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# On stability robustness with respect to LTV uncertainties

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# On stability robustness with respect to LTV uncertainties

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**Abstract**

It is shown that the well-known $(D, G)$-scaling upper bound of the structured singular value is a nonconservative test for robust stability with respect to certain linear time-varying uncertainties.

**Keywords**

Mixed structured singular values, Duality, Linear matrix inequalities, Time-varying systems, Robustness, IQC.

**1991 Mathematics Subject Classification:** 93B36, 93C50, 93D09, 93D25.

# 1 Introduction

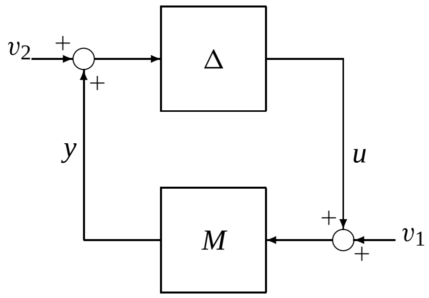


Figure 1: The closed loop.

Is the above closed loop stable for all $\Delta$'s in a given set of stable operators $B$? That, roughly, is the fundamental robust stability problem.

There is an intriguing result by Megretski and Treil [4] and Shamma [8] which says, loosely speaking, that if $M$ is a stable LTI operator and the set of $\Delta$'s is the set of contractive linear time-varying operators of some fixed block diagonal structure

$$\Delta = \text{diag}\,(\Delta_1, \Delta_2, \ldots, \Delta_{m_F}), \tag{1}$$

that then the closed loop is robustly stable—that is, stable for all such $\Delta$'s—if and only if the $H_\infty$-norm of $DMD^{-1}$ is less than one for some constant diagonal matrix $D$ that commutes with the $\Delta$'s. The problem can be decided in polynomial time, and it is a problem that has since long been associated with an *upper bound* of the structured singular value. The intriguing part is that the result holds for any number of LTV blocks $\Delta_i$, which is in stark contrast with the case that the $\Delta_i$'s are assumed time-invariant.

Paganini [6] extended this result by allowing for the more general block diagonal structure

$$\Delta = \text{diag}\,(\delta_1 I_{n_1}, \ldots, \delta_{m_c} I_{n_{m_c}}, \Delta_1, \ldots, \Delta_{m_F}). \tag{2}$$

A precise definition is given in Section 2. Paganini's result is an exact generalization and leads, again, to a convex optimization problem over the constant matrices $D$ that commute with $\Delta$.

In view of the connection of these results with the upper bounds of the structured singular it is natural to ask if the well known $(D, G)$-scaling upper bound of the *mixed* structured singular value also has a similar interpretation. In this note we show that that is indeed the case.

The $(D, G)$-scaling upper bound of the structured singular value was originally defined as a means to provide an easy-to-verify condition that guarantees robust stability with respect to the contractive linear *time-invariant* operators $\Delta$ of the form

$$\Delta = \text{diag}\,(\tilde{\delta}_1 I_{\tilde{n}_1}, \ldots, \tilde{\delta}_{m_r} I_{\tilde{n}_{m_r}}, \delta_1 I_{n_1}, \ldots, \delta_{m_c} I_{n_{m_c}}, \Delta_1, \ldots, \Delta_{m_F}), \tag{3}$$

with $\tilde{\delta}_i$ denoting real-valued constants [1]. It is known that for general LTI plants $M$ this sufficient condition is *necessary* as well if and only if,

$$2(m_r + m_c) + m_F \leq 3.$$

(See [5].) In this note we show that the $(D, G)$-scaling condition is in fact both necessary and sufficient for robust stability with respect to the contractive LTV operators $\Delta$ of the form (3) with now $\tilde{\delta}_i$ denoting linear time-varying *self-adjoint* operators on $\ell_2$. A precise definition follows. Paganini [7] has gone through considerable trouble to show that for his structure (2) one may assume causality of $\Delta$ without changing the condition. In the extended structure (3) with self-adjoint $\tilde{\delta}_i$ this is no longer possible.

