4th Twente Workshop on Cooperative Game Theory joint with 3rd Dutch–Russian Symposium

WCGT2005, June 28–30, 2005

edited by
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Foreword

This workshop volume contains the program, in its global and detailed version, as well as the contributions by the invited speakers as well as the extended abstracts by the Russian participants.

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Sponsors

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Enschede, June 2005

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Cubicus building, room C-238

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On Bargaining Sets and Voting Games

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On noncooperative games and minimax theory

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A SURVEY OF BICOOPERATIVE GAMES

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Abstract. The aim of the present paper is to study several solution concepts for bicooperative games. For these games introduced by Bilbao [1], we define a one-point solution called the Shapley value, since this value can be interpreted in a similar way to the classical Shapley value for cooperative games. The firs result of the paper is an axiomatic characterization of this value. Next, we define the core and the Weber set and prove that the core of a bicooperative game is always contained in its Weber set. Finally, we introduce an special class of bicooperative games, the so-called bisupermodular games, and show that these games are the only ones in which their core and the Weber set coincide.

Keywords Bicooperative games, Bisupermodular games, Shapley value, Core, Weber set

1. Introduction

The theory of cooperative games studies situations where a group of people/agents are associated to obtain a profit as a result of their cooperation. Thus, a cooperative game is defined as a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a function verifying that $v(\emptyset) = 0$. For each $S \in 2^N$, the worth $v(S)$ can be interpreted as the maximal gain or minimal cost that the players which form coalition S can achieve themselves against the best offensive threat by the complementary coalition $N \setminus S$. Classical market games for economies with private goods are examples of cooperative games. Hence, we can say that a cooperative game has *orthogonal coalitions* (see Myerson [10]).

Games with non-orthogonal coalitions are games in which the worth of coalition S are not independent of the actions of coalition $N \setminus S$. Clearly, social situations involving externalities and public goods are such cases. For instance, we consider a group of agents with a common good which is causing them expenses or costs. In a external or internal way, a modification (sale, buying, etc.) of this good is proposed to them. This action will suppose a greater profit to them in case they all agree with the change proposed about the actual situation of the good. Moreover, even though the patrimonial good can be divisible, we suppose that the greatest value of the selling operation is reached if we consider all the common good.

A possibility of modeling these situations may be the following. We consider pairs (S, T), with $S, T \subseteq N$ and $S \cap T = \emptyset$. Thus, (S, T) is a partition of N in three groups. Players in S are defenders of modifying the *statu quo* and they want to accept a proposal; players in T do not agree with modifying the situation and they will take action against any change. Finally, the members of $N \setminus (S \cup T)$ are not convinced of the profits derived from the proposal and they vote abstention.

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Thus, in our model we consider the set of all ordered pairs of disjoint coalitions $3^N = \{(S,T) : S,T \subseteq N, \ \ S \cap T = \emptyset\}$, and define a function $b : 3^N \to \mathbb{R}$. For each $(S,T) \in 3^N$, the worth $b(S,T)$ can be interpreted as the maximal gain (whenever $b(S,T) > 0$) or minimal loss (whenever $b(S,T) < 0$) that the players of the coalition S can achieve when they decide to play together against the players of T and the players of $N \setminus (S \cup T)$ not taking part. This leads us in a natural way into the concept of bicooperative game introduced by Bilbao [1].

Definition 1. *A bicooperative game is a pair (N,b) with N a finite set and b is a function $b : 3^N \to \mathbb{R}$ with $b(\emptyset,\emptyset) = 0$.*

An especial kind of bicooperative games has been studied by Felsenthal and Machover [5] who consider *ternary voting games*. This concept is a generalization of voting games which recognizes abstention as an option alongside *yes* and *no* votes. These games are given by mappings $u : 3^N \to \{-1, 1\}$ satisfying the following three conditions: $u(N,\emptyset) = 1$, $u(\emptyset,N) = -1$, and $\mathbf{1}_{(S,T)}(i) \leq \mathbf{1}_{(S',T')}(i)$ for all $i \in N$, implies $u(S,T) \leq u(S',T')$. A negative outcome, -1, is interpreted as defeat and a positive outcome, 1, as passage of a bill.

In Chua and Huang [3] the Shapley-Shubik index for ternary voting games is considered. More recently, several works by Freixas [6, 7] and Freixas and Zwicker [8] have been devoted to the study of voting systems with several ordered levels of approval in the input and in the output. In their model, the abstention is a level of input approval intermediate between *yes* and *no* votes.

A one-point solution concept for cooperative games is a function which assigns to every cooperative game a n-dimensional real vector which represents a payoff distribution over the players. The study of solution concepts is central in cooperative game theory. The most important solution concept is the *Shapley value* as proposed by Shapley [12]. A solution concept for cooperative games is a function which assigns to every cooperative game (N,v) with $|N| = n$, a subset of n-dimensional real vectors which represent the payoff distribution over the players. The *core* is one of the most studied solution concepts. Weber [14] proposed as a solution concept for a cooperative game, a set that contains the core, is always nonempty and easier to compute. Its definition is based in the marginal worth vectors. Each permutation of the elements of N, $\pi = (i_1, i_2, \ldots, i_n)$, can be interpreted as a sequential process of formation of the grand coalition N. Beginning from the emptyset, first the player i_1 is incorporated, next the player i_2 and so sucessively until the incorporation of the player i_n give rise to the coalition N. In each one of these processes, the corresponding *marginal worth vector*, $a^\pi(v) \in \mathbb{R}^n$, evaluates the marginal contribution of every player to the coalition formed by his predecessors, that is,

$$a_i^\pi(v) = v\left(\pi^i \cup \{i\}\right) - v\left(\pi^i\right) \quad \text{for all } i \in N,$$

where π^i is the set of the predecessors of player i in the order π. The *Weber set* of game v is the convex hull of all marginal worth vectors, that is,

$$W(N,v) = conv\,\{a^\pi(v) : \pi \in \Pi_n\}.$$

Let us outline the contents of our work. In the next section, we study some properties and characteristics of the lattice 3^N. The aim of the third section is to introduce the Shapley value for a bicooperative game. We obtain an axiomatization of the Shapley value in this context as well as a nice formula to compute it. This value is the only one that satisfies our five axioms. Four of them are extensions of the classical axioms for the Shapley value: linearity, symmetry, dummy and efficiency. The fifth axiom is refereed to the structure of the family of signed coalitions. In the fourth section we define the above solutions concepts for bicooperative games and prove that the core is always contained in the Weber set. In the relation between the Weber set and the core, the bisupermodular games, which are defined in the fifth section, play an important role. We see that the bisupermodular games are the only ones for which their Weber set and the core coincide, establishing a characterization of these games. Throughout this paper, we will write $S\cup i$ and $S\setminus i$ instead of $S\cup\{i\}$ and $S\setminus\{i\}$ respectively.

2. The lattice 3^N

Let $N=\{1,\ldots,n\}$ be a finite set and let $3^N=\{(A,B):A,B\subseteq N,\ A\cap B=\emptyset\}$. Grabisch and Labreuche [9] proposed a relation in 3^N given by

$$(A,B)\sqsubseteq(C,D)\Longleftrightarrow A\subseteq C,\ B\supseteq D.$$

The set $\left(3^N,\sqsubseteq\right)$ is a partially ordered set (or poset) with the following properties:
1. $(\emptyset,N)$ is the first element: $(\emptyset,N)\sqsubseteq(A,B)$ for all $(A,B)\in 3^N$.
2. $(N,\emptyset)$ is the last element: $(A,B)\sqsubseteq(N,\emptyset)$ for all $(A,B)\in 3^N$.
3. Every pair of elements of 3^N has a join

$$(A,B)\vee(C,D)=(A\cup C,B\cap D)$$

and a meet

$$(A,B)\wedge(C,D)=(A\cap C,B\cup D).$$

Moreover, $\left(3^N,\sqsubseteq\right)$ is a finite distributive lattice. Two pairs (A,B) and (C,D) are comparable if $(A,B)\sqsubseteq(C,D)$ or $(C,D)\sqsubseteq(A,B)$; otherwise, (A,B) and (C,D) are incomparable. A chain of 3^N is an induced subposet of 3^N in which any two elements are comparable. In $\left(3^N,\sqsubseteq\right)$, all maximal chains have the same number of elements and this number is $2n+1$. Thus, we can consider the rank function

$$\rho:3^N\to\{0,1,\ldots,2n\}$$

such that $\rho\left[(\emptyset,N)\right]=0$ and $\rho\left[(S,T)\right]=\rho\left[(A,B)\right]+1$ if (S,T) covers (A,B), that is, if $(A,B)\sqsubset(S,T)$ and there no exists $(H,J)\in 3^N$ such that $(A,B)\sqsubset(H,J)\sqsubset(S,T)$.

For the distributive lattice 3^N, let P denote the set of all nonzero $\vee$-irreducible elements. Then P is the disjoint union $C_1+C_2+\cdots+C_n$ of the chains

$$C_i=\{(\emptyset,N\setminus i),\ (i,N\setminus i)\},\quad 1\le i\le n=|N|.$$

An order ideal of P is a subset I of P such that if $x\in I$ and $y\le x$, then $y\in I$. The set of all order ideals of P, ordered by inclusion, is the distributive lattice $J(P)$, where the lattice operations $\vee$ and $\wedge$ are just ordinary union and intersection. The fundamental theorem for finite distributive lattices (see [13, Theorem 3.4.1]) states that the map $\varphi : 3^N \to J(P)$ given by $(A, B) \mapsto \{(X, Y) \in P : (X, Y) \sqsubseteq (A, B)\}$ is an isomorphism (see Figure 1).

Example. Let $N = \{1, 2\}$. Then $P = \{(\emptyset, \{1\}), (\emptyset, \{2\}), (\{2\}, \{1\}), (\{1\}, \{2\})\}$ is the disjoint union of the chains $(\emptyset, \{1\}) \sqsubset (\{2\}, \{1\})$ and $(\emptyset, \{2\}) \sqsubset (\{1\}, \{2\})$. We will denote $a = (\emptyset, \{1\}),\ b = (\{2\}, \{1\}),\ c = (\emptyset, \{2\}),\ d = (\{1\}, \{2\})\}$, and hence

$$J(P) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}$$

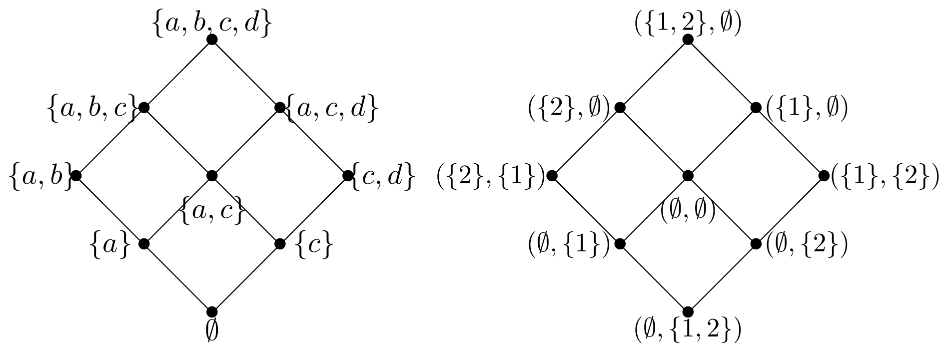

Figure 1.

