbb{C}[(u^{(i)}_{\alpha,j})_{\rho\sigma}, (l^{(i)}_j)_{\rho\sigma}].$$

There is a unique $\mathbb{C}$-linear derivation $\tilde{\partial} : A \to A$ such that

$$\tilde{\partial}((u^{(i)}_{\alpha,j})_{\rho\sigma}) = (u^{(i+1)}_{\alpha,j})_{\rho\sigma} \text{ and } \tilde{\partial}((l^{(i)}_j)_{\rho\sigma}) = (l^{(i+1)}_j)_{\rho\sigma}$$

for all $i \geq 0$, $j \geq 1$ and all α, ρ and σ in $\{1, \ldots, n\}$. Any derivation Δ of a commutative $\mathbb{C}$-algebra B extends in a natural way to a derivation Δ of the $\mathbb{C}$-algebra $M_n(B)$ by putting $\Delta(b)_{ij} = \Delta(b_{ij})$ for all $b = (b_{ij}) \in M_n(B)$. In the ring $M_n(A)[\tilde{\partial}]$ of differential operators in $\tilde{\partial}$ with coefficients from $M_n(A)$, one can, in general, not take roots of monic operators. Thereto one passes to the extension $M_n(A)[\tilde{\partial}, \tilde{\partial}^{-1})$ of all pseudodifferential operators in $\tilde{\partial}$ with coefficients from $M_n(A)$. It consists of all expressions

$$\sum_{i=-\infty}^{N} a_i \tilde{\partial}^i, a_i \in M_n(A) \text{ for all } i,$$

that are added coefficientwise and that are multiplied according to

$$\tilde{\partial}^j \cdot a\tilde{\partial}^i = \sum_{k=0}^{\infty} \begin{pmatrix} j \\ k \end{pmatrix} \tilde{\partial}^k(a)\tilde{\partial}^{i+j-k}$$

Each operator $A = \sum a_j \tilde{\partial}^j$ decomposes as $A = A_+ + A_-$ with $A_+ = \sum_{j\geq 0} a_j \tilde{\partial}^j$ its differential operator part and $A_- = \sum_{j<0} a_j \tilde{\partial}^j$ its pure integral operator part. Note that in order to define a derivation of A that commutes with $\tilde{\partial}$, it suffices to prescribe the image of all the $\{(u^{(0)}_{\alpha,j})_{\rho\sigma}\}$ and all the $\{(l^{(i)}_j)_{\rho\sigma}\}$ and this can be done freely. The same thing holds for derivations of $M_n(A)$ that should commute with $\tilde{\partial}$ in $\mathrm{Der}(M_n(A))$. The choice that we are interested in here, can in a condense way be expressed in the operators $\tilde{L}$ and $\tilde{U_\alpha}$ from $M_n(A)[\tilde{\partial}, \tilde{\partial}^{-1})$ defined by

$$\tilde{L} = \tilde{\partial} + \sum_{j\geq 1} l^{(0)}_j \tilde{\partial}^{-j} \text{ and } \tilde{U_\alpha} = E_{\alpha\alpha} + \sum_{j\geq 1} u^{(0)}_{\alpha,j} \tilde{\partial}^{-j}.,$$

Here $E_{\alpha\beta}$ is the $n \times n$-matrix with its (α, β)-th entry equal to 1 and the other entries equal to zero and $l^{(0)}_j$ resp. $u^{(0)}_{\alpha,j}$ are the elements of $M_n(A)$ with the (ρ, σ)-entry equal to $(l_j^{(0)})_{\rho\sigma}$ resp. $(u_{\alpha,j}^{(0)})_{\rho\sigma}$. For each $i \geq 1$ and each α, $1 \geq \alpha \geq n$, let $\tilde{\partial}_{i\alpha}$ be the unique derivation of A resp. $M_n(A)$ that commutes with $\tilde{\partial}$ and that is fixed by the following operator identities in $M_n(A)[\tilde{\partial}, \tilde{\partial}^{-1})$

$$\tilde{\partial}_{i\alpha}(\tilde{L}) := 0.\tilde{\partial} + \sum_{j>0} \tilde{\partial}_{i\alpha}(l_j^{(0)})\tilde{\partial}^{-j} = [((\tilde{L})^i \tilde{U}_\alpha)_+, \tilde{L}], \tag{3}$$

$$\tilde{\partial}_{i\alpha}(\tilde{U}_\beta) := \tilde{\partial}_{i\alpha}(E_{\beta\beta}) + \sum_{j>0} \tilde{\partial}_{i\alpha}(u_{\beta,j}^{(0)})\tilde{\partial}^{-j} = [((\tilde{L})^i \tilde{U}_\alpha)_+, \tilde{U}_\beta]. \tag{4}$$

These derivations commute among each other, if we have for all α and β

$$[\tilde{L}, \tilde{U}_\alpha] = [\tilde{U}_\alpha, \tilde{U}_\beta] = 0 \tag{5}$$

The complex of equations (3), (4) and (5) is called the n-component KP-hierarchy. One is mainly interested in the equations (3) and (4), because they render the non linear differential equations, and they are called the Lax equations of the hierarchy. To give sense to this exercition, the next step has to be to make concrete realizations of the relations in (3), (4) and (5). This means that we are looking first of all for commutative $\mathbb{C}$-algebras R equiped with a privileged derivation $\partial : R \to R$ and a collection of derivations $\{\partial_{i\alpha}, i \geq 1\, 1 \leq \alpha \leq n\}$ of R that commute all with ∂ and also among each other. Further there should be a $\mathbb{C}$-algebra morphism $\pi : A \to R$ that is compatible with all these derivations i.e. it should satisfy

$$\pi \circ \tilde{\partial} = \partial \circ \pi \ \text{ and } \ \pi \circ \tilde{\partial}_{i\alpha} = \partial_{i\alpha} \circ \pi, \ \text{ with } \ i \geq 1, 1 \leq \alpha \leq n. \tag{6}$$

One should think of R as some ring of functions depending of the parameters $\{x, t_{i\alpha} | i \geq 1, 1 \leq \alpha \leq n\}$, where the derivation ∂ is differentiation w.r.t. the parameter x and likewise $\partial_{i\alpha}$ is differentiation w.r.t. $t_{i\alpha}$. Since all the unknowns $\{(l_j^{(i)})_{\rho\sigma}\}$ and $\{(u_{\alpha,j}^{(i)})_{\rho\sigma}\}$ are independent, this allows you to pick their images freely to get a $\mathbb{C}$-algebra morphism $\pi : A \to R$.