## 2 Notation and preliminaries

$\ell_2 := \{x : \mathbb{Z} \mapsto \mathbb{R} : \sum_{k\in\mathbb{Z}} x^2(k) < \infty\}$. The norm $\|v\|_2$ of $v \in \ell_2$ is the usual norm on $\ell_2$ and for vector-valued signals $v \in \ell_2^n$ the norm $\|v\|_2$ is defined as $(\|v_1\|_2^2 + \cdots + \|v_n\|_2^2)^{1/2}$. The induced norm is denoted by $\|\cdot\|$. So, for $F : \ell_2^n \mapsto \ell_2^n$ it is defined as $\|F\| := \sup_{u\in\ell_2^n} \|Fu\|_2/\|u\|_2$. For matrices $F \in \mathbb{C}^{n\times m}$ the induced norm will be the spectral norm, and for vectors this reduces to the Euclidean norm.

$F^{\mathrm{H}}$ is the complex conjugate transpose of $F$, and $\operatorname{He} F$ is the Hermitian part $F$ defined as $\operatorname{He} F = \frac{1}{2}(F + F^{\mathrm{H}})$.

An operator $\Delta : \ell_2^n \mapsto \ell_2^n$ is said to be *contractive* if $\|\Delta v\|_2 \leq \|v\|_2$ for every $v \in \ell_2^n$. Lower case $\delta$'s always denote operators from $\ell_2^1$ to $\ell_2^1$. Then for $u, y \in \ell_2^n$ the expression $y = \delta I_n u$ is defined to mean that the entries $y_k$ of $y$ satisfy $y_k = \delta u_k$. An operator $\delta : \ell_2 \mapsto \ell_2$ is *self-adjoint* if $\langle u, \delta v\rangle = \langle \delta u, v\rangle$ for all $u, v \in \ell_2$.

The $M$ and $\Delta$ throughout denote bounded operators from $\ell_2^n$ to $\ell_2^n$ and $M$ is assumed linear time invariant (LTI). Bounded operators on $\ell_2^n$ are also called *stable*.

Hats will denote $Z$-transforms, so if $y \in \ell_2$ then $\hat{y}(z)$ is defined as $\hat{y}(z) = \sum_{k\in\mathbb{Z}} y(k)z^{-k}$. To avoid clutter we shall use for functions $\hat{f}$ of frequency the notation

$$\hat{f}_\omega := \hat{f}(e^{i\omega}).$$

### 2.1 Stability

The closed loop depicted in Fig. 1 is considered *internally stable* if the map from $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ is bounded as a map from $l_2^{2n}$ to $l_2^{2n}$. Because of stability of $M$ and $\Delta$ the closed loop is internally stable iff $(I - \Delta M)^{-1}$ is bounded. The closed loop will be called *uniformly robustly stable* with respect to some set $B$ of stable LTV operators if there is an $\gamma > 0$ such that

$$\left\| \begin{bmatrix} u \\ y \end{bmatrix} \right\|_2 \leq \gamma \left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\|_2 \quad \forall \Delta \in B, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \ell_2^{2n}. \tag{4}$$

We only consider $\Delta$'s with norm at most one and stable $M$. In that case (4) holds if and only if there is an $\epsilon > 0$ such that

$$\|(I - \Delta M)u\|_2 \geq \epsilon \|u\|_2 \quad \forall \Delta \in B, \, u \in \ell_2^n.$$

### 2.2 The $\Delta$'s and the $(D, G)$-scaling matrices

Throughout we assume that $\Delta : \ell_2^n \mapsto \ell_2^n$ and that $\Delta$ is of the form

$$\Delta = \operatorname{diag}(\tilde{\delta}_1 I_{\tilde{n}_1}, \ldots, \tilde{\delta}_{m_r} I_{\tilde{n}_{m_r}}, \delta_1 I_{n_1}, \ldots, \delta_{m_c} I_{n_{m_c}}, \Delta_1, \ldots, \Delta_{m_F}) \tag{5}$$

with

$$\begin{cases} \tilde\delta_i & : \quad \ell_2 \mapsto \ell_2 \quad \text{LTV, self-adjoint and } \|\tilde\delta_i\| \le 1, \\ \delta_i & : \quad \ell_2 \mapsto \ell_2 \quad \text{LTV and } \|\delta_i\| \le 1, \\ \Delta_i & : \quad \ell_2^{q_i} \mapsto \ell_2^{q_i} \quad \text{LTV and } \|\Delta_i\| \le 1. \end{cases} \tag{6}$$

