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THE NORM OF AN AVERAGING OPERATOR

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ABSTRACT. We consider the operator $A : \ell^2 \to \ell^2$ defined by $A(a) = b$ for $a = (a_n)$ and $b = (b_n)$ with $b_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$.
We prove that A has norm 2.

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1. Introduction

We consider the Hilbert space ℓ^2 of square-summable sequences, i.e. sequences $a = (a_1, a_2, a_3, \dots)$, such that

$$\|a\| = \left\{ \sum_{i=1}^{\infty} |a_i|^2 \right\}^{\frac{1}{2}} < \infty,$$

and corresponding inner product

$$(a, b) = \sum_{i=1}^{\infty} a_i \overline{b_i},$$

for $a = (a_1, a_2, a_3, \dots)$ and $b = (b_1, b_2, b_3, \dots)$ both in ℓ^2.

For a sequence $a = (a_1, a_2, a_3, \dots)$, we can define a new sequence $A(a)$ of averages

$$A(a) = (a_1, \frac{1}{2}(a_1 + a_2), \frac{1}{3}(a_1 + a_2 + a_3), \dots, \frac{1}{n}(a_1 + a_2 + \cdots + a_n), \dots).$$

Clearly, A is a linear operator. We will prove that $A(a)$ is again square-summable, and in fact that for $A : \ell^2 \to \ell^2$, we have

$$\|A\| = \sup_{\|a\| \neq 0} \frac{\|A(a)\|}{\|a\|} = 2.$$

2. A is bounded

We start by proving that A is bounded. Hence we take any $a \in \ell^2$, and consider $\|A(a)\|^2$.

$$\|A(a)\|^2 = \sum_{i=1}^{\infty} |\frac{1}{i} \sum_{j=1}^{i} a_j|^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} (\sum_{j=1}^{i} |a_j|)^2$$

$$(*) \qquad = \sum_{i=1}^{\infty} \frac{1}{i^2} (\sum_{j=1}^{i} \frac{1}{j^{\frac{1}{4}}} \cdot j^{\frac{1}{4}} |a_j|)^2 \leq \sum_{i=1}^{\infty} \frac{1}{i^2} \left(\sum_{j=1}^{i} \frac{1}{j^{\frac{1}{2}}} \cdot \sum_{j=1}^{i} j^{\frac{1}{2}} |a_j|^2 \right)$$

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In the last step we used the Cauchy-Schwarz inequality. Now we estimate the middle term.

$$\sum_{j=1}^{i} \frac{1}{j^{\frac{1}{2}}} \leq 1 + \int_1^i \frac{dx}{x^{\frac{1}{2}}} = 2i^{\frac{1}{2}} - 1.$$

This result we substitute in (*) and we interchange the order of summation:

$$\begin{aligned} \|A(a)\|^2 &\leq \sum_{i=1}^{\infty} \frac{1}{i^2}(2i^{\frac{1}{2}} - 1) \sum_{j=1}^{i} j^{\frac{1}{2}} |a_j|^2 \\ &= \sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} (\frac{2}{i^{\frac{3}{2}}} - \frac{1}{i^2}) \right) j^{\frac{1}{2}} |a_j|^2 \end{aligned} \tag{**}$$

We recall the Riemann zeta-function $\zeta(s)$ for $s > 1$ defined by

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s}.$$

For $j = 1$ we have that

$$\sum_{i=1}^{\infty} \frac{2}{i^{\frac{3}{2}}} - \frac{1}{i^2} = 2\zeta(\frac{3}{2}) - \zeta(2) \approx 3.58 < \frac{4}{\sqrt{1}},$$

for $j = 2$:

$$\sum_{i=2}^{\infty} \frac{2}{i^{\frac{3}{2}}} - \frac{1}{i^2} = 2\zeta(\frac{3}{2}) - \zeta(2) - 1 \approx 2.58 < \frac{4}{\sqrt{2}},$$

while for $j = 3$,

$$\sum_{i=3}^{\infty} \frac{2}{i^{\frac{3}{2}}} - \frac{1}{i^2} = 2\zeta(\frac{3}{2}) - \zeta(2) - \frac{3}{4} - \frac{1}{\sqrt{2}} \approx 2.12 < \frac{4}{\sqrt{3}}.$$