In the following, we will denote by $c\left(3^N\right)$ the number of maximal chains in 3^N and by $c\left([(A, B), (C, D)]\right)$ the number of maximal chains in the sublattice $[(A, B), (C, D)]$.

Proposition 1. *The number of maximal chains of 3^N is $(2n)!/2^n$, where $n = |N|$.*

Proposition 2. *For all $(A, B) \in 3^N$, the number of maximal chains of the sublattice $[(\emptyset, N), (A, B)]$ is $(n + a - b)!/2^a$, where $a = |A|$ and $b = |B|$.*

Proposition 3. *Let $(A, B),\ (C, D) \in 3^N$ with $(A, B) \sqsubseteq (C, D)$. The number of maximal chains of the sublattice $[(A, B), (C, D)]$ is equal to the number of maximal chains of the sublattice $[(D, C), (B, A)]$.*

3. The Shapley value for bicooperative games

We denote by $\mathcal{BG}^N$ the real vector space of all bicooperative games on N. A *value* on $\mathcal{BG}^N$ is a function $\Phi : \mathcal{BG}^N \to \mathbb{R}^n$, which associates to each bicooperative game b a vector $(\Phi_1(b), \ldots, \Phi_n(b))$ which represents the 'a priori' value that every player has in the game b. In order to define a reasonable value for a bicooperative game and following the same issue and interpretation of the Shapley value in the cooperative case, we consider that a player i estimates his participation in game b, evaluating his marginal contributions $b(S \cup i, T) - b(S, T)$ in those signed coalitions $(S \cup i, T)$ that are formed from others (S, T) when i is incorporated to S and his marginal contributions $b(S, T) - b(S, T \cup i)$ in those (S, T) that are formed when i leaves the coalition $T \cup i$. Thus, a value for player i can be written as

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \left[\overline{p}^i_{(S,T)}\left(b(S \cup i, T) - b(S, T)\right) + \underline{p}^i_{(S,T)}\left(b(S, T) - b\left(S, T \cup i\right)\right)\right],$$

where for every (S,T), the coefficient $\overline{p}^i_{(S,T)}$ can be interpreted as the subjective probability that the player i has of joining the coalition S and $\underline{p}^i_{(S,T)}$ as the subjective probability that the player i has of leaving the coalition $T \cup i$. Thus, $\Phi_i(b)$ is the value that the player i can expect in the game b.

If we assume that all sequential orders or chains have the same probability, we can deduce formulas for these probabilities $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ in terms of the number of chains which contain to these coalitions. Applying Propositions 2 and 3, we obtain

$$\overline{p}^i_{(S,T)} = \frac{(n+s-t)!\,(n+t-s-1)!}{(2n)!}2^{n-s-t},$$

$$\underline{p}^i_{(S,T)} = \frac{(n+t-s)!\,(n+s-t-1)!}{(2n)!}2^{n-s-t}.$$

Taking into account that $\overline{p}^i_{(S,T)}$ and $\underline{p}^i_{(S,T)}$ are independent of player i, and only depend of $s = |S|$ and $t = |T|$, we can establish the following definition.

Definition 2. *The Shapley value for the bicooperative game $b \in \mathcal{BG}^N$ is given, for each $i \in N$, by*

$$\Phi_i(b) = \sum_{(S,T)\in 3^{N\backslash i}} \left[\overline{p}_{s,t}\left(b(S\cup i,T) - b(S,T)\right) + \underline{p}_{s,t}\left(b(S,T) - b\left(S,T\cup i\right)\right)\right]$$

where, for all $(S,T) \in 3^{N\backslash i}$,

$$\overline{p}_{s,t} = \frac{(n+s-t)!\,(n+t-s-1)!}{(2n)!}2^{n-s-t}$$

and

$$\underline{p}_{s,t} = \frac{(n+t-s)!\,(n+s-t-1)!}{(2n)!}2^{n-s-t}.$$

With the aim to characterize the Shapley value for bicooperative games, we consider a set of reasonable axioms and we prove that the Shapley value is the unique value on $\mathcal{BG}^N$ which satisfies these axioms.

Linearity axiom. For all $\alpha, \beta \in \mathbb{R}$, and $b, w \in \mathcal{BG}^N$,

$$\Phi_i(\alpha b + \beta w) = \alpha\Phi_i(b) + \beta\Phi_i(w).$$

We now introduce the dummy axiom, understanding that a player is a *dummy player* when his contributions to signed coalitions $(S \cup i, T)$ formed with his incorporation to S and his contributions to signed coalitions (S,T) formed with his desertion of $T \cup i$ coincide exactly with his individual contributions, that is, a player $i \in N$ is a dummy in $b \in \mathcal{BG}^N$ if, for every $(S,T) \in 3^{N\backslash i}$, it holds

$$b(S\cup i,T) - b(S,T)) = b\left(\{i\},\emptyset\right), \quad b(S,T) - b\left(S,T\cup i\right) = -b\left(\emptyset,\{i\}\right).$$

Note that if $i \in N$ is a dummy in $b \in \mathcal{BG}^N$ then, for all $(S,T) \in 3^{N\backslash i}$,

$$b(S\cup i,T) - b\left(S,T\cup i\right) = b(\{i\},\emptyset) - b\left(\emptyset,\{i\}\right).$$

Since a dummy player i in a game b has no meaningful strategic role in the game, the value that this player should expect in the game b must exactly be the sum up of his contributions.

Dummy axiom. If player $i \in N$ is dummy in $b \in \mathcal{BG}^N$, then

$$\Phi_i(b) = b(\{i\}, \emptyset) - b(\emptyset, \{i\}).$$

In the similar way to the cooperative case, for the comparison of roles in a game to be meaningful, the evaluation of a particular position should depend on the structure of the game but not on the labels of the players.

Symmetry axiom. For all $b \in \mathcal{BG}^N$ and for any permutation π over N, it holds that $\Phi_{\pi i}(\pi b) = \Phi_i(b)$ for all $i \in N$, where $\pi b(\pi S, \pi T) = b(S, T)$ and $\pi S = \{\pi i : i \in S\}$.

In a cooperative game, it is assumed that all players decide to cooperate among them and form the grand coalition N. This leads to the problem of distributing the amount $v(N)$ among them. Taking into account different situations that can be modelled by a bicooperative game b, we suppose that the amount $b(N, \emptyset)$ is the maximal gain and $b(\emptyset, N)$ is the minimal loss obtained by the players when they decide full cooperation. Then the maximal global gain is given by $b(N, \emptyset) - b(\emptyset, N)$. From this perspective, the value Φ must satisfy the following axiom.

Efficiency axiom. For every $b \in \mathcal{BG}^N$, it holds

$$\sum_{i \in N} \Phi_i(b) = b(N, \emptyset) - b(\emptyset, N).$$

It is easy to check that our Shapley value for bicooperative games verifies the above axioms. But this value is not the unique value which satisfies these four axioms. For instance, the value $\Phi(b)$ defined, for $b \in \mathcal{BG}^N$ and $i \in N$, by

$$\Phi_i(b) = \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!} \left[b(S \cup i, N \setminus (S \cup i)) - b(S, N \setminus S) \right],$$

also verifies these axioms. However, note that, for any bicooperative game $b \in \mathcal{BG}^N$, this value is the Shapley value corresponding to the cooperative game (N, v), where $v : 2^N \to \mathbb{R}$ is defined by $v(A) = b(A, N \setminus A)$ if $A \neq \emptyset$, and $v(\emptyset) = 0$. This value is not satisfactory for any bicooperative game in the sense that only consider the contributions to signed coalitions in which all players take part. Moreover, there is an infinity of different bicooperative games which give rise to the same cooperative game. For these reasons, if we want to obtain an axiomatic characterization of our Shapley value for bicooperative games, we need to introduce an additional axiom. This new axiom will take into account the structure of the set of the signed coalitions.

Structural axiom. For every $(S, T) \in 3^{N \setminus i}$, $j \in S$ and $k \in T$, it holds

$$\frac{c([(\emptyset, N), (S \setminus j, T)])}{c([(\emptyset, N), (S, T \cup i)])} = -\frac{\Phi_j(\delta_{(S,T)})}{\Phi_i\left(\delta_{(S,T \cup i)}\right)}, \quad \frac{c([(S, T \setminus k), (N, \emptyset)])}{c([(S \cup i, T), (N, \emptyset)])} = -\frac{\Phi_k(\delta_{(S,T)})}{\Phi_i\left(\delta_{(S \cup i,T)}\right)}.$$

Theorem 4. *Let Φ be a value on $\mathcal{BG}^N$. The value Φ is the Shapley value if and only if Φ satisfies the efficiency axiom and each component satisfies linearity, dummy, symmetry and structural axioms.*

4. The core and the Weber set

Now, some solution concepts for bicooperative games are introduced, understanding as a solution concept any subset of vectors in $\mathbb{R}^n$ that provide an equitable distribution of the total saving among the players. A vector $x \in \mathbb{R}^n$ which satisfies $\sum_{i \in N} x_i = b(N, \emptyset) - b(\emptyset, N)$ is called *efficient vector* and the set of all efficient vectors is called *preimputation set* which is defined by

$$I^*(N,b) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = b(N, \emptyset) - b(\emptyset, N) \right\}.$$

The *imputations* for game b are the preimputations that satisfy the *individual rationality principle* for all players, that is, each player gets at least the difference between the amount that he can attain for himself taking the rest of players against and the value of the signed coalition $(\emptyset, N)$,

$$I(N,b) = \{ x \in I^*(N,b) : x_i \geq b(i, N \setminus i) - b(\emptyset, N) \text{ for all } i \in N \}.$$

A satisfactory distribution criterium could be that every signed coalition $(S,T) \in 3^N$ receives at least the amount it can contribute to the coalition $(\emptyset, N)$, that is, the amount $b(S,T) - b(\emptyset, N)$. It leads us to define the notion of the core of the game b as the set

$$C(N,b) = \left\{ \begin{array}{ll} x \in I^*(N,b) : & x = y + z \text{ with} \\ & y(S) + z(N \setminus T) \geq b(S,T) - b(\emptyset, N) \;\; \forall \;\; (S,T) \in 3^N \end{array} \right\}.$$

This definition can be interpreted in the following manner. For each $(S,T) \in 3^N$, the players who are not in the coalition T have contributed to the formation of (S,T) since they will not act against the player of the coalition S and for this, they must be received a payoff given by the vector z. Moreover, those players of $N \setminus T$ who are in the coalition S must get a different payoff to the rest of players, given by the vector y since these players have contributed to the formation of (S,T) in a different way.

