Faculty of Mathematical Sciences

University of Twente

University for Technical and Social Sciences

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl

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Symmetries of the WDVV equations and Chazy-type equations

R. Martini and G.F. Post

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SYMMETRIES OF THE WDVV EQUATIONS AND CHAZY-TYPE EQUATIONS

R. MARTINI AND G.F. POST

ABSTRACT. We investigate the symmetry structure of the WDVV equations. We obtain an r-parameter group of symmetries, where $r = \frac{1}{2}(n^2 + 7n + 2) + \lfloor \frac{n}{2} \rfloor$. Moreover it is proved that for $n = 3$ and $n = 4$ these comprise all symmetries.

We determine a subgroup, which defines an SL_2-action on the space of solutions. For the special case $n = 3$ this action is compared to the SL_2-symmetry of the Chazy equation. For $n = 4$ and $n = 5$ we construct new, Chazy-type, solutions.

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1. INTRODUCTION.

In the physics literature on two-dimensional topological field theory, a remarkable and rather complicated system of partial differential equations emerged [1, 2]. Roughly speaking the equations describe the conditions for a function $F = F(t)$ of the variable $t = (t_1, \ldots, t_n)$, such that the third-order derivatives define a set of structure constants of an associative algebra. Commonly this system of nonlinear equations is known as the Witten–Dijkgraaf–H. Verlinde–E. Verlinde (WDVV) system.

Recently the equations also appeared in four-dimensional supersymmetric gauge theory, see e.g. [3, 4, 5], suggesting an interesting relationship between 4-dimensional $N = 2$ supersymmetric gauge theories and 2-dimensional topological field theories.

Although extremely difficult to solve in general, these equations admit exact solutions and other features of integrability. It is therefore interesting to investigate its symmetry structure; this is the subject of the present paper. For all n we obtain an extensive symmetry group generalizing the results for $n = 3$ and $n = 4$. The results for $n = 3$ and $n = 4$ were obtained in [6] by computer programs written in REDUCE [7], using methods which are for example described in [8]; solving these overdetermined systems of partial differential equations is already difficult for $n = 4$.

The symmetry group contains $SL_2(\mathbb{C})$. We discuss its intriguing action especially for the Chazy equation [9, 10] that is contained in the case $n = 3$.

2. THE WDVV EQUATIONS.

2.1. **Definition.** Following the conventions in Dubrovin [10], the WDVV equations are a system of partial differential equations for a function $F(t) =$

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$F(t_1, \dots, t_n)$ such that the third order derivatives,

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F(t)}{\partial t_\alpha \partial t_\beta \partial t_\gamma}$$

obey the following conditions, cf. [10]

1. (Normalization)

$$c_{1\alpha\beta} = \begin{cases} 0 & \text{if } \alpha + \beta \neq n+1 \\ 1 & \text{if } \alpha + \beta = n+1 \end{cases} \tag{2.1}$$

   We introduce the metric $\eta^{\alpha\beta} = \eta_{\alpha\beta} = c_{1\alpha\beta}$.

2. (Associativity)
   The functions

$$c^\gamma_{\alpha\beta}(t) = \sum_{\epsilon=1}^{n} \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}$$

   for any t, must define an n-dimensional associative algebra with basis $e_1, \dots, e_n$ and product given by

$$e_\alpha \cdot e_\beta = \sum_{\gamma=1}^{n} c^\gamma_{\alpha\beta} e_\gamma$$

The normalization expresses the existence of a unity in the corresponding algebra. Throughout we use subscripts instead of superscripts.

This system of conditions, we call, following [10], the WDVV equations, [2, 1]. (We omit quasi-homogeneity). We assume the metric $\eta_{\alpha\beta}$ to be in standard form, corresponding to an algebra with unity ∂_{t_1}.