To get the first property in (6) for such a morphism π, one still has the freedom of choosing the image of the $\{(l_j^{(0)})_{\rho\sigma}\}$ and the $\{(u_{\alpha,j}^{(0)})_{\rho\sigma}\}$'s freely. In order that the remaining properties hold, the morphism π has to factorize over the relations in (3), (4) and (5). In other words, the operators L and U_α, $1 \leq \alpha \leq n$, in $M_n(R)[\partial, \partial^{-1})$ given by

$$L = \partial + \sum_{j \geq 1} \pi(l_j^{(0)})\partial^{-j} \ \text{ and } \ U_\alpha = E_{\alpha\alpha} + \sum_{j \geq 1} \pi(u_{\alpha,j}^{(0)})\partial^{-j}. \tag{7}$$

should satisfy the following system of nonlinear equations inside $M_n(R)[\partial, \partial^{-1})$

$$[L, U_\alpha] = [U_\alpha, U_\beta] = 0, \tag{8}$$

$$\partial_{i\alpha}(L) = [(L^i U_\alpha)_+, L], \tag{9}$$

$$\partial_{i\alpha}(U_\beta) = [(L^i U_\alpha)_+, U_\beta], \tag{10}$$

for all α and β in $\{1, \ldots, n\}$. Given R, δ and the $\delta_{i\alpha}$, we call a set of operators L and U_α, $1 \leq \alpha \leq n$, in $M_n(R)[\partial, \partial^{-1})$ of the form (7) a solution of the n-component KP-hierarchy. One can see a solution L and $\{U_\alpha\}$ of the n-component KP-hierarchy as a deformation of the trivial solution $L = \partial$ and $U_\alpha = E_{\alpha\alpha}, 1 \leq \alpha \leq n$.

As in the one-dimensional case, there exists a linearization of the n-component KP-hierarchy from which the Lax equations (9) and (10) follow as compatibility conditions. Consider namely

$$L\psi = z\psi \quad , \quad U_\alpha \psi = \psi E_{\alpha\alpha} \tag{11}$$

and

$$\partial_{i\alpha}(\psi) = (L^i U_\alpha)_+(\psi) =: P_{i\alpha}(\psi) \qquad \text{for all } \; i \geq 1, 1 \leq \alpha \leq n. \tag{12}$$

If one applies the operator $\partial_{i\alpha}$ to both sides of both equations in (11) and performs the following manipulations

$$\partial_{i\alpha}(L)\psi + L\partial_{i\alpha}(\psi) = \{\partial_{i\alpha}(L) + LP_{i\alpha}\}\psi = z\partial_{i\alpha}(\psi) = P_{i\alpha}(z\psi) = \{P_{i\alpha}L\}(\psi), \tag{13}$$

$$\partial_{i\alpha}(U_\beta)\psi + U_\beta\partial_{i\alpha}(\psi) = \{\partial_{i\alpha}(U_\beta) + U_\beta P_{i\alpha}\}\psi = \partial_{i\alpha}(\psi)E_{\beta\beta} = \{P_{i\alpha}U_\beta\}(\psi), \tag{14}$$

then one ends up with the equations

$$(\partial_{i\alpha}(L) - [(L^i U_\alpha)_+, L])(\psi) = 0(\partial_{i\alpha}(U_\beta) - [(L^i U_\alpha)_+, U_\beta])(\psi) = 0 \tag{15}$$

So, if we can scratch the function ψ in both expressions, we get the equations (9) and (10) for L and the $\{U_\alpha\}$. Once one has obtained with this procedure operators L and U_α that satisfy (9) and (10), one still has to verify the equations (8). This will be immediate from our construction. We proceed now with specifying somewhat more the functions ψ in the linearization. This is done in the next section.

3 Oscillating matrix functions

From the equations (11) and (12), we see that the functions ψ must be susceptible for left actions of various pseudodifferential operators with coefficients from $M_n(R)$ and for right actions with elements from $M_n(\mathbb{C})$. Therefore it is not surprising that they will be matrix functions. To motivate the form of the matrix functions ψ that we will choose, we consider the trivial solution $L = \partial$ and $U_\alpha = E_{\alpha\alpha}$. Then the equations (11) reduce to

$$\partial(\psi) = z\psi \quad , \quad E_{\alpha\alpha}\psi = \psi E_{\alpha\alpha} \quad \text{and} \quad \partial_{i\alpha}(\psi) = z^i E_{\alpha\alpha}\psi. \tag{16}$$

Hence, for the trivial solution of the n-component KP-hierarchy, the matrix function

$$\gamma(t) = \exp\left(\sum_{\alpha=1}^{n}\sum_{i=1}^{\infty} t_{i\alpha}E_{\alpha\alpha}z^i\right) = \sum_{k=0}^{\infty} \underline{p}_k z^k, \underline{p}_k \in M_n(R) \tag{17}$$

satisfies the equation in (11). Next we introduce a space of matrix functions for which all the manipulations at the end of section 3 make sense and whose elements can be seen as perturbations of the solution (17). This space M of so-called oscillating matrix functions, is defined as

$$M = \left\{ \left(\sum_{j \leq N} a_j z^j\right) \exp\left(\sum_{\alpha=1}^{n}\sum_{i=1}^{\infty} t_{i\alpha}E_{\alpha\alpha}z^i\right) \middle| a_j \in M_n(R) \right\}.$$

The product in the elements of M is formal, unless one can make sense in $M_n(R)$ of all the

$$\sum_{k\in\mathbb{Z}} a_{l-k}\underline{p}_k, l \in \mathbb{Z}, \tag{18}$$

and this clearly requires convergence considerations. They have been given in ([HP]) and we will recall the necessary ingredients in section 5.