In order to extend the idea of the Weber set to a bicooperative game (N, b), it is assumed that all players estimate that $(N, \emptyset)$ is formed as a sequential process where in each step a different player is incorporated to the *defender coalition* or a different player leaves the *detractor coalition.* These sequential processes are obtained considering the different chains from $(\emptyset, N)$ to $(N, \emptyset)$. In each one of these processes, a player can evaluate his contribution when is incorporated to the defenders or his contribution when leaves the detractors. This can be reflected in the vectors of $\mathbb{R}^n$ denominated *superior marginal worth vectors* and *inferior marginal worth vectors.* With the aim to formalize this idea, we introduce the following notation.

Given $N = \{1, \dots, n\}$, let $\overline{N} = \{-n, \dots, -1, 1, \dots, n\}$. We can define an isophorfim $\Lambda : 3^N \longrightarrow 2^{\overline{N}}$ as follows: For each $(S,T) \in 3^N$, $\Lambda(S,T) = S \cup \{-i : i \in N \setminus T\} \in 2^{\overline{N}}$. For instance, $\Lambda(\emptyset, N) = \emptyset$ and $\Lambda(N, \emptyset) = \overline{N}$. Since $S \cap T = \emptyset \Leftrightarrow S \subseteq N \setminus T$ we see that $i \in \Lambda(S,T)$ and $i > 0$ imply $-i \in \Lambda(S,T)$.

In the lattice $(3^N, \sqsubseteq)$, we consider the set of all maximal chains which going from $(\emptyset, N)$ to $(N, \emptyset)$ and denote this set by $\Theta(3^N)$. If $\theta \in \Theta(3^N)$ is the maximal chain

$$(\emptyset, N) \sqsubset (S_1, T_1) \sqsubset \cdots \sqsubset (S_j, T_j) \sqsubset \cdots \sqsubset (S_{2n-1}, T_{2n-1}) \sqsubset (N, \emptyset),$$

we can write the associated chain of sets in $2^{\overline{N}}$

$$\emptyset \subset \{i_1\} \subset \cdots \subset \{i_1, \ldots, i_j\} \subset \cdots \subset \{i_1, \ldots, i_{2n-1}\} \subset \overline{N}.$$

where $\{i_1, \ldots, i_j\} = \Lambda(S_j, T_j)$ for $j = 1, \ldots, 2n$. We define the vector $\theta(i_j) = (i_1, \ldots, i_j)$, where the last component $i_j \in \overline{N}$ satisfies the following property: if $i_j > 0$ then the player $i_j \in S_j$ and $i_j \notin S_{j-1}$, that is, i_j is the last player who joins S_j and if $i_j < 0$, then the player $-i_j \notin T_j$ and $-i_j \in T_{j-1}$, that is, $-i_j$ is the last player who leaves T_{j-1}. Equivalently, the elements in $\theta(i_j) = (i_1, \ldots, i_j)$ are written following the order of incorporation in the defenders coalitions or desertion of the detractors coalition (depending on the sign of each i_k) in the signed coalitions in chain θ . Moreover, we write $\theta(i_j) \setminus i_j = (i_1, i_2, \ldots, i_{j-1}) = \theta(i_{j-1})$ and $i_k \in \theta(i_j)$ when i_k is one component of the vector $\theta(i_j)$, that is $1 \leq k \leq j$. Note that an equivalence between maximal chains and vectors $\theta = (i_1, \ldots, i_{2n})$ is obtained. Fix an order $\theta = (i_1, \ldots, i_{2n})$, we also define $\alpha[\theta(i_j)] = (S_j, T_j)$ such that $\Lambda(S_j, T_j) = \{i_1, \ldots, i_j\}$. Moreover, $\alpha[\theta(i_j) \setminus i_j] = \alpha[\theta(i_{j-1})] = (S_{j-1}, T_{j-1})$. In particular, $\alpha[\theta(i_{2n})] = (N, \emptyset)$ and $\alpha[\theta(i_1) \setminus i_1] = (\emptyset, N)$.

For example, let $N = \{1, 2, 3\}$ and $\theta \in \Theta\left(3^N\right)$ given by

$$(\emptyset, N) \sqsubset (\emptyset, \{1, 3\}) \sqsubset (\{2\}, \{1, 3\}) \sqsubset (\{2\}, \{1\}) \sqsubset (\{2\}, \emptyset) \sqsubset (\{2, 3\}, \emptyset) \sqsubset (N, \emptyset).$$

Its associated chain of sets in $2^{\overline{N}}$ is given by

$$\emptyset \subset \{-2\} \subset \{-2, 2\} \subset \{-2, 2, -3\} \subset \{-2, 2, -3, -1\} \subset \{-2, 2, -3, -1, 3\} \subset \overline{N}.$$

and the maximal chain can be also represented by the order $\theta = (-2, 2, -3, -1, 3, 1)$. One signed coalition, for example $(\{2\}, \emptyset)$, can be also represented by $\alpha[\theta(-1)]$ and by $\Lambda^{-1}(\{-2, 2, -3, -1\})$

Definition 3. *Let $\theta \in \Theta\left(3^N\right)$ and $b \in \mathcal{BG}^N$. We call inferior and superior marginal worth vectors with respect to θ to the vectors $m^\theta(b)$, $M^\theta(b) \in \mathbb{R}^n$ respectively where*

$$m_i^\theta(b) = b(\alpha[\theta(-i)]) - b(\alpha[\theta(-i) \setminus -i]),$$
$$M_i^\theta(b) = b(\alpha[\theta(i)]) - b(\alpha[\theta(i) \setminus i]),$$

for all $i \in N$. We call marginal worth vector respect to θ, $a^\theta(b) \in \mathbb{R}^n$, to the vector obtained as the sum of inferior and superior marginal worth vectors, that is,

$$a_i^\theta(b) = m_i^\theta(b) + M_i^\theta(b), \qquad \text{for } i \in N.$$

The following result show that the marginal worth vectors are efficients.

Proposition 5. *For any $b \in \mathcal{BG}^N$ and $\theta \in \Theta\left(3^N\right)$ we have*

$$\sum_{i \in N} a_i^\theta(b) = b(N, \emptyset) - b(\emptyset, N).$$

Proposition 6. *Let $b \in \mathcal{BG}^N$ and $\theta \in \Theta\left(3^N\right)$. Then,*

$$\sum_{j \in S} M_j^\theta(b) + \sum_{j \in N \setminus T} m_j^\theta(b) = b(S,T) - b(\emptyset, N),$$

for every (S,T) in the chain θ.

Definition 4. *Let $b \in \mathcal{BG}^N$ be a bicooperative game. The Weber set of b is the convex hull of the marginal worth vectors, that is*

$$W(N,b) = conv\left\{a^\theta(b) : \theta \in \Theta\left(3^N\right)\right\}$$

As the preimputation set is a convex set, it is evident that $W(N,b) \subseteq I^*(N,b)$. However, in general, the vectors of the Weber set are not imputations. For example, let (N,b) with $N = \{1,2\}$ and $b : 3^N \longrightarrow \mathbb{R}$ defined as

$$b(\emptyset, N) = -5, \quad b(\emptyset, i) = -4, \quad b(i,j) = -1, \quad b(i,\emptyset) = 1, \quad b(N,\emptyset) = 2,$$

for all $i,j \in N$. If we consider $\theta = (-2,2,-1,1)$, then $a_1^\theta(b) = m_1^\theta(b) + M_1^\theta(b) = 3$. As $b(1,2) - b(\emptyset,N) = 4$, then $a_1^\theta(b) < b(1, N \setminus 1) - b(\emptyset, N)$ and $a^\theta(b) \notin I(N,b)$.

It is easy to see, taking into account that $I(N,b)$ is a convex set, that $W(N,b) \subseteq I(N,b)$ if all marginal worth vectors are imputations. For this, a sufficient condition is that the game b is *zero-monotonic*, a concept that is defined as follows.

Definition 5. *A bicooperative game $b \in \mathcal{BG}^N$ is monotonic when for all signed coalitions $(S_1,T_1),(S_2,T_2)$ with $(S_1,T_1) \sqsubseteq (S_2,T_2)$ it holds that $b(S_1,T_1) \leq b(S_2,T_2)$.*

Definition 6. *The zero-normalization of a bicooperative game $b \in \mathcal{BG}^N$ is the game $b_0 \in \mathcal{BG}^N$ defined by*

$$b_0(S,T) = b(S,T) - \sum_{j \in S}\left[b(j, N \setminus j) - b(\emptyset, N)\right], \quad \text{for all } (S,T) \in 3^N.$$

Definition 7. *A bicooperative game $b \in \mathcal{BG}^N$ is called zero-monotonic if its zero-normalization is monotonic.*

Proposition 7. *Let $b \in \mathcal{BG}^N$ be a zero-monotonic bicooperative game. Then, for every $\theta \in \Theta\left(3^N\right)$, the marginal worth vector associated to θ is an imputation for the game b.*

Now we prove that the core of a bicooperative game is always included in its Weber set. It should be noted that the proof of this result is closely related to the proof in [4] of the inclusion of the core in the Weber set for cooperative games.

Theorem 8. *If $b \in \mathcal{BG}^N$, then $C(N,b) \subseteq W(N,b)$*

5. Bisupermodular games

Now we introduce a special class of bicooperative games.

Definition 8. *A bicooperative game $b \in \mathcal{BG}^N$ is called bisupermodular if, for all (S_1, T_1) and (S_2, T_2) it holds*

$$b((S_1, T_1) \vee (S_2, T_2)) + b((S_1, T_1) \wedge (S_2, T_2)) \geq b(S_1, T_1) + b(S_2, T_2),$$

or equivalently

$$b(S_1 \cup S_2, T_1 \cap T_2) + b(S_1 \cap S_2, T_1 \cup T_2) \geq b(S_1, T_1) + b(S_2, T_2).$$

The next proposition characterizes the bisupermodular games as those bicooperative games for which the marginal contributions of a player to one signed coalition is never less that the marginal contribution of this player to any signed coalition contained in it. This characterization will be used in the proves of the following results.