The dimensions and numbers $\tilde n_i, n_i, q_i, m_r, m_c, m_F$ of the various identity matrices and $\Delta_i$ blocks are fixed, but otherwise $\Delta$ may vary over all possible $n \times n$ LTV operators of the form (5), (6). Given that, the sets $\mathcal{D}$ and $\mathcal{G}$ of $D$ and $G$-scales are defined accordingly as

$$\begin{aligned} \mathcal{D} &= \{D = \operatorname{diag}(\tilde D_1, \dots, \tilde D_{m_r},\ D_1, \dots, D_{m_c},\ d_1 I_{q_1}, \dots, d_{m_F} I_{q_{m_F}}) \\ &\qquad : 0 < D = D^{\mathrm T} \in \mathbb{R}^{n\times n}\}, \\ \mathcal{G} &= \{G = \operatorname{diag}(\tilde G_1, \dots, \tilde G_{m_r}, 0, \dots, 0, 0, \dots, 0) \\ &\qquad : G = G^{\mathrm H} \in j\mathbb{R}^{n\times n}\}. \end{aligned}$$

Note that the $D$-scales are assumed real-valued and that the $G$-scales are taken to be purely imaginary. As it turns out there is no need to consider a wider class of $D$ and $G$-scales.

# 3 The discrete-time result

**Theorem 3.1.** *The discrete-time closed-loop in Fig. 1 with stable LTI plant with transfer matrix $M$ is uniformly robustly stable with respect to $\Delta$'s of the form (5, 6) if and only if there is a constant matrix $D \in \mathcal{D}$ and a constant matrix $G \in \mathcal{G}$ such that*

$$M_\omega^{\mathrm H} D M_\omega + j(G M_\omega - M_\omega^{\mathrm H} G) - D < 0 \quad \forall \omega \in [0, 2\pi]. \tag{7}$$

□

The existence of such $D$ and $G$ can be tested in polynomial time. The remainder of this paper is devoted to a proof of this result. Megretski [3] showed this for the full blocks case (1), Paganini [6] derived this result for the case that the $\Delta$'s are of the form (2). The proof of the general case (5) follows the same lines as that of [6] and [5]. A key idea is to replace the condition of the contractive $\Delta$-blocks with an integral quadratic condition independent of $\Delta$:

**Lemma 3.2.** *Let $u, y \in \ell_2^q$ and consider the quadratic integral*

$$\Sigma(u, y) := \int_0^{2\pi} (\hat y_\omega - \hat u_\omega)(\hat y_\omega + \hat u_\omega)^{\mathrm H}\, d\omega \ \in\ \mathbb{R}^{q\times q}. \tag{8}$$

*The following holds.*

1. *There is a contractive self-adjoint LTV $\tilde\delta : \ell_2 \mapsto \ell_2$ such that $u = \tilde\delta I_q y$ if and only if $\Sigma(u, y)$ is Hermitian and nonnegative definite.*

2. *There is a contractive LTV $\delta : \ell_2 \mapsto \ell_2$ such that $u = \delta I_q y$ if and only if the Hermitian part of $\Sigma(u, y)$ is nonnegative definite.*

3. *There is a contractive LTV $\Delta : \ell_2^q \mapsto \ell_2^q$ such that $u = \Delta y$ if and only if the trace of $\Sigma(u, y)$ is nonnegative.*

*Proof.* See appendix. ∎

A consequence of this result is the following.

**Lemma 3.3.** *Let $u$ be a nonzero element of $\ell_2^n$. Then $(I - \Delta M)u = 0$ for some $\Delta$ of the form (5, 6) if-and-only-if*

$$\Sigma(u, Mu) := \int_0^{2\pi} (M_\omega - I)\hat{u}_\omega \hat{u}_\omega^{\mathrm{H}} (M_\omega + I)^{\mathrm{H}} \, d\omega \tag{9}$$

*is of the form*

$$\left[\begin{array}{ccc|ccc|ccc} \tilde{Z}_1 & ? & ? & ? & ? & ? & ? & ? & ? \\ ? & \tilde{Z}_2 & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & \ddots & ? & ? & ? & ? & ? & ? \\ \hline ? & ? & ? & \bar{Z}_1 & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & \bar{Z}_2 & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & \ddots & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? & Z_1 & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & Z_2 & ? \\ ? & ? & ? & ? & ? & ? & ? & ? & \ddots \end{array}\right] \in \mathbb{R}^{n \times n}, \tag{10}$$

*with* $\tilde{Z}_i = \tilde{Z}_i^{\mathrm{T}} \geq 0$, $\operatorname{He} \bar{Z}_i \geq 0$, $\operatorname{Tr} Z_i \geq 0$, *and with "?" denoting an irrelevant entry. Here the partitioning of (10) is compatible with that of* $\Delta$.