The space M can be made into a module for the ring $M_n(R)[\partial,\partial^{-1})$ of pseudodifferential operators in ∂ with coefficients from $M_n(R)$. If one puts the actions of $b \in M_n(R)$ and $\partial_{k\beta}$ as follows

$$b\left[\left(\sum_j a_j z^j\right)\exp\left(\sum_{\alpha=1}^{n}\sum_{i\geq 1} t_{i\alpha}E_{\alpha\alpha}z^i\right)\right] = \left(\sum_j ba_j z^j\right)\gamma(t)$$

$$\partial_{k\beta}\left\{\left(\sum_j a_j z^j\right)\exp\left(\sum_{\alpha=1}^{n}\sum_{i\geq 1} t_{i\alpha}E_{\alpha\alpha}z^i\right)\right\} = \left(\sum_j \partial_{k\beta}(a_j) + \sum_j a_j E_{\beta\beta}z^{k+j}\right)\gamma(t),$$

then one sees immediately that ∂ acts by an invertible transformation on M. Hence these actions extend in a natural way to one of $M_n(R)[\partial,\partial^{-1})$. Thus M becomes even a free left $M_n(R)[\partial,\partial^{-1})$-module, for

$$\left(\sum p_j\partial^j\right)\cdot\exp\left(\sum_{\alpha,i} t_{i\alpha}E_{\alpha\alpha}z^i\right) = \left(\sum p_j z^j\right)\exp\left(\sum_{\alpha,i} t_{i\alpha}E_{\alpha\alpha}z^i\right).$$

M is also in a natural way a right module for the diagonal matrices in $M_n(\mathbb{C})$. For, if $A = \sum_{i=1}^n a_i E_{ii}, a_i \in \mathbb{C}$, then we put

$$\left\{\sum a_j z^j\right\}\exp\left(\sum t_{i\alpha}z^i E_{\alpha\alpha}\right)\cdot A := \left\{\sum_j a_j A z^j\right\}\exp\left(\sum t_{i\alpha}z^i E_{\alpha\alpha}\right).$$

Since all the operations in the formulae (11) and (12) now make sense in M, one can look for solutions of these equations inside M. Note that the equations (11) impose still another relation on the operators U_α. For, we have

$$(\sum_{\alpha=1}^{n} U_\alpha)\psi = \psi(\sum_{\alpha=1}^{n} E_{\alpha\alpha}) = \psi, \tag{19}$$

and, since M is a free $M_n(R)[\partial,\partial^{-1})$-module, this implies that

$$\sum_{\alpha=1}^{n} U_\alpha = Id. \tag{20}$$

Hence the solutions of the n-component KP-hierarchy that one obtains via the linearization satisfy still this additional condition.

There are two types of transformations that map solutions ψ in M of the equations (11) and (12) into new ones for the same operators L and U_α. First there is the right action of the diagonal matrices in $M_n(\mathbb{C})$. Secondly there is an action of a discrete group. For, if $\delta = (\delta_1,\dots,\delta_n) \in \mathbb{Z}^n$ and we write z^δ for the element

$$\operatorname{diag}\left(z^{\delta_1},\dots,z^{\delta_n}\right) = \begin{pmatrix} z^{\delta_1} & & 0 \\ & \ddots & \\ 0 & & z^{\delta_n}\end{pmatrix}.$$

Then the element $z^\delta\exp(\sum t_{i\alpha}z^i E_{\alpha\alpha})$ belongs to M and we have

$$z^\delta\gamma(t) = \left(\sum_{\alpha=1}^{n} E_{\alpha\alpha}\partial_{1\alpha}^{\delta_\alpha}\right)\exp(\sum t_{i\alpha}z^i E_{\alpha\alpha}) =: R(\delta)\cdot\gamma(t)$$

Clearly the element $R(\delta)$ in $M_n(R)[\partial,\partial^{-1})$ is invertible with inverse $R(\delta)^{-1} = R(-\delta)$. Note that, if $\psi = P \cdot \exp(\sum t_{i\alpha} z^i E_{\alpha\alpha}) \in M$ satisfies the equations in (11) and (12), then also

$$\psi^{(\delta)} = P \cdot R(\delta) \cdot \exp(\sum t_{i\alpha} z^i E_{\alpha\alpha}). \tag{21}$$

Any matrix function ψ in M can without any restriction be written in the form (21) with P of the form $P = \sum_{j\le 0} p_j \partial^j$. We assume now that the leading coefficient of P belongs to $GL_n(R)$. Just like $R(\delta)$, the operator P is then invertible and we look in that way at the group orbit of the trivial solution (16). The equations (11) for ψ tells you

$$\begin{array}{lllllll} L\psi & = & LPR(\delta)\cdot\exp(\sum t_{i\alpha} z^i E_{\alpha\alpha}) & = & z\psi & = & PR(\delta)\partial\cdot\gamma(t) \\ U_\alpha\psi & = & U_\alpha PR(\delta)\cdot\exp(\sum t_{i\alpha} z^i E_{\alpha\alpha}) & = & \psi E_{\alpha\alpha} & = & PR(\delta)E_{\alpha\alpha}\cdot\gamma(t). \end{array}$$

Hence we have that L and the $\{U_\alpha\}$ are completely determined by ψ and are given by

$$L = PR(\delta)\partial R(-\delta)P^{-1} = P\partial P^{-1} \text{ and } U_\alpha = PR(\delta)E_{\alpha\alpha}R(-\delta)P^{-1} = PE_{\alpha\alpha}P^{-1}. \tag{22}$$

From these expressions it is clear that the operators L and the U_α all commute. Moreover we see from these equations that $p_0\partial p_o^{-1} = p_0$ and $p_0 E_{\alpha\alpha} p_o^{-1} = E_{\alpha\alpha}$, so that $\partial(p_0) = 0$ and p_0 commutes with the diagonal matrices. Therefore p_0 has to be itself a diagonal matrix. Now we pass to the equation (12) for ψ. The equality

$$\partial_{i\alpha}(\psi) = \{\partial_{i\alpha}(P)R(\delta) + PR(\delta)\partial^i E_{\alpha\alpha}\}R(\delta)^{-1}P^{-1}(\psi) = P_{i\alpha}(\psi) \tag{23}$$

is equivalent to

$$\partial_{i\alpha}(P)P^{-1} + P\partial^i E_{\alpha\alpha}P^{-1} = \partial_{i\alpha}(P)P^{-1} + L^i U_\alpha = P_{i\alpha}. \tag{24}$$

If we look at the coefficient of ∂^0 in this equation we find that $\partial_{i\alpha}(p_0)p_0^{-1} = 0$. Hence $\partial_{i\alpha}(p_0) = 0$ for all $i \ge 1$ and all α, $1 \le \alpha \le n$, so that p_0 is a constant for all these derivations and an invertible matrix. Therefore we might just as well take it equal to the identity. This clarifies the following notion: an element ψ in M is called an oscillating matrix function of type z^δ if it has the form

$$\psi = P \cdot z^\delta \gamma(t) = P \cdot R(\delta) \cdot \exp(\sum t_{i\alpha} z^i E_{\alpha\alpha}) \text{ with } P = \mathrm{Id} + \sum_{j<0} p_j \partial^j.$$

We formulate now the connection between oscillating matrix functions of type z^δ satisfying (11) and (12) and solutions of the n-component KP-hierarchy.