Proposition 9. *Let $b \in \mathcal{BG}^N$. The bicooperative game b is bisupermodular if and only if for all $i \in N$ and (S_1, T_1), $(S_2, T_2) \in 3^{N \setminus i}$ such that $(S_1, T_1) \sqsubseteq (S_2, T_2)$, it holds*

$$b(S_2 \cup i, T_2) - b(S_2, T_2) \geq b(S_1 \cup i, T_1) - b(S_1, T_1)$$

and

$$b(S_2, T_2) - b(S_2, T_2 \cup i) \geq b(S_1, T_1) - b(S_1, T_1 \cup i)$$

The following result permits the identification of the games for which the marginal worth vectors are distributions of the core.

Theorem 10. *A necessary and sufficient condition so that all marginal worth vectors of a bicooperative game $b \in \mathcal{BG}^N$ are vectors of the core is that the game b is bisupermodular*

As the core of a bicooperative game $b \in \mathcal{BG}^N$ is a convex set, an immediate consequence of this theorem is the following result.

Corollary 11. *Let $b \in \mathcal{BG}^N$. A necessary and sufficient condition so that $W(N, b) = C(N, b)$ is that the bicooperative game b is bisupermodular.*

Note that the Shapley value of a bicooperative game b is given by

$$\Phi_i(N, b) = \frac{1}{c(3^N)} \sum_{\theta \in \Theta(3^N)} \left[m_i^\theta(b) + M_i^\theta(b) \right],$$

for all $i \in N$. Then the Shapley value of a bisupermodular game b is in $C(N, b)$ and hence, the core of a bisupermodular game is non-empty.

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On the computation of Semivalues for TU games via Shapley value

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In an earlier paper (I.Dragan,2004) we proved that every Least Square Value is the Shapley value of a game obtained by rescaling from the given game. In the paper where the Least Square Values were introduced (L.Ruiz, F.Valenciano and J.M. Zarzuelo,1998), the authors have shown that the efficient normalization of a Semivalue is a Least Square Value, (briefly LS-value).

In the present paper, we developed the idea suggested by these two results and we obtained a direct relationship between the efficient normalization of a Semivalue and the Shapley value. The main tools for proofs were the so-called the Average per capita formulas we proved earlier for the Shapley value (I.Dragan,1992) and for the Semivalues (1999), as well as the formula for the Power game of a given game relative to Semivalues (I.Dragan and J.E.Martinez-Legaz,2001). The last one was needed to compute the efficiency term and to derive an algorithm for computing any Semivalue via the Shapley value. The present paper is containing results from various sources, so that in order to make this paper self contained, we shall be proving below our previous results together with the new results, which appear here for the first time. All proofs are algebraic, in opposition to the RVZ proofs which are axiomatic. The direct connection between a Semivalue and the Shapley value does not need any reference to LS-values, which may well be unknown to the reader of the present paper.

In the first section, we prove the Average per capita formula for Semivalues, (Theorem 1), from which we derive our formula for the Shapley value, (Corollary 2), to be used later. In the second section, we give the formula for the efficiency term in the efficient normalization of a Semivalue ,(Theorem 3), as well as the main results showing the connection between the efficient normalization of a Semivalue and the Shapley value, (Theorems 4 and 5). In the last section we discuss the algorithm for computing a Semivalue via the Shapley value, after noticing that the Average per capita formula for Semivalues proposed as a basic tool in computing a Semivalue, is doubling the number of weighting operations. A small game is used for illustrating how this algorithm works (Example 1). The motivation for the present work was the fact that in Mathematica there is a program for computing the Shapley value and there is some experience in computing the Shapley value, while we do not know of any computational work relative to the Semivalues. Further, we consider the inverse problem for Semivalues which was solved in an earlier paper, (I.Dragan,2002), by extending to Semivalues our procedure used for the Shapley value (I.Dragan,1991). It is interesting to note that here the solution set of the inverse problem depends on a unique basis, the basis for the inverse problem of the Shapley value, (Theorem 6), in opposition to what has happened in the previous paper on the inverse problem for Semivalues, where there was an infinite set of bases, each one being singled out by the dependence of the weight vector of the Semivalue. An example is also shown here, (Example 2).

Keywords: Shapley value, Semivalues, Average per capita formulas, Efficient normalization, Banzhaf value, the inverse problem.

1 Average per capita formula for Semivalues

Let G^N denote the space of cooperative TU games with a fixed set of players N. The Semivalues associated with a weight vector $p^n \in R^n$ satisfying the normalization condition

$$\sum_{s=1}^{n} \begin{pmatrix} n-1 \\ s-1 \end{pmatrix} p_s^n = 1, \tag{1}$$

have been introduced axiomatically by P.Dubey, A.Neyman and R.J.Weber (1981), as values on G^N, and even on more general structures. For G^N they proved that a Semivalue associated with a weight vector p^n is given by

$$SE_i(N,\nu) = \sum_{S: i \in S \subseteq N} p_s^n[\nu(S) - \nu(S-i)], \quad \forall i \in N. \tag{2}$$

where $s = |S|$, and p_s^n is the common weight of all coalitions of size s. We take this formula as the definition of Semivalues on G^N. To define the Semivalues on the union of all spaces G^N, when N is arbitrary, we need a sequence of weight vectors $p^1, p^2, ..., p^n, ...,$ all satisfying the above normalization condition, that is $p_1^1 = 1, p_1^2 + p_2^2 = 1, p_1^3 + p_2^3 + p_3^3 = 1, ...$ and so on. The definition of Semivalues on $G^T$is given by the same formula as above, where N is replaced by T, $n$by$t$, and p^n by p^t. However, the sequence of weight vectors are supposed to satisfy what we call the inverse Pascal triangle relations

$$p_s^{t-1} = p_s^t + p_{s+1}^t, \quad s = 1, 2, ..., t-1. \tag{3}$$

It is easy to see that if the normalization condition for G^N holds, then from the inverse Pascal triangle relations we get the normalization condition satisfied for G^T , and all coalitions$T \subseteq N$

Note the important fact that among the Semivalues we get the Shapley value for $p_s^n = /frac(s-1)!(n-s)!n!$, the Banzhaf value for $p_s^n = 2^{1-n}, s = 1, 2, ..., n$, and many other well known values. Therefore, if we prove what we call the Average per capita formula for Semivalues, (I.Dragan,1999, and I.Dragan and J.E.Martinez-Legaz,2001), we get also the formula for the Shapley value (I.Dragan,1992), by taking these particular weights (see also I.Dragan,T.Driessen and Y.Funaki,1996). This will be used later.

We call an Average per capita formula any formula in which occur only the average worth of various coalitions defined as follows:

$$\nu_s = \begin{pmatrix} n \\ s \end{pmatrix}^{-1} \sum_{|S|=s} \nu(s), \nu_s^i = \begin{pmatrix} n-1 \\ s \end{pmatrix}^{-1} \sum_{|S|=s, i \notin S} \nu(s), s = 1, 2, ...n{-}1, \forall i \in N. \tag{4}$$

Clearly, ν_s is the average worth of coalitions of size s , while ν_s^i is the average worth of coalitions of size s which do not contain player i. If we denote $\nu_n = \nu(N)$, then there are n averages ν_s, and $n(n-1)$ averages ν_s^i, hence n^2 averages all together. Let us introduce also the new weights, defined for all t by

$$q_s^i = \frac{p_s^t}{\gamma_s^t}, \quad s = 1, 2, ..., t, \tag{5}$$

where $\gamma_s^t = (t!)^{-1}(s-1)!(t-s)!$, that is the weights for the Shapley value on G^T.

Theorem 1 (I.Dragan,1999): Let SE be a Semivalue associated with a non- negative weight vector p^n satisfying the normalization condition. Let q^n be the nonnegative weight vector defined above. Then, SE is given by the formula

$$SE(N, \nu) = q_n^n \frac{\nu_n}{n} + \sum_{s=1} n \frac{q_s^n \nu_s - q_s^{n-1} \nu_s^i}{s}, \quad \forall i \in N. \tag{6}$$

For $q_s^n = 1, \quad s = 1, 2, ..., n$, that is $p_s^n = \gamma_s^n, \quad s = 1, 2, ..., n$, we obtain:

Corollary 2 (I.Dragan,1992): The Shapley value of the game (N, ν) is given by

$$SH_i(N, \nu) = \frac{\nu_n}{n} + \sum_{s=1} n \frac{\nu_s - \nu_s^i}{s}, \quad \forall i \in N. \tag{7}$$

Proof of Theorem 1: For $i \in N$ fixed, rewrite (2) as

$$SE_i(N, \nu) = p_n^n \nu(N) + \sum_{S: i \in S \subset N} p_s^n \nu(S) - \sum_{S: i \in S \subseteq N} p_s^n \nu(S - i); \tag{8}$$

now, write the two sums separately as

$$\sum_{S:i\in S\subset N} p_s^n \nu(S) = \sum_{s=1}^{n-1} p_s^n \left(\sum_{|S|=s, i\in S} \nu(S) \right) = \sum_{s=1}^{n-1} p_s^n \left(\sum_{|S|=s} \nu(S) - \sum_{|S|=s, i\notin S} \nu(S) \right), \tag{9}$$

and

$$\sum_{S:i\in S\subseteq N} p_s^n \nu(S-i) = \sum_{s=1}^{n-1} p_{s+1}^n \left(\sum_{|S|=s} \nu(S) \right). \tag{10}$$

From (8), (9), and (10), with notations (4), we obtain

$$SE_i(N,\nu) = p_n^n \nu_n + \sum_{s=1} n-1 [p_s^n \begin{pmatrix} n \\ s \end{pmatrix} \nu_s - p_s^{n-1} \begin{pmatrix} n-1 \\ s \end{pmatrix}], \tag{11}$$

where we have also used (3) for $t = n$. If in (11) we introduce the new weights by noticing that $p_s^n \begin{pmatrix} n \\ s \end{pmatrix} = s^{-1} q_s^n, \quad s-1, 2, ..., n-1$, we get (6).

Note that the weights q_s^n should satisfy the normalization condition

$$\sum_{s=1}^{n} q_s^n = n, \tag{12}$$

derived from (1) and (5), and the Pascal triangle conditions (3) become

$$q_s^{i-1} = (1 - st^{-1}) q_s^t + st^{-1} q_{s+t}^t, \quad s = 1, 2, ..., t-1. \tag{13}$$

In the next section, we shall derive a new Average per capita formula for the term which should be added to the Semivalue, to get the efficient normalization. This formula will be needed later in the computation of Semivalues via Shapley value.