*Proof (sketch).* The equation $(I - \Delta M)u = 0$ is the same as

$$u = \Delta M u.$$

With appropriate partitionings, the expression $u = \Delta M u$ can be written row-block by row-block as

$$\begin{array}{ccl} u_1 & = & \tilde{\delta}_1 M_1 u \\ u_2 & = & \tilde{\delta}_2 M_2 u \\ \vdots & \vdots & \vdots \\ u_K & = & \Delta_{m_F} M_K u. \end{array}$$

By Lemma 3.2 there exist contractive $\tilde{\delta}_i$, $\delta_i$ and $\Delta_i$ of the form (6) for which the above equalities hold iff certain quadratic integrals $\Sigma_i$ have certain properties. It is not to difficult to figure out that these quadratic integrals $\Sigma_i$ are exactly the blocks on the diagonal of $\Sigma(u, Mu)$, and that the conditions on these blocks are that they satisfy $\Sigma_i = \Sigma_i^{\mathrm{T}} \geq 0$, $\operatorname{He} \Sigma_i \geq 0$, or $\operatorname{Tr} \Sigma_i \geq 0$, corresponding to the three types of uncertainties. ∎

*Proof of Theorem 3.1.* Suppose such $D \in \mathcal{D}$ and $G \in \mathcal{G}$ exist. Then a standard argument will show that there is an $\epsilon > 0$ such that $\|(I - \Delta M)u\|_2 \geq \epsilon \|u\|_2$ for all $u$ and contractive $\Delta$ of the form (5). This is the definition of uniformly robustly stable.

Conversely suppose the closed loop is uniformly robustly stable. For some $\epsilon > 0$, then, $\|(I - \Delta M)u\|_2 \geq \epsilon$ for every $u$ of unit norm. Define

$$\mathcal{W} := \{\, \Sigma(u, Mu) \;:\; \|u\|_2 = 1\} \subset \mathbb{R}^{n\times n}. \tag{11}$$

By application of Lemma 3.3, the set $\mathcal{W}$ does not intersect the convex cone $\mathcal{Z}$ defined as

$$\mathcal{Z} := \{Z \;:\; Z \text{ is of the form (10) with } \tilde{Z}_i = \tilde{Z}_i^{\mathrm{T}} \geq 0, \text{He}\, \bar{Z}_i \geq 0, \text{Tr}\, Z_i \geq 0\,\}.$$

In the appendix we show that in fact $\mathcal{W}$ is bounded away from $\mathcal{Z}$. Remarkably the closure $\overline{\mathcal{W}}$ of $\mathcal{W}$ is convex. This observation is from Megretski & Treil [4], and for completeness a proof is listed in the appendix, Lemma 5.1. Because $\mathcal{W}$ is bounded away from $\mathcal{Z}$, also the closure $\overline{\mathcal{W}}$ is bounded away from $\mathcal{Z}$, so there is a $\gamma > 0$ such that $\overline{\mathcal{W}}$ also does not intersect

$$\mathcal{Z}_\gamma := \mathcal{Z} + \{Z \in \mathbb{R}^{n\times n} \;:\; \|Z\| \leq \gamma\}.$$

Both $\overline{\mathcal{W}}$ and $\mathcal{Z}_\gamma$ are convex and have empty intersection, and therefore a hyper-plane exists that separates the two sets [2, p.133]. In other words there is a nonzero matrix $E \in \mathbb{R}^{n\times n}$ (say of unit norm) such that[1]

$$\langle E, \overline{\mathcal{W}}\rangle \leq \langle E, \mathcal{Z}_\gamma\rangle. \tag{12}$$