The associativity condition may be written as an overdetermined system of partial differential equations, namely

$$\sum_{\lambda=1}^{n} \frac{\partial^3 F(t)}{\partial t_\alpha \partial t_\beta \partial t_\lambda} \cdot \frac{\partial^3 F(t)}{\partial t_\gamma \partial t_\delta \partial t_{\bar\lambda}} = \sum_{\lambda=1}^{n} \frac{\partial^3 F(t)}{\partial t_\gamma \partial t_\beta \partial t_\lambda} \cdot \frac{\partial^3 F(t)}{\partial t_\alpha \partial t_\delta \partial t_{\bar\lambda}} \tag{2.2}$$

for any α, β, γ and δ. We denoted $\bar\lambda = n+1-\lambda$. From (2.1), the dependency of F on t_1 can be solved completely:

$$F(t) = \frac{1}{2} t_1^2 t_n + \frac{1}{2} t_1 \sum_{\alpha=2}^{n-1} t_\alpha t_{\bar\alpha} + f(t_2, \dots, t_n). \tag{2.3}$$

Substituting this expression for F in (2.2) we obtain the following system for f (subscripts denote partial derivatives)

$$\sum_{\lambda=2}^{n-1} \left(f_{\alpha\beta\lambda} \cdot f_{\gamma\delta\bar\lambda} - f_{\gamma\beta\lambda} \cdot f_{\alpha\delta\bar\lambda} \right)$$
$$+\eta_{\alpha\beta} f_{\gamma\delta n} - \eta_{\gamma\beta} f_{\alpha\delta n} + f_{\alpha\beta n}\eta_{\gamma\delta} - f_{\gamma\beta n}\eta_{\alpha\delta} = 0 \tag{2.4}$$

Hence the WDVV equations (2.1) and (2.2) are equivalent to (2.4).

The number of (generating) equations in (2.4) grows rapidly with n. This number is $\binom{n}{3}$ for $n \leq 7$.

3. Description of the symmetries

We split the discussion of symmetries in 4 parts.

3.1. **Translations.** The simplest symmetries of the WDVV equations are the translations. Since the associativity equations (2.4) does not explicitly depend on $(t_1, \dots, t_n)$ we have the infinitesimal symmetry

$$\mathbf{v} = \partial_{t_a} \qquad (a = 1, \dots, n) \tag{3.1}$$

with corresponding symmetry

$$t_a \mapsto t_a + \varepsilon \tag{3.1'}$$

(all non-occurring variables remain fixed).

Similarly, we have the translation-type infinitesimal symmetries

$$\mathbf{v} = P(t_1, \dots, t_n)\partial_F \tag{3.2}$$

where P is a polynomial of degree at most 2. Obviously, the corresponding symmetry is

$$F \mapsto F + \varepsilon P(t_1, \dots, t_n) \tag{3.2'}$$

Hence (3.1) and (3.2) yield $n + \binom{n+2}{2}$ infinitesimal symmetries.

3.2. **Scalings.** The scalings are most easily described in terms of f instead of F. For instance the system (2.4) is independent of t_1. Hence one can rescale t_1. For f nothing changes, while F should be corrected to maintain the normalization (2.1). So the first scaling is infinitesimally

$$\mathbf{v} = t_1\partial_{t_1} + t_1 Q\partial_F. \tag{3.3}$$

Here and hereafter,

$$Q = \frac{1}{2}\sum_{s=1}^{n} t_s t_{\bar{s}} \tag{3.4}$$

The corresponding scaling is

$$t_1 \mapsto e^{\varepsilon} t_1; \qquad f \mapsto f \tag{3.3'}$$

Next we have for $2 \le a < \frac{n+1}{2}$ the generator

$$\mathbf{v} = t_a\partial_{t_a} - t_{\bar{a}}\partial_{t_{\bar{a}}} \tag{3.5}$$

with corresponding symmetry

$$t_a \mapsto e^{\varepsilon} t_a; \qquad t_{\bar{a}} \mapsto e^{-\varepsilon} t_{\bar{a}} \tag{3.5'}$$