2 Average per capita formula for the efficiency term

In the paper where the Least Square Values (briefly LS-values) were introduced by L.Ruiz, F.Valenciano and J.M.Zarzuelo, (1998), the authors defined what they called the Efficient normalization of a Semivalue SE associated with a nonnegative weight vector $p^n = (p_s^n)$. This is the value $ESE : G^N \rightarrow R^n$ written componentwise as

$$ESE_i(N,\nu) = SE_i(N,\nu) + \alpha, \quad \forall i \in N, \tag{14}$$

with α such that ESE is efficient, that is

$$\alpha = \frac{1}{n}[\nu(N) - \sum_{j\in N} SE_j(N,\nu)]. \tag{15}$$

We call α the efficiency term and we intend to derive an Average per capita formula for α . This can be derived from our formula for a Power Game relative to a Semivalue by introducing the averages (4) and our new weights (5), (I.Dragan and J.E.Martinez- Legaz,2001). However, we cut somehow the work by using (6). From the last formula, we obtain

$$\sum_{j\in N} SE_j(N,\nu) = q_n^n\nu_n + \sum_{s=1}^{n-1} \frac{nq_s^n\nu_s - q_s^{n-1}\sum_{j\in N}\nu_s^j}{s} = q_n^n\nu_n + n\sum_{s=1}^{n-1}\frac{(q_s^n - q_s^{n-1})\nu_s}{s}, \tag{16}$$

where we have used the equality $\sum_{j\in N}\nu_s^j = n\nu_s$, holding for all $s = 1,2,...,n-1$. In this way, from (15) and (16), we proved:

Theorem 3: The efficiency term for the additive normalization of a Semivalue is given by the Average per capita formula

$$\alpha = \frac{\nu_n}{n} - [\frac{q_n^n\nu_n}{n} + \sum_{s=1}^{n-1}\frac{(q_s^n - q_s^{n-1})}{s}]. \tag{17}$$

Putting together the Average per capita formulas (6) and (17) of the Theorems 1 and 3, we prove algebraically for the efficient normalization of a Semivalue the main result:

Theorem 4: The Efficient normalization of a Semivalue associated with a non- negative weight vector $p^n = (p_s^n)$ is given by

$$ESE_i(N,\nu) = \frac{\nu_n}{n} + \sum_{s=1}^{n-1} q_s^{n-1}\frac{\nu_s - \nu_s^i}{s}, \quad \forall i \in N, \tag{18}$$

where q_s^{n-1} are expressed in terms of p^n as

$$q_s^{n-1} = \frac{p_s^n + p_{s+1}^n}{\gamma_s^n + \gamma_{s+1}^n}, \quad s = 1,2,...n-1, \tag{19}$$

with γ_s^n and γ_{s+1}^n denoting the corresponding Shapley weights.

Note that (19) is derived from (5) for $t = n-1$ and (3) for $t = n$, taking into account that the weights for the Shapley value satisfy also (3). Note also that for the Banzhaf value (19) becomes

$$q_s^{n-1} = \frac{1}{2^{n-2}\gamma_s^{n-1}}, \quad s = 1,2,...n-1, \tag{20}$$

Note that Theorem 4 could be derived from the relationship axiomatically proved by Ruiz, Valenciano and Zarzuelo (1998) between the efficient normalization of a Semi- value and the LS-values, together with our relationship between the LS-values and the Shapley value (I.Dragan,2004). In the present paper, as it was shown, there is no need of LS-values, and this was the reason why we have chosen the above proof.

Consider a game (N, ν) and rescale it by introducing the new game (N, w) :

$$w(N) = \nu(N), \quad w(S) = q_s^{n-1}\nu(S), \quad \forall S \subset N. \tag{21}$$

By (4) we have

$$w_s = q_s^{n-1}\nu_s, \quad w_s^i = q_s^{n-1}\nu_s^i, \quad \forall i \in N, \quad s = 1, 2, ..., n-1. \tag{22}$$

Therefore, from (18) and (22), we get the right hand side in (7), for the new game (N, w). We proved:

Theorem 5. The Efficient normalization of the Semivalue of a game (N, ν), associated to the weight vector $p^n \in R^n$, is the Shapley value of a new game (N, w) obtained by rescaling of (N, ν) with factors q_s^{n-1} , for the worth of coalitions of size s, $s = 1, 2, ..., n-1$, derived from the weight vector p^n and the Shapley weights by means of (19).

This last result is helpful in computing the Semivalues of the TU games via the Shapley value, as it will be discussed in the next section, where we shall also discuss an application of Theorem 5 to the Inverse problem for Semivalues.

3 Applications to the computation of Semivalues and to the Inverse problem

In an earlier paper (I.Dragan,1999), we have shown that a Semivalue for a TU game may be computed by means of the Average per capita formula in the same way as the Shapley value was shown to be computable from its Average per capita formula, (I.Dragan,1992). The difference is that the averages should be weighted, as could be seen in (6). Based upon Theorem 5, we may modify first the game, then use the algorithm for computing the Shapley value. However, as shown in Theorem 4, we may better compute the usual terms which appear in the Shapley value formula, then go on and rescale the term by q_s^{n-1} , as seen in formula (18), to get the Efficient normalization. Now, we have also two alternatives: if we rescale the game, then we may still modify it by subtracting an additive game (N, α), in which $\alpha(S) = s\alpha$ for all $S \subset N$, and $\alpha(N) = N\alpha$, where α is the efficiency term. Due to the linearity of the Shapley value, and to Theorem 5, we get the game for which the Shapley value is exactly the Semivalue to be computed. However, this entails to subtract from each value of the characteristic function the corresponding value of $\alpha(S)$ so that the number of operations needed is increasing dramatically. Therefore, we prefer the second alternative, namely after we computed the Efficient normalization of the Semivalue, we shall subtract from each of the components the number α. This entire procedure will be shown in the next example of a four person simple game. The first alternative is useful in discussing further the inverse problem for Semivalues, as it will become clear below.

Example 1: Consider the four person simple game with winning coalitions {1}, {2}, {1,2}, {1,3}, {2,3}, {1,2,3}, {1,2,4}, and {1,2,3,4}, and the weight vector $p^4 \in R^4$, given by $p^4 = (\frac{1}{8}, \frac{1}{8}, \frac{1}{18}, \frac{1}{3})$ which obviously satisfies the normalization condition (1). From (3) for $t = 4$, we get the weight vector $p^3 = (\frac{1}{4}, \frac{13}{72}, \frac{7}{18})$ which satisfies also (1), and the Shapley weight vector $\gamma^3 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{3})$ gives the weight vector $q^3 = \frac{p^3}{\gamma^3}$ containing the factors needed in (18), to compute the ESE. Now, the usual computation of terms $\frac{\nu_s - \nu_s^i}{s}$, $\quad s = 1, 2, ..., n-1$, in the Shapley value formula gives:

$$\begin{array}{lllll} \nu_1 = \frac{1}{2}, & \nu_1^1 = \nu_1^2 = \frac{1}{3}, & \nu_1^3 = \nu_1^4 = \frac{2}{3} & \rightarrow & (\nu_1 - \nu_1^i) = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\right) \\ \nu_2 = \frac{1}{2}, & \nu_2^1 = \nu_2^2 = \nu_2^3 = \frac{1}{3}, & \nu_2^4 = 1 & \rightarrow & \frac{1}{2}(\nu_2 - \nu_2^i) = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, -\frac{1}{4}\right) \\ \nu_3 = \frac{1}{2} & \nu_3^1 = \nu_3^2 = 0, & \nu_3^3 = \nu_3^4 = 1 & \rightarrow & \frac{1}{3}(\nu_3 - \nu_3^i) = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\right) \end{array} \tag{23}$$

and weighting these vectors with q^3 leads to

$$q_1^3(\nu_1 - \nu_1^i) + q_2^3 \frac{\nu_2 - \nu_2^i}{2} + q_3^3 \frac{\nu_3 - \nu_3^i}{3} = \left(\frac{59}{144}, \frac{59}{144}, -\frac{33}{144}, -\frac{85}{144}\right) \tag{24}$$

By adding 1/4 to each component, we obtain

$$ESE(N, \nu) = \left(\frac{59}{144}, \frac{59}{144}, \frac{1}{48}, -\frac{49}{144}\right) \tag{25}$$

Now, we compute α by means of (17); we need $q^4 = \frac{p^4}{\gamma^4} = \left(\frac{1}{2}, \frac{3}{2}, \frac{2}{3}, \frac{4}{3}\right)$ and q^3 that was computed above, to use in (17) $q_4^4 = \frac{4}{3}$ and

$$\left(q_1^4 - q_+1^3, q_2^4 - q_2^3, q_3^4 - q_3^3,\right) = \left(-\frac{1}{4}, \frac{5}{12}, -\frac{1}{2}\right), \tag{26}$$

together with the averages ν_1, ν_2, ν_3 to obtain $\alpha = \frac{1}{48}$. We got

$$SE(N, \nu) = ESE(N, \nu) - \alpha = \left(\frac{23}{36}, \frac{23}{36}, 0, -\frac{13}{36}\right). \tag{27}$$

Of course, we may verify this answer by using the definition (2) of the Semivalue.

Turn to what we called the inverse problem; recall that for a value $\Psi : G^N \rightarrow R^n$, the inverse problem can be stated as follows: an n-vector

L being given, find out all games in G^N such that $\Psi(N,\nu)$ =L. This problem has been solved for the Shapley value and the weighted Shapley value (I.Dragan,1991). Recently, the problem was also solved for Semivalues (I.Dragan,2002), extending the procedure used for the Shapley value to Semivalues. Here we have an alternative solution based upon the remark made at the beginning of this section, precisely

$$SE(N,\nu) = SH(N, w-\alpha). \tag{28}$$

Let us state the result which solves the inverse problem for the Shapley value. Consider the following basis for G^N : $B = \left\{B_s \in G^N : S \subseteq N, S \neq \emptyset\right\}$, where for $S \subset N$ we have $B_s(T) = |S|$, if $T = S$, and $B_s(T) = -1$, if $T = S \cup \{j\}, j \notin S$, and $B_s(T) = 0$, otherwise; $B_N(N) = n$, and $B_N(T) = 0$, otherwise.

Theorem 6 (I.Dragan,1991): For any Ł $\in R^n$, the set of games in G^N with the Shapley value $SH(N,\nu) =$ Ł is given by the formula:

$$\nu = \sum_{|S|\leq n-2} \beta_S B_S + \beta_N (B_N + \sum_{j\in N} B_{N-\{j\}}) - \sum_{j \in N} L_j B_{N-\{j\}}, \tag{29}$$

where β_N and β_S, $|S| \leq n-2$, are any real numbers.