Proposition 3.1. *Let $\psi = P \cdot R(\delta) \cdot \exp(\sum t_{i\alpha} z^i E_{\alpha\alpha})$ in M be an oscillating matrix function of type z^δ . Assume, we have operators L and $U_\alpha, 1 \le \alpha \le n$, in $M_n(R)[\partial,\partial^{-1})$ of the form (7), such that the equations in (11) hold and such that for $i \ge 1$ and all $\beta, 1 \le \beta \le n$, there holds*

$$\partial_{i\beta}(\psi) = B_{i\beta}(\psi) \text{ for some } B_{i\beta} \in M_n(R)[\partial].$$

Then we have $L = P\partial P^{-1}$, $U_\alpha = PE_{\alpha\alpha}P^{-1}$ and $B_{i\beta} = P_{i\beta} = (L^i U_\beta)_+$ for all i and β. Moreover the operators L and the $\{U_\alpha\}$ are a solution of the n-component KP-hierarchy.

Proof. We have seen above that L and the U_α must have the form described in the proposition and they clearly satisfy (8). To get the Lax equations of the n-component KP-hierarchy for these L and $\{U_\alpha\}$, we apply as at the end of section 2, the operator $\partial_{i\alpha}$ to both sides of the equations in (11). Since M is a free $M_n(R)[\partial, \partial^{-1})$-module, we may scratch the functions ψ in the resulting equalities and thus conclude for all i and α

$$\partial_{i\alpha}(L) = [B_{i\alpha}, L] \text{ and } \partial_{i\alpha}(U_\beta) = [B_{i\alpha}, U_\beta].$$

Hence we merely have to show $B_{i\alpha} = (L^i U_\alpha)_+$. The given equation for $\partial_{i\alpha}(\psi)$ boils down to an equality between pseudodifferential operators, as M is free. More concretely, the equality

$$\partial_{i\alpha}(\psi) = \{\partial_{i\alpha}(P)R(\delta) + PR(\delta)\partial^i E_{\alpha\alpha}\}R(\delta)^{-1}P^{-1}(\psi) = B_{i\alpha}(\psi) \tag{25}$$

implies that

$$\partial_{i\alpha}(P)P^{-1} + P\partial^i E_{\alpha\alpha}P^{-1} = \partial_{i\alpha}(P)P^{-1} + L^i U_\alpha = B_{i\alpha}. \tag{26}$$

By taking the differential operator part of both sides, we arrive at $B_{i\alpha} = (L^i U_\alpha)_+$. This concludes the proof of the proposition. □

Corollary 3.1. *An oscillating matrix function $\psi = PR(\delta)\exp(\sum t_{i\alpha} z^i E_{\alpha\alpha})$ of type z^δ, satisfies equation (12) if and only if P satisfies the so-called Sato-Wilson equations:*

$$\partial_{i\alpha}(P)P^{-1} = -(P\partial^i E_{\alpha\alpha}P^{-1})_-, \text{ for all } i \geq 1, 1 \leq \alpha \leq n. \tag{27}$$

Such a function ψ is called a wavefunction of the n-component KP-hierarchy.

The Sato-Wilson equations are equivalent to the equations in (24) and these are an operator version of those in (12) and this proves the corollary. The next section brings us the main object of study in this paper.

4 The dual wavefunction

At the description of the bilinear form of the n-component KP-hierarchy and at the construction of Bäcklund-Darboux transformations of this system of equations it is convenient to have besides M also at one's disposal its adjoint space M^* consisting of all formal products

$$\{\sum_{j \leq N} a_j z^j\} \exp(-\sum_{\alpha=1}^{n}\sum_{i=1}^{\infty} t_{i\alpha} z^i E_{\alpha\alpha}), \text{ where } a_j \in M_n(R) \text{ for all } j.$$

On M^* we define the following actions of $M_n(R)$ and the $\{\partial_{k\beta}\}$

$$\begin{aligned} b\{(\textstyle\sum_j a_j z^j)\exp(-\sum t_{i\alpha} z^i E_{\alpha\alpha})\} &= (\textstyle\sum_j b a_j z^j)\exp(-\sum t_{i\alpha} z^i E_{\alpha\alpha}) \\ \partial_{k\beta}\{(\textstyle\sum_j a_j z^j)\exp(-\sum t_{i\alpha} z^i E_{\alpha\alpha})\} &= (\textstyle\sum_j \partial_{k\beta}(a_j) z^j - \sum_j a_j E_{\beta\beta} z^{k+j})\gamma(t) \end{aligned}$$

In particular we see that $\partial = \sum_\alpha \partial_{1\alpha}$ acts invertible on M^*, so that the natural extension of these actions leads to an $M_n(R)[\partial, \partial^{-1})$-module structure on M^*, which is again free, since we have

$$\sum_j P_j(-\partial)^j \cdot \exp(-\sum t_{i\alpha} z^i E_{\alpha\alpha}) = \{\sum_j P_j z^j\}\exp(-\sum t_{i\alpha} z^i E_{\alpha\alpha}).$$

Now we have a bilinear pairing $\mathcal{R} : M \times M^* \to M_n(R)$ defined as follows: if $\varphi = \varphi(t,z) = (\sum_j a_j(t)z^j)\exp(\sum t_{i\alpha}z^iE_{\alpha\alpha})$ belongs to M and $\psi = \psi(t,z) = (\sum_k b_k(t)z^k)\exp(-\sum t_{i\alpha}z^iE_{\alpha\alpha})$ is an element of M^*, then we put

$$\begin{aligned}\mathcal{R}(\varphi,\psi): &= \mathrm{Res}_{z=0}((\textstyle\sum_j a_j(t)z^j)(\sum b_k(t)^T z^k))\\ &= \textstyle\sum_{k\in\mathbb{Z}} a_{-k-1}(t)b_k(t)^T,\end{aligned}$$

where a^T for each a in $M_n(R)$ denotes the transposed of a. Clearly the defining sum in $\mathcal{R}$ is finite, hence belongs to $M_n(R)$. On the algebra $M_n(R)[\partial,\partial^{-1})$ we have a $\mathbb{C}$-linear anti-algebra morphism called "taking the adjoint". The adjoint of $P = \sum P_j\partial^j$ is given by

$$\begin{aligned}P^* &= \textstyle\sum_j(-\partial)^jP_j^T = \sum_j(-1)^j\sum_{k\geq 0}\begin{pmatrix} j\\ k\end{pmatrix}\partial^k(P_j^T)\partial^{j-k}\\ &= \textstyle\sum_l\left\{\sum_{k=0}(-1)^{l+k}\begin{pmatrix} l+k\\ k\end{pmatrix}\partial^k(P_{l+k}^T)\right\}\partial^l.\end{aligned}$$

