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Disproof of two conjectures of George Weiss

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Abstract

Two conjectures that were posed by Weiss almost ten years ago are shown not to hold. The first conjecture states that a scalar operator is admissible if and only a certain resolvent estimate holds. The second was posed by Weiss together with Russell and states that a system is exactly observable if and only if a test similar to the Hautus test for finite-dimensional systems holds. The C_0-semigroup in both counter-examples is analytic and possesses a basis of eigenfunctions.

Keywords: Infinite-dimensional system, admissible observation operator, exact observability, conditional basis, C_0-semigroup.

Mathematics Subject Classification: 93C25, 93A05, 93B07, 47D60

1 Introduction

Consider the abstract system

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t), \qquad x(0) = x_0 \tag{1}$$

on a Hilbert space H. For this abstract differential equation one would like to obtain conditions in terms of A and C such that it has a solution with certain properties. If one only considers the differential equation $\dot{x}(t) = Ax(t)$, then it is well-known that it has a unique (weak) solution which is strongly continuous and depends continuously on the initial condition x_0 if and only if A satisfies the estimates of the Hille-Yosida Theorem, see e.g. [4]. Since $\dot{x}(t) = Ax(t)$ is a part of (1) we have to assume that A satisfies the estimates of the Hille-Yosida Theorem, or equivalently that A generates a C_0-semigroup. If in addition C is a bounded linear operator from H to a second Hilbert space Y, then it is straightforward to see that $y(\cdot)$ in (1) is well-defined, and continuous. However, many p.d.e.'s rewritten in the form (1) do not have a bounded C operator, although the output is a well-defined square integrable function. If the output is locally square integrable, then C is called an *admissible observation operator*, see Weiss [17]. In other words, C is an admissible observation operator if and only if for some $t_0 > 0$ (and hence any $t_0 > 0$) there exists a constant $L > 0$ such that

$$\int_0^{t_0} \|CT(t)x\|^2 \, dt \leq L\|x\|^2, \qquad x \in D(A).$$

Here $T(t)$ is the C_0-semigroup generated by A, and $D(A)$ denotes the domain of A. If the C_0-semigroup is exponentially stable, then t_0 can be replaced by ∞. Now an interesting question is if there are simple conditions on C (and A) such that C is an admissible observation operator.

Dual to the concept of admissible observation operators is the concept of admissible control operator. An operator B is said to be an admissible control operator when $\dot{x}(t) = Ax(t) + Bu(t)$ has a continuous (weak) solution for every locally square integrable input u. It is well-known that C is an admissible observation operator for A if and only if C^* is an admissible control operator for A^*. Here $*$ denotes the adjoint operator. Because of this duality any result for admissible observation operators has an equivalent counterpart for admissible control operators, and visa versa. Hence if we refer to a paper which only deals with control operators, we trust that the reader can make the equivalent statement for observation operators. Basically, it boils down to replacing B by C^* and replacing the infinitesimal generator by its dual one.

In Weiss [18] it is shown that if C is admissible, then there exists a constant $M > 0$ such that

$$\|C(sI - A)^{-1}\| \leq \frac{M}{\sqrt{\mathrm{Re}(s)}}, \tag{2}$$

for all s in some right-half plane. He conjectured in [18], see also [19], that this condition is also sufficient. The sufficiency of condition (2) was proved for surjective semigroups in Weiss [18], for normal, analytic semigroups in Weiss [18, 19], for the right shift semigroup with scalar output in Partington and Weiss [12] and for contraction semigroups with scalar output by Jacob and Partington [6]. Recently, Zwart and Jacob [22] and Jacob, Partington and Pott [7] showed that in general estimate (2) is not sufficient. Their observation operator is infinite-dimensional. Here we use similar techniques as in [22] to show that (2) is not sufficient for scalar outputs. Note that in [5] a necessary and sufficient condition has been obtained. This condition involves all powers of the resolvent, as in the Hille-Yosida theorem.

Apart from the well-posedness of the abstract differential equation (1) one would like to characterize other properties in terms of the pair (A, C). One property that has received a lot of attention is the property of exact observability. Assuming that the observation operator C is admissible, the system (1) is said to be exactly observable if there a bounded mapping from the output trajectory to the initial condition, that is, for some $t_0 > 0$ (and hence any $t_0 > 0$) there exists a constant $l > 0$ such that

$$\int_0^{t_0} \|CT(t)x\|^2 \, dt \geq l\|x\|^2, \qquad x \in D(A).$$

If the C_0-semigroup is exponentially stable, then t_0 can be replaced by ∞. Note that admissibility gives that the mapping from initial condition to output trajectory is bounded. If the state space H is finite-dimensional, and thus A and C are just matrices, then it is well-known that (1) is exactly observable if and only if

$$\operatorname{rank} \begin{bmatrix} C \\ sI - A \end{bmatrix}$$

is full for all complex s. For infinite-dimensional systems Russell and Weiss [14] proposed the following test for exact observability of an exponentially stable system

$$\|(sI - A)x_0\|^2 + |\mathrm{Re}(s)|\|Cx_0\|^2 \geq m|\mathrm{Re}(s)|^2\|x_0\|^2 \tag{3}$$

for all complex s with negative real part, for all $x_0 \in D(A)$, and for some positive m independent of s and x_0. In [14] they proved that this condition is always necessary, and that for A and C bounded this condition is sufficient as well. In the same paper they showed that if A has a Riesz basis of eigenfunctions, and an extra conditions on the eigenvalues is satisfied, then (3) is sufficient. In Zhou and Yamamoto [20] it was show that (3) is sufficient if A is skew adjoint and C is bounded. For Riesz spectral systems with finite-dimensional output space inequality (3) is sufficient as well, see Jacob and Zwart [8]. Grabowski and Callier [5] proved that if m in (3) is equal to one, then this estimate implies exact observability. In Section 4 we show that for general m estimate (3) is not sufficient. Note that in our counterexample the output is one-dimensional and that A generates an analytic semigroup.

We conclude this paper with a section on left-invertibility of C_0-semigroups. It is known that uniform left-invertibility of the semigroup implies uniform left-invertibility of the generator on the open left half plane. We show that in general the inverse implication does not hold.