As inner product take $\langle X, Y\rangle = \text{Tr}\, X^{\mathrm{T}} Y$. In particular (12) says that $\langle E, \mathcal{Z}\rangle$ is bounded from below. By Lemma 5.3 that is the case if and only if $E$ is of the form

$$E = \text{diag}\,(\tilde{E}_1, \ldots, \tilde{E}_{m_r},\; E_1, \ldots, E_{m_c},\; e_1 I, \ldots, e_{m_F} I)$$

with $\tilde{E}_i + \tilde{E}_i^{\mathrm{T}} \geq 0$, $E_i = E_i^{\mathrm{T}} \geq 0$ and $0 \leq e_i \in \mathbb{R}$, that is, if and only if $E \in \overline{\mathcal{D} + j\mathcal{G}}$. In that case $\inf\langle E, \mathcal{Z}\rangle = 0$, and so

$$a_\gamma := \inf\langle E, \mathcal{Z}_\gamma\rangle < 0.$$

From (12) we thus see that $\langle E, \overline{\mathcal{W}}\rangle \leq a_\gamma < 0$. If $\|u\|_2 = 1$, then

$$\begin{aligned} \int_0^{2\pi} & \hat{u}_\omega^{\mathrm{H}} \big(\text{He}\,(M_\omega + I)^{\mathrm{H}} E (M_\omega - I)\big) \hat{u}_\omega \, d\omega \\ &= \text{Re}\,\text{Tr} \int_0^{2\pi} E(M_\omega - I)\hat{u}_\omega \hat{u}_\omega^{\mathrm{H}} (M_\omega + I)^{\mathrm{H}} \, d\omega \\ &= \langle E, \Sigma(u, Mu)\rangle \leq \sup\langle E, \mathcal{W}\rangle \leq a_\gamma < 0. \end{aligned} \tag{13}$$

This being at most $a_\gamma < 0$ for every $u \in \ell_2^n$, $\|u\|_2 = 1$ implies that

$$\text{He}\,(M_\omega + I)^{\mathrm{H}} (E + \epsilon I)(M_\omega - I) < 0 \quad \forall \omega \in [0, 2\pi], \tag{14}$$

for some small enough $\epsilon > 0$. Express $E + \epsilon I$ as $E + \epsilon I = D + jG$ for some $D \in \mathcal{D}$ and $G \in \mathcal{G}$. Then Equation (14) becomes (7). ∎

[1] In (12) the expression $\langle E, \overline{\mathcal{W}}\rangle$ denotes the set $\{x : x = \langle E, Y\rangle, Y \in \overline{\mathcal{W}}\}$ and the inequality in (12) is defined to mean that every element of the set on the left-hand side, $\langle E, \overline{\mathcal{W}}\rangle$, is less than or equal to every element of the set on the right-hand side, $\langle E, \mathcal{Z}_\gamma\rangle$.

## 4 The continuous-time result

Analagous to the discrete-time case we say that a continuous-time system is *uniformly robustly stable* if there is a $\gamma > 0$ such that (4) holds for all $v_1, v_2 \in L_2$. Completely analagous to the discrete-time case it can be shown that:

**Theorem 4.1.** *The continuous-time closed-loop in Fig. 1 with stable LTI plant with transfer matrix $M$ is uniformly robustly stable with respect to $\Delta$'s of the form (5) with*

$$\begin{cases} \tilde{\delta}_i & : \quad L_2 \mapsto L_2 \quad \text{LTV, self-adjoint and } \|\tilde{\delta}_i\| \leq 1, \\ \delta_i & : \quad L_2 \mapsto L_2 \quad \text{LTV and } \|\delta_i\| \leq 1, \\ \Delta_i & : \quad L_2^{q_i} \mapsto L_2^{q_i} \quad \text{LTV and } \|\Delta_i\| \leq 1. \end{cases}$$

*if and only if there is a constant matrix $D \in \mathcal{D}$ and a constant matrix $G \in \mathcal{G}$ such that*

$$M(j\omega)^{\mathrm{H}} D M(j\omega) + j(GM(j\omega) - M(j\omega)^{\mathrm{H}} G) - D < 0$$

*for all $\omega \in \mathbb{R} \cup \infty$.* □

## 5 Appendix

*Proof of Lemma 3.2.* Items 2 and 3 are proved in [6] (note that the Hermitian part of (8) is $\int_0^{2\pi} \hat{y}_\omega \hat{y}_\omega^{\mathrm{H}} - \hat{u}_\omega \hat{u}_\omega^{\mathrm{H}} \, d\omega$, and its trace equals $2\pi(\|y\|_2^2 - \|u\|_2^2)$).