Taking into account (28), the solution of the inverse problem for a Semivalue can be obtained from (29), where in the left hand side we should take $\nu = w - \alpha$, with the game (N,α) defined at the beginning of this section, and (N,w) defined in (21), then the equation should be solved for w. The $2^n - 1$ dimensional games (N,w) and (N,α) should be expressed in terms of the original game and L, by using (21); the last game is via (15) given by

$$\alpha(S) = \frac{s}{n}[\nu(N) - \sum_{j\in N} L_j], \quad \forall S \subseteq N. \tag{30}$$

(see below).

Example 2: Consider the given vector $L = \left(\frac{23}{36}, \frac{23}{36}, 0, -\frac{13}{36}\right)$, and find out all games for which the Semivalue associated with the weight vector $p^4 = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{18}, \frac{1}{3}\right)$ equals L. For our four person game, from (29) we obtain:

$$w = \alpha + \sum_{|S|=1,2} \beta_S B_S + \beta_N (B_N + \sum_{j\in N} B_{N-\{j\}}) - \sum_{j\in N} L_j B_{N-\{j\}}, \tag{31}$$

This is a vector-equation, in which the left hand side is providing the coalitional form of all the games we are trying to find (each $\nu(S)$ is multiplied by γ_s^3, $s = 1,2,3$) and in the right hand side appear linear combinations of the 12 parameters defining the set of the solution games, precisely, the 10 parameters β_S with $|S| = 1, 2$, then β_N and α. For

$$\begin{array}{c}\beta_1 = \frac{35}{48}, \beta_2 = \frac{35}{48}, \beta_3 = -\frac{1}{48}, \beta_4 = -\frac{1}{48}. \\ \beta_{12} = \frac{5}{4}, \beta_{13} = \frac{7}{8}, \beta_{14} = \frac{1}{3}, \beta_{23} = \frac{7}{8}, \beta_{24} = \frac{1}{3}, \beta_{34} = -\frac{1}{24} \\ \beta_{1234} = \frac{71}{72}, \alpha = \frac{1}{48},\end{array} \tag{32}$$

we obtain our game considered in Example 1. It is interesting to notice that the basis does not depend on the weight vector, which appear only in the left hand side, as was explained above. Hence, in fact the entire set of games solving the inverse problem is depending on 15 parameters, namely beside the 12 parameters mentioned above, we have also $\gamma_1^3, \gamma_2^3, \gamma_3^3$. In the paper where we solved the inverse problem relative to the Semivalues, we had an infinite set of bases depending on the chosen weight vector p^4 .

Returning to the general case, the similar procedure is providing the characteristic function of all solution games as functions of $2^n - n - 1$ parameters β_N and β_S for $|S| \leq n-2$, and α. The other parameters $\gamma_s^{n-1}, \quad s = 1, 2, ...n-1$, appear in the left hand sides.

Aknowledgement. This is a survey on the research of the author on the Semivalues, to be presented at The 4th Twente Workshop on Cooperative Game Theory, Enschede, The Netherlands, June 28-30,2005.

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Game Theoretic Analysis of Transportation Problems

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Abstract

This paper presents some game theoretical approaches to railway problems. The main topic is the definition of a fair access fee to the European railway network, which matches the directives of the European Union.

1 Introduction

In the last fifteen years game theory found many applications to railway sector in Europe, after the European Community directives 440/91, 18/95 and 19/95, later confirmed and/or modified as European Union directives 12/01, 13/01 and 14/01. These directives deal with the reorganization of the European railway system; in particular they state the separation between infrastructure management and transport operations and allow the access to the infrastructure also to private railway undertakers.

Since the railway industry traditionally has been organized as vertically integrated firms, the allocation of scarce resources (track capacity) may results in inefficiencies arising from differential information, inappropriate incentives and the existence of priority groups. The EC/EU directives increased the importance of an efficient capacity allocation, jointly with the necessity of a fair tariff system that guarantees a non discriminatory access to the infrastructure, minimizing the government subsidizations to the railway system.

The EU directive 14/01 explicitly refers to the cooperation between the infrastructure managers and railway undertakers in order to enhance the exploitation of the track capacity, maximizing the number of requests satisfied as best as possible.

More precisely, the directive 14/01/UE gives some useful suggestions and guidelines. A tariff system should favor transparency and non-discriminatory access, impose equivalent tariffs for equivalent services, averaging the costs, encourage an optimal use of the network, reduce the scarcity of the capacity of the network, coordinating the requests of railway undertakers, enhance the available infrastructure capacity, incentivating the investments by the infrastructure managers. Moreover it is possible to charge the transport operators for infrastructure maintenance.

The issue of track allocation has been treated, for instance, in Brewer and Plott (1996), Bassanini and Nastasi (1997) and Nilsson (1999).

Brewer and Plott (1996) proposed a decentralized allocation process based on a binary conflict ascending price (BICAP) mechanisms in which each agent submits bids for trains in a continuous time auction. The highest bid on a train prevails as the potential winner and cancels all lower bids for the train. Since the potential allocation is of higher value than any allocation possible from the excluded trains, the final allocation must necessarily be efficient if the excluded agents are fully revealing their willingness to pay. Experiments indicate that the mechanism operates at near 100% efficiency.

Bassanini and Nastasi (1997) presented a three-stage model: in the first stage the railway undertaker ask for their preferred tracks, specifying a monetary evaluation; in the second stage the infrastructure manager assigns the available capacity of the network, maximizing the total assigned value in a non-discriminatory mechanism, based on a non cooperative market game; the third stage deals with the service prices for the users.

Nilsson (1999) suggested a Vickrey-type mechanism to handle incentive aspects of this technically complex optimization task. Here, the price for operating a train will correspond to the bids foregone by other operators who are pushed off their preferred routes. The main advantage of a second price auction is that bidders can confine their attention to appraising the value of an item in their own hands rather than deliberating over value or bidding strategy by others. Consequently, more bidders are induced to participate in the process, resulting in a better allocation of resources and a higher selling price. Experimental solutions capture 90-100% of the potential benefits.

Here we are interested in the second issue, the design of a tariff system.

A fair infrastructure access fee, i.e. the amount paid by the railway undertakers to the firm in charge of the infrastructure management for a particular journey, should take into account several aspects such as the a priori profitability and social utility of the journey, congestion issues, the number of passengers and/or goods transported, the services required by the operator, infrastructure costs, etc. The tariff is conceived in an additive way, i.e. as the sum of various tariffs corresponding to the different aspects to be considered.

The problem can be informally described as follows. A given railway path is used by different types of trains belonging to several operators, and the infrastructure costs have to be divided among these trains. Clearly it is a problem of joint cost allocation (see Tijs and Driessen, 1986 and Young, 1994).

The infrastructure can be considered as consisting of some kinds of "facilities" (track, signaling system, stations, etc.). Different groups of trains need these facilities at different levels: for example, fast trains need a more sophisticated track and signaling system, compared to local trains, for which instead station services are more important (particularly in small stations).

So the infrastructure can be viewed as the "sum" of different facilities, each of them required by the trains at a different level of cost.

Furthermore, for each facility, infrastructure costs can be seen as the sum of "building" costs and "maintenance" costs. The first can be seen as a fixed part, because they depend on the level of the facility, but are independent from the number of trains; the latter represent a variable part, because they are proportional to the number of trains (and depend on the level of the facility).

In a game theoretical setting Fragnelli et al. (2000) proposed as a solution for this problem the Shapley value (see Shapley, 1953) that results especially appropriate because of the following two reasons:

1. It is well-known that the Shapley value is an additive solution. This feature fits well with the "additive nature" of the access tariff, as commented above.

2. The infrastructure access tariff based on the Shapley value can be computed very easily (using, once more, the additivity of the Shapley value). As a very big amount of fees will have to be computed by the infrastructure manager every new season, computational issues become highly relevant.

Similar considerations extend to other problems: for example the costs for a bridge, to be used by small and big cars. There are building costs, which are different in the case of a bridge for small or big cars, and maintenance costs, which can be assumed to be proportional to the number of vehicles using the bridge and to the kind of bridge needed.

Another situation (see Remark 1) refers to the allocation of the operating costs for a consortium for urban solid wastes collection and disposal (Fragnelli and Iandolino, 2004). This application fully exploits the structure of fixed and variable costs; in fact operating costs apparently refer only to "maintenance interventions", but those costs that depend only on the type of users, e.g. environmental monitoring, can be classified as "building" costs, while the costs that take into account also the number of users, e.g. Raw materials, can be classified as "maintenance" costs.

Maintenance and building cost games were used also in Garcia and Garcia-Jurado (2000) in queue management and in González and Herrero (2004) for sharing the costs related to the operating-theatre in a hospital.

In Section 2 we introduce the infrastructure cost games for one facility and provide a simple expression of the Shapley value for this class of games. In Section 3 we briefly study the balancedness of the infrastructure cost games and give a simple expression for the nucleolus. Section 4 deals with infrastructure cost games for more than one facility. In Section 5 we present a simple case-study. Section 6 concludes.

2 One Facility Infrastructure Cost Games

For simplicity, we concentrate first on infrastructure cost games when we are dealing with the building and maintenance costs of one facility. To begin with, we recall the definition of an "airport game" (see Littlechild and Owen, 1973).

Definition 1 *Suppose we are given k groups of players $g_1, ..., g_k$ with n_1,...,n_k players respectively and k non-negative numbers $b_1, ..., b_k$. The* airport game *corresponding to $g_1, ..., g_k$ and $b_1, ..., b_k$ is the cooperative (cost) game (N, c) with $N = \cup_{i=1,...,k} g_i$ and cost function c defined by*

$$c(S) = b_1 + \cdots + b_{j(S)}$$

for every $S \subseteq N$, where $j(S) = \max\{j : S \cap g_j \neq \emptyset\}$.

Airport games match the characteristics of building cost games for one facility, where the groups of players represent the trains requiring a certain level of the facility and b_i represents the extra cost in order that a facility that can be used by players in groups $g_1, ...g_{i-1}$ can also be used by the more sophisticated players in group g_i; consequently, the cost of a facility of level i is given by $b_1 + \cdots + b_i$.

The Shapley value of a building cost game (N, c^b) for a player in the group $g_i, i = 1, ..., k$ is given by:

$$\phi_i(c^b) = \sum_{j=1,...,i} \frac{b_j}{G_{jk}}$$

where $G_{jk} = |\cup_{h=j,...,k}\ g_h|$.

Now we consider the maintenance cost games (see Fragnelli et al., 2000) for one facility, starting from the basic assumptions that maintenance costs are proportional to the number of users and increasing with the level of the facility.

