Faculty of Mathematical Sciences

University of Twente

University for Technical and Social Sciences

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl

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The AKNS-hierarchy

G.F. Helminck

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G.F. Helminck

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Abstract

We present here an overview for the Encyclopaedia of Mathematics of the various forms and properties of this system of equations together with its geometric and Lie algebraic background.

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The AKNS-hierarchy is an infinite tower of non linear evolution equations that derives its name from the simplest non trivial system of equations contained in it, the AKNS-equations

$$\begin{aligned} i\frac{\partial}{\partial t}q(x,t) &:= iq_t = -\frac{1}{2}q_{xx} + q^2 r \\ i\frac{\partial}{\partial t}r(x,t) &:= ir_t = \frac{1}{2}r_{xx} - qr^2. \end{aligned} \tag{1}$$

It were Ablowitz, Kaup, Newell and Segur who showed that the **initial value problem** of this system of equations could be solved with the Inverse Scattering Transform. To get a natural embedding of the AKNS-equations in a larger system we rewrite (1) in the zero curvature form as

$$\frac{\partial}{\partial t}(P_1) - \frac{\partial}{\partial x}Q_2 + [P_1, Q_2] = 0, \ where \tag{2}$$

$$P_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \ and \ Q_2 = \begin{pmatrix} -\frac{i}{2}qr & \frac{i}{2}q_x \\ -\frac{i}{2}r_x & \frac{i}{2}qr \end{pmatrix}. \tag{3}$$

Consider now the polynomial expressions in the parameter z

$$\begin{aligned} P = P_0 z + P_1 &:= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} z + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \ and \\ Q^{(n)} &:= Q_0 z^n + Q_1 z^{n-1} \cdots Q_n, \end{aligned} \tag{4}$$

where the Q_i are sl_2-valued functions depending of the variables $x, t = (t_n)$ and $Q_0 = P_0$ and $Q_1 = P_1$. For these data we have the zero curvature equations

$$\frac{\partial}{\partial t_n}(P) - \frac{\partial}{\partial x}Q^{(n)} + [P, Q^{(n)}] = 0 \Leftrightarrow \left[\frac{\partial}{\partial x} - P, \frac{\partial}{\partial t_n} - Q^{(n)}\right] = 0, \tag{5}$$

which is an infinite tower of equations extending the system (1). The system (5) generalizes from $Sl_2(\mathbb{C})$ to a general simple complex **Lie algebra** $\mathfrak{g}$, a **regular element** P_0 in a **Cartan subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ and an element Q_0 in $\mathfrak{h}$, see [DS] and [W1]. Solutions of the equations (5) can be obtained by the dressing method of Zacharov and Shabat. Consider namely the function

$$\phi(x,t,z) = exp(xP_0z + \sum_{r=1}^{\infty} Q_0 z^r) g(z) exp(-xP_0 z - \sum_{r=1}^{\infty} Q_0 z^r), \tag{6}$$

with $g(z)$ belonging to the loop group $C^\infty(S^1, Sl_2(\mathbb{C}))$. If this ϕ factorizes as $\phi = \phi_- \phi_+$, with

$$\phi_-(x,t,z) = exp(\sum_{i=1}^{\infty} \chi_i(x,t) z^{-i}) \;\; and \;\; \phi_+ = exp(\sum_{j=1}^{\infty} \phi_j(x,t) z^j), \tag{7}$$

then conjugating the trivial **connections** $\frac{\partial}{\partial x} - P_0 z$ and $\frac{\partial}{\partial t_n} - Q_0 z^n$ with ϕ_- gives you the connections of the required form

$$\phi_-^{-1}(\frac{\partial}{\partial x} - P_0 z)\phi_- = \frac{\partial}{\partial x} - P \;\; and \;\; \phi_-^{-1}\frac{\partial}{\partial t_n} - Q_0 z^n \phi_- = \frac{\partial}{\partial t_n} - Q^{(n)}. \tag{8}$$

Since flatness is preserved by this procedure, this leads to solutions of (5). If $Q_0 = P_0$, as in the AKNS-case, one can take just as well $\frac{\partial}{\partial x} = \frac{\partial}{\partial t_1}$.

It was observed in ([FNR]) that the equations (5) for the $Sl_2(\mathbb{C})$-case could be catched in the following system

$$\frac{\partial}{\partial t_n}(Q) = [Q^{(n)}, Q], \; n \geq 1, \tag{9}$$

for the single series

$$Q = \sum_{j=0}^{\infty} Q_j z^{-j}, \;\; Q_j = \begin{pmatrix} h_j & e_j \\ f_j & -h_j \end{pmatrix} \;\; and \;\; Q^{(n)} = \sum_{j=0}^{n} Q_j z^{n-j}. \tag{10}$$

They showed that these equations are commuting **Hamiltonian flows** on the Lie algebra $\sum_{i=0}^{\infty} X_i z^{-i}, X_i \in sl_2(\mathbb{C})$, w.r.t. a natural **Poisson bracket**. Further they introduced the flux tensor F_{jk} by

$$F_{jk} = \frac{1}{2} Trace \left(\sum_{r=0}^{j} (j-r) Q_r Q_{k+j-r} + \frac{1}{2} \sum_{r=0}^{j} (r-k) Q_r Q_{k+j-r} \right) \tag{11}$$

and proved the local conservation laws of the system, namely

$$\frac{\partial}{\partial t_k}(F_{ij}) = \frac{\partial}{\partial t_i}(F_{jk}). \tag{12}$$

The left hand side of (12) is in fact even symmetric under permutations of the indices i, j, k and this property made them introduce a potential τ by

$$F_{jk} = \frac{\partial}{\partial t_j} \frac{\partial}{\partial t_k} log(\tau). \tag{13}$$

The equations (10) are called the Lax equations of the AKNS-hierarchy. As such the AKNS-hierarchy is a natural reduction of the two-component KP-hierarchy, see ([HP]), a fact that enables a description in the Grassmanian of that hierarchy.