Finally we have the generators

$$\mathbf{v} = \sum_{i=1}^{n-1} t_i\partial_{t_i} + 4t_n\partial_{t_n} \tag{3.6}$$

and

$$\mathbf{v} = t_n\partial_{t_n} + (t_1 Q - F)\partial_F \tag{3.7}$$

with corresponding scalings

$$t_i \mapsto e^{\varepsilon} t_i \quad (i = 1, \dots, n-1); \qquad t_n \mapsto e^{4\varepsilon} t_n \tag{3.6'}$$

and

$$t_n \mapsto e^{\varepsilon} t_n; \qquad f \mapsto f e^{-\varepsilon} \tag{3.7'}$$

Using the equations (2.4), these last two scalings are easily checked. Totally we find $3 + \lfloor \frac{n-2}{2} \rfloor$ scaling symmetries.

3.3. **Linear-type transformations.** The associativity condition is invariant under linear changes of the variables $t_1, \dots, t_n$. However η_{ij} (defined as $\eta_{ij} = \frac{\partial^3 F(t)}{\partial t_1 \partial t_i \partial t_j}$) changes. Hence the normalization spoils most of these symmetries. Some, with a slight modification, remain; apart from the scalings, which were discussed above, there are two classes. The first class is easy to check:

$$\mathbf{v} = t_a \partial_{t_1} + t_a Q \partial_F \quad (a = 2, \dots, n) \tag{3.8}$$

with 1-parameter group

$$t_1 \mapsto t_1 + \varepsilon t_a; \qquad f \mapsto f \tag{3.8'}$$

The second class is not so obvious

$$\mathbf{v} = t_n \partial_{t_a} + t_{\bar{a}} Q \partial_F \quad (a = 1, \dots, n-1). \tag{3.9}$$

The corresponding transformation does not leave f invariant. In terms of F it is

$$t_a \mapsto t_a + \varepsilon t_n; \qquad F \mapsto F + \varepsilon t_{\bar{a}} Q + \frac{1}{2} \varepsilon^2 t_n t_{\bar{a}}^2 \tag{3.9'}$$

To check that (3.9') is a symmetry, we use the infinitesimal criterion, see e.g. [8], section 2.3. One calculates the prolongation coefficients $\Phi^{(k,l,m)}$ of the vector field (3.9)

$$\begin{aligned} \Phi^{(k,\ell,m)} &= \delta_{k,\bar{a}} \eta_{\ell m} + \delta_{\ell,\bar{a}} \eta_{km} + \delta_{m,\bar{a}} \eta_{k\ell} \\ &\quad - \delta_{k,n} F_{a\ell m} - \delta_{\ell,n} F_{akm} - \delta_{m,n} F_{ak\ell} \end{aligned}$$

and by direct calculation one checks that

$$\sum_{\lambda=1}^{n} \Phi^{(\alpha,\beta,\lambda)} F_{\gamma\delta\bar{\lambda}} + F_{\alpha,\beta,\lambda} \Phi^{(\gamma,\delta,\bar{\lambda})} - \Phi^{(\gamma,\beta,\lambda)} F_{\alpha\delta\bar{\lambda}} + F_{\gamma,\beta,\lambda} \Phi^{(\alpha,\delta,\bar{\lambda})} = 0 \tag{3.10}$$

holds for solutions F of (2.2). That the normalization is not spoilt, can be checked directly from (3.9').

3.4. **Quadratic symmetry.** Calculations for $n = 3$ and $n = 4$ show that the WDVV equations admit one highly non-trivial symmetry. It can be generalized to all n. The generator is

$$\mathbf{v} = t_n \sum_{i=1}^{n} t_i \partial_{t_i} + (\frac{1}{2} Q^2 + 2 t_n F) \partial_F \tag{3.11}$$

The corresponding 1-parameter group is

$$t_i \mapsto \frac{t_i}{\varepsilon t_n + 1} \quad (i = 1, \dots n); \qquad F \mapsto \frac{F}{(\varepsilon t_n + 1)^2} - \frac{1}{2} \varepsilon \frac{Q^2}{(\varepsilon t_n + 1)^3} \tag{3.12}$$