2 General results

Let H be a separable Hilbert space with a conditional basis $\{\varphi_n\}_{n\in\mathbb{N}}$. Since $\{\varphi_n\}_{n\in\mathbb{N}}$ is a conditional basis, we have that for every $x \in H$ there exists a unique sequence of complex numbers α_n such that

$$x = \lim_{k\to\infty} \sum_{n=1}^{k} \alpha_n \varphi_n. \tag{4}$$

Hence, we can write

$$x = \sum_{n=1}^{\infty} \alpha_n \varphi_n.$$

Using (4) it is not hard to see that the following holds (see also Singer [15, pages 18–20]).

Lemma 2.1 *If $\{\varphi_n\}_{n\in\mathbb{N}}$ is a conditional basis, then the following mappings are uniformly bounded*

$$P_n x = \sum_{k=1}^{n} \alpha_k \varphi_k, \tag{5}$$

and

$$\tilde{P}_n x = \alpha_n \varphi_n, \tag{6}$$

where $x = \sum_{n=1}^{\infty} \alpha_n \varphi_n$.
Furthermore, if $\inf_{n\in\mathbb{N}} \|\varphi_n\| > 0$, then

$$\sup_{n\in\mathbb{N}} |\alpha_n| \le \kappa \|x\|, \tag{7}$$

for some $\kappa > 0$ independent of x.

The following two properties of a conditional basis are important for the constructions of our counterexamples.

Definition 2.2 *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a conditional basis.*

1. *$\{\varphi_n\}_{n\in\mathbb{N}}$ is* Besselian *if there exists a constant $c > 0$ such that*

$$\sum_{k=1}^{n} |a_k|^2 \le c \left\| \sum_{k=1}^{n} a_k \varphi_k \right\|^2,$$

for all finite sequences of scalars $a_1, \ldots, a_n$.

2. $\{\varphi_n\}_{n\in\mathbb{N}}$ *is* Hilbertian *if there exists a constant* $c > 0$ *such that*

$$\left\| \sum_{k=1}^{n} a_k \varphi_k \right\|^2 \leq c \sum_{k=1}^{n} |a_k|^2,$$

for all finite sequences of scalars $a_1, \ldots, a_n$.

Equivalently, $\{\varphi_n\}_{n\in\mathbb{N}}$ is Besselian if and only if there exists a bounded linear operator S such that $v_n := S\varphi_n$ is an orthonormal basis for H. More information on conditional bases can be found in Singer [15].

For diagonal operators on a conditional basis of H there is the following nice result, which can be found in Benamara and Nikolski [1, Lemma 3.2.5].

Lemma 2.3 *Let* $\{\varphi_n\}_n$ *be a conditional basis of* H. *If* Q *is defined as*

$$Q\varphi_n = q_n \varphi_n$$

with $\{q_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, *and the total variation of the sequence* $\{q_n\}$ *is finite, i.e.,*

$$Var(q_n) := \sum_{n=1}^{\infty} |q_{n+1} - q_n| < \infty,$$

then Q *can be extended to a linear bounded operator on* H, *and*

$$\|Q\| \leq K(Var(q_n) + \limsup |q_n|), \tag{8}$$

where K *is the supremum of* $\|P_n\|$, *see Lemma 2.1.*

In order to calculate the total variation, the following observation is useful. If f is a continuous function which is non-decreasing or non-increasing on the interval (a, b), and if the sequence $\{q_n\}_n \subset (a, b)$ is non-decreasing or non-increasing, then

$$Var(f(q_n)) \leq |f(a) - f(b)|.$$

Using this, it is not hard to prove the following result.

Lemma 2.4 *Let* $\{\mu_n\}_n \subset (-\infty, -1]$ *be a monotonically decreasing sequence with* $\lim_{n\to\infty} \mu_n = -\infty$. *Furthermore, let* $\{\varphi_n\}_{n\in\mathbb{N}}$ *be a conditional basis for the Hilbert space* H.

For $t \geq 0$, *we define* $T(t)$ *by*

$$T(t)\varphi_n := e^{\mu_n t}\varphi_n, \qquad n \in \mathbb{N}. \tag{9}$$

The operator valued function $T(t)$ *defines an analytic, exponentially stable* C_0-*semigroup on* H.

Proof: Using the fact that the sequence $\{\mu_n\}_n$ is monotonically decreasing and that $\lim_{n\to\infty} \mu_n = -\infty$, we get by Lemma 2.3 that $T(t)$ has a linear bounded extension to H. Thus $T(t) \in \mathcal{L}(H)$, and

$$\|T(t)\| \leq Ke^{-t}, \qquad t \geq 0. \tag{10}$$

Clearly, $T(0) = I$ and $T(t)T(s) = T(t+s)$ for $t, s \geq 0$. We shall show that $T(t)$ is strongly continuous. For $x \in H$, there exists a sequence $\{\alpha_n\}_n$ of scalars such that

$$x = \sum_{n=1}^{\infty} \alpha_n \varphi_n.$$

Choose $\varepsilon > 0$ and choose N such that $\|x - x_N\|_H < \varepsilon$, where $x_N := \sum_{n=1}^{N} \alpha_n \varphi_n$. Next choose $t_0 > 0$ such that $\sum_{n=1}^{N} |e^{\mu_n t_0} - 1|\,|\alpha_n|\,\|\varphi_n\| \leq \varepsilon$. Then we have for $t \in (0, t_0)$ that

$$\begin{aligned}
\|T(t)x - x\| &\leq \|T(t)x - T(t)x_N\| + \|T(t)x_N - x_N\| + \|x_N - x\| \\
&\leq Ke^{-t}\varepsilon + \sum_{n=1}^{N} |e^{\mu_n t_0} - 1|\,|\alpha_n|\,\|\varphi_n\| + \varepsilon \\
&\leq [K+2]\varepsilon.
\end{aligned}$$

Thus $T(t)$ is a C_0-semigroup on H. From (10) we see that $T(t)$ is exponentially stable.

It remains to show that $T(t)$ is analytic.