If $u := \tilde{\delta} I_q y$ with $\tilde{\delta}$ self-adjoint and contractive then (8) is easily seen to be Hermitian and $\geq 0$. Conversely suppose (8) is Hermitian and nonnegative. Now let $\{f_i\}_{i=0,1,2,\ldots}$ be an orthonormal basis of $\ell_2$, and expand $y \in \ell_2^q$ in this basis:

$$y = \sum_{j=0,1,\ldots} \gamma(j) f_j, \qquad \gamma(j) \in \mathbb{R}^q.$$

We may associate with this expansion the matrix $Y \in \mathbb{R}^{\infty \times q}$ of coefficients

$$Y = \begin{bmatrix} \gamma_1(0) & \gamma_2(0) & \cdots & \gamma_q(0) \\ \gamma_1(1) & \gamma_2(1) & \cdots & \gamma_q(1) \\ \gamma_1(2) & \gamma_2(2) & \cdots & \gamma_q(2) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The matrix $U$ is likewise defined from $u$. In this matrix notation the expression $u = \tilde{\delta} I_q y$ becomes $U = \tilde{\Delta} Y$, and the quadratic integral (8) becomes

$$\Sigma(u, y) = (Y^{\mathrm{T}} - U^{\mathrm{T}})(Y + U).$$

By assumption the above is Hermitian and nonnegative definite, that is,

$$Y^{\mathrm{T}} U = U^{\mathrm{T}} Y \quad \text{and} \quad U^{\mathrm{T}} U \leq Y^{\mathrm{T}} Y. \tag{15}$$

We may assume without loss of generality that the orthonormal basis $\{f_j\}$ was chosen such that the first, say $p$, elements $\{f_1, \ldots, f_p\}$ span the space spanned by the entries $\{y_1, \ldots, y_q\}$ of $y$. Then $Y$ is of the form

$$Y = \begin{bmatrix} I_p \\ 0_{\infty \times p} \end{bmatrix} C \quad \text{for some full row rank } C \in \mathbb{R}^{p \times q}.$$

Then the second inequality of (15) is that $U^{\mathrm{T}}U \le C^{\mathrm{T}}C$. This implies that $U$ is of the form $U = VC$ for some $V$. Partition $V$ as $\left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right]$ with $V_1 \in \mathbb{R}^{p\times p}$. The two formulas of (15) then become

$$C^{\mathrm{T}}V_1C = C^{\mathrm{T}}V_1^{\mathrm{T}}C \quad \text{and} \quad C^{\mathrm{T}}(V_1^{\mathrm{T}}V_1 + V_2^{\mathrm{T}}V_2)C \le C^{\mathrm{T}}I_pC. \tag{16}$$

As $C$ has full row rank, (16) is equivalent to that

$$V_1 = V_1^{\mathrm{T}} \quad \text{and} \quad V_1^{\mathrm{T}}V_1 + V_2^{\mathrm{T}}V_2 \le I_p.$$

It is now immediate that $U$ equals $U = \tilde{\Delta}Y$ for $\tilde{\Delta}$ defined as

$$\tilde{\Delta} := \begin{bmatrix} V_1 & V_2^{\mathrm{T}} \\ V_2 & -V_2V_1(I - V_1^2)^{-1}V_2^{\mathrm{T}} \end{bmatrix}. \tag{17}$$

It is easy to verify that $\tilde{\Delta}$ is contractive. Furthermore $\tilde{\Delta}$ is symmetric and so the corresponding operator $\tilde{\delta}$ is self-adjoint.

(It may happen that $I - V_1^2$ is singular. In that case the inverse in (17) may be replaced with the Moore-Penrose inverse.) ■

**Lemma 5.1.** *The closure of (11) is convex.*

*Proof.* The proof hinges on the fact that $\lim_{N\to\infty}\langle u, \sigma^N v\rangle = 0$ for every pair $u, v \in \ell_2^n$ and with $\sigma^N$ denoting the $N$-step delay.