The normalization is maintained. The corresponding transformation for f is

$$f \mapsto \frac{f}{(\varepsilon t_n + 1)^2} - \frac{1}{2} \varepsilon \frac{\left(\sum_{i=2}^{n-1} \frac{1}{2} t_i t_{\bar{i}} \right)^2}{(\varepsilon t_n + 1)^3} \tag{3.13}$$

To check that (3.12) is a symmetry, we again use the infinitesimal criterion. The prolongation coefficients of (3.11) are

$$\begin{aligned}\Phi^{(k,\ell,m)} &= -\delta_{k,n}E(F_{\ell m}) - \delta_{\ell,n}E(F_{km}) + \delta_{m,n}E(F_{k\ell}) \\ &\quad +\eta_{k\ell}t_m + \eta_{km}t_\ell + \eta_{\ell m}t_k - t_nF_{k\ell m}\end{aligned} \tag{3.14}$$

Here we denoted the Euler vector field $\sum_i t_i\partial_{t_i}$ by E. One can substitute (3.14) in (3.10). A straightforward calculation yields the result. Hence we have proved

3.5. **Proposition.**

a. The WDVV equations (2.1) and (2.2) admit an r-parameter group of symmetries, where $r = \frac{1}{2}(n^2 + 9n + 2) + \lfloor \frac{n}{2} \rfloor$.
b. For $n = 3$ and $n = 4$ these are all (continuous) symmetries.

Part b. was proved by explicit calculation, see [6].

4. SL_2-action and the Chazy equation

4.1. **Lie algebra structure.** We have investigated the structure of the Lie algebra of infinitesimal symmetries described above. It turned out to be a semi-direct product. The elements of the form $P\partial_F$, corresponding to the symmetries (3.2), form a commutative ideal. Dividing out this ideal leaves us essentially with vector fields of the form

$$X = \sum_{i=1}^{n} x_i(t_1, \dots, t_n)\partial_{t_i}.$$

Here the coefficients x_i have degree at most two (in case of (3.11)). The elements for which all x_i are linear in $t_1, \dots, t_n$ form a subalgebra of $g\ell_n(\mathbb{C})$ by the isomorphism $\sum a_{ij}t_i\partial_{t_j} \mapsto (a_{ij})$. In our case this subalgebra is isomorphic to a set of lower-triangular matrices, with only non-zero elements in the first column, the last row and on the diagonal. However, using the quadratic element (3.11) we obtain a subalgebra, isomorphic to $s\ell_2(\mathbb{C})$.

4.2. We consider the three infinitesimal symmetries,

$$Y = t_n\sum_{i=1}^{n} t_i\partial_{t_i} + (\frac{1}{2}Q^2 + 2t_nF)\partial_F$$

$$H = -\sum_{i=1}^{n} t_i\partial_{t_i} - t_n\partial_{t_n} - (t_1Q + 2F)\partial_F$$

$$X = \partial_{t_n} + \frac{1}{2}t_1^2\partial_F$$

One can check that

$$[H, X] = 2X; \qquad [X, Y] = H \text{ and } [H, Y] = -2Y$$

These vector fields may be integrated to yield an SL_2-action on the solution space. To describe this action, it is again convenient to shift from F to f: integrating X, Y and H above, we have

$$\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a}\end{pmatrix}\cdot f = \frac{f}{a^2} \qquad \begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\cdot f = f \tag{4.1}$$

while the action of $\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$ is given by (3.13).