There is an important connection between the bilinear form $\mathcal{R}$ and taking the adjoint

Theorem 4.1. *Let $\varphi(t,z) = P(t,\partial)\cdot\exp(\sum t_{i\alpha}z^iE_{\alpha\alpha}) \in M$ and $\psi(t,z) = Q(t,\partial)\cdot\exp(-\sum t_{i\alpha}z^iE_{\alpha\alpha}) \in M^*$, then*

$$\begin{aligned}(P(t,\partial)Q(t,\partial)^*)_- &= -\textstyle\sum_{n=0}^\infty \mathcal{R}(\varphi(t,z),\partial^n(\psi(t,z)))(-\partial)^{-n-1}\\ &=: \mathcal{R}(\varphi(t,z),\partial^{-1}\circ\psi(t,z))\end{aligned}\tag{28}$$

Proof. The proof consists of showing, by a direct computation, that the coefficients of ∂^{-n-1} for $n\geq 0$ on both sides of (28) are equal. Let $P = \sum_j a_j\partial^j$ and $Q = \sum_k b_k\partial^k$. First we compute the coefficient $\mathcal{R}(\varphi(t,z),\partial^n(\psi(t,z)))$. Since

$$\begin{aligned}\partial^n(\psi) &= \left\{\textstyle\sum_{l=0}^n\begin{pmatrix} n\\ l\end{pmatrix}\sum_k\partial^{n-l}(b_k)(-z)^{k+l}\right\}\exp(-\textstyle\sum t_{i\alpha}z^iE_{\alpha\alpha})\\ &= \left\{\textstyle\sum_p\left(\sum_{l=0}^n\begin{pmatrix} n\\ l\end{pmatrix}(-1)^p\partial^{n-l}(b_{p-l})\right)z^p\right\}\exp(-\textstyle\sum t_{i\alpha}z^iE_{\alpha\alpha}),\end{aligned}$$

we get that

$$\begin{aligned}\mathcal{R}(\varphi(t,z),\partial^n(\psi(t,z))) &= \textstyle\sum_{p\in\mathbb{Z}}a_{-p-1}\left\{\sum_{l=0}^n\begin{pmatrix} n\\ l\end{pmatrix}(-1)^p\partial^{n-l}(b_{p-l})\right\}^T\\ &= \textstyle\sum_{p\in\mathbb{Z}}a_{-p-1}\sum_{l=0}^n\begin{pmatrix} n\\ l\end{pmatrix}(-1)^p\partial^{n-l}(b_{p-l}^T).\end{aligned}$$

On the other hand we have by definition

$$\begin{aligned}P\cdot Q^* &= \textstyle\sum_{j,k}a_j\partial^j(-\partial)^kb_k^T = \sum_{j,k}(-1)^ka_j\partial^{j+k}b_k^T\\ &= \textstyle\sum_{j,k}(-1)^ka_j\sum_{r=0}^\infty\begin{pmatrix} j+k\\ r\end{pmatrix}\partial^r(b_k^T)\partial^{j+k-r}\\ &= \textstyle\sum_s(-1)^s\left\{\sum_j(-1)^j\sum_{r=0}^\infty\begin{pmatrix} s+r\\ r\end{pmatrix}(-1)^ra_j\partial^r(b_{r+s-j}^T)\right\}\partial^s.\end{aligned}$$

Hence the coefficient of $\partial^{-n-1}, n\geq 0$, in $P\cdot Q^*$ equals

$$\begin{aligned}&(-1)^{n+1}\textstyle\sum_j\sum_{r=0}^\infty\begin{pmatrix} r-n-1\\ r\end{pmatrix}(-1)^{r+j}a_j\partial^r(b_{r-n-1-j}^T) =\\ &= (-1)^{n+1}\textstyle\sum_j(-1)^ja_j\sum_{l=0}^n\begin{pmatrix} n\\ l\end{pmatrix}\partial^{n-l}(b_{-l-1-j}^T)\\ &= (-1)^n\textstyle\sum_p(-1)^pa_{p-1}\sum_{l=0}^n\begin{pmatrix} n\\ l\end{pmatrix}\partial^{n-l}(b_{-p-l}^T)\\ &= (-1)^n\mathcal{R}(\varphi(t,z),\partial^n(\psi(t,z)))\end{aligned}$$

This proves the claim in the theorem. □

Like we have in M the notion of oscillating matrix function of type z^δ, we have in M^* that of dual oscillating matrix function of type $z^{-\delta}$. This is by definition an element ψ in M^* of the form

$$\begin{aligned}\psi(t,z) &= Q(t,z)\cdot R(-\delta)^*\cdot \exp(-\sum t_{i\alpha}z^iE_{\alpha\alpha})\\ &= \left\{(1+\textstyle\sum_{j<0} g_jz^j)z^{-\delta}\right\}\exp(-\sum t_{i\alpha}z^iE_{\alpha\alpha})\end{aligned}$$

If $\varphi = P(t,\partial)\cdot\exp(\sum t_{i\alpha}z^iE_{\alpha\alpha})$ in M is an oscillating matrix function of type z^δ, then

$$\varphi^* = (P(t,\partial)^*)^{-1}\cdot\exp(-\sum t_{i\alpha}z^iE_{\alpha\alpha}) \tag{29}$$

is clearly a dual oscillating matrix function of type $z^{-\delta}$. It is called the adjoint of φ. Since $(P(t,\partial)((P(t,\partial)^*)^{-1})^*)_- = 0$, the foregoing theorem shows that for all $n \geq 0$

$$\mathcal{R}(\varphi, \partial^n(\varphi^*)(t,z)) = 0. \tag{30}$$

This property even characterizes φ^* among the dual oscillating matrix functions of type $z^{-\delta}$. For, if $\psi = Q(t,\partial)R(-\delta)^*\exp(-\sum t_{i\alpha}z^iE_{\alpha\alpha}) \in M^*$ is such a function satisfying (30) with φ^* replaced by ψ, then we have according to the theorem

$$PR(\delta)(R(-\delta)^*)^*Q^* = PQ^* = \partial^0 + (PQ^*)_- = 1.$$

In other words, $Q = (P^*)^{-1}$ and ψ is the adjoint of φ. We will use this criterion later on and therefore we resume it in a