Let $u, v \in \ell_2^n$ both have unit norm, i.e., $\Sigma(u, Mu), \Sigma(v, Mv) \in \mathcal{W}$. Given $N \in \mathbb{N}$ and $\lambda \in [0, 1]$ define $x$ as

$$x := \sqrt{\lambda}u + \sqrt{1-\lambda}\sigma^N v.$$

Since $\Sigma$ is linear in its two arguments, we have that

$$\begin{aligned}\Sigma(x, Mx) =& \lambda\Sigma(u, Mu) + \sqrt{1-\lambda}\sqrt{\lambda}\Sigma(u, M\sigma^N v) \\ &+ \sqrt{1-\lambda}\sqrt{\lambda}\Sigma(\sigma^N v, Mu) + (1-\lambda)\Sigma(v, Mv).\end{aligned}$$

As $N \to \infty$ the contributions of $\Sigma(u, M\sigma^N v)$ and $\Sigma(\sigma^N v, Mu)$ tend to zero, so

$$\lim_{N\to\infty} \Sigma(x, Mx) = \lambda\Sigma(u, Mu) + (1-\lambda)\Sigma(v, Mv).$$

That this is an element of the closure of (11) follows from the fact that. $\lim_{N\to\infty}\|x\|_2^2 = \lambda\|u\|_2^2 + (1-\lambda)\|v\|_2^2 = 1$. ■

**Lemma 5.2.** *Uniform robust stability implies that $\mathcal{W}$ is bounded away from $\mathcal{Z}$.*

*Proof.* Suppose to the contrary that

$$\inf_{u\in\ell_2^n, \|u\|_2=1, Z\in\mathcal{Z}} \|\Sigma(u, Mu) - Z\| = 0.$$

This means that there is a sequence $\{u^k, Q_k\}_{k\in\mathbb{N}} \subset \ell_2^n \times \mathbb{R}^{n\times n}$ such that

$$\Sigma(u^k, Mu^k) + Q_k \in \mathcal{Z}, \quad \|u^k\|_2 = 1, \quad \lim_{k\to\infty}\|Q_k\| = 0.$$

For each $k$ define $y^k := Mu^k \in \ell_2^n$ and take $z^k$ to be any element of $\ell_2^n$ whose entries are mutually orthogonal and have unit norm, $\langle z_i^k, z_j^k \rangle = \delta_{ij}$, and whose entries are also orthogonal to all entries of $u^k$ and $y^k$. With it define

$$\begin{aligned}\bar{u}^k : &= u^k + \frac{1}{2}(\sqrt{\|Q_k\|}I_n - \frac{1}{\sqrt{\|Q_k\|}}Q_k)z^k,\\ \bar{y}^k : &= y^k + \frac{1}{2}(\sqrt{\|Q_k\|}I_n + \frac{1}{\sqrt{\|Q_k\|}}Q_k)z^k.\end{aligned}$$

The reason for this definition is that now

$$\begin{aligned}\Sigma(\bar{u}^k, \bar{y}^k) &= \int_0^{2\pi} (\hat{y}_\omega^k - \hat{u}_\omega^k + \frac{1}{\sqrt{\|Q_k\|}}Q_k\hat{z}_\omega^k)\ (\hat{y}_\omega^k + \hat{u}_\omega^k + \sqrt{\|Q_k\|}\hat{z}_\omega^k)^{\mathrm{H}}\, d\omega\\ &= \Sigma(u^k, y^k) + Q_k \in \mathcal{Z}.\end{aligned}$$

So we see that $\Sigma(\bar{u}^k, \bar{y}^k)$ is an element of $\mathcal{Z}$ and, hence, $\bar{u}^k = \Delta^k\bar{y}^k$ for some contractive $\Delta^k$ of the form (5,6). Finally consider

$$\begin{aligned}(I - \Delta^k M)\bar{u}^k &= \bar{u}^k - \Delta^k M(u^k + (\bar{u}^k - u^k))\\ &= \bar{u}^k - \Delta^k(y^k + M(\bar{u}^k - u^k))\\ &= \bar{u}^k - \Delta^k(\bar{y}^k + (y^k - \bar{y}^k)) - \Delta^k M(\bar{u}^k - u^k)\\ &= -\Delta^k(y^k - \bar{y}^k) - \Delta^k M(\bar{u}^k - u^k).\end{aligned} \tag{18}$$