Definition 2 *Suppose we are given k groups of players g_1,...,g_k with n_1,...,n_k players respectively and $k(k+1)/2$ non-negative numbers $\{\alpha_{ij}\}_{i,j \in \{1,...,k\}, j \geq i}$. The* maintenance cost game *corresponding to g_1,...,g_k and $\{\alpha_{ij}\}_{i,j \in \{1,...,k\}, j \geq i}$ is the cooperative (cost) game (N, c^m) with $N = \cup_{i=1,...,k}\ g_i$ and cost function c^m defined by:*

$$c^m(S) = \sum_{i=1}^{j(S)} |S \cap g_i| A_{ij(S)}$$

for every $S \subseteq N$, where $A_{ij} = \alpha_{ii} + ... + \alpha_{ij}$ for all $i, j \in \{1, ..., k\}$ with $j \geq i$.

The interpretation of the numbers α_{ij} and A_{ij} is the following. Suppose that one player in g_i has used the facility. In order to restore the facility up to level i the maintenance costs are $A_{ii} = \alpha_{ii}$. If, however, the facility is going to be restored up to level $i+1$, then extra maintenance costs $\alpha_{i,i+1}$ will be made. So, in order to restore the facility up to level $j \geq i$ the maintenance costs are $A_{ij} = \alpha_{ii} + ... + \alpha_{ij}$.

Note that, for every $i \leq j$, the more sophisticated the facility is, i.e. the larger j is, the higher the maintenance costs produced by a player in g_i are.

A maintenance cost game (N, c^m) can be decomposed as:

$$c^m(S) = \sum_{i=1,...,k} \sum_{j=i,...,k} \alpha_{ij} c^{ij}(S), S \subseteq N$$

where

$$c^{ij}(S) = \begin{cases} |S \cap g_i| & \text{if } j \leq j(S) \\ 0 & \text{if } j > j(S) \end{cases}$$

for all $i, j \in \{1, ..., k\}$ with $j \geq i$.

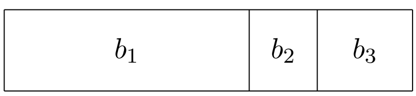
The previous decomposition allows us stating the following theorem (see Theorem 3.1 in Fragnelli et al., 2000) that provides a simple expression of the Shapley value for a maintenance cost game.

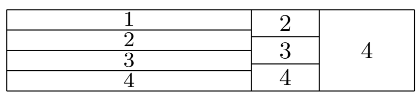
Theorem 1 *Let (N, c^m) be the maintenance cost game corresponding to the groups g_1,...,g_k, with n_1,...,n_k players respectively and to non-negative numbers $\{\alpha_{lm}\}_{l,m\in\{1,...,k\},m\geq l}$. Then the Shapley value for a player in the group $g_i, i = 1, ..., k$ is:*

$$\phi_i(c^m) = \alpha_{ii} + \sum_{l=i+1,...,k} \alpha_{il} \frac{G_{lk}}{G_{lk}+1} + \sum_{l=2,...,i} \sum_{j=1,...,l-1} \alpha_{jl} \frac{|g_j|}{(G_{lk})(G_{lk}+1)}$$

The following graphical example may make clearer the previous formulas for the Shapley value.

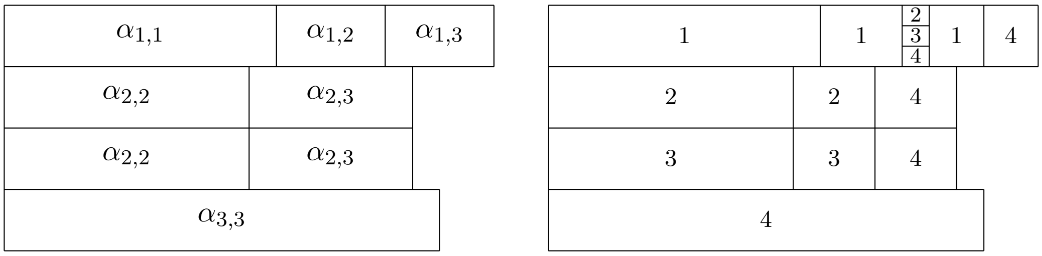
Example 1 *Let $N = g_1 \cup g_2 \cup g_3$ where $g_1 = \{1\}; g_2 = \{2, 3\}; g_3 = \{4\}$ and let the building cost game represented as in the figure on the left; the Shapley value divides the cost as in the figure on the right:*





$$\begin{aligned}
\phi_1(c^b) &= \tfrac{1}{4}b_1 \\
\phi_2(c^b) &= \phi_3(c^b) = \tfrac{1}{4}b_1 + \tfrac{1}{3}b_2 \\
\phi_4(c^b) &= \tfrac{1}{4}b_1 + \tfrac{1}{3}b_2 + b_3
\end{aligned}$$

Analogously, let the maintenance cost game represented as in the figure on the left; the Shapley value divides the cost as in the figure on the right:



$$\phi_1(c^m) = \alpha_{1,1} + \tfrac{3}{4}\alpha_{1,2} + \tfrac{1}{2}\alpha_{1,3}$$
$$\phi_2(c^m) = \phi_3(c^m) = \alpha_{2,2} + \tfrac{1}{2}\alpha_{2,3} + \tfrac{1}{3} \cdot \tfrac{1}{4}\alpha_{1,2}$$
$$\phi_4(c^m) = \alpha_{3,3} + \tfrac{1}{3} \cdot \tfrac{1}{4}\alpha_{1,2} + \tfrac{1}{2}\alpha_{1,3} + 2 \cdot \tfrac{1}{2}\alpha_{2,3}$$

Remark 1 *Applying the infrastructure cost games to a different kind of facility, namely a consortium for collection and disposal of urban solid wastes (see Fragnelli and Iandolino, 2004), an undesired behavior of the Shapley value showed up (see also the discussion on the monotonicity of the Shapley value in Young, 1994). When the number of players in each group increases proportionally (for example the population in each group doubles) the Shapley value of the maintenance cost game smooths the differences among the amount charged to the different groups (in the case analyzed in Fragnelli and Iandolino (2004) they converge to a unique value for all the players). This characteristic may be avoided using the Owen value (Owen, 1977), for which a simple expression exists, as stated in the following theorem (see Proposition 1 in Fragnelli and Iandolino, 2004).*

Theorem 2 *Let (N, c^m) be the maintenance cost game corresponding to the groups $g_1,\ldots,g_k$, with $n_1,\ldots,n_k$ players respectively and to the non-negative numbers $\{\alpha_{lm}\}_{l,m\in\{1,\ldots,k\},m\geq l}$. If the groups $g_1,\ldots,g_k$ correspond to the* a priori unions*, then the Owen value for a player in the group $g_i, i = 1, \ldots, k$ is:*

$$\Omega_i(c^m) = \sum_{H\subset G_{i-1}} \frac{h!(k-h-1)!}{k!}\left(\frac{1}{|g_i|}\beta(H) + \alpha_{ii}\right) + \sum_{H\not\subset G_{i-1}} \frac{h!(k-h-1)!}{k!} A_{i,j(H)}$$

where $h = |H|$, $G_{i-1} = \{g_1, \ldots, g_{i-1}\}$, $\beta(H)$ is the cost for "upgrading" the players in H, i.e. $\beta(H) = A_{j(H),i}\sum_{j|g_j\in H}|g_j|$ and $j(H) = max\{j|g_j \in H\}$.

Analogously, for a building cost game (N, c^b) corresponding to the groups $g_1,\ldots,g_k$, with $n_1,\ldots,n_k$ players respectively and to the non-negative numbers $b_1,\ldots,b_k$, with a priori unions *$g_1,\ldots,g_k$, then the Owen value for a player in the group $g_i, i = 1, \ldots, k$ is:*

$$\Omega_i(c^b) = \sum_{H\subset G_{i-1}} \frac{h!(k-h-1)!}{k!}\left(\frac{b_i - b_{j(H)}}{|g_i|}\right)$$

3 Balancedness of One Facility Infrastructure Cost Games

In this section we provide a characterization of the balancedness for one facility infrastructure cost games and a formula for computing the nucleolus.

The balancedness conditions may be stated by the following proposition (see Proposition 3.1 in Norde et al., 2002)

Proposition 1 *Let (N, c) be a one facility infrastructure cost game with groups g_1,...,g_k, with n_1,...,n_k players respectively and non-negative numbers b_1,...,b_k and $\{\alpha_{ij}\}_{i,j\in\{1,...,k\},j\geq i}$. Then (N, c) is balanced iff:*

$$\sum_{i=1,...,j} n_i (A_{ik} - A_{ij}) \leq \sum_{i=1,...,j} b_i$$

for every $j = 1, ..., k - 1$.

The balancedness conditions are obtained by considering minimal balanced collections which correspond to "splits" of N into two groups of the following kind: $g_1 \cup \cdots \cup g_j$ and $g_{j+1} \cup \cdots \cup g_k$. The interpretation of these conditions is the following: the maintenance costs that the players in $g_1 \cup \cdots \cup g_j$ have to pay for the level of the "needs" of the other players should be less than or equal to the building costs for the facility at the level needed by these groups themselves.

Remark 2 *The computation of the Shapley values and the check of the balancedness conditions may be done using the package ShRInC (Sharing Railways Infrastructure Costs), created with the collaboration of Luisa Carpente and Claudia Viale.*

The nucleolus of a one facility infrastructure game (see Schmeidler, 1969) can be computed according to the following definition and proposition (see Definition 3.4 and Proposition 3.5 in Norde et al., 2002).

Definition 3 *Let (N, c) be a balanced one facility infrastructure cost game with groups g_1,...,g_k, with n_1,...,n_k players respectively and non-negative numbers b_1,...,b_k and $\{\alpha_{ij}\}_{i,j\in\{1,...,k\},j\geq i}$. Let the numbers $\hat{b}_1$,...,$\hat{b}_k$ be defined by the linear system:*

$$\begin{cases} \hat{b}_1 & = & b_1 - n_1 (A_{1k} - A_{11}) \\ \hat{b}_1 + \hat{b}_2 & = & \sum\limits_{i=1,2} b_i - \sum\limits_{i=1,2} n_i (A_{ik} - A_{i2}) \\ ... & & ... \\ \hat{b}_1 + \hat{b}_2 + ... + \hat{b}_{k-1} & = & \sum\limits_{i=1,...,k-1} b_i - \sum\limits_{i=1,...,k-1} n_i (A_{ik} - A_{i,k-1}) \\ \hat{b}_1 + \hat{b}_2 + ... + \hat{b}_{k-1} + \hat{b}_k & = & \sum\limits_{i=1,...,k} b_i \end{cases}$$

Let the vector $(z_1, ..., z_k)$ *be defined recursively by:*

$$z_1 = \min_{1\leq j\leq k} \left\{ \frac{\sum_{l=1,...,j} \hat{b}_l}{W_j} \right\}$$
$$z_i = \min_{i\leq j\leq k} \left\{ \frac{\sum_{l=1,...,j} \hat{b}_l - (n_1 z_1 + ... + n_{i-1} z_{i-1})}{W_j - \sum_{l=1,...,i-1} n_l} \right\},\ i = 2, ..., k$$

where $W_j = \sum_{l=1,...,j} n_l + 1$ *for* $j = 1, ..., k-1$ *and* $W_k = \sum_{l=1,...,k} n_l$.
Define the allocation $\Phi(c) = (\Phi_1(c), ..., \Phi_k(c))$ *by:*

$$\Phi_i(c) = A_{ik} + z_i,\ i = 1, ..., k$$

Proposition 2 *Let* (N, c) *be a balanced one facility infrastructure cost game with groups* g_1,...,g_k, *with* n_1,...,n_k *players respectively and non-negative numbers* b_1,...,b_k *and* $\{\alpha_{ij}\}_{i,j\in\{1,...,k\},j\geq i}$. *Then* $\Phi(c)$ *is the nucleolus of* (N, c).