Hence totally we obtain for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$

$$A \cdot t_i = \frac{t_i}{ct_n + d} \quad (i = 1, \dots, n-1); \qquad A \cdot t_n = \frac{at_n + b}{ct_n + d} \tag{4.2}$$

$$A \cdot f = \frac{f}{(ct_n + d)^2} - \frac{1}{2} c \frac{\left(\sum_{i=2}^{n-1} \frac{1}{2} t_i t_{\bar{i}} \right)^2}{(ct_n + d)^3} \tag{4.2a}$$

4.3. **Quasi-homogeneity.** In Dubrovin [10] the WDVV equations above are supplemented with the requirement that any solution should be quasi-homogeneous. This means that there must exist constants $d_1, d_2, \dots, d_n$ and d_F such that

$$\sum_{i=1}^{n} d_i t^i \frac{\partial F}{\partial t^i} = d_F F \tag{4.3}$$

Most symmetries above do not respect this quasi-homogeneity. However, we did not include (4.3) in our system, as we don't mind that a symmetry *changes* the quasi-homogeneity, i.e. the transformed solution is again quasi-homogeneous, but now with different d_i and d_F. For the SL_2-action above another phenomenon occurs. For the special choice $d_n = 0$, and hence $d_F = 2d_1$, see (2.3), quasi-homogeneity is preserved. This case is discussed for $n = 3$ in Appendix C of [10]. Quasi-homogeneity requires $d_2 = \frac{1}{2}$ (if we normalize $d_1 = 1$), and we obtain

$$F(t_1, t_2, t_3) = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 - \frac{1}{16} t_2^4 \gamma(t_3).$$

The associativity equation now turns into

$$\gamma''' - 6\gamma\gamma'' + 9\gamma'^2 = 0 \tag{4.4}$$

which is the Chazy equation for γ, see [9]. The transformation (4.2a) boils down to the well-known $SL_2(\mathbb{C})$-symmetry for γ

$$\gamma \mapsto (ct_3 + d)^2 \gamma + 2c(ct_3 + d).$$

Using the $SL_2(\mathbb{C})$-action, one can find a 2-parameter family of rational solutions to the Chazy equation, starting with the seed solution $\gamma = 1$, see e.g. [11].

5. Chazy-type equations

It turns out that for higher n and special choices of the degrees $d_1, d_2, \dots, d_n$ the system (2.4) reduces to a system of equations, resembling the Chazy equation. We investigated $n = 4$ and $n = 5$. The calculations for $n = 4$ are already very complicated, while $n = 5$ is prohibitive.

5.1. **Chazy-type equations for $n = 4$.** We consider the case $n = 4$ with $d_1 = 1$, $d_4 = 0$ and hence $d_F = 2$, and $d_2 > d_3 > 0$. We will denote $x = t_2, y = t_3$ and $z = t_4$. Note that $d_2 + d_3 = 1$. Consequently, F is polynomial in x and y, at most quartic in x. We put

$$F(t) = \frac{1}{2}t_1^2 t_4 + t_1 t_2 t_3 - \frac{1}{4}x^2 y^2 \gamma(z) + g(x, y, z)$$

where g does not contain the term with x^2y^2. Now look at (2.4) for $\alpha = \beta = 2$ and $\gamma = \delta = 3$. It reads

$$2f_{xyz} = f_{xxx}f_{yyy} - f_{xxy}f_{xyy}.$$

Expressed in γ and g it yields

$$-2xy\gamma' + 2g_{xyz} = g_{xxx}g_{yyy} - (-y\gamma + g_{xxy})(-x\gamma + g_{xyy})$$

We compare the coefficients of xy left and right. To obtain $\gamma' - \frac{1}{2}\gamma^2 \neq 0$ we have only 2 possibilities:

a.) $d_2 = d_3 = \frac{1}{2}$.
b.) $d_2 = \frac{2}{3}$; $d_3 = \frac{1}{3}$.

Let us discuss these 2 cases.