Corollary 4.1. *Let φ be an oscillating matrix function of type z^δ and let ψ be a dual oscillating matrix function of type $z^{-\delta}$. Then ψ is the adjoint of φ if and only if it satisfies*

$$\mathcal{R}(\varphi(t,z), \partial^n(\psi(t,z))) = 0 \ \textit{ for all } \ n \geq 0.$$

Associated to the adjoint of an oscillating matrix function $\varphi = PR(\delta)\cdot\exp(\sum t_{i\alpha}z^iE_{\alpha\alpha})$ of type z^δ are the operators

$$(P^*)^{-1}(-\partial)P^* = (P\partial P^{-1})^* = L^* \ \text{ and } \ U^*_\alpha = (P^*)^{-1}E_{\alpha\alpha}P^*.$$

They satisfy by definition

$$L^*\varphi^* = z\varphi^* \ \text{ and } \ U^*_\alpha\varphi^* = \varphi^*E_{\alpha\alpha}.$$

If φ is a wavefunction of the n-component KP-hierarchy, then its adjoint φ^* is called a dual wavefunction of the n-component KP-hierarchy. It satisfies a set of linear equations similar to (12), namely for all $j \geq 1$ and all $\alpha, 1 \leq \alpha \leq n$,

$$\partial_{j\alpha}(\varphi^*) = -(L^jU_\alpha)^*_+(\varphi^*). \tag{31}$$

These equations follow from the Sato-Wilson equations for P. By taking the adjoint of (27) we get

$$(\partial_{j\alpha}(P)P^{-1})^* = P^{*-1}\partial_{j\alpha}(P^*) = -(P^{*-1}(-\partial)^jE_{\alpha\alpha}P^*)_- = -((L^j)^*U^*_\alpha)_-.$$

Since $\partial_{j\alpha}(P^{*-1}) = -P^{*-1}\partial_{j\alpha}(P^*)P^{*-1}$, these equations combine to give

$$\partial_{j\alpha}(P^{*-1})P^* = ((L^*)^jU^*_\alpha)_-. \tag{32}$$

The equations (32) immediately give (31) for φ^*

$$\begin{aligned}&\partial_{j\alpha}(\varphi^*) = \partial_{j\alpha}(P^{*-1}\cdot \exp(-\sum t_{i\alpha}z^iE_{\alpha\alpha})) = \\ &= \left\{\partial_{j\alpha}(P^{*-1}) - \cdot P^{*-1}E_{\alpha\alpha}\cdot z^j\right\}\exp(-\textstyle\sum_{i,\alpha} t_{i\alpha}z^iE_{\alpha\alpha}) \\ &= \left\{\partial_{j\alpha}(P^{*-1})P^* - P^{*-1}(-\partial)^jE_{\alpha\alpha}P^*\right\}\varphi^* \\ &= -((L^*)^jU^*_\alpha)_+\cdot\varphi^*.\end{aligned}$$

Reversely, if the adjoint of an oscillating matrix function φ of type z^δ satisfies (31), then φ is a wavefunction of the n-component KP-hierarchy. The equations (31) are namely equivalent to (32) and by taking the adjoint of (32) we obtain the Sato-Wilson equations for P. They in their turn yield the equations (12). Thus we have obtained the following duality

Proposition 4.1. *Let $\psi = P\cdot R(\delta)\cdot\exp(\sum t_{i\alpha}z^iE_{\alpha\alpha})$ in M be an oscillating matrix function of type z^δ. Then ψ satisfies the equations (21) with $L = P\partial P^{-1}$ and $U_\alpha = PE_{\alpha\alpha}P^{-1}$ if and only if its adjoint ψ^* satisfies the equations (31).*

5 The Grassmannian picture

We recall here shortly the main ingredients of the analytic approach to construct wavefunctions of the n-component KP-hierarchy such as it was given in [HP]. Consider on $\mathbb{C}^n$ the standard inner product

$$((a_1,\dots,a_n)|(b_1,\dots,b_n)) = \sum_{i=1}^n a_i\overline{b_i} \tag{33}$$

Let H be the Hilbert space $L^2(S^1,\mathbb{C}^n)$ with inner product

$$<\sum a_nz^n|\sum b_nz^n> = \sum_{n\in\mathbb{Z}}(a_n|b_n).$$

The space H decomposes as $H = H_+\oplus H_-$, where

$$H_+ = \{\sum_{n\geq 0} a_nz^n\in H\} \text{ and } H_- = \{\sum_{n<0} a_nz^n \in H\}.$$

To this decomposition is associated the Grassmannian $Gr(H)$ consisting of all closed subspaces W of H such that the orthogonal projection $p_+ : W\to H_+$ is Fredholm and the orthogonal projection $p_- : W\to H_-$ is Hilbert-Schmidt. The connected components of $Gr(H)$ are given by

$$Gr^{(l)}(H) = \{W\in Gr(H)|p_+ : z^{-l}W\to H_+ \text{ has index zero}\}$$

Each of these components is a homogeneous space for the group $Gl^{(0)}_{\text{res}}(H)$ of all bounded invertible operators $g : H\to H$ that decompose with respect ot $H = H_+\oplus H_-$ as

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix},$$

where a and d are Fredholm operators and b and c are Hilbert-Schmidt. The group of commuting flows relevant for the n-component KP-hierarchy can be read off from the exponential part in the oscillating matrix functions. For each $N > 1$, we consider the multiplicative group

$$\Gamma_+(N) = \left\{\exp(\sum t_{i\alpha}z^iE_{\alpha\alpha})|t_{i\alpha}\in\mathbb{C}, \sum_{i,\alpha}|t_{i\alpha}|N^i<\infty\right\}$$

equiped with the uniform norm. These groups are nested in a natural way and the inductive limit is denoted by Γ_+. The group Γ_+ acts by multiplication on H and this gives a continuous injection of Γ_+ into $Gl_{\text{res}}^{(0)}(H)$. Another relevant group that acts on H by multiplication is the group

$$\Delta = \left\{ z^\delta = \begin{pmatrix} z^{\delta_1} & & 0 \\ & \ddots & \\ 0 & & z^{\delta_n} \end{pmatrix} \;,\; \delta_i \in \mathbb{C} \right\}.$$

Because of the form of the oscillating matrix functions, we let both Γ_+ and Δ act from the right on H.