4 Infrastructure cost games

In this section we consider infrastructure cost games with an arbitrary number m of facilities, where no special requirements upon the ordering of the wishes of the coalitions for the several facilities will be made, but the groups are the same for each facility.

Suppose we are given an infrastructure cost game (N, c) with groups of players g_1,...,g_k, with n_1,...,n_k players respectively. Let $c = c^1 + ... + c^m$ be such that, for every $l \in M := \{1, ..., m\}$, (N, c^l) is a one facility infrastructure cost game with groups of players $g_{\pi^l(1)}$,...,$g_{\pi^l(k)}$, where π^l is a permutation of the set $K = \{1, ..., k\}$. Let $(b_i^l)_{i\in K}$ and $\{\alpha_{ij}^l\}_{i,j\in K,j\geq i}$ be the non-negative numbers which define the one facility infrastructure game (N, c^l) and let $n_i^l := n_{\pi^l(i)}$ be the number of players in the group ranked at the i-th place for facility l.

If an infrastructure cost game is the sum of balanced one facility infrastructure cost games then clearly this game is balanced. The following example (see Example 4.2 in Norde et al., 2002) shows that the converse statement is not true.

Example 2 *Consider the infrastructure cost game* (N, c), *dealing with the building and maintenance costs of two facilities, where the ordering of the wishes of the three groups involved for the facilities are given by:*

facility 1	$g_1\ g_2\ g_3$
facility 2	$g_2\ g_1\ g_3$

Suppose that every group has precisely one player, say $g_1 = \{1\}$*,* $g_2 = \{2\}$*, and* $g_3 = \{3\}$*. The one facility infrastructure cost games* $< N, c^1 >$ *and* $< N, c^2 >$ *are defined by the numbers*

$$\begin{array}{lll} b_1^1 = b_1^2 = 1 & b_2^1 = b_2^2 = 9 & b_3^1 = b_3^2 = 1 \\ \alpha_{1,1}^1 = \alpha_{1,1}^2 = 1 & \alpha_{1,2}^1 = \alpha_{1,2}^2 = 1 & \alpha_{1,3}^1 = \alpha_{1,3}^2 = 1 \\ & \alpha_{2,2}^1 = \alpha_{2,2}^2 = 2 & \alpha_{2,3}^1 = \alpha_{2,3}^2 = 1 \\ & & \alpha_{3,3}^1 = \alpha_{3,3}^2 = 3 \end{array}$$

One easily verifies that $c^1(1) = 2$*,* $c^1(2) = 12$*,* $c^1(3) = c^1(12) = 14$*,* $c^1(13) = c^1(23) = 17$*, and* $c^1(123) = 20$*. Since* $c^1(123) > c^1(1) + c^1(23)$ *we conclude that* (N, c^1) *is not balanced. Moreover, we have* $c^2(1) = 12$*,* $c^2(2) = 2$*,* $c^2(3) = c^2(12) = 14$*,* $c^2(13) = c^2(23) = 17$*, and* $c^2(123) = 20$*. From* $c^2(123) > c^2(2) + c^2(13)$ *we infer that* (N, c^2) *is not balanced. The game* (N, c) *is specified by the data* $c(1) = 14$*,* $c(2) = 14$*,* $c(3) = c(12) = 28$*,* $c(13) = c(23) = 34$*, and* $c(123) = 40$*. One easily verifies that* $(6, 6, 28)$ *is a core element of* (N, c)*, so it is balanced.*

Remark 3 *The different ordering of the groups of players for the various facilities makes the monotonicity condition for the costs associated to the different groups no longer valid. This leads to the definition of generalized airport games (Norde et al. 2002). Generalized airport games have been applied to deterministic auction situations for computing the Shapley value in an easy way (see Branzei et al. 2005).*

5 An Example

In this section we compute the Shapley value for a case study elaborated on data taken from Baumgartner (1997). The example (see Fragnelli et al. 2000) concentrates on a single element (the track), even if Baumgartner provides data also for other elements (line, catenary, signaling and security system, etc.), that can be analyzed in a similar fashion. Consider one kilometer of track, we get two kinds of costs[1], that depend on the type of train (slow or fast) and on the number of trains running. More precisely, we have both *renewal costs* and *repairing costs* and accordingly we divide the track into two facilities: "track renewal" and "track repairing".

Renewal costs can be approximated by the following formula:

$$RWC = 0.001125X + 11,250$$

where RWC are the renewal costs per kilometer and per year (expressed in Swiss Francs) and X measures the "number" of trains, expressed in yearly TGCK (Tons Gross and Complete per Kilometer).

[1]We assumed the weight of 50Kg for a meter of rail and made a linear approximation of the costs given in table 2 of Baumgartner (1997).

Assuming for simplicity that all of the trains running are of the same weight, the facility "track renewal" has a fixed component (building costs) and a part proportional to the number of trains running (maintenance costs). If the assumption of equal weight cannot be sustained, it suffices to divide the trains into groups of similar weight. In such a case each group will have different unitary maintenance costs.

Similarly, for the facility "track repairing", costs can be given by analogous formulas:

$$RPC_s = 0.001X + 10,000$$

$$RPC_f = 0.00125X + 12,500.$$

RPC_s denotes the repairing costs (in Swiss Francs) per kilometer and per year of a track prepared only for slow trains, whereas RPC_f denotes the repairing costs (in Swiss Francs) per kilometer and per year of a track prepared for all trains. X denotes the same as before.

So, consider one kilometer of line, which will be used this year by a total weight of 10^7 TGCK (corresponding to 20,000 trains, assuming a weight per train of approximately 500 tons). Assume that 5,000 trains are fast and 15,000 are slow. The infrastructure cost game that can be used to allocate the costs is (N, c) given by:

- $N = g_1 \cup g_2$, g_1 being the set of slow trains ($n_1 = 15,000$) and g_2 being the set of fast trains ($n_2 = 5,000$).
- $c = c^1 + c^2$, c^1 and c^2 being one facility infrastructure cost games both having the same groups of players and ordered in the same way: g_1, g_2.

Now, c^1 and c^2 are characterized by the following parameters.

- $c^1 : b_1^1 = 11,250$; $b_2^1 = 0$; $\alpha_{1,1}^1 = 0.5625$; $\alpha_{1,2}^1 = 0$; $\alpha_{2,2}^1 = 0.5625$.
- $c^2 : b_1^2 = 10,000$; $b_2^2 = 2,500$; $\alpha_{1,1}^2 = 0.5$; $\alpha_{1,2}^2 = 0.125$; $\alpha_{2,2}^2 = 0.625$.

Denoting the Shapley value of a slow and a fast train by $\phi_s(c)$ and $\phi_f(c)$ respectively, then:

- $\phi_s(c) = \frac{b_1^1}{n_1+n_2} + \alpha_{1,1}^1 + \frac{b_1^2}{n_1+n_2} + \alpha_{1,1}^2 + \alpha_{1,2}^2 \frac{n_2}{n_2+1} = 2.25$
- $\phi_f(c) = \frac{b_1^1}{n_1+n_2} + \alpha_{2,2}^1 + \frac{b_1^2}{n_1+n_2} + \frac{b_2^2}{n_2} + \alpha_{2,2}^2 + \alpha_{1,2}^2 \frac{n_1}{n_2(n_2+1)} = 2.75.$

These are the fees, in Swiss Francs, that every slow and fast train (respectively) should pay per kilometer of track used, according to our solution. Clearly, in front of a specific allocation problem regarding a specific line, with specific transport operators and trains, appropriate data should be collected.

6 Concluding Remarks

The interactions among railways cost allocation and game theory continue with the analysis of the problem of a tariff system for the freight trains. Again, the directive 14/01/UE suggests that a tariff system that favors sustainable mobility, a better balance of transport between modes, efficient use of international freight corridors, discounts for efficient use of the underutilized lines, direct charge of direct costs; moreover appropriate charges for paths allocated but not used are suggested and, similarly, incentives for reducing scarcity and limiting environmental impact, mainly acoustic pollution. The aim of the present researches is to develop a unified formula for all European infrastructure managers, in order to simplify the procedures for the railway undertakers.

Other possibilities are offered by the railway scheduling where the cooperation among the different agents may improve the gains and the timetable. In this last case the concept of homotachicity may improve the exploitation of the capacity of a line, with higher regularity of trains.

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Sequentially Stable Coalition Structures *

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Abstract

In this paper, we examine the questions of which coalition structure is formed and how payoff is distributed among players in cooperative games with externalities. We introduce a stability concept called a sequentially stable coalition structure in a game with coalition structures by extending the concept of the equilibrium binding agreements by Ray and Vohra (1997). In their definition, only breaking up is allowed for coalitions. However, in our stability concept, coalitions can both break up and merge into. A sequentially stable payoff configuration is defined as a payoff configuration which sequentially dominates all other payoff configurations. Diamantaudi and Xue (2002) also extend the concept of the Ray and Vohra, but the domination is very different. As an application of our stability notion, we study a common pool resource game. We show that if the number of players is between 4 and 48, then for some concave production function, the payoff configuration related to the grand coalition structure is sequentially stable in the common pool resource game.

1 Introduction

In this paper, we examine the questions of which coalition structure is formed and how payoff is distributed among players in cooperative games with externalities. We introduce a stability concept called a sequentially stable coalition structure in a game with coalition structures by extending the concept of the equilibrium binding agreements(EBA) by Ray and Vohra (1997). Ray and Vohra capture explicitly

*Preliminaries

the credibility of blocking coalitions, and then induce a recursive definition of the stable coalition structures in a game with externalities. However, in their definition, only breaking up is allowed for coalitions. This means that for example the coalition structure containing only singletons is always EBA.

Diamantoudi