Using the fact that $\|\bar{u}^k - u^k\|_2 = O(\sqrt{\|Q_k\|})$, $\|\bar{y}^k - y^k\|_2 = O(\sqrt{\|Q_k\|})$ and that $\lim_{k\to\infty}\|Q_k\| = 0$, we obtain from (18) that

$$\lim_{k\to\infty}(I - \Delta^k M)\bar{u}^k = 0, \quad \lim_{k\to\infty}\|\bar{u}^k\|_2 = 1.$$

This contradicts uniform robust stability. ■

**Lemma 5.3.** $\inf_{Z\in\mathcal{Z}} \operatorname{Tr} E^{\mathrm{T}} Z$ *is bounded from below for some* $E \in \mathbb{R}^{n\times n}$ *if and only if* $E$ *is of the form*

$$E = diag\,(\tilde{E}_1, \dots, \tilde{E}_{m_r},\ E_1, \dots, E_{m_c},\ e_1 I, \dots, e_{m_F} I)$$

*with* $\tilde{E}_i + \tilde{E}_i^{\mathrm{T}} \geq 0$, $E_i = E_i^{\mathrm{T}} \geq 0$ *and* $e_i \geq 0$.

*Proof.* Suppose that $\inf_{Z\in\mathcal{Z}} \operatorname{Tr} E^{\mathrm{T}} Z$ is bounded from below. The off-diagonal blocks of $E$ are then zero for the following reason: Let $F$ be equal to $E$ but with its blocks on the diagonal equal to zero. The off-diagonal blocks of $Z \in \mathcal{Z}$ are not restricted in any way so $Z := \lambda F$ is an element of $\mathcal{Z}$ for every $\lambda \in \mathbb{R}$. If $F$ is nonzero then $\operatorname{Tr} E^{\mathrm{T}} Z = \operatorname{Tr} E^{\mathrm{T}}(\lambda F) = \lambda \operatorname{Tr} F^{\mathrm{T}} F$ and this is unbounded from below as a function of $\lambda$. Therefore $F$ must be zero, i.e., $E$ is block-diagonal.

The general form of a block-diagonal $E$ is

$$E = \operatorname{diag}\,(\tilde{E}_1, \dots, \tilde{E}_{m_r},\ E_1, \dots, E_{m_c},\ \bar{E}_1, \dots, \bar{E}_{m_F})$$

Express $Z$ as in (10). Then

$$\operatorname{Tr} E^{\mathrm{T}} Z = \sum \operatorname{Tr} \tilde{E}_i^{\mathrm{T}} \tilde{Z}_i + \sum \operatorname{Tr} E_i^{\mathrm{T}} \bar{Z}_i + \sum \operatorname{Tr} \bar{E}_i^{\mathrm{T}} Z_i.$$

Each block of $Z \in \mathcal{Z}$ can vary independently of all other blocks of $Z$, so the only way that the above is bounded from below is that all

$$\inf_{\tilde{Z}_i=\tilde{Z}_i^{\mathrm{T}}\geq 0} \operatorname{Tr} \tilde{E}_i^{\mathrm{T}} \tilde{Z}_i, \qquad \inf_{\operatorname{He} \bar{Z}_i \geq 0} \operatorname{Tr} E_i^{\mathrm{T}} \bar{Z}_i \quad \text{and} \quad \inf_{\operatorname{Tr} Z_i \geq 0} \operatorname{Tr} \bar{E}_i^{\mathrm{T}} Z_i$$

are bounded from below. It is fairly easy to show that

$$\begin{aligned} \inf_{\tilde{Z}_i=\tilde{Z}_i^{\mathrm{T}}\geq 0} \operatorname{Tr} \tilde{E}_i^{\mathrm{T}} \tilde{Z}_i > -\infty \quad &\Leftrightarrow \quad \operatorname{He} \tilde{E}_i \geq 0 \\ \inf_{\operatorname{He} \bar{Z}_i \geq 0} \operatorname{Tr} E_i^{\mathrm{T}} \bar{Z}_i > -\infty \quad &\Leftrightarrow \quad E_i = E_i^{\mathrm{T}} \geq 0 \\ \inf_{\operatorname{Tr} Z_i \geq 0} \operatorname{Tr} \bar{E}_i^{\mathrm{T}} Z_i > -\infty \quad &\Leftrightarrow \quad \bar{E}_i = e_i I, \; 0 < e_i \in \mathbb{R}. \end{aligned}$$

(This is considered in more detail in [5].) ■

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## References

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