Now we take for R the ring of meromorphic fuctions on Γ_+ and for $\partial_{i\alpha}$ the partial derivative w.r.t. the parameter $t_{i\alpha}$ of Γ_+. For each $W \in Gr(H)$ and each $z^\delta \in \Delta$, let $\Gamma_+^W(\delta)$ be given by

$$\Gamma_+^W(\delta) = \{\gamma(t) = \exp(\sum t_{i\alpha} z^i E_{\alpha\alpha}) \in \Gamma_+ | p_+ : W z^{-\delta}\gamma^{-1} \to H_+ \text{ is a bijection}\}$$

These sets might be empty, but similar to the proof in [SW] for the KP-hierarchy one shows that for each W in $Gr(H)$, there exists a z^δ such that $\Gamma_+^W(\delta)$ is non-zero. We see $\mathbb{C}^n$ as a part of H by the embedding $\alpha \mapsto az^0$. Let $\{e_i | 1 \le i \le n\}$ be the standard basis of $\mathbb{C}^n$, i.e. $e_i = (\dots 0, 1, 0, \dots)$, where the one is at the i-th entry. For $\gamma \in \Gamma_+^W(\delta)$, let $P_{W,i}(\gamma, z)$ be the inverse image of e_i under the orthogonal projection $p_+ : W z^{-\delta}\gamma^{-1} \to H_+$ and we write $\psi_{W,i}^{(\delta)}(\gamma, z) = P_{W,i}(\gamma, z) z^\delta \gamma$. Then $\psi_{W,i}^{(\delta)}$ is a function on an open dense part of Γ_+ with values in W. If we define, moreover, $\psi_W^{(\delta)}$ as the matrix whose i-th row is $\psi_{W,i}^{(\delta)}$, then $\psi_W^{(\delta)}$ is an oscillating matrix function of type z^δ. By construction the product in $\psi_W^{(\delta)}$ is no longer formal and we can write

$$\psi_W^{(\delta)} = \sum_{n \in \mathbb{Z}} a_n(\gamma, \delta) z^n \quad ,$$

with a well-defined $a_n(\gamma, \delta) \in M_n(\mathbb{C})$ and such that

$$\sum \| a_n \|^2 < \infty \;\;,\;\; \text{where} \;\; \| a \|^2 = \sum_{i,j} |a_{ij}|^2 \;\text{ for }\; a \in M_n(\mathbb{C}). \tag{34}$$

It is shown in [HP] that $\psi_W^{(\delta)}$ satisfies the linear equations in (3.1), hence

Theorem 5.1. *The matrix function $\psi_W^{(\delta)}$ is a wavefunction for the n-component KP-hierarchy.*

What remains to be shown is how one can obtain $(\psi_W^{(\delta)})^*$ in this Grassmannian setting.

6 The construction of the dual wavefunction

Thanks to the construction from section 5 we have now a wide class of wavefunctions of the n-component KP-hierarchy, namely the $\{\psi_W^{(\delta)} | W \in Gr(H)\}$. We will show here how the dual wavefunctions $(\psi_W^{(\delta)})^*$ fit into the geometric picture. From the formal definition

$$(\psi_W^{(\delta)})^* = (P(t,\delta)^*)^{-1} \cdot \exp(-\sum t_{i\alpha} z^i E_{\alpha\alpha}),$$

where $\psi_W^{(\delta)} = P(t,\delta) \cdot \gamma(t)$, it is not at all clear that the coefficients of the powers of z in $(\psi_W^{(\delta)})^*$ satisfy a convergence condition like (34).

A consequence of the geometric construction of $(\psi_W^{(\delta)})^*$ that we present here, will be that also the formal product in $(\psi_W^{(\delta)})^*$ of an exponential factor and a part that has at the utmost a pole at $z=\infty$ satisfies a convergence condition similar to (34). Hence $(\psi_W^{(\delta)})^*$ is of the same class as $\psi_W^{(\delta)}$.

Let W belong to $Gr^{(l)}(H)$, then $W^\perp$ is a closed subspace of H for which the orthogonal projection $p_-:W^\perp\to H_-$ is a Fredholm operator of index $-l$ and such that $p_+:W^\perp\to H_+$ is a Hilbert-Schmidt operator. Interchanging the role of H_+ and H_-, we see that $W^\perp$ is a plane in the adjoint Grassmanian $Gr^*(H)$ consisting of planes U for which $p_-|_U$ is a Fredholm operator and $p_+|_U$ is a Hilbert-Schmidt operator. The connected components of $Gr^*(H)$ are also homogeneous spaces for the group $Gl_{\text{res}}^{(0)}(H)$ introduced above. On $Gr^*(H)$ we consider the commuting flows that are the adjoints of the ones from Γ_+. Namely, for each $N>1$, we take the group

$$\Gamma_-(N)=\{g(r)=\exp(\sum r_{i\alpha}z^{-i}E_{\alpha\alpha})|r_{i\alpha}\in\mathbb{C},\sum_{i,\alpha}|r_{i\alpha}|N^i<\infty\}$$

and its inductive limit Γ_- w.r.t. N. The action of Γ_- and Δ on H is again from the right. As above we consider for each $z^\delta\in\Delta$ and $W^\perp\in Gr^*(H)$

$$\Gamma_-^{W^\perp}(\delta)=\{g(r)\in\Gamma_-|p_-:W^\perp z^{-\delta}g(r)^{-1}\to H_-\ \text{ is a bijection}\}$$

By using the facts that for each $V\in Gr(H)$ the projection $p_+:V\to H_+$ is a bijection if and only if $p_-:V^\perp\to H_-$ is a bijection and that for each bounded invertible linear operator $A:H\to H, A(V^\perp)^\perp={A^*}^{-1}(V)$, one sees that

$$g(r)\in\Gamma_-^{W^\perp}(\delta)\Leftrightarrow(g(r)^*)^{-1}\in\Gamma_+^W(\delta).$$