Since the semigroup is uniformly bounded, it is sufficient, [13, Theorem 2.5.2], to show that

$$\|(sI - A)^{-1}\| \leq \frac{M}{|\operatorname{Im}(s)|}, \qquad s \in \mathbb{C}_+,$$

for some $M > 0$ independent of s. Let $s = s_r + is_i \in \mathbb{C}_+$. Clearly,

$$(sI - A)^{-1}\varphi_n = \frac{1}{s - \mu_n}\varphi_n, \qquad n \in \mathbb{N}.$$

In order to show the above estimate, we first prove that

$$\gamma_n := \frac{1}{s - \mu_n}, \qquad n \in \mathbb{N},$$

is of bounded variation. We get

$$\gamma_n = \frac{1}{s - \mu_n} = \frac{\bar{s}}{|s - \mu_n|^2} - \frac{\mu_n}{|s - \mu_n|^2},$$

and we define

$$h_1 : \mathbb{R}_- \to \mathbb{R}_+, \qquad h_1(x) := \frac{1}{|s-x|^2},$$
$$h_2 : \mathbb{R}_- \to \mathbb{R}_-, \qquad h_2(x) := \frac{x}{|s-x|^2}.$$

Clearly, h_1 is monotonically increasing on $(-\infty, 0)$, and $h_1(-\infty) = 0$ and $h_1(0) = \frac{1}{|s|^2}$. Moreover, we have

$$h_2'(x) = \frac{|s-x|^2 + 2x(s_r - x)}{|s-x|^4}.$$

Thus

$$h_2'(x) = 0 \Leftrightarrow |s-x|^2 + 2x(s_r - x) = 0 \Leftrightarrow -x^2 + |s|^2 = 0 \Leftrightarrow x = -|s|.$$

Thus h_2 is monotonically decreasing on $(-\infty, -|s|)$ and monotonically increasing on $(-|s|, 0)$. Moreover, $h_2(-\infty) = h_2(0) = 0$, and thus $|h_2|$ has its maximum in $-|s|$. Note that

$$|h_2(-|s|)| = \frac{|s|}{|s+|s||^2} \le \frac{1}{|s|}.$$

Using Lemma 2.3 we get the following estimate for $\|(sI-A)^{-1}\|$.

$$\begin{aligned}
&\|(sI-A)^{-1}\| \\
&\le \; K(\mathrm{Var}(\{\gamma_n\}) + |\lim_{n\to\infty} \gamma_n|) \le K\left(\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|\right) \\
&\le \; K\left(|s| \sum_{n=1}^{\infty} |h_1(\mu_{n+1}) - h_1(\mu_n)| + \sum_{n=1}^{\infty} |h_2(\mu_{n+1}) - h_2(\mu_n)|\right) \\
&\le \; \frac{3K}{|s|} \le \frac{3K}{|\operatorname{Im}(s)|},
\end{aligned}$$

where $K > 0$ is independent of s. Thus the statement is proved. ■

3 Counterexample on admissibility

In this section we show that the conjecture of George Weiss for admissibility of scalar observation operators (see [18, 19]) does not hold. That means we construct an exponentially stable C_0-semigroup $T(t)$ with infinitesimal generator A and an operator $C \in \mathcal{L}(D(A), \mathbb{C})$ such that

$$\|C(sI-A)^{-1}\| \le \frac{M}{\sqrt{\operatorname{Re}(s)}},$$

for all s in some right-half plane and some constant $M > 0$, but C is not an admissible observation operator for $T(t)$.

Let $\{e_n\}_{n\in\mathbb{N}}$ be a conditional basis on H which has the following properties:

1. $\inf_{n\in\mathbb{N}} \|e_n\| > 0$.
2. $\{e_n\}_{n\in\mathbb{N}}$ is not Besselian.

Those Hilbert spaces and bases do exists, see for example Singer [15, Example 11.2 on page 351].

We define the sequence μ_n as

$$\mu_n := -4^n, \qquad n \in \mathbb{N}, \tag{11}$$

and the C_0-semigroup $T(t)$ as

$$T(t)e_n = e^{\mu_n t} e_n. \tag{12}$$

By Lemma 2.4 we know that $T(t)$ is an exponentially stable analytic semigroup. By A we denote the infinitesimal generator of $T(t)$. It is easy to see that A satisfies

$$Ae_n = \mu_n e_n, \qquad n \in \mathbb{N}.$$

For $x \in D(A)$, $x = \sum_{n=1}^{\infty} x_n e_n$, we further define

$$Cx = \sum_{n=1}^{\infty} \sqrt{-\mu_n} x_n. \tag{13}$$

First of all we show that C is a bounded linear operator from the domain of A into $\mathbb{C}$.

Proposition 3.1 *Let C be given as in (13) and let A be the infinitesimal generator of the C_0-semigroup (12). Then we have $C \in \mathcal{L}(D(A), \mathbb{C})$.*

Proof It is enough to show that there exists a constant $c > 0$ such that

$$|CA^{-1}x| \leq c, \qquad x \in H, \|x\| = 1.$$

Let $x \in H$ with $\|x\| = 1$. Then there exist scalars x_n, $n \in \mathbb{N}$, such that

$$x = \sum_{n=1}^{\infty} x_n e_n.$$

Using that $\inf_{n\in\mathbb{N}} \|e_n\| > 0$, we get from Lemma 2.1 that $\sup_{n\in\mathbb{N}} |x_n| \leq \kappa < \infty$. Note that κ is independent of $x \in H$ with $\|x\| = 1$. Now we have

$$|CA^{-1}x| = \left| \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{-\mu_n}} \right| \leq \kappa \sum_{n=1}^{\infty} 2^{-n} = \kappa.$$

Thus the proposition is proved. ■

Next we show that C satisfies the estimate (2).