It was shown by Bergvelt and ten Kroode, see ([BtK1]), that it is natural to consider a system of zero curvature relations (5) labeled by the root lattice of the Lie algebra, where the operators at different sites of the lattice are linked by Toda type differential-difference equations. For example, for nearest neighbour sites there holds

$$q^{(l+1)} = -(q^{(l)})^2 r^{(l)} + q^{(l)} log(q^{(l)}) \ \ and \ \ r^{(l+1)} = \frac{1}{q^{(l)}}, \tag{14}$$

$$where \ \ P^{(l)} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} z + \begin{pmatrix} 0 & q^{(l)} \\ r^{(l)} & 0 \end{pmatrix}. \tag{15}$$

This phenomenon is due to the fact that there is a natural lattice group that commutes with the commuting flows corresponding to the parameters t_n.

In the representation theoretic approach to soliton equations, see ([DKJM]) and ([K]) the soliton equations occur as the equations describing the group orbit of the highest weight vector. A similar description holds for these combined differential-difference equations. Let $L(\Lambda_0)$ be the basic representation of the **Kac-Moody Lie algebra** $A_1^{(1)}$. In $A_1^{(1)}$ one takes the homogeneous Heisenberg algebra

$$s = \sum_{i>0} \mathbb{C}\lambda^i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \sum_{i>0} \mathbb{C}\lambda^{-i} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{C}c,$$

where c is the central element of $A_1^{(1)}$ that is in the kernel of the projection of $A_1^{(1)}$ onto the loop algebra of $Sl_2(\mathbb{C})$. The $A_1^{(1)}$-module $L(\Lambda_0)$ decomposes with respect to the action of the homogeneous Heisenberg algebra s as a direct sum of irreducible s-modules $\mathbb{C}[t] = \mathbb{C}[t_1, \ t_2, \ \ldots]$ labeled by the root lattice. Thus we can write each element of $L(\Lambda_0)$ as $\tau(t) = (\tau_l(t))_{l\in\mathbb{Z}}$. The group orbit of the highest weight vector can then be characterized by a set of so-called Hirota bilinear relations for the components (τ_l). By using the representation theory one constructs a series of elements (g_l) in a suitable completion of the Kac-Moody group associated with $A_1^{(1)}$ such that the vacuum expectation value of g_l is exactly τ_l. The Birkhoff decomposition of the (g_l) in that group enables you then to construct solutions of the lattice zero curvature equations, see ([BtK2]). In particular, the operators $P^{(l)}$ from (14) one obtains in this way, can be expressed in the components (τ_l) by

$$q^{(l)} = 2i\frac{\tau_{l+1}}{\tau_l} \ \ and \ \ r^{(l)} = -2i\frac{\tau_{l-1}}{\tau_l}. \tag{16}$$

By using the adjoint action of the Kac-Moody group Bergvelt and ten Kroode also showed in ([BtK2]) that the

$$F_{jk}^{(l)} := \frac{\partial}{\partial t_j}\frac{\partial}{\partial t_k} log(\tau_l) \tag{17}$$

give you exactly the flux tensor from (11), thus furnishing a representation theoretic basis for the results in ([FNR]).

A geometric way to look at τ-functions, see e.g.([SW]), is to consider a homogeneous space over the relevant loop group L, a holomorphic line bundle $\mathcal{L}$ over this space and its pull-back over the corresponding central extension $\hat{L}$ of L. If this last line bundle has a global holomorphic section, then τ-functions measure the failure of equivariance of partial liftings from L to $\hat{L}$ w.r.t. this section. With this point of view, one can also arrive at the formulae in (16) by lifting the discrete group of transformations that commute with the flows from (8) appropriately, see ([W2]).

An important class of equations associated with the AKNS-hierarchy are the so-called stationary AKNS-equations. These are the differential equations for the functions q and r from the first order differential operator

$$L := P_0 \frac{d}{dx} + P_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \tag{18}$$

resulting from the existence of a 2×2-matrix valued differential operator

$$\mathcal{P}_{n+1} = \sum_{i=0}^{n+1} u_i (\frac{d}{dx})^i$$

that commutes with L. Such a pair is naturally associated with a hyperelliptic curve of genus n and that is why one calls P_1 an algebro-geometric AKNS-potential. The elliptic algebro-geometric AKNS-potentials have been characterized in ([GW]). They correspond exactly to the potentials for which the equation $L(\psi) = z\psi$ has a meromorphic **fundamental system of solutions** with respect to x for all values of the spectral parameter $z \in \mathbb{C}$.

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