In particular one sees that there are z^δ's for which $\Gamma_-^{W^\perp}(\delta)$ is non-empty. We fix such a δ and take a $g(r)$ in $\Gamma^{W^\perp}(\delta)$. By definition we know that $p_-:W^\perp z^{-\delta}g(r)^{-1}\to H_-$ is a bijection. For all $i,1\le i\le n$, let $Q_{W^\perp,i}(g(r),z)$ be the inverse image of the element e_iz^{-1} under this projection and put

$$\psi_{W^\perp,i}^{(\delta)}(g(r),z)=Q_{W^\perp,i}(g(r),z)z^\delta g(r).$$

Then the map $g(r)\mapsto\psi_{W^\perp}^{(\delta)},i(g(r),z)$ is an analytic function on an open dense part of Γ_- with values in $W^\perp$. Finally we build the matrix function $\psi_{W^\perp}^{(\delta)}$ by taking $\psi_{W^\perp}^{(\delta)},i$ as its i-th row. It has the form

$$\psi_{W^\perp}^{(\delta)}(g(r),z)=\{z^{-1}Id+\sum_{j\ge0}b_j(g(r))z^j\}z^\delta\exp(\sum r_{i\alpha}z^{-i}E_{\alpha\alpha})\ ,$$

with $b_j\in M_n(R^*)$, where R^* denotes the ring of meromorphic functions on Γ_-. Hence, if we take $g(r)=\gamma(-t)^*$, with $\gamma(t)\in\Gamma_+$, then we get the following dual oscillating matrix function

$$\overline{z\psi_{W^\perp}^{(\delta)}(\gamma(-t)^*,z)}=\{Id+\sum_{j\ge0}\overline{b_j(\gamma(-t)^*)}z^{-j-1}\}z^{-\delta}\gamma(-t)$$

and this is the function we are looking for.

Theorem 6.1. *The dual wavefunction* $(\psi_W^{(\delta)})^*$ *of the wavefunction* $\psi_W^{(\delta)}$ *of the n-component KP-hierarchy is equal to* $\overline{z\psi_{W^\perp}^{(\delta)}(\gamma(-t)^*,z)}$.

Proof. Thanks to corollary 4.1, we merely have to show for all $m \geq 0$

$$\mathcal{R}(\psi_W^\delta(\gamma(t),z), \partial^m \overline{(z\psi_{W^\perp}^{(\delta)}(\gamma(-t)^*,z))}) = 0. \tag{35}$$

Now is $\partial^m \overline{(z\psi_{W^\perp}^{(\delta)}(\gamma(-t)^*,z))} = \overline{(-1)^m z(\sum_{\alpha=1}^n \frac{\partial}{\partial r_{1\alpha}})^m(\psi_{W^\perp}^\delta)(\gamma(-t)^*,z)}$. Since $\psi_{W^\perp}^\delta$ depends analytically of $g(r)$, we have for all $m \geq 0$ that $(\sum_{\alpha=1}^n \frac{\partial}{\partial r_{1\alpha}})^m(\psi_{W^\perp}^\delta)$ has rows that take values in $W^\perp$. Now we combine this with the fact that the rows of ψ_W^δ take values in W, so that there holds for all k and $l, 1 \leq k \leq n, 1 \leq l \leq n$, and all $\gamma(t) \in \Gamma_+^W(\delta)$ and all $g(r) \in \Gamma_-^{W^\perp}(\delta)$

$$< \psi_{W,k}^{(\delta)}(\gamma(t),z) | (\sum_{\alpha=1}^n \frac{\partial}{\partial r_{1\alpha}})^m (\psi_{W^\perp}^{(\delta)}(g(r),z)) >= 0. \tag{36}$$

This relation will prove the desired property (35). The following notation is convenient at that: if $a \in M_n(R)$, then a_{*j} is the j-th column of a and a_{j*} is the j-th row of a. If the matrix functions ψ_W^δ and $(\sum_{\alpha=1}^n \frac{\partial}{\partial r_{1\alpha}})^m \psi_{W^\perp}^\delta$ decompose as follows

$$\psi_W^{(\delta)} = \{\sum_{i\leq 0} a_i z^i\} z^\delta \gamma(t) \text{ and } (\sum_{\alpha=1}^n \frac{\partial}{\partial r_{1\alpha}})^m \psi_{W^\perp}^{(\delta)} = \{\sum_{j\geq -m-1} b_j z^j\} z^\delta g(r),$$

then relation (36) means by definiton of the inner product that the constant term of the powerseries in z

$$\{\sum_{i\leq 0} (a_i)_{k*} z^i\} z^\delta \gamma(t) g(r)^* z^{-\delta} \{\sum_{j\geq -m-1} ((b_j)^*)_{*l} z^{-j}\}$$

has to be zero. In particular, if we choose $g(r)$ equal to $\gamma(-t)^*$, then the middle term in this expression cancels and we get for all k and l that

$$\sum_j (a_j)_{k*}((b_j)^*)_{*l} = 0. \tag{37}$$

Since $\partial^m \overline{z\psi_{W^\perp}^\delta(\gamma(t)^*,z)} = \{\sum c_j z^j\} z^\delta \gamma(-t)$, with $b_j = c_{-j-1}$, the relations (37) clearly imply that property (35) holds. This concludes the proof of the theorem. □

Thanks to this theorem we can say now that we have

$$\psi_W^{(\delta)} = \sum_{n\in\mathbb{Z}} a_n(\gamma(t),\delta) z^n \quad \text{with} \quad \sum_{n\in\mathbb{Z}} \| a_n(\gamma(t),\delta) \|^2 < \infty \tag{38}$$

$$(\psi_W^{(\delta)})^* = \sum_{n\in\mathbb{Z}} b_n(\gamma(s),\delta) z^n \quad \text{with} \quad \sum_{n\in\mathbb{Z}} \| b_n(\gamma(s),\delta) \|^2 < \infty. \tag{39}$$

One has in the present context a convergent interpretation of the well-known bilinear identity between a wavefunction ψ and its adjoint

$$\text{Res}_{z=0}\, \psi(t,z)\psi^*(s,z) = 0. \tag{40}$$

For, from equation (36) follows that for all $\gamma(t)$ and all $\gamma(s)$ in $\Gamma_+^W(\delta)$ that

$$\sum_{n\in\mathbb{Z}} a_n(\gamma(t),\delta)) b_{-n-1}(\gamma(s),\delta)) = 0, \tag{41}$$

and the left hand side of this equation is exactly the coefficient of z^{-1} in the product $\psi_W^{(\delta)}(t,z)(\psi_W^{(\delta)})^*(s,z)$. This proves equation (40) in the present context.

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