Proposition 3.2 *For C given by (13) and A the infinitesimal generator of the semigroup (12) the following holds. There exists a constant $M > 0$ such that*

$$\|C(sI - A)^{-1}\| \leq \frac{M}{\sqrt{\mathrm{Re}(s)}}, \qquad s \in \mathbb{C}_+.$$

Proof Let s be an element of $\mathbb{C}_+$, and let $x \in H$ have norm one. We have the following estimate

$$\begin{aligned} \sqrt{\mathrm{Re}(s)}|C(sI-A)^{-1}x| &= \sqrt{\mathrm{Re}(s)} \left| \sum_{k=1}^{\infty} \frac{2^k}{s+4^k} x_k \right| \\ &\leq \sqrt{\mathrm{Re}(s)} \sum_{k=1}^{\infty} \frac{2^k}{|\,\mathrm{Re}(s)+4^k|} |x_k| \\ &\leq \kappa \sqrt{\mathrm{Re}(s)} \sum_{k=1}^{\infty} \frac{2^k}{\mathrm{Re}(s)+4^k}, \end{aligned}$$

where we have used Lemma 2.1. Note that κ is independent of x. In order to estimate this last expression we introduce the monotonically decreasing sequence $a_k := \frac{1}{\mathrm{Re}(s)+k^2}$. Then for $N \geq 2^K$ we have

$$\begin{aligned} \sum_{k=1}^{N} a_k &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{K-1}+1} + \cdots + a_{2^K}) \\ &\geq a_2 + 2a_4 + \cdots + 2^{K-1} a_{2^K} \\ &= \frac{1}{2} \sum_{k=1}^{K} 2^k a_{2^k}, \end{aligned}$$

and so

$$\sum_{k=1}^{\infty} \frac{2^k}{\mathrm{Re}(s)+4^k} \leq 2 \sum_{k=1}^{\infty} \frac{1}{\mathrm{Re}(s)+k^2}.$$

Using this in our estimate of $\sqrt{\mathrm{Re}(s)}|C(sI-A)^{-1}x|$, we obtain that

$$\begin{aligned}
\sqrt{\mathrm{Re}(s)}|C(sI-A)^{-1}x| &\leq 2\kappa\sqrt{\mathrm{Re}(s)}\sum_{k=1}^{\infty}\frac{1}{\mathrm{Re}(s)+k^2} \\
&\leq 2\kappa\sqrt{\mathrm{Re}(s)}\int_0^{\infty}\frac{1}{\mathrm{Re}(s)+t^2}\,dt \\
&\leq 2\kappa\sqrt{\mathrm{Re}(s)}\left(\frac{1}{\sqrt{\mathrm{Re}(s)}}\arctan\left(\frac{t}{\sqrt{\mathrm{Re}(s)}}\right)\Big|_0^{\infty}\right) \\
&\leq 2\kappa\frac{\pi}{2} = \kappa\pi,
\end{aligned}$$

which proves our assertion. ■

Suppose now that C is an admissible observation operator for $T(t)$, then there would exists a constant $L > 0$ such that

$$\int_0^{\infty}|CT(t)x|^2dt \leq L\|x\|^2, \qquad x \in D(A).$$

Now take a finite sequence of α_k's and consider

$$x := \sum_{k=1}^{n}\alpha_k e_k$$

Then the above estimate gives that

$$\int_0^{\infty}\left|\sum_{k=1}^{n}\sqrt{-\mu_k}e^{\mu_k t}\alpha_k\right|^2 dt \leq L\|x\|^2.$$

However, from Nikolskii and Pavlov [11] (see also Jacob and Zwart [9]), we know that there exists a constant $L_1 > 0$, independent of x, such that

$$\int_0^{\infty}\left|\sum_{k=1}^{n}\sqrt{-\mu_k}e^{\mu_k t}\alpha_k\right|^2 dt \geq L_1\sum_{k=1}^{n}|\alpha_k|^2.$$

Thus we have that for any finite sequence

$$\|x\|^2 \geq \frac{L_1}{L}\sum_{k=1}^{n}|\alpha_k|^2.$$

However, this implies that $\{e_n\}$ is Besselian, providing the contradiction.

Thus we have disproved the scalar admissibility conjecture of George Weiss.

4 Counterexample on exact observability

In this section we disprove the conjecture of Russell and Weiss [14] on exact observability. That means we construct an exponentially stable C_0-semigroup $T(t)$ with infinitesimal generator A and an operator $C \in \mathcal{L}(D(A), \mathbb{C})$ such that

$$\|(sI - A)x_0\|^2 + |\mathrm{Re}(s)|\|Cx_0\|^2 \geq m|\mathrm{Re}(s)|^2\|x_0\|^2, \quad s \in \mathbb{C}_-, x_0 \in D(A),$$

for some constant $m > 0$, but the pair (A, C) is not exactly observable. The operators A and C are similar to the ones used in the previous section, but now they are defined via a Besselian basis.

Let $\{e_n\}_{n\in\mathbb{N}}$ be a conditional basis on H which is Besselian, normalized, that is, $\|e_n\| = 1$, but not Hilbertian. Those Hilbert spaces and bases do exists, see for example Singer [15, Example 11.2 on page 351].

We define the sequence μ_n as

$$\mu_n := -4^n, \qquad n \in \mathbb{N}, \tag{14}$$

and the C_0-semigroup as

$$T(t)e_n = e^{\mu_n t}e_n. \tag{15}$$

By Lemma 2.4 we know that this is an exponentially stable analytic C_0-semigroup. By A we denote the infinitesimal generator of $T(t)$. It is easy to see that A satisfies

$$Ae_n = \mu_n e_n, \qquad n \in \mathbb{N}.$$

Since $\{e_n\}_{n\in\mathbb{N}}$ is Besselian, we know that there exists a bounded linear operator S such that $v_n := Se_n$ is an orthonormal basis for H. On this new basis we define

$$\tilde{A}v_n = \mu_n v_n.$$

It is easy to see that $\tilde{A}$ generates a C_0-semigroup $\tilde{T}(t)$, and that

$$ST(t) = \tilde{T}(t)S. \tag{16}$$

Now define the operator $\tilde{C}$ as

$$\tilde{C}v_n = \sqrt{-\mu_n},$$

It is easy to see that we can extend $\tilde{C}$ as a bounded operator form the domain of $\tilde{A}$ to $\mathbb{C}$. We denote this extension again by $\tilde{C}$. We shall prove that $\tilde{C}$ is an admissible observation operator for $\tilde{T}(t)$. Since $\tilde{T}(t)$ has an orthonormal basis of eigenfunctions, we can use the result of Weiss [16], which tells that $\tilde{C}$ is admissible if and only if

$$\sum_{-\mu_n \in R(h,\omega)} |\mu_n| \leq \beta h,$$

where

$$R(h, \omega) := \{s \in \mathbb{C}_+ \mid \mathrm{Re}(s) \leq h, |\mathrm{Im}(s) - \omega| \leq h\}$$

and β is independent of h. Using the definition of μ_n this is easy to prove. Now we define for $x \in D(A)$,

$$Cx = \tilde{C}Sx. \tag{17}$$

From this and (16) we see that for $x \in D(A)$

$$CT(t)x = \tilde{C}\tilde{T}(t)Sx.$$

Since S is bounded, and since $\tilde{C}$ is admissible for $\tilde{T}(t)$, we obtain that C is an admissible output operator for $T(t)$.

In several steps we shall prove that the pair (A, C) satisfies the estimate of Russell and Weiss, but that it is not exactly observable. In our proof we follow closely the proof of Theorem 4.4 of Russell and Weiss [14]. As in [14] we define $N : \mathbb{C}_- \to \mathbb{N}$ as the integer such that

$$|s - \mu_{N(s)}| = \min_{k \in \mathbb{N}} |s - \mu_k|. \tag{18}$$

This number is well-defined if the real part of s is unequal to $(\mu_k + \mu_{k+1})/2$ for all k. We define the set for which this mapping is well-defined as $\mathbb{C}_g$.

Lemma 4.1 *There exists a constant $c > 0$ such that for all $s \in \mathbb{C}_g$ we have that*

$$\left| \frac{\mathrm{Re}(s)}{s - \mu_k} \right| \leq c, \qquad s \in \mathbb{C}_g, k \neq N(s).$$

and

$$\left| \frac{\mathrm{Re}(s)}{\mathrm{Re}(s) - \mu_k} \right| \leq c, \qquad s \in \mathbb{C}_g, k \neq N(s).$$

Proof: In Weiss and Russell [14] it is shown that the first estimate holds. Since $\{\mu_k\}$ is a real sequence, it is easy to see that $N(s) = N(\mathrm{Re}(s))$. Taking s to be real in the first inequality, and using this observation, proves the second inequality. ■

For $s \in \mathbb{C}_g$, we define

$$V(s) := \overline{\mathrm{span}_{n \neq N(s)}\{e_n\}}. \tag{19}$$

Clearly, $V(s)$ is again a Hilbert space and in Singer [15, Proposition 4.1 on page 26] it is shown that $\{e_n\}_{n \neq N(s)}$ is a conditional basis of $V(s)$. By $P_{V(s)}$ we denote the projection from H onto $V(s)$ given by

$$P_{V(s)} := I - \tilde{P}_{N(s)}.$$

Using Lemma 2.1 we see that the projections $P_{V(s)}$ are uniformly bounded. For $s \in \mathbb{C}_g$, we introduce the following notation

$$e_n^s := \begin{cases} e_n & , n < N(s) \\ e_{n+1} & , n \geq N(s) \end{cases} , \tag{20}$$

and

$$\mu_n^s := \begin{cases} \mu_n & , n < N(s) \\ \mu_{n+1} & , n \geq N(s) \end{cases} . \tag{21}$$

The constant K in Lemma 2.4 is given by $K := \sup_{n \in \mathbb{N}} \|P_n\|$. Let $K(s)$ be the corresponding constant for $V(s)$ with conditional basis $\{e_n^s\}$, for $s \in \mathbb{C}_g$. Then it follows easily that $K(s) \leq K$.

Let $s \in \mathbb{C}_g$. We denote by A_s the part of A in $V(s)$, that is

$$A_s x := Ax, \qquad x \in D(A_s).$$

and $D(A_s) := D(A) \cap V(s)$. Note that $V(s)$ is a $T(t)$-invariant subspace. Thus it is easy to see that C_s, defined by

$$C_s x := Cx, \qquad x \in D(A_s),$$

is an admissible observation operator for $T_s(t)$. Here $T_s(t)$ is the C_0-semigroup generated by A_s. Now we shall prove two important estimates.

Lemma 4.2 *Let A_s, C_s and $V(s)$ denote the objects defined above. The following two estimates hold.*

1. *There exists a constant $M > 0$ such that*

$$\|(sI - A_s)^{-1}\|_{V(s)} \leq \frac{M}{|\operatorname{Re}(s)|}, \qquad s \in \mathbb{C}_g.$$

2. *There exists a constant $d > 0$ such that*

$$\|C_s(sI - A_s)^{-1}\| \leq \frac{d}{\sqrt{|\operatorname{Re}(s)|}}, \qquad s \in \mathbb{C}_g.$$

Proof: *Part 1.* Let $s = s_r + is_i \in \mathbb{C}_g$. Clearly,

$$(sI - A_s)^{-1} e_n^s = \frac{1}{s - \mu_n^s} e_n^s, \qquad n \in \mathbb{N}.$$

This is an operator of the form as discussed in Lemma 2.3, and thus we have to show that $1/(s - \mu_n^s)$ is of bounded variation. We begin with the following simple observation,

$$\begin{aligned}\left|\frac{1}{s-\mu^s_{n+1}}-\frac{1}{s-\mu^s_n}\right| &= \left|\frac{\mu^s_{n+1}-\mu^s_n}{(s-\mu^s_{n+1})(s-\mu^s_n)}\right| \\ &\le \left|\frac{\mu^s_{n+1}-\mu^s_n}{(s_r-\mu^s_{n+1})(s_r-\mu^s_n)}\right| \\ &= \left|\frac{1}{s_r-\mu^s_{n+1}}-\frac{1}{s_r-\mu^s_n}\right|, \qquad (22)\end{aligned}$$

where we have used the fact that μ^s_n is real.

Next we define

$$h:\mathbb{R}_-\setminus\{s_r\}\to\mathbb{R}, \qquad h(x):=\frac{1}{s_r-x}.$$

Then we have $h(-\infty)=0$, $h(0)=\frac{1}{s_r}$ and h is monotonically increasing on $(-\infty,s_r)$ and on $(s_r,0)$. Combining the above results with Lemma 2.3 we get the following estimate for $\|(sI-A_s)^{-1}\|$.

$$\begin{aligned}&\|(sI-A_s)^{-1}\| \\ &\quad\le K\left(\operatorname{Var}\left(\frac{1}{s-\mu^s_n}\right)+\left|\lim_{n\to\infty}\frac{1}{s-\mu^s_n}\right|\right)=K\sum_{n=1}^{\infty}\left|\frac{1}{s-\mu^s_{n+1}}-\frac{1}{s-\mu^s_n}\right| \\ &\quad\le K\sum_{n=1}^{\infty}\left|\frac{1}{s_r-\mu^s_{n+1}}-\frac{1}{s_r-\mu^s_n}\right| \\ &\quad\le K\left[\left[0+\frac{1}{s_r-\mu_{N(s)+1}}\right]+\left[\frac{1}{s_r-\mu_{N(s)+1}}-\frac{1}{s_r-\mu_{N(s)-1}}\right]\right. \\ &\qquad\left.+\left[\frac{1}{s_r}-\frac{1}{s_r-\mu_{N(s)-1}}\right]\right] \\ &\quad\le \frac{(4c+1)K}{|\operatorname{Re}(s)|},\end{aligned}$$

where we have used Lemma 2.3, Lemma 4.1 and (22). Since c and K are independent of s we have proved the statement.

Part 2. In order to prove this statement we follow Lemma 4.6 of Russell and Weiss [14]. Let $s\in\mathbb{C}_g$. Using the resolvent identity, we have

$$C_s(sI-A_s)^{-1}=C_s(-\bar{s}I-A_s)^{-1}[I-(\bar{s}+s)(sI-A_s)^{-1}].$$

Since C_s is an admissible observation operator for $T_s(t)$ there exists a constant $\tilde{d}>0$, independent of s, such that

$$\|C_s(-\bar{s}I-A_s)^{-1}\|\le\frac{\tilde{d}}{\sqrt{|\operatorname{Re}(s)|}}$$

(see for example Weiss [19]). Combining this with Part 1, the statement is proved. ■

Now we can prove the estimate of Russell and Weiss [14].

Lemma 4.3 *For C defined by (17) and A the infinitesimal generator of (15) the following holds. There exists a constant $m > 0$ such that for every $s \in \mathbb{C}_-$ and every $x \in D(A)$ we have*

$$\frac{1}{|\operatorname{Re}(s)|^2}\|(sI - A)x\|^2 + \frac{1}{|\operatorname{Re}(s)|}\|Cx\|^2 \geq m\|x\|^2. \tag{23}$$

Proof: The proof of this lemma is divided in two steps. First we show that the estimate holds for $s \in \mathbb{C}_- \setminus \mathbb{C}_g$. Secondly, we prove the estimate for $s \in \mathbb{C}_g$.

Part 1 If s is not in $\mathbb{C}_g$, then there exists an $k_0 \in \mathbb{N}$, such that $\operatorname{Re}(s) = (\mu_{k_0+1} + \mu_{k_0})/2$. It is easy to see that

$$(sI - A)^{-1}e_n = \frac{1}{s - \mu_n}e_n.$$

We use Lemma 2.3 to estimate the norm of this operator. Using (22) we see that it is sufficient to show that $\{\frac{1}{\operatorname{Re}(s)-\mu_n}\}$ is of bounded variation. Similar as in the proof of Part 1 of Lemma 4.2, we obtain that

$$\|(sI - A)^{-1}\| \leq K\sum_{n=1}^{\infty}\left|\frac{1}{\operatorname{Re}(s) - \mu_{n+1}} - \frac{1}{\operatorname{Re}(s) - \mu_n}\right|.$$

Now we have that $\operatorname{Re}(s) = (\mu_{k_0+1} + \mu_{k_0})/2$, and thus we obtain

$$\begin{aligned}
&\|(sI - A)^{-1}\| \\
&\leq K\left[\left[0 + \frac{1}{\operatorname{Re}(s) - \mu_{k_0+1}}\right] + \left[\frac{1}{\operatorname{Re}(s) - \mu_{k_0+1}} - \frac{1}{\operatorname{Re}(s) - \mu_{k_0}}\right]\right. \\
&\quad \left. + \left[\frac{1}{\operatorname{Re}(s)} - \frac{1}{\operatorname{Re}(s) - \mu_{k_0}}\right]\right] \\
&\leq K\left[\frac{8}{\mu_{k_0} - \mu_{k_0+1}} + \frac{1}{|\operatorname{Re}(s)|}\right].
\end{aligned}$$

Now the sequence $\{\mu_n\} = \{-4^n\}$ satisfies

$$\frac{1}{\mu_n - \mu_{n+1}} = \frac{5/3}{|\mu_n + \mu_{n+1}|}.$$

So we see that

$$\|(sI - A)^{-1}\| \leq \frac{40K}{3|\mu_{k_0} + \mu_{k_0+1}|} + \frac{K}{|\operatorname{Re}(s)|} = \frac{23K}{3|\operatorname{Re}(s)|}.$$

This is equivalent to

$$|\operatorname{Re}(s)|^{-1}\|(sI-A)x\| \geq \frac{3}{23K}\|x\|,$$

and so (23) holds for $s \in \mathbb{C}_- \setminus \mathbb{C}_g$.

Part 2 In order to prove this statement we follow Theorem 4.4 of Russell and Weiss.

If (23) would not hold, then there would exist sequences $\{s_n\}$ and $\{z^n\}$ such that $s_n \in \mathbb{C}_g$, $z^n \in D(A)$, $\|z^n\| = 1$ and

$$\frac{1}{|\operatorname{Re}(s_n)|^2}\|(s_nI-A)z^n\|^2 + \frac{1}{|\operatorname{Re}(s_n)|}|Cz^n|^2 = \varepsilon_n^2, \tag{24}$$

where $\varepsilon_n \geq 0$ and $\varepsilon_n \to 0$.

Now define

$$q^n := \frac{1}{|\operatorname{Re}(s_n)|}(s_nI - A_{s_n})P_{V(s_n)}z^n.$$

and the scalar α_n such that

$$\alpha_n e_{N(s_n)} = \tilde{P}_{N(s_n)}z^n = (I - P_{V(s_n)})z^n.$$

Thus we have that

$$\frac{1}{|\operatorname{Re}(s_n)|}(s_nI-A)z^n = \frac{s_n - \mu_{N(s_n)}}{|\operatorname{Re}(s_n)|}\alpha_n e_{N(s_n)} + q_n.$$

Now we have that

$$\|q^n\| = \|P_{V(s_n)}\frac{1}{|\operatorname{Re}(s_n)|}(s_nI-A)z^n\| \leq K\frac{1}{|\operatorname{Re}(s_n)|}\|(s_nI-A)z^n\| \leq K\varepsilon_n, \tag{25}$$

by (24). For α_n, we obtain,

$$\begin{aligned}
\left|\frac{s_n - \mu_{N(s_n)}}{\operatorname{Re}(s_n)}\alpha_n\right| &= \left\|\frac{s_n - \mu_{N(s_n)}}{\operatorname{Re}(s_n)}\alpha_n e_{N(s_n)}\right\| = \left\|\frac{s_n - \mu_{N(s_n)}}{\operatorname{Re}(s_n)}\tilde{P}_{N(s_n)}z^n\right\| \\
&= \frac{1}{|\operatorname{Re}(s_n)|}\|\tilde{P}_{N(s_n)}(s_n - A)z^n\| \\
&\leq 2K\frac{1}{|\operatorname{Re}(s_n)|}\|(s_n - A)z^n\| \leq 2K\varepsilon_n.
\end{aligned} \tag{26}$$

By definition of q^n, we have that

$$P_{V(s_n)}z^n = |\operatorname{Re}(s_n)|(s_nI - A_{s_n})^{-1}q^n.$$

Using (25) and Lemma 4.2, we get

$$\|P_{V(s_n)}z^n\| \leq MK\varepsilon_n,$$

whence $P_{V(s_n)}z^n \to 0$. Since $\|z^n\| = 1$, it follows that $\|(I - P_{V(s_n)})z^n\| \to 1$, i.e.,

$$\lim_{n\to\infty} |\alpha_n| = 1. \tag{27}$$

Together with (26) this implies that

$$\lim_{n\to\infty} \left| \frac{s_n - \mu_{N(s_n)}}{\mathrm{Re}(s_n)} \right| = 0.$$

It is now easy to see that

$$\lim_{n\to\infty} \left| \frac{\mu_{N(s_n)}}{\mathrm{Re}(s_n)} \right| = 1. \tag{28}$$

Now we turn our attention to the second term of (24). We have

$$\begin{aligned} Cz^n &= C(I - P_{V(s_n)})z^n + CP_{V(s_n)}z^n \\ &= \alpha_n Ce_{N(s_n)} + C_{s_n}(s_n I - A_{s_n})^{-1}(s_n I - A_{s_n})P_{V(s_n)}z^n \\ &= \alpha_n \sqrt{-\mu_{N(s_n)}} + |\,\mathrm{Re}(s_n)|C_{s_n}(s_n I - A_{s_n})^{-1}q^n. \end{aligned}$$

Thus we can estimate the norm of this number as

$$|Cz^n| \geq |\alpha_n \sqrt{-\mu_{N(s_n)}}| - |\,\mathrm{Re}(s_n)||C_s(s_n I - A_{s_n})^{-1}q^n|.$$

Hence using Lemma 4.2, Part 2, we obtain that

$$\frac{1}{\sqrt{|\,\mathrm{Re}(s_n)|}}|Cz^n| \geq |\alpha_n| \left| \frac{\mu_{N(s_n)}}{\mathrm{Re}(s_n)} \right|^{\frac{1}{2}} - d\|q^n\|. \tag{29}$$

By (25), and (27)-(29), we conclude that there exists a positive number κ, such that for n sufficiently large,

$$\frac{1}{|\,\mathrm{Re}(s_n)|}|Cz^n|^2 \geq \kappa.$$

On the other hand, (24) implies that for each $n \in \mathbb{N}$,

$$\frac{1}{|\,\mathrm{Re}(s_n)|}|Cz^n|^2 \leq \varepsilon_n^2$$

which is a contradiction. Therefore, (23) must be true. ■

So we know that the system (A, C) as defined in the beginning of this section satisfies the estimate of Russell and Weiss. Suppose now that the pair would be exactly observable, then there would exists a constant $l > 0$ such that

$$\int_0^\infty |CT(t)x|^2 dt \geq l\|x\|^2, \qquad x \in D(A).$$

Now take a finite sequence of α_k's and consider

$$x := \sum_{k=1}^{n} \alpha_k e_k$$

Then the above estimate gives that

$$\int_0^\infty \left| \sum_{k=1}^{n} \sqrt{-\mu_k} e^{\mu_k t} \alpha_k \right|^2 dt \geq l \|x\|^2.$$

However, from Nikolskii and Pavlov [11] (see also Russell and Weiss [14]), we know that there exists a constant $l_1 > 0$ such that

$$\int_0^\infty \left| \sum_{k=1}^{n} \sqrt{-\mu_k} e^{\mu_k t} \alpha_k \right|^2 dt \leq l_1 \sum_{k=1}^{n} |\alpha_k|^2.$$

Thus we have that for any finite sequence

$$\|x\|^2 \leq \frac{l_1}{l} \sum_{k=1}^{n} |\alpha_k|^2.$$

However, this implies that $\{e_n\}$ is Hilbertian, providing the contradiction.

Thus we have disproved the conjecture of Russell and Weiss on exact observability.

5 On left-invertibility of C_0-semigroups

We consider a bounded C_0-semigroup $T_e(t)$ with infinitesimal generator A_e on a separable Hilbert space Z. A natural question is whether uniform left-invertibility of the C_0-semigroup, that is,

$$\|T_e(t)x\| \geq c_1 \|x\|, \quad x \in Z, \tag{30}$$

for some $c_1 > 0$, is equivalent to uniform left-invertibility of $sI - A_e$ on the open left half plane, that is,

$$\|(sI - A_e)x\| \geq c_2 |\operatorname{Re}(s)| \, \|x\|, \quad x \in D(A_e), s \in \mathbb{C}_-, \tag{31}$$

for some constant $c_2 > 0$.

In van Neerven [10] it is shown that (30) implies (31). Van Neerven only considered the case of a semigroup of isometries, but the general case can be proved in a similar way. If $T_e(t)$ can be extended to a group or if $\mathbb{C}_-$ is contained in the resolvent set of A, then (31) implies (30), see van Casteren [2], [3] or Zwart [21].

We now show that in general (31) does not imply (30). Consider the operators A and C of Section 4, and let $T(t)$ denote the exponentially stable C_0-semigroup generated by A. We now define the semigroup $T_e(t)$ on $H \oplus L^2(0,\infty)$ by

$$T_e(t)\begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} T(t)x \\ CT(t-\cdot)x|_{[0,t]} + f(\cdot - t)|_{[t,\infty)} \end{pmatrix}.$$

In Grabowski and Callier [5] it is shown that $T_e(t)$ is a uniformly bounded C_0-semigroup on $H \oplus L^2(0,\infty)$, and that the infinitesimal generator A_e of $T_e(t)$ is given by

$$\begin{aligned} A_e\begin{pmatrix} x \\ f \end{pmatrix} &:= \begin{pmatrix} Ax \\ -\dot{f} \end{pmatrix}, \quad \begin{pmatrix} x \\ f \end{pmatrix} \in D(A_e), \\ D(A_e) &:= \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \mid x \in D(A), f, \dot{f} \in L^2(0,\infty), \right. \\ &\qquad\qquad f \text{ is abs. cont. and } f(0) = Cx \Big\}. \end{aligned}$$

Next we calculate the norm of $\left\|(sI - A_e)\begin{pmatrix} x \\ f \end{pmatrix}\right\|$. For $s = s_r + is_i \in \mathbb{C}_-$ we have

$$\begin{aligned} &\left\|(sI - A_e)\begin{pmatrix} x \\ f \end{pmatrix}\right\|^2 \\ &= \|(sI-A)x\|^2 + \|sf + \dot{f}\|^2_{L^2(0,\infty)} \\ &= \|(sI-A)x\|^2 + |s|^2\|f\|^2_{L^2(0,\infty)} + \|\dot{f}\|^2_{L^2(0,\infty)} \\ &\quad + 2s_r \operatorname{Re}(\langle f, \dot{f}\rangle_{L^2(0,\infty)}) + is_i(\langle f, \dot{f}\rangle_{L^2(0,\infty)} - \langle \dot{f}, f\rangle_{L^2(0,\infty)}) \\ &= \|(sI-A)x\|^2 + \|is_i f + \dot{f}\|^2_{L^2(0,\infty)} + s_r^2\|f\|^2_{L^2(0,\infty)} + 2s_r \operatorname{Re}(\langle f, \dot{f}\rangle_{L^2(0,\infty)}) \\ &= \|(sI-A)x\|^2 + \|is_i f + \dot{f}\|^2_{L^2(0,\infty)} + s_r^2\|f\|^2_{L^2(0,\infty)} \\ &\quad + s_r \int_0^\infty \frac{d}{dt}\langle f(t), f(t)\rangle\, dt \\ &= \|(sI-A)x\|^2 + \|is_i f + \dot{f}\|^2_{L^2(0,\infty)} + s_r^2\|f\|^2_{L^2(0,\infty)} - s_r\|Cx\|^2, \end{aligned}$$

because $f(0) = Cx$ and $f, \dot{f} \in L^2(0,\infty)$. Thus

$$\begin{aligned} &\left\|(sI - A_e)\begin{pmatrix} x \\ f \end{pmatrix}\right\|^2 \\ &\geq \|(sI-A)x\|^2 + |\operatorname{Re}(s)|^2\|f\|^2_{L^2(0,\infty)} + |\operatorname{Re}(s)|\|Cx\|^2 \\ &\geq c_2|\operatorname{Re}(s)|^2\left\|\begin{pmatrix} x \\ f \end{pmatrix}\right\|^2, \qquad \text{(using Lemma 4.3)} \end{aligned}$$

where c_2 is independent of x and f. This shows that (31) holds. Assuming (30) holds as well, we get

$$\left\|T_e(t)\begin{pmatrix} x \\ f \end{pmatrix}\right\| \geq c_1\left\|\begin{pmatrix} x \\ f \end{pmatrix}\right\|, \qquad t \geq 0, x \in H, f \in L^2(0,\infty),$$

for some constant $c_1 > 0$. Thus

$$\|T(t)x\|^2 + \|CT(\cdot)x\|^2_{L^2(0,t)} = \left\| T_e(t) \begin{pmatrix} x \\ 0 \end{pmatrix} \right\| \geq c_1 \|x\|^2, \qquad x \in H, t \geq 0. \quad (32)$$

Using that $T(t)$ is exponentially stable, we get $\lim_{t\to\infty} \|T(t)x\|^2 = 0$, and so letting t to infinity in (32) gives

$$\|CT(\cdot)x\|_{L^2(0,\infty)} \geq \sqrt{c_1}\|x\|, \qquad x \in H,$$

which says that the pair (A, C) is exactly observable. However, this is in contradiction with Section 4, where we showed that the pair (A, C) is not exactly observable. Thus (31) holds, but (30) is not valid.

We conclude this section with a positive result; it shows that (31) implies (30) if the constant c_2 satisfies $c_2 \geq 1$.

Proposition 5.1 *Let $T_e(t)$ be a bounded C_0-semigroup with infinitesimal generator A_e on a separable Hilbert space Z. If (31) holds with $c_2 \geq 1$, then (30) holds as well.*

Proof If $c_2 \geq 1$, then it is easy to see that (31) implies that

$$\|(sI - A_e)x\| \geq |\operatorname{Re} s|\|x\|, \quad s \in \mathbb{C}_-,$$

for all $x \in D(A)$. Choosing $s < 0$, and taking the square of the above equation gives

$$\|(sI - A_e)x\|^2 \geq s^2\|x\|^2.$$

Using the fact that Z is a Hilbert space, gives that the above inequality is equivalent to

$$s^2\|x\|^2 - 2s \operatorname{Re}\langle x, A_e x\rangle + \|A_e x\|^2 \geq s^2\|x\|^2,$$

which is equivalent to

$$-2s \operatorname{Re}\langle x, A_e x\rangle + \|A_e x\|^2 \geq 0.$$

Since this must hold for all negative s, we see that

$$\operatorname{Re}\langle x, A_e x\rangle \geq 0.$$

We now consider the function $f(t) := \|T_e(t)x\|^2$. Taking the derivative of f gives

$$\dot{f}(t) = 2 \operatorname{Re}\langle T_e(t)x, A_e T_e(t)x\rangle \geq 0.$$

Hence f is non-decreasing, and thus

$$\|T_e(t)x\|^2 = f(t) \geq f(0) = \|x\|^2.$$

Since x was arbitrary, we have showed the result. ■

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