Moreover for $j \geq 4$:

$$\begin{aligned} \sum_{i=j}^{\infty} \frac{2}{i^{\frac{3}{2}}} - \frac{1}{i^2} &\leq \frac{2}{j^{\frac{3}{2}}} + \int_j^{\infty} \frac{2dx}{x^{\frac{3}{2}}} - \int_j^{\infty} \frac{dx}{x^2} \\ &= \frac{2}{j^{\frac{3}{2}}} + \frac{4}{j^{\frac{1}{2}}} - \frac{1}{j} \leq \frac{4}{j^{\frac{1}{2}}}, \end{aligned}$$

where we used in the last step that $\frac{2}{j\sqrt{j}} - \frac{1}{j} \leq 0$, due to $j \geq 4$. Substituting these results in (**), we obtain

$$\|A(a)\|^2 \leq \sum_{j=1}^{\infty} \frac{4}{j^{\frac{1}{2}}} j^{\frac{1}{2}} |a_j|^2 = 4\|a\|^2.$$

It follows that $\|A\| \leq 2$.

3. A HAS NORM 2

It remains to show that $\|A\| \geq 2$. For this we consider the sequence $a = (a_j)$ with $a_j = \frac{1}{j^\alpha}$, where $\alpha > \frac{1}{2}$. Then

$$\|A(a)\|^2 = \sum_{i=1}^{\infty} \frac{1}{i^2} (\sum_{j=1}^{i} \frac{1}{j^\alpha})^2. \tag{***}$$

Now

$$\sum_{j=1}^{i} \frac{1}{j^\alpha} > \int_1^i \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}(i^{1-\alpha} - 1)$$

If we substitute this estimate in (***), we obtain

$$\begin{aligned}\|A(a)\|^2 &> \sum_{i=1}^{\infty} \frac{1}{i^2} \left(\frac{1}{1-\alpha}(i^{1-\alpha} - 1)\right)^2 = \frac{1}{(1-\alpha)^2} \sum_{i=1}^{\infty} \frac{i^{2-2\alpha} - 2i^{1-\alpha} + 1}{i^2} \\ &> \frac{1}{(1-\alpha)^2} \sum_{i=1}^{\infty} (\frac{1}{i^{2\alpha}} - \frac{2}{i^{1+\alpha}}) > \frac{1}{(1-\alpha)^2} \sum_{i=1}^{\infty} (|a_i|^2 - \frac{2}{i^{\frac{3}{2}}}) \\ &= \frac{1}{(1-\alpha)^2} (\|a\|^2 - 2\zeta(\frac{3}{2})).\end{aligned}$$

Consequently

$$\frac{\|A(a)\|^2}{\|a\|^2} > \frac{1}{(1-\alpha)^2} - \frac{2\zeta(\frac{3}{2})}{\|a\|^2(1-\alpha^2)}.$$

Now let $\alpha \downarrow \frac{1}{2}$, so that $\|a\| \to \infty$. Therefore

$$\frac{\|A(a)\|^2}{\|a\|^2} \to \frac{1}{(1-\frac{1}{2})^2} = 4.$$

Hence we obtain that $\|A\|^2 \geq 4$.

Remembering that we already had that $\|A\| \leq 2$, we obtain that $\|A\| = 2$, as desired.

4. REMARK

We can consider the same problem for $A : \ell^p \to \ell^p$, so now with norm

$$\|a\| = \left\{ \sum_{i=1}^{\infty} |a_i|^p \right\}^{\frac{1}{p}},$$

for $1 \leq p < \infty$ and

$$\|a\| = \sup_i |a_i|$$

for $p = \infty$. Some preliminary calculations suggest that $\|A\| = q$ in this case, where q is given by $\frac{1}{p} + \frac{1}{q} = 1$. Only for the cases $p = 1$ and $p = \infty$ (and $p = 2$, the situation above), we are able to give a full proof.

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