Case a.) In this case

$$g(x, y, z) = g_1(z)y^4 + g_2(z)xy^3 + g_3(z)x^3y + g_4x^4$$

Substitution of this g in (2.4) we obtain that $g_3 = Kg_1$, $g_4 = \frac{K}{4}g_2$ with K a constant, and

$$g_1' = 6g_2^2 + \gamma g_1; \tag{5.1}$$

$$g_2' = 24Kg_1^2 + \gamma g_2; \tag{5.2}$$

$$\gamma' = \frac{1}{2}\gamma^2 - 72Kg_1g_2 \tag{5.3}$$

Following [10], we denote $\nabla g_i = g_i' - \gamma g_i \quad (i = 1, 2)$ and $\nabla\gamma = \gamma' - \frac{1}{2}\gamma^2$ and $\Omega = -72Kg_1g_2$. Then we are led to the equivalent system

$$\nabla g_1 = 6g_2^2; \qquad \nabla g_2 = 24Kg_1^2; \qquad \nabla\gamma = \Omega \tag{5.4}$$

Like Dubrovin, we put

$$\nabla\Omega = \Omega' - 2\gamma\Omega$$

and

$$\nabla^2\Omega = (\nabla\Omega)' - 3\gamma\Omega.$$

From (5.4) it now follows that

$$\nabla^2\Omega + 12\Omega^2 = 0 \tag{5.5}$$

(The easiest way is to calculate first $\nabla(\Omega^3)$). Expressed in γ, using $\Omega = \nabla\gamma$, we exactly obtain the Chazy equation (4.4).

Conversely, suppose γ satisfies (4.4), fix a K and define g_1 and g_2 by

$$\Omega^3 = -(72)^3K^3g_1^3g_2^3 \quad \text{and} \quad \nabla\Omega = -432Kg_2^3 - 1728K^2g_1^2.$$

Then one can check that g_1 and g_2 satisfy (5.1) en (5.2).

Case b.) In case that $d_2 = \frac{2}{3}$ and $d_3 = \frac{1}{3}$ we have

$$g(x, y, z) = g_3(z)y^6 + g_2(z)xy^4 + g_1(z)x^3$$

Substition in (2.4) now gives that $g_2 = Kg_1^3$ and $g_3 = Kg_1(g_1 - \frac{1}{2}\gamma g_1)$, where again K is a constant. Moreover putting $\Omega = -72Kg_1^4$ we find

$$g_1'' - 2\gamma g_1' - \gamma' g_1 = 0 \quad \text{and} \quad \nabla\gamma = \Omega \tag{5.6}$$

Eliminating g_1 from the first equation (using $g_1^4 = -\frac{1}{72K}\nabla\gamma$) we obtain a third order equation for γ that can be expressed elegantly in Ω by

$$(\nabla\Omega)^2 = \frac{4}{3}\Omega\nabla^2\Omega + 8\Omega^3 \tag{5.7}$$

5.2. **Chazy equation for $n = 5$.** The analysis for $n = 5$ is much more difficult; while the system (2.4) is determined by 4 equations in case $n = 4$, for $n = 5$ we have 10 equations, namely equations for f_{ab5} for $a, b \in \{2, 3, 4, 5\}$. We consider the case $d_2 = d_3 = d_4 = \frac{1}{2}$, and again $d_1 = 1$ and $d_5 = 0$. Each of these 10 equations splits into several equations, once the polynomial Ansatz is made; one is left with a system of approximately 100 nonlinear ordinary differential equations for 15 functions in t_5. We were not able to solve the system in general. However we found a solution similar to the cases $n = 3$ and $n = 4$. We denote $z = t_5$. For f we have

$$f = -\frac{1}{4}(t_2t_4 + \frac{1}{2}t_3^2)^2\gamma(z) + \left(t_2^4 - 2t_2t_3^3 - 2t_2t_4^3 + 3t_3^2t_4^2\right) g_1(z)$$
$$-\frac{1}{2}\left(t_4^4 - 2t_3^3t_4 - 2t_2^3t_4 + 3t_2^2t_3^2\right) g_2(z)$$

and the remaining system of equations for g_1, g_2 and γ is exactly the system (5.1), (5.2) and (5.3) with $K = -2$.

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Faculty of Applied Mathematics, University Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands.