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Super Darboux-Egoroff equations and solutions

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Abstract

Super Darboux-Egoroff equations are discussed. First of all linearity of the potential φ with respect to odd variables is proved.
Solutions of Darboux-Egoroff equations in dimension $(2|2)$ including flatness of the unit vector field are constructed.
Moreover solutions of Darboux-Egoroff equations in the presence of an Euler vector field are given.
In dimension $(3|3)$ solutions of Darboux-Egoroff equations including flatness of unit and presence of Euler field are given.

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1 Introduction

The notion of Frobenius manifolds M has been introduced and studied by B. Dubrovin in his comprehensive paper [].
This notion requires the existence of a function φ, defined on this manifold, in such a way, that there can be defined a multiplication structure on the tangent space TM where structure constants for this multiplication arise from third order partial derivatives of the function φ.
Moreover there should be a flat metric g which is compatible with this multiplication structure; i.e.,

$$g(X \circ Y\ ,\ Z) = g(X, Y \circ Z)$$

The requirement that the defined multiplication is associative, leads to an (overdetermined) system of partial differential equations for the potential φ. This system of partial differential equations, the equations of associativity, are often called Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations.
Just these equations play a fundamental role in 2D topological field theory.
For a special type of Frobenius manifolds, the semisimple Frobenius manifolds, this implies that there exist local coordinates, called canonical, such that on the tangent space TM multiplication is like

$$\partial_a \circ \partial_b = \delta_{ab}\partial_b$$

All essential information is then encoded in the metric g which in these coordinates is diagonal.
Flatness of this metric g is described by Darboux-Egoroff equations.
In his paper [2], arising from a series of lectures, Manin discussed amongst others a super version of the notion of Frobenius manifolds. In this framework the notion of semisymplicity is nontrivial.
Finally Manin & Merkulov [] proposed a superversion of semisimple Frobenius and corresponding Darboux-Egoroff equations, in order to avoid superization of semisimple Frobenius manifolds.

In section 2 we give an outline of several notions involved.
In section 3 we prove linearity of the odd (potential) function φ.
Darboux-Egoroff equations and flatness of the unit vector field e are given for component functions in canonical coordinates in section 4.
In section 5 we give solutions of Darboux-Egoroff equations in $(2|2)$ dimensions including flatness of e moreover the solutions of Darboux-Egoroff equations in the presence of the Euler vector field E (homogeneous case) are given.
Finally in section 6 we construct solutions of Darboux-Egoroff equations in $(3|3)$ dimensions including flatness of e and in presence of the Euler vector field E as well.

2 General notions

In the extensive paper [1] B. Dubrovin developped the theory of Frobenius manifolds of extreme importance to physics, which puts the Witten-Dijkgraaf-Verlinde-Verlinde equations (WDVV) as associativity conditions of a specific multiplication structure of a tangent manifold. In his survey paper [2] Manin honoured him by writing *The geometric version of this theory was almost singlehandedly created by B. Dubrovin.* In [2] he showed the equivalence of

(a) Flat coordinates, flat metric $g_{a,b}$, and function ψ satisfying associativity conditions and semi simplicity.

(b) Canonical coordinates $(u^1, \dots, u^n)$, function $h(u)$ such that the metric $g = \sum e_i(h)(du^i)^2$ is flat. Flatness of metric g results in Darboux-Egoroff equations.

For convenience we are going to review the Darboux-Egoroff picture here following [3], where only those notions of importance to us will be given. For further definitions and details the reader is referred to [3].

In the following all considerations will be local.
We start from an $(n|n)$ dimensional supermanifold M and assume the existence of a canonical system of local coordinates $(u^\alpha, \vartheta^{\dot\alpha})(\alpha, \alpha = 1, \dots, n)$ where $u^\alpha(\alpha = 1, \dots, n)$ are even while $\vartheta^{\dot\alpha}(\dot\alpha = 1, \dots, n)$ are odd coordinates.

- *Local functions on M* are given by

$$f(u, \vartheta) = \sum_{k=0}^{n} \sum_{\dot\alpha, \dots, \dot\alpha_k = 1}^{n} f_{\dot\alpha, \dots, \dot\alpha_k}(u) \vartheta^{\dot\alpha_1} \dots \vartheta^{\dot\alpha_k} \tag{1}$$

- As a *basis for TM* we choose

$$\begin{aligned} e_{\dot\alpha} &= \frac{\partial}{\partial \vartheta^{\dot\alpha}} + \vartheta^{\dot\alpha} \frac{\partial}{\partial u^\alpha} = \partial_\alpha + \vartheta^{\dot\alpha} \partial_\alpha \quad & \dot\alpha = 1, \dots, n \\ e_\alpha &= \frac{\partial}{\partial u^\alpha} = \partial_\alpha & \alpha = 1, \dots, n \end{aligned} \tag{2}$$

- The subspaces $T_1, T_0 \subset TM$ over the $\mathbb{Z}_2$-graded ring of local functions are generated by

$$\begin{aligned} T_1 &: e_{\dot 1}, \dots, e_{\dot n} \\ T_0 &: e_1, \dots, e_n \end{aligned} \tag{3}$$

- The *Frobenius form* $\Phi : \Lambda^2 T_1 \to T_0$ is given by

$$X \otimes Y \mapsto \frac{1}{2}[X, Y] \mod T_1 \tag{4}$$

  whereas in (4), $[*, *]$ represents the $\mathbb{Z}_2$-graded Lie bracket of vector fields

- *Left and right odd involutions* π_ℓ, π_r, both denoted by π exist such that

  (i) $\pi(T_0) = T_1,\ \pi(T_1) = T_0,\ (T_0)\pi = T_1,\ (T_1)\pi = T_0,\ \pi^2 = id$

  (ii) $\pi(e_{\dot\alpha}) = (e_{\dot\alpha})\pi = e_\alpha$
  $\pi(e_\alpha) = (e_\alpha)\pi = e_{\dot\alpha}$

(iii)

$$\begin{array}{ll} \pi(aX) = (-1)^a \pi(X) & \pi(Xa) = \pi(X)a \\ (aX)\pi = a(X)\pi & (Xa)\pi = (X)\pi \cdot a \cdot (-1)^a \end{array} \quad (5)$$

- An *associative product* $\circ$ on TM is defined by

$$X \circ Y = \begin{cases} \Phi(X,Y) & X \in T_1, Y \in T_1 \\ \pi\Phi(\pi X, Y) & X \in T_0, Y \in T_1 \\ \Phi(X, Y\pi)\pi & X \in T_1, Y \in T_0 \\ \Phi(X\pi, \pi Y) & X \in T_0, Y \in T_0 \end{cases} \quad (6)$$

- An *Egoroff metric* g is defined on TM, i.e. an *odd* metric g on TM, which is *isotropic* on T_1 such that

$$g(\partial_\alpha, \partial_\beta) = -\delta_{\alpha\beta}\eta_\beta \quad g(\partial_\alpha, e_{\dot\beta}) = \delta_{\alpha\dot\beta}\eta_{\dot\alpha} \quad (7)$$

where in (7)

$$\eta_\alpha = \partial_\alpha \varphi \quad \eta_{\dot\alpha} = e_{\dot\alpha}\varphi = (\partial_{\dot\alpha} + \vartheta^{\dot\alpha}\partial_\alpha)\varphi$$

for some *odd* function φ, called *potential*.

From the above ingredients which define the notion of a *semisimple pre-Frobenius structure on the supermanifold M* we write the mulitplication table

$$\begin{array}{rcl} e_{\dot\alpha} \circ e_{\dot\beta} & = & \delta_{\dot\alpha\dot\beta} e_\alpha \\ e_{\dot\alpha} \circ e_\beta & = & e_\alpha \circ e_{\dot\beta} = \delta_{\alpha\beta} e_\alpha \\ e_\alpha \circ e_\beta & = & \delta_{\alpha\beta} e_\alpha \end{array} \quad (8)$$

or equivalently on basis $(\partial_\alpha, \partial_{\dot\alpha})$ $(\alpha, \dot\alpha = 1, \ldots, n)$

$$\begin{array}{rcl} \partial_{\dot\alpha} \circ \partial_{\dot\beta} & = & \delta_{\dot\alpha\dot\beta} \partial_\alpha \\ \partial_{\dot\alpha} \circ \partial_\beta & = & \partial_\alpha \circ \partial_\beta = \delta_{\alpha\beta} \partial_{\dot\alpha} \\ \partial_\alpha \circ \partial_\beta & = & \delta_{\alpha\beta} \partial_\alpha \end{array} \quad (9)$$

while the identity e is given by

$$e = \sum_{\alpha=1}^{n} e_\alpha = \sum_{\alpha=1}^{n} \partial_\alpha \quad (10)$$

The term canonical refers to the specific structure of multiplication on TM in these coordinates (8), (9).

Definition 2.1 *M is called* **semisimple Frobenius** *if g is flat.*

Theorem 2.1 *(Darboux-Egoroff equations)*

The metric g is flat if and only if φ satisfies

$$e_{\dot\mu}\gamma_{\dot\alpha\dot\beta} = \gamma_{\dot\mu\dot\alpha}\gamma_{\dot\mu\dot\beta} \quad (\dot\mu \neq \dot\alpha \neq \dot\beta \neq \dot\mu) \quad (11)$$

$$e(\gamma_{\dot\alpha\dot\beta}) = 0 \quad (12)$$

where

$$\gamma_{\dot\alpha\dot\beta} = \frac{e_{\dot\alpha}\eta_{\dot\beta}}{2\sqrt{\eta_{\dot\alpha}\eta_{\dot\beta}}} \quad (13)$$

Proposition 2.1 *The identity e is flat if and only if the potential φ satisfies*

$$e(\eta_{\dot{\alpha}}) = 0\ , \tag{14}$$

or equivalently

$$\sum_{\alpha=1}^{n} \eta_{\alpha} = 0. \tag{15}$$

The Euler vector field E on M is defined by

$$E = \sum_{\alpha,\dot{\alpha}=1}^{n} (u^{\alpha}\partial_{\alpha} + \frac{1}{2}\vartheta^{\dot{\alpha}}\partial_{\dot{\alpha}}) \tag{16}$$

and has to satisfy [3]

$$E(\eta_{\dot{\alpha}}) = (d - \frac{3}{2})\eta_{\dot{\alpha}} \tag{17}$$

3 Linearity of the potential φ

After writing down equations (11), part of Darboux-Egoroff equations (11 - 12), more explicitly, we shall prove that the potential φ defining the odd metric g (7) is linear with respect to the odd variables $\vartheta^{\dot\alpha}(\dot\alpha = 1,\dots,n)$, when we require flatness of g.
We start from (13)

$$\gamma_{\dot\alpha\dot\beta} = \frac{1}{2}e_{\dot\alpha}e_{\dot\beta}\varphi \cdot (e_{\dot\alpha}\varphi)^{-\frac{1}{2}}(e_{\dot\beta}\varphi)^{-\frac{1}{2}}.$$

Then (11) results in

$$\begin{aligned} 2\eta_{\dot\mu}\eta_{\dot\alpha}\eta_{\dot\beta}(e_{\dot\mu}e_{\dot\alpha}e_{\dot\beta}\varphi) &+ \eta_{\dot\mu}\eta_{\dot\beta}(e_{\dot\alpha}e_{\dot\beta}\varphi)(e_{\dot\mu}e_{\dot\alpha}\varphi) \\ &+ \eta_{\dot\mu}\eta_{\dot\alpha}(e_{\dot\alpha}e_{\dot\beta}\varphi)(e_{\dot\mu}e_{\dot\beta}\varphi) \\ &- \eta_{\dot\alpha}\eta_{\dot\beta}(e_{\dot\mu}e_{\dot\alpha}\varphi)(e_{\dot\mu}e_{\dot\beta}\varphi) = 0 \qquad (\dot\mu \neq \dot\alpha \neq \dot\beta \neq \dot\mu) \end{aligned} \tag{18}$$

whereas in (18)

$$\eta_{\dot\mu} = e_{\dot\mu}\varphi \quad , \quad e_{\dot\mu} = \partial_{\vartheta^{\dot\mu}} + \vartheta^{\dot\mu}\partial_\mu$$

We now start from a general *odd* function (potential) φ

$$\varphi = \sum_{1\leq|\dot A|\leq n} F^A\vartheta^{\dot A} \tag{19}$$

where $\dot A$ is a multi index

$$\dot A = (i_1,\dots,i_k), \;\; i_1 < i_2 < i_3 \dots < i_k \;, \;\; |\dot A| = k \quad (k \text{ is odd})$$

$$\vartheta^{\dot A} = \vartheta^{i_1}\vartheta^{i_2}\dots\vartheta^{i_k}\;;\; F^A = F^{i_1 i_2 \dots i_k}(u).$$

We shall omit summation notation in (19) and write

$$\varphi = F^A\vartheta^{\dot A} \tag{20}$$

We further introduce $-\dot\alpha + \dot A, \dot\alpha + \dot A$ as those multi indices arising in φ by differentiation of φ with respect to $\vartheta^{\dot\alpha}$ or multiplication of φ by $\vartheta^{\dot\alpha}$ respectively. Note further that $-\dot\alpha - \dot\beta + \dot A$ should be interpreted as 0 if $|\dot A| = 1$ or $\dot\alpha, \dot\beta$ are not in multi index $\dot A$.

Remark 3.1 *We do not take into account signs here in this notation which arise, due to*

$$\vartheta^{\dot\alpha}\vartheta^{\dot\beta} = -\vartheta^{\dot\beta}\vartheta^{\dot\alpha}$$

and the strict ordering of $\dot A$, since these signs play no role in the proof of the linearity of φ with respect to $\vartheta^{\dot\alpha}$.

With the above notations in mind we derive

$$\eta_{\dot\beta} = e_{\dot\beta}\varphi = (\partial_{\dot\beta} + \vartheta^{\dot\beta}\partial_\beta)(F^A\vartheta^{\dot A}) = F^A\vartheta^{-\dot\beta+\dot A} + (F^A)_\beta\vartheta^{+\dot\beta+\dot A} \tag{21}$$

where

$$\begin{aligned} (F^A)_\beta &= \partial_\beta F^A \\ e_{\dot\alpha} e_{\dot\beta}\varphi &= F^A \vartheta^{-\dot\alpha-\dot\beta+\dot A} + (F^A)_\beta \vartheta^{-\dot\alpha+\dot\beta+\dot A} + (F^A)_\alpha \vartheta^{+\dot\alpha-\dot\beta+\dot A} \\ &+ (F^A)_{\alpha\beta}\vartheta^{+\dot\alpha+\dot\beta+\dot A} \end{aligned} \tag{22}$$

$$\begin{aligned} e_{\dot\mu} e_{\dot\alpha} e_{\dot\beta}\varphi &= F^A \vartheta^{-\dot\mu-\dot\alpha-\dot\beta+\dot A} + (F^A)_\beta \vartheta^{-\dot\mu-\dot\alpha+\dot\beta+A} + (F^A)_\alpha \vartheta^{-\dot\mu+\dot\alpha-\dot\beta+\dot A} \\ &+ (F^A)_{\alpha\beta}\vartheta^{-\dot\mu+\dot\alpha+\dot\beta+\dot A} + (F^A)_\mu \vartheta^{+\dot\mu-\dot\alpha-\dot\beta+\dot A} + (F^A)_{\mu\beta}\vartheta^{+\dot\mu-\dot\alpha+\dot\beta+\dot A} \\ &+ (F^A)_{\mu\alpha}\vartheta^{+\dot\mu+\dot\alpha-\dot\beta+\dot A} + (F^A)_{\mu\alpha\beta}\vartheta^{+\dot\mu+\dot\alpha+\dot\beta+\dot A} \end{aligned} \tag{23}$$

Since (18), resulting from (11) is required for $\dot\mu \neq \dot\alpha \neq \dot\beta \neq \dot\mu$ no problems arise in notation (21), (22), (23).
Moreover, since we are only interested in those terms in (18) containing neither $\vartheta^{\dot\mu}$, nor $\vartheta^{\dot\alpha}$, nor $\vartheta^{\dot\beta}$ the only terms delivering a contribution in (18) are

$$\begin{aligned} \eta_{\dot\alpha} = (e_{\dot\alpha}\varphi) &\to F^A\vartheta^{-\dot\alpha+\dot A} \qquad \text{similar for } \eta_{\dot\mu}\ ,\ \eta_{\dot\beta} \\ e_{\dot\alpha}e_{\dot\beta}\varphi &\to F^A\vartheta^{-\dot\alpha-\dot\beta+\dot A} \\ e_{\dot\mu}e_{\dot\alpha}\varphi &\to F^A\vartheta^{-\dot\mu-\dot\alpha+\dot A} \\ e_{\dot\mu}e_{\dot\beta}\varphi &\to F^A\vartheta^{-\dot\mu-\dot\beta+\dot A} \\ e_{\dot\mu}e_{\dot\alpha}e_{\dot\beta}\varphi &\to F^A\vartheta^{-\dot\mu-\dot\alpha-\dot\beta+A} \end{aligned} \tag{24}$$

First of all, we consider terms in (18) of degree 0 with respect to the odd variables $\vartheta^{\dot\gamma}(\dot\gamma = 1, \dots, n)$.
Due to the fact that φ is required to be odd, i.e., degree $1, 3, 5, \dots$ in $\vartheta^{\dot\gamma}$ may occur, $|-\dot\epsilon - \dot\delta + \dot A|$ is *odd* and there are *no* contributions from the second, third or fourth term in (18). Therefore we only have a contribution to degree 0 terms due to the first term in (18) i.e., $F^{\alpha\beta\mu}$; leading to

$$F^{\alpha\beta\mu} = 0 \tag{25}$$

Since (18) has to hold for all $\dot\mu \neq \dot\alpha \neq \dot\beta \neq \dot\mu$, all terms in the representation of F (19) of degree 3 in $\vartheta^{\dot\gamma}(\dot\gamma = 1, \dots, n)$ cancel (have to be zero).
In order to prove linearity of φ, i.e. φ is of degree 1 in $\vartheta^{\dot\gamma}(\dot\gamma = 1, \dots, n)$ we proceed by induction with respect to k. We define

$$\varphi_k = F^\gamma\vartheta^{\dot\gamma} + \sum_{2k+1\leq|\dot A|\leq n} F^A\vartheta^{\dot A} \quad (k = 1, \dots) \tag{26}$$

We note that the most general form for φ is $\varphi = \varphi_1$ (i.e. $k = 1$).
We already proved above (19), i.e. (25)

$$\varphi = \varphi_2 \quad (k = 2) \tag{27}$$

Suppose $\varphi = \varphi_k$ for some $k \geq 2$. We are searching for contributions in (18) containing neither $\vartheta^{\dot\alpha}$, nor $\vartheta^{\dot\beta}$, nor $\vartheta^{\dot\mu}$ and of degree : $2k - 2$ in $\vartheta^{\dot\delta}(\dot\delta = 1, \dots)$.
In the first term of (18) there is only one contribution resulting from
$|A| = 2k + 1$, leading to terms

$$F^A\vartheta^{-\dot\mu-\dot\alpha-\dot\beta+\dot A} \quad , \quad |-\dot\mu - \dot\alpha - \dot\beta + \dot A| = 2k - 2 \tag{28}$$

In the second, third and fourth term, contributions *not containing* $\vartheta^{\dot{\alpha}}, \vartheta^{\dot{\beta}}, \vartheta^{\dot{\mu}}$ at lowest degree are

$$(2k+1-2)+(2k+1-2)=4k-2 \tag{29}$$

due to $e_{\dot{\alpha}} e_{\dot{\beta}} \varphi$ (24).
But

$$4k-2>2k-2 \qquad (k>0) \tag{30}$$

meaning that there are no contributions from the second, third or fourth term at degree : $2k-2$. From this observation we have (28) (18)

$$F^{A}=0 \quad \text{if} \;\; |\dot{A}|=2k+1 \tag{31}$$

From this we have: If $\varphi=\varphi_k$ then $\varphi=\varphi_{k+1}$.
Finally we have

$$\varphi=F^{\gamma} \vartheta^{\dot{\gamma}} \tag{32}$$

thus proving linearity of φ. □

4 Darboux-Egoroff equations and flatness of the unit vector field e

In section 3 we proved linearity of φ with respect to the odd variables $\vartheta^{\dot{\gamma}}(\dot{\gamma} = 1, \dots, n)$. Since we want to construct solutions of Darboux-Egoroff equations under additional assumption of the flatness of identity e, and the presence of an Euler vector field E, we discuss consequences of these facts for the partial differential equations for the component functions $F^{\gamma}(u)$ in the representation

$$\varphi = F^{\gamma}(u)\vartheta^{\dot{\gamma}} \tag{33}$$

Proposition 4.1 *If φ satisfies (14) then φ satisfies (12), so (12) is a consequence of (14) i.e.,*

$$e(\eta_{\dot{\alpha}}) = 0 \Rightarrow e(\gamma_{\dot{\alpha}\dot{\beta}}) = 0$$

Proof The proof is obvious due to the fact that

$$[e, e_{\dot{\alpha}}] = 0 \quad \dot{\alpha} = 1, \dots, n \tag{34}$$

□

Due to linearity of φ with respect to $\vartheta^{\dot{\gamma}}$ (33) we derive ($\dot{\mu} \neq \dot{\alpha} \neq \dot{\beta} \neq \dot{\mu}$)

$$\begin{aligned} e_{\dot{\alpha}}\varphi &= \eta_{\dot{\alpha}} = (\partial_{\dot{\alpha}} + \vartheta^{\dot{\alpha}}\partial_{\alpha})\vartheta = F^{\alpha} + (F^{\gamma})_{\alpha}\vartheta^{\dot{\alpha}+\dot{\gamma}} \\ e_{\dot{\alpha}}e_{\dot{\beta}}\varphi &= (F^{\beta})_{\alpha}\vartheta^{\dot{\alpha}} - (F^{\alpha})_{\beta}\vartheta^{\dot{\beta}} + (F^{\gamma})_{\alpha\beta}\vartheta^{\dot{\alpha}+\dot{\beta}+\dot{\gamma}} \\ e_{\dot{\mu}}e_{\dot{\alpha}}e_{\dot{\beta}}\varphi &= (F^{\mu})_{\alpha\beta}\vartheta^{\dot{\alpha}+\dot{\beta}} + \\ & (F^{\alpha})_{\beta\mu}\vartheta^{\dot{\beta}+\dot{\mu}} + (F^{\beta})_{\mu\alpha}\vartheta^{\dot{\mu}+\dot{\alpha}} + (F^{\gamma})_{\mu\alpha\beta}\vartheta^{\dot{\mu}+\dot{\alpha}+\dot{\beta}+\dot{\gamma}} \end{aligned} \tag{35}$$

We now substitute (35) into (18) i.e.,

$$\begin{aligned} 2\eta_{\dot{\mu}}\eta_{\dot{\alpha}}\eta_{\dot{\beta}}(e_{\dot{\mu}}e_{\dot{\alpha}}e_{\dot{\beta}}\varphi) + \eta_{\dot{\mu}}\eta_{\dot{\beta}}(e_{\dot{\alpha}}e_{\dot{\beta}}\varphi)(e_{\dot{\mu}}e_{\dot{\alpha}}\varphi) \\ +\eta_{\dot{\mu}}\eta_{\dot{\alpha}}(e_{\dot{\alpha}}e_{\dot{\beta}}\varphi)(e_{\dot{\mu}}e_{\dot{\beta}}\varphi) - \eta_{\dot{\alpha}}\eta_{\dot{\beta}}(e_{\dot{\mu}}e_{\dot{\alpha}}\varphi)(e_{\dot{\mu}}e_{\dot{\beta}}\varphi) = 0 \end{aligned}$$

Keeping in mind that $e_{\dot{\alpha}}\varphi$ is even and just contains terms of degree 0 and $2, e_{\dot{\alpha}}e_{\dot{\beta}}\varphi$ is odd and contains term of degree 1 and 3, $e_{\dot{\mu}}e_{\dot{\alpha}}e_{\dot{\beta}}\varphi$ is even and contains terms of degree 2 and 4, we observe that (18) is even, containing terms of degree $2, 4, 6, 8, 10$.
By inspection of (18) we observe that all terms of degree 10 drop out due to double appearances of $\vartheta^{\alpha}, \vartheta^{\dot{\beta}}$ *and* $\vartheta^{\dot{\mu}}$. By a similar inspection we see that terms of degree 8 drop out, due to double appearances of at least *two* of $\vartheta^{\dot{\alpha}}, \vartheta^{\dot{\beta}}, \vartheta^{\dot{\mu}}$. Finally terms of degree 6 drop out, due to double appearance of at least *one* of $\vartheta^{\dot{\alpha}}, \vartheta^{\dot{\beta}}, \vartheta^{\dot{\mu}}$. So the remaining terms in (18) are those of degree 2 and 4.
For terms of degree 2 i.e., $\vartheta^{\dot{\alpha}+\dot{\beta}}$ we derive

$$\begin{aligned} 2F^{\mu}F^{\alpha}F^{\beta}(F^{\mu})_{\alpha\beta} \quad &- \quad F^{\mu}F^{\beta}F^{\mu}_{\alpha}F^{\alpha}_{\beta} \\ &- \quad F^{\mu}F^{\alpha}F^{\mu}_{\beta}F^{\beta}_{\alpha} \\ &- \quad F^{\alpha}F^{\beta}F^{\mu}_{\alpha}F^{\mu}_{\beta} = 0 \qquad (\mu \neq \alpha \neq \beta \neq \mu) \end{aligned} \tag{36}$$

Now the condition for flatness of e (14) i.e.

$$e(\eta_{\dot{\alpha}}) = 0 \qquad (\alpha = 1, \dots, n)$$

results in

$$e(F^{\alpha} + (F^{\gamma})_{\alpha}\vartheta^{\dot{\alpha}+\dot{\gamma}}) = 0 \qquad (\alpha = 1, \dots, n)$$

from which we have

$$e(F^{\alpha}) = \mathrm{Div}(F^{\alpha}) = 0 \qquad (\alpha = 1, \dots, n) \tag{37}$$

A tedious calculation shows that the equations resulting from (18) by looking at terms of degree 4, consist of 16 terms and are differential consequences of (36) and (37). So in order to investigate (11), (12), (14) we may study the equivalent system (36), (37).
Summarizing the forgoing we have the following

Theorem 4.1 *Let $\varphi = F^{\gamma}(u)\vartheta^{\dot{\gamma}}$.*
Then the system of partial differential equations

$$\begin{array}{lcll} e_{\dot{\mu}}\gamma_{\dot{\alpha}\dot{\beta}} & = & \gamma_{\dot{\mu}\dot{\alpha}}\gamma_{\dot{\mu}\dot{\beta}} & \\ e(\gamma_{\dot{\alpha}\dot{\beta}}) & = & 0 & \textit{(Darboux-Egoroff)} \\ e(\eta_{\dot{\alpha}}) & = & 0 & \textit{(flatness of } e\textit{)} \end{array}$$

is equivalent to (36), (37) for the component functions F^{γ} i.e.,

$$\begin{array}{rcl} 2F^{\mu}F^{\alpha}F^{\beta}(F^{\mu})_{\alpha\beta} & - & F^{\mu}F^{\beta}F^{\mu}_{\alpha}F^{\alpha}_{\beta} \\ & - & F^{\mu}F^{\alpha}F^{\mu}_{\beta}F^{\beta}_{\alpha} \\ & - & F^{\alpha}F^{\beta}F^{\mu}_{\alpha}F^{\mu}_{\beta} = 0 \quad (\mu \neq \alpha \neq \beta \neq \mu) \end{array}$$

and

$$Div(F^{\alpha}) = 0 \qquad (\alpha = 1, \dots, n)$$

□

5 Solutions of Darboux-Egoroff equations in $(2|2)$ dimensions

We shall investigate the coordinate presentation of Darboux-Egoroff equations, flatness condition of e and construct solutions under presence of Euler vector field E, in $(2|2)$ dimensions .
First of all, note that in $(2|2)$ dimensions equations (11) are empty, while in canonical coordinates $(x, y, \vartheta^1, \vartheta^2)$ and the linearity of φ

$$\varphi = F^1(x,y)\vartheta^1 + F^2(x,y)\vartheta^2 \tag{38}$$

(12) results in

$$\begin{aligned} -2(eF^1_y)F^1F^2 + F^1_y(eF^1)F^2 + F^1_yF^1(eF^2) &= 0 \\ -2(eF^2_x)F^2F^1 + F^2_x(eF^2)F^1 + F^2_xF^2(eF^1) &= 0 \end{aligned} \tag{39}$$

where

$$e = \partial_x + \partial_y.$$

Flatness condition for e (14) i.e.

$$e(\eta_{\dot\alpha}) = 0,$$

results in (37)

$$e(F^1) = e(F^2) = 0\ , \tag{40}$$

as we already observed in section 4 (39) is a consequence of (40). So we have the following

Proposition 5.1 *The general solution of (39), (40) is given by*

$$\varphi = G_1(x-y)\vartheta^1 + G_2(x-y)\vartheta^2 \tag{41}$$

for arbitrary G_1, G_2.

We now discard the condition for flatness of e.
We want to construct solutions of (39) in the presence of an Euler vector field E, i.e.

$$E = x\partial_x + y\partial_y + \frac{1}{2}\vartheta^1\partial_{\vartheta^1} + \frac{1}{2}\vartheta^2\partial_{\vartheta^2} \tag{42}$$

which satisfies $E(\eta_{\dot\alpha}) = (d - \frac{3}{2})\eta_\alpha$.
From the conditions (17) for an Euler vector field, (homogeneous solutions) we have additional conditions to (39) i.e.,

$$\begin{aligned} xF^1_x + yF^1_y - (d - \tfrac{3}{2})F^1 &= 0 \\ xF^2_x + yF^2_y - (d - \tfrac{3}{2})F^2 &= 0 \end{aligned} \tag{43}$$

In order to solve (39), (43) we transform the system by

$$\begin{aligned} x &= v \\ y &= uv \end{aligned} \tag{44}$$

i.e.,

$$\begin{array}{rcl} v & = & x \\ u & = & \dfrac{y}{x} \end{array} \quad (45)$$

The general solution to (43) is then given by

$$\begin{array}{rcccl} F^1 & = & F^1(u,v) & = & F^3(u)v^{d-\frac{3}{2}} \\ F^2 & = & F^2(u,v) & = & F^4(u)v^{d-\frac{3}{2}} \end{array} \quad (46)$$

System (39) can be rewritten by (46) as

$$-F^4_u F^3_u F^3(u-1) + 2F^3_{uu}F^4F^3(u-1) - (F^3_u)^2F^4(u-1) + 2F^3_uF^3F^4 = 0 \quad (47)$$

and

$$\begin{array}{lclcl} 4F^4_{uu}F^4F^3(u^2-u) & - & 2(F^4_u)^2F^3(u^2-u) & - & 2F^4_uF^4F^3_u(u^2-u) \\ -2dF^4_uF^4F^3(u-1) & + & F^4_uF^4F^3(11u-7) & + & (F^4)^2F^3_u(u-1)(2d-3) \\ +(F^4)^2F^3(-4d+6) & = & 0 & & \end{array}$$

(48)

First integrals of (47) and (48) are

$$\frac{F^4F^3}{(F^3_u)^2(u-1)^2} = C_1 \quad (49)$$

and

$$(u-1)[2uF^4_u + (-2d+3)F^4]/(F^4F^3)^{\frac{1}{2}} = C_2 \quad (50)$$

respectively.
We solve F^4 from (49), i.e.,

$$F^4 = C_1(F^3_u)^2(u-1)^2/F^3 \quad (51)$$

and substitute the result into (50), which yields

$$4\frac{F^3_{uu}}{F^3}u(u-1)^2 - 2\left(\frac{F^3_u}{F^3}\right)^2 u(u-1)^2 + \frac{F^3_u}{F^3}(-2d(u-1)^2 + 7u^2 - 10u + 3) = C_2$$

(52)

being a first integral of (48).
Substitution

$$\frac{F^3_u}{F^3} = H \quad (53)$$

into (52) leads to

$$4u(u-1)^2H' + 2H^2u(u-1)^2 + H(-2d(u-1)^2 + 7u^2 - 10u + 3) = C_2 \quad (54)$$

For $C_2 = 0$ we obtain the solution

$$\frac{1}{H} = \left(2du^{\frac{3}{4}}(u-1) + u^{\frac{3}{4}}(u-1)\int \frac{u^{\frac{d}{2}}}{u^{\frac{3}{4}}(u-1)}du\right)/(2u^{\frac{d}{2}}) \qquad (55)$$

Now F^3, F^4 follow from (55) by quadrature of (53) and (51) directly, leading to the functions F^1, F^2 in terms of x, y from (44), (45), (46).
In general
Let $H(u)$ be a solution of (54), then the potential φ is given by

$$\varphi(x, y, \vartheta^1, \vartheta^2) = F^1(x, y)\vartheta^1 + F^2(x, y)\vartheta^2$$

where, by (54), (51), (46) $F^1(x, y), F^2(x, y)$ are given by

$$\begin{aligned} F^1(x, y) &= x^{d-\frac{3}{2}} \exp[\textstyle\int^{\frac{y}{x}} H(u)du] \\ F^2(x, y) &= x^{d-\frac{3}{2}} C_1 \left(\frac{y-x}{x}\right)^2 H(\frac{y}{x})^2 \exp[\textstyle\int^{\frac{y}{x}} H(u)du] \end{aligned}$$

6 Darboux-Egoroff equations in $(3|3)$ dimensions

In the first part of this section we discuss Darboux-Egoroff equations in $(3|3)$ dimensions, *together* with both the flatness condition for e, and the presence of an Euler vector field E given by

$$E = x\partial_x + y\partial_y + z\partial_z + \frac{1}{2}\vartheta^1\partial_{\vartheta^1} + \frac{1}{2}\vartheta^2\partial_{\vartheta^2} + \frac{1}{2}\vartheta^3\partial_{\vartheta^3} \tag{56}$$

where $(x, y, z, \vartheta^1, \vartheta^2, \vartheta^3)$ are canonical coordinates.
In the second part we skip the flatness condition for e and discuss integrability of the system of equations and give the number of parametric variables.
We start from the potential function φ, which is linear in $\vartheta^1, \vartheta^2, \vartheta^3$, and is given by

$$\varphi = F^1\vartheta^1 + F^2\vartheta^2 + F^3\vartheta^3 \tag{57}$$

where F^1, F^2, F^3 are functions of the even variables x, y, z.
Darboux-Egoroff equations (11), (12) are

$$e_{\dot\mu}\gamma_{\dot\alpha\dot\beta} = \gamma_{\dot\mu\dot\alpha}\gamma_{\dot\mu\dot\beta} \qquad (\dot\mu \neq \dot\alpha \neq \dot\beta \neq \dot\mu) \tag{58}$$

$$e(\gamma_{\dot\alpha\dot\beta}) = 0 \qquad (\dot\alpha \neq \dot\beta) \tag{59}$$

while *flatness for e* yields

$$e(e_{\dot\alpha}\varphi) = 0 \tag{60}$$

and *the presence of the Euler vector field*, homogeniety of φ,

$$E(e_{\dot\alpha}\varphi) = (d - \frac{3}{2})(e_\alpha\varphi) \tag{61}$$

From (57) and (58), (59) we derive the following system of nine partial differential equatons of second order, i.e.:

$$\begin{aligned}
2F^1_{yz} - F^1_yF^1_z(F^1)^{-1} - F^1_yF^2_z(F^2)^{-1} - F^1_zF^3_y(F^3)^{-1} &= 0\\
2F^2_{zx} - F^2_zF^2_x(F^2)^{-1} - F^2_zF^3_x(F^3)^{-1} - F^2_xF^1_z(F^1)^{-1} &= 0\\
2F^3_{xy} - F^3_xF^3_y(F^3)^{-1} - F^3_xF^1_y(F^1)^{-1} - F^3_yF^2_x(F^2)^{-1} &= 0
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
2e(F^1_y)F^1F^2 - F^1_ye(F^1)F^3 - F^1_yF^1e(F^2) &= 0\\
2e(F^1_z)F^1F^3 - F^1_ze(F^1)F^3 - F^1_zF^1e(F^3) &= 0\\
2e(F^2_x)F^2F^1 - F^2_xe(F^2)F^1 - F^2_xF^2e(F^1) &= 0\\
2e(F^2_z)F^2F^3 - F^2_ze(F^2)F^3 - F^2_zF^2e(F^3) &= 0\\
2e(F^3_x)F^3F^1 - F^3_xe(F^3)F^1 - F^3_xF^3e(F^1) &= 0\\
2e(F^3_y)F^3F^2 - F^3_ye(F^3)F^2 - F^3_yF^3e(F^2) &= 0
\end{aligned} \tag{63}$$

The other conditions arising from Darboux-Egoroff equations (58),(59) are differential consequences of (62, 63).
From (62),(63) we can write

$$F^1_{yz}, F^1_{yy}, F^1_{zz}, F^2_{xz}, F^2_{xx}, F^2_{zz}, F^3_{xy}, F^3_{xx}, F^3_{yy}$$

in terms of the remaining partial derivatives, without *additional compatibility* conditions.

Remark 6.1 *The remaining equations resulting from (58) are consequences of (62) while the remaining condition from (59) are consequences of (62) and (63).*

We note further that flatness of e (60), and (61) result in

$$e(F^i) = 0 \qquad (i = 1, 2, 3) \tag{64}$$

and

$$(x\partial_x + y\partial_y + z\partial_z)F^i = (d - \frac{3}{2})F^i \qquad (i = 1, 2, 3) \tag{65}$$

Due to the invariances defined by (64) and (65) we carry through the following transformation of coordinates

$$x = v \ , \ y = uv \ , \ z = wv \tag{66}$$

or equivalently

$$v = x \ , \ u = \frac{y}{x} \ , \ w = \frac{z}{x} \tag{67}$$

Condition (65) transforms to

$$vF^i_v - (d - \frac{3}{2})F^i = 0, \qquad (i = 1, 2, 3) \tag{68}$$

the solution of which is given by

$$F^i(x, y, z) \equiv F^i(v, u, w) = v^{(d-\frac{3}{2})}\overline{F}^i(u, w)(i = 1, 2, 3) \tag{69}$$

Substitution of (69) into (64) i.e.,

$$F^i_x + F^i_y + F^i_z = 0 \qquad (i = 1, 2, 3)$$

leads to

$$((u-1)\partial_u + (w-1)\partial_w)\overline{F}^i(u, w) = (d - \frac{3}{2})\overline{F}^i(u, w) \tag{70}$$

Similar to (66) we transform

$$u - 1 = r \ , \ w - 1 = sr \tag{71}$$

or

$$r = u - 1 \ , \ s = \frac{w-1}{u-1}$$

while (70) reduces to

$$r\partial_r \overline{F}^i = (d - \frac{3}{2})\overline{F}^i \qquad (i = 1, 2, 3) \tag{72}$$

Now $\overline{F}^i(u, w)(i = 1, 2, 3)$ is given by

$$\overline{F}^i(r, s) = r^{(d - \frac{3}{2})} H^i(s). \tag{73}$$

Substitution of the result obtained on $F^i(i = 1, 2, 3)$ thusfar does transform (62) into system 3 ordinary second order equations for the functions $H^i(s)(i = 1, 2, 3)$i.e.,

$$\begin{aligned} - &\ 4sH_1''H_1H_2H_3 + 2sH_1'H_3'H_1H_2 + 2sH_1'H_2'H_1H_3 - (2d-3)H_2'(H_1)^2H_3 \\ + &\ 2s(H_1')^2H_2H_3 - 4H_1'H_1H_2H_3 = 0 \end{aligned} \tag{74}$$

$$\begin{aligned} - &\ 4(s-1)H_2''H_1H_2H_3 + 2(s-1)H_2'H_3'H_1H_2 + 2(s-1)(H_2')^2H_1H_3 \\ + &\ 2(s-1)H_1'H_2'H_2H_3 - 4H_2'H_1H_2H_3 - (2d-3)H_1'(H_2)^2H_3 = 0 \end{aligned}$$

$$\begin{aligned} &\ 8s(s-1)H_3''H_1H_2H_3 - 4s(s-1)(H_3')^2H_1H_2 - 4s(s-1)H_2'H_3'H_1H_3 \\ - &\ 4s(s-1)H_1'H_3'H_2H_3 + (16s-8)H_3'H_1H_2H_3 + 2(2d-3)(s-1)H_2'H_3^2H_1 \\ + &\ 2(2d-3)sH_1'H_2H_3^2 + (-4d^2+4d+3)H_1H_2H_3^2 = 0 \end{aligned}$$

The general solution of this system contains 6 parameters.
Finally by transformations

$$H_1(s) = e^{\int G_1} \;,\; H_2(s) = e^{\int G_2} \;,\; H_3(s) = e^{\int G_3} \tag{75}$$

we arrive at a system of three first order ordinary differential equations

$$\begin{aligned} -4sG_1' + 2sG_1(-G_1 + G_2 + G_3) - (2d-3)G_2 - 4G_1 &= 0 \\ -4(s-1)G_2' + 2(s-1)G_2(G_1 - G_2 + G_3) - 4G_2 - (2d-3)G_1 &= 0 \\ 8s(s-1)G_3' + 4s(s-1)G_3(G_3 - G_2 - G_1) + (16s-8)G_3 & \\ +2(2d-3)(s-1)G_2 + 2s(2d-3)G_1 - 4d^2 + 4d + 3 &= 0 \end{aligned} \tag{76}$$

A special solution of (76) is given by $(a = \frac{2d-3}{4})$

$$G_1 = \frac{a}{s} \;,\; G_2 = \frac{a}{s-1} \;,\; G_3 = a\frac{(2s-1)}{s(s-1)} \tag{77}$$

From (77) we obtain the solution to the original problem

$$\begin{aligned} F^1 &= p(1)^a \\ F^2 &= p(2)^a \\ F^3 &= p(3)^a \end{aligned}$$

$$\begin{aligned} p_1 &= x^2 - xy - xz + yz &&= (y-x)(z-x) \\ p^2 &= xy - xz - y^2 + yz &&= (z-y)(y-x) \\ p_3 &= xy - xz - yz + z^2 &&= (z-x)(z-y) \end{aligned}$$

A number of special solutions is given in the Appendix.
In the last part of this section we shall discuss Darboux-Egoroff equations together with the condition for the Euler vector field E but without flatness of e.
Since (59) is a consequence of the flatness of e the system we now discuss (58), (59), (61) will be more general. In order to construct solutions to this system we again start at transformation (66), (67) i.e.,

$$x = v \quad y = uv \quad z = wv$$

Although system (59) is more complicated than (60) the vector field e plays a special role, and after solving (61) for $F^i(v,u,w)$ in terms of v i.e.,

$$F^i(v,u,w) = v^{d-\frac{3}{2}}\overline{F}^i(u,w) \qquad (i = 1,2,3) \tag{78}$$

and thus eliminating (61) we transform (58), (59) by

$$u = r + 1 \;\;,\;\; w = sr + 1$$

We note that

$$\begin{aligned} u &= \frac{y}{x} \quad , & w &= \frac{z}{x} \\ r &= \frac{y-x}{x} & s &= \frac{z-x}{y-x} \end{aligned} \tag{79}$$

The resulting system consists of nine partial differential equations for the three functions $\overline{F}^i (i = 1,2,3)$, for symplicity denoted by F^i. Since dependency is on

r, s no confusion with original F^i will arise. This new system is given by

$$
\begin{aligned}
\text{equ (1)} &= (F^2)_r(F^1)_rF^1r - (F^2)_r(F^1)_sF^1s \\
&+ 2(F^1)_{rs}F^2F^1s - 2(F^1)_{rr}F^2F^1r \\
&+ (F^1)_r^2F^2r - (F^1)_r(F^1)_sF^2s - 2(F^1)_rF^2F^1 \\
\\
\text{equ (2)} &= (F^3)_r(F^1)_sF^2F^1r - (F^3)_s(F^1)_sF^2F^1s + (F^2)_s(F^1)_rF^3F^1r \\
&- (F^2)_s(F^1)_sF^3F^1s - 2(F^1)_{rs}F^3F^2F^1r + (F^1)_r(F^1)_sF^3F^2r \\
&+ 2(F^1)_{ss}F^3F^2F^1s - (F^1)_s^2F^3F^2s + 2(F^1)_sF^3F^2F^1 \\
\\
\text{equ (3)} &= -(F^3)_r(F^1)_sF^1 + 2(F^1)_{rs}F^3F^1 - (F^1)_r(F^1)_sF^3 \\
\\
\text{equ (4)} &= -4(F^2)_{rs}F^2F^1s + 4(F^2)_{rs}F^2F^1 + 4(F^2)_{rr}F^2F^1r^2 + 4(F^2)_{rr}F^2F^1r \\
&- 2(F^2)_r^2F^1r^2 - 2(F^2)_r^2F^1r + 2(F^2)_r(F^2)_sF^1s - 2(F^2)_r(F^2)_sF^1 \\
&- 2(F^2)_r(F^1)_rF^2r^2 - 2(F^2)_r(F^1)_rF^2r - 2(F^2)_rF^2F^1dr \\
&+ 11(F^2)_rF^2F^1r + 4(F^2)_rF^2F^1 + 2(F^2)_s(F^1)_rF^2s - 2(F^2)_s(F^1)_rF^2 \\
&+ 2(F^1)_r(F^2)^2dr - 3(F^1)_rF^2F^2r - 4(F^2)^2F^1d + 6(F^2)^2F^1 \\
\\
\text{equ (5)} &= 2(F^3)_r(F^2)_sF^2F^1r^2 + 2(F^3)_r(F^2)_sF^2F^1r - 2(F^3)_s(F^2)_sF^2F^1s \\
&+ 2(F^3)_s(F^2)_sF^2F^1 - 4(F^2)_{rs}F^3F^2F^1r^2 - 4(F^2)_{rs}F^3F^2F^1r \\
&+ 2(F^2)_r(F^2)_sF^3F^1r^2 + 2(F^2)_r(F^2)_sF^3F^1r + 2(F^2)_r(F^1)_sF^3F^2r^2 \\
&+ 2(F^2)_r(F^1)_sF^3F^2r + 4(F^2)_{ss}F^3F^2F^1s - 4(F^2)_{ss}F^3F^2F^1 \\
&- 2(F^2)_s^2F^3F^1s + 2(F^2)^2F^3F^1 - 2(F^2)_s(F^1)_sF^3F^2s + 2(F^2)_s(F^1)_sF^3F^2 \\
&+ 4(F^2)_sF^3F^2F^1 - 2(F^1)_sF^3(F^2)^2dr + 3(F^1)_sF^3(F^2)^2r \\
\\
\text{equ (6)} &= -(F^3)_r(F^2)_sF^2 + 2(F^2)_{rs}F^3F^2 - (F^2)_r(F^2)_sF^3 \\
\\
\text{equ (7)} &= -4(F^3)_{rs}F^3F^1s + 4(F^3)_{rs}F^3F^1 + 4(F^3)_{rr}F^3F^1r^2 + 4(F^3)_{rr}F^3F^1r \\
&- 2(F^3)_r^2F^1r^2 - 2(F^3)_r^2F^1r + 2(F^3)_r(F^3)_sF^1s - 2(F^3)_r(F^3)_sF^1 \\
&- 2(F^3)_r(F^1)_rF^3r^2 - 2(F^3)_r(F^1)_rF^3r - 2(F^3)_rF^3F^1dr + 11(F^3)_rF^3F^1r \\
&+ 4(F^3)_rF^3F^1 + 2(F^3)_s(F^1)_rF^3s - 2(F^3)_s(F^1)_rF^3 + 2(F^1)_r(F^3)^2dr \\
&- 3(F^1)_r(F^3)^2r - 4(F^3)^2F^1d + 6(F^3)^2F^1
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
\text{equ (8)} &= 4(F^3)_{rs}F^3F^2F^1r^2s + 8(F^3)_{rs}F^3F^2F^1rs - 4(F^3)_{rs}F^3F^2F^1r \\
&- 4(F^3)_{rr}F^3F^2F^1r^3 - 4(F^3)_{rr}F^3F^2F^1r^2 + 2(F^3)_r^2F^2F^1r^3 \\
&+ 2(F^3)_r^2F^2F^1r^2 - 2(F^3)_r(F^3)_sF^2F^1r^2s - 4(F^3)_r(F^3)_sF^2F^1rs \\
&+ 2(F^3)_r(F^3)_sF^2F^1r + 2(F^3)_r(F^2)_rF^3F^1r^3 + 2(F^3)_r(F^2)_rF^3F^1r^2 \\
&- 2(F^3)_r(F^2)_sF^3F^1rs + 2(F^3)_r(F^2)_sF^3F^1r + 2(F^3)_r(F^1)_rF^3F^2r^3 \\
&+ 2(F^3)_r(F^1)_rF^3F^2r^2 - 2(F^3)_r(F^1)_sF^3F^2r^2s - 2(F^3)_r(F^1)_sF^3F^2rs \\
&- 4(F^3)_rF^3F^2F^1r^2 - 4(F^3)_{ss}F^3F^2F^1s^2 + 4(F^3)_{ss}F^3F^2F^1s + 2(F^3)_s^2F^2F^1s^2 \\
&- 2(F^3)_s^2F^2F^1s - 2(F^3)_s(F^2)_rF^3F^1r^2s - 2(F^3)_s(F^2)_rF^3F^1rs \\
&+ 2(F^3)_s(F^2)_sF^3F^1s^2 - 2(F^3)_s(F^2)_sF^3F^1s - 2(F^3)_s(F^1)_rF^3F^2rs \\
&+ 2(F^3)_s(F^1)_rF^3F^2r + 2(F^3)_s(F^1)_sF^3F^2s^2 - 2(F^3)_s(F^1)_sF^3F^2s \\
&- 8(F^3)_sF^3F^2F^1s + 4(F^3)_sF^3F^2F^1 - 2(F^1)_r(F^3)^2F^2dr^2 + 3(F^1)_r(F^3)^2F^2r^2 \\
&+ 2(F^1)_s(F^3)^2F^2drs - 3(F^1)_s(F^3)^2F^2rs
\end{aligned}
$$

$$
\begin{aligned}
\text{equ (9)} &= 2(F^3)_{rs}F^3F^2s - 2(F^3)_{rr}F^3F^2r + (F^3)_r^2F^2r - (F^3)_r(F^3)_sF^2s \\
&+ (F^3)_r(F^2)_rF^3r - 2(F^3)_rF^3F^2 - (F^3)_s(F^2)_rF^3s
\end{aligned}
$$

From this system of equations (80) we construct six first integrals with respect to r

$$
\begin{array}{l}
\{-rF_r^1 + sF_s^1\}(F^1)^{-\frac{1}{2}}(F^2)^{-\frac{1}{2}} = h^1(s) \\
F_s^1(F^1)^{-\frac{1}{2}}(F^3)^{-\frac{1}{2}} = h^2(s) \\
\{2(r^2+r)F_r^2 + 2(1-s)F_s^2 - 2(d-\frac{3}{2})rF^2\}(F^4)^{-\frac{1}{2}}(F^2)^{-\frac{1}{2}} = h^3(s) \\
F_s^2(F^2)^{-\frac{1}{2}}(F^3)^{-\frac{1}{2}} = h^4(s) \\
\{2(r^2+r)F_r^3 + 2(1-s)F_s^3 - 2(d-\frac{3}{2})rF^3\}(F^1)^{-\frac{1}{2}}(F^3)^{-\frac{1}{2}} = h^5(s) \\
\{-rF_r^3 + sF_s^3\}(F^2)^{-\frac{1}{2}}(F^3)^{-\frac{1}{2}} = h^6(s)
\end{array}
\tag{81}
$$

associated to equations $1, 3, 4, 6, 7, 9$ in (80) respectively.
In order to construct solutions to the original problem (58), (59), (61) we solve (81-1,2) for F_r^1, F_s^1 respectively

$$
\begin{aligned}
F_r^1 &= \frac{1}{r}(F^1)^{\frac{1}{2}}[s(F^3)^{\frac{1}{2}}h^2 - (F^2)^{\frac{1}{2}}h^1] \\
F_s^1 &= h^2(F^1F^3)^{\frac{1}{2}}
\end{aligned}
\tag{82}
$$

From (80-2,3,5,8) we obtain expressions for $F_s^2, F_s^3, F_r^2, F_r^3$ respectively. It turns out that compatibility conditons on these expressions

$$\partial_r(F_s^i) = \partial_s(F_r^i)$$

are just equivalent to the remaining system of two ordinary third order equations for h^1, h^2

$$
\begin{array}{llllll}
& (s^2-s)h_2'''(h_1')^2h_2 & - & (s^2-s)h_2''h_2'h_1^2 & - & 2(s^2-s)h_2''h_1'h_1h_2 \\
+ & (5s-3)h_2''h_1^2h_2 & + & 2(s^2-s)(h_2')^2h_1'h_1 & + & (-3s+2)(h_2')^2h_1^2 \\
- & 2(s^2-s)h_2'h_1'h_1h_2 & + & 2(s^2-s)h_2'(h_1')^2h_2 & + & (-5s+2)h_1'h_2'h_1h_2 \\
+ & 3h_2'h_2h_1^2 & - & 2(s-1)h_1''h_1h_2^2 & + & 2(s-1)(h_1')^2h_2^2 \\
- & 3h_1'h_1(h_2)^2 & = & 0 & & \\
\\
- & (s^2-s)h_1'''h_1h_2^2 & + & 2(s^2-s)h_2''h_1'h_1h_2 & - & 2(s^2-s)(h_2')^2h_1'h_1 \\
+ & 2(s^2-s)h_1''h_2'h_1h_2 & - & 2(s^2-s)(h_1')^2h_2'h_2 & + & (6s-3)h_1'h_2'h_1h_2 \\
+ & (s^2-s)h_1''h_1'h_2^2 & + & (-3s+1)h_1''h_1h_2^2 & + & (h_1')^2h_2^2 = 0
\end{array}
\tag{83}
$$

We now put

$$
\begin{array}{lll}
h_1 & = & e^{h_3} \\
h_2 & = & e^{h_4}
\end{array}
\tag{84}
$$

which results in

$$
\begin{array}{llllll}
& (s^2-s)h_4''' & + & 2(s^2-s)h_4''h_4' & - & 2(s^2-s)h_4''h_3' \\
+ & (5s-3)h_4'' & + & (2s-1)(h_4')^2 & - & (s^2-s)h_3''h_4' \\
+ & (-5s+2)h_3'h_4' & + & 3h_4'-2(s-1)h_3'' & - & 3h_3' = 0 \\
- & (s^2-s)h_3''' & + & 2(s^2-s)h_4''h_3' & + & 2(s^2-s)h_3''h_4' \\
+ & (6s-3)h_3'h_4' & - & 2(s^2-s)h_3''h_3' & + & (-3s+1)h_3'' \\
+ & (-3s+2)(h_3')^2 & = & 0 & &
\end{array}
\tag{85}
$$

while after introduction of

$$
h_3' = H_1 \qquad h_4' = H_2 \tag{86}
$$

we arrive at

$$
\begin{array}{l}
-(s^2-s)H_1'' + 2(s^2-s)H_2'H_1 + 2(s^2-s)H_1'H_2 + (6s-3)H_1H_2 \\
-2(s^2-s)H_1'H_1 + (-3s+1)H_1' + (-3s+2)H_1^2 = 0 \\
(s^2-s)H_2'' + 2(s^2-s)H_2'H_2 - 2(s^2-s)H_2'H_1 + (5s-3)H_2' + (2s-1)H_2^2 \\
-s(s^2-s)H_1'H_2 + (-5s+2)H_1H_2 + 3H_2 - 2(s-1)H_1' - 3H_1 = 0
\end{array}
\tag{87}
$$

Now from (82), compatibilities of partial derivatives $\partial_r F^i, \partial_s F^i$ and the resulting system of two third order ordinary differential equations (83) we conclude that the total number of parametric variables of system (58), (59), (61) is just nine. These parameters arise from:

1. expressions for $F^1_{r'} F^1_{s'} F^2_{r'} F^2_{s'} F^3_{r'} F^3_{s'}$; i.e., three

2. from the system of third order equations (83); i.e., six.

Example 6.1 *We now construct special solutions of the system (58),(59), (61) by imposing an additional conditon on H_1, H_2 by putting (81-3) to zero i.e.*

$$H_1' \equiv \frac{H_1}{s-1}((H_2 - H_1)(s-1) - 1) \tag{88}$$

From the first equation in (87)

$$H_2' = (-s(s-1)H_2(H_2 - H_1) + (-3s+2)H_2 + (s-1)H_1 - 1)/s(s-1) \tag{89}$$

This system finally can be reduced to a single second order differential equations for the function $\overline{H}_2$ defined by

$$H_2 = \frac{\overline{H}_2 - 1}{s-1}, \tag{90}$$

while the ordinary differential equation is given by

$$s(s-1)\overline{H}_2'' + 2s\overline{H}_2'\overline{H}_2 + (2s-3)\overline{H}_2' + \overline{H}_2^2 = 0 \tag{91}$$

which is equivalent to

$$s(s-1)\overline{H}_2' + s\overline{H}_2^2 - 2\overline{H}_2 + C = 0 \tag{92}$$

Now we summarize the result.
Let $\overline{H}_2$ be the solution of (92) then $H_1(s), H_2(s)$ are obtained from (88), (90). From (86) we obtain

$$h_3 = \int H_1 \quad , \quad h_4 = \int^s H_2 \tag{93}$$

while (84)

$$h_1 = e^{\int^s H_1} \quad , \quad h_2 = e^{\int^s H_2} \tag{94}$$

and $F^1(r,s), F^2(r,s), F^3(r,s)$ satisfy

$$\begin{aligned}
F_r^1 &= \tfrac{1}{s}(F^1)^{\frac{1}{2}} s \cdot h_2 (F^3)^{\frac{1}{2}} - h_1 (F^2)^{\frac{1}{2}} \\
F_s^1 &= h_2 (F^1 F^3)^{\frac{1}{2}} \\
F_r^2 &= (F^2)^{\frac{1}{2}}[(2d-3) r h_1 (F^2)^{\frac{1}{2}} + 4 h_2 (H_2 s - 1)(F^3)^{\frac{1}{2}}]/r(r+1) \\
F_s^2 &= 2h_2(sH_2 - 1)(F^3)^{\frac{1}{2}}/(h_1(s-1)) \\
F_r^3 &= (F^3)^{\frac{1}{2}}[h_2(2d-3)(F^3)^{\frac{1}{2}} rs(H_2 s - 1) \\
&\quad -4h_1(\text{const} - 1)(F^2)^{\frac{1}{2}}]/(2h_2 r(H_2 s - 1)(rs+1)) \\
F_s^3 &= (F^3)^{\frac{1}{2}}[h_2(F^3)^{\frac{1}{2}}(sH_2 - 1)(2d-3)r(s-1) \\
&\quad -4h_1(F^2)^{\frac{1}{2}}(c-1)(r+1)] \\
&\quad /(2h_2(sH_2 - 1)(s-1)(rs+1))
\end{aligned} \tag{95}$$

The solution of (95) together with (78), (79) yields the final solution to (58), (59), (62-63).

Example 6.2 *As a last application we put*

$$H_2 = -\frac{1}{s}. \tag{96}$$

Then (87) reduces to

$$\begin{array}{cccccc} - & H_1''s^2(s-1) & - & 2s^2(s-1)H_1'H_1 & + & (-5s^2+3s)H_1' \\ + & (-3s^2+2s)H_1^2 & + & H_1(-4s+1) & = & 0 \end{array} \tag{97}$$

A first integral of this equation is

$$-s^2(s-1)H_1' - s^2(s-1)H_1^2 + s(-2s+1)H_1 + C = 0 \tag{98}$$

Remark: For $C = 0$

$$(H_1{}^{-1} = s(s-1)[\alpha + \log(s-1) - \log s]$$

Now to summarize the result here, we have

$$h_1 = e^{\int H_1} \qquad h_2 = e^{\int -\frac{1}{s}} = -\frac{a}{s} \tag{99}$$

$$\begin{aligned} F_r^1 &= \frac{1}{r}(F^1)^{\frac{1}{2}}[sh_2(F^3)^{\frac{1}{2}} - h_1(F^2)^{\frac{1}{2}}] \\ F_s^1 &= h_2(F^1F^3)^{\frac{1}{2}} \\ F_r^2 &= F^2(2d-3)/2(r+1) \\ F_s^2 &= 0 \\ F_r^3 &= (F^3)^{\frac{1}{2}}[h_2(2d-3)rs^2(F^3)^{\frac{1}{2}} + 4h_1(s^2-s)H_1(F^2)^{\frac{1}{2}} + 8(F^1)^{\frac{1}{2}}C] \\ &\quad /2h_2rs(rs+1) \\ F_s^3 &= (F^2)^{\frac{1}{2}}[h_2(2d-3)rs^2(F^3)^{\frac{1}{2}} + 4h_1s^2(r+1)H_1(F^2)^{\frac{1}{2}} + 8(F^1)^{\frac{1}{2}}C] \\ &\quad /2h_2s^2(rs+1) \end{aligned}$$

from which the final solution of (58), (59), (61) with additional condition (96) is obtained.

Similar special solutions of (58), (59), (61) are obtained by specific choices of partial first integrals in (81).

Appendix

Other special solutions to (76) are obtained by

1. $G_1 = G_3$

$$G_1(s) = \frac{2d-3}{2s} \quad G_3 = \frac{2d-3}{2s} \tag{100}$$

G_2 being a solution of

$$-8s(s-1)G_2' - 4s(s-1)G_z^2 + 8d(s-1)G_2 - (20s-12)G_2 - 4d^2 + 12d - 9 = 0 \tag{101}$$

2. $G_2 = G_3$

$$G_2(s) = \frac{2d-3}{2(s-1)} \quad G_3 = \frac{2d-3}{2(s-1)} \tag{102}$$

while G_1 satisfies

$$-8s(s-1)G_1' - 4s(s-1)G_1^2 + 8dsG_1 + 8G_1 - 20sG_1 - 4d^2 + 12d - 9 = 0 \tag{103}$$

3.

$$G_1 = G_2 \Rightarrow d = -\frac{1}{2} \Rightarrow \tag{104}$$

$$G_3 = 2\frac{G_2'}{G_2} \tag{105}$$

and G_2 satisfies

$$2s(s-1)G_2'' - 2s(s-1)G_2'G_2 + 2(2s-1)G_2' - (2s-1)G_2^2 = 0 \tag{106}$$

A first integral of (106) is

$$2s(s-1)G_2' - s(s-1)G_2^2 = \text{const} \tag{107}$$

The final result is obtained, using (70), (71) i.e.,

$$r = \frac{y-x}{x} \qquad s = \frac{z-x}{y-x}$$
$$v = x, u = \frac{y}{x}, w = \frac{z}{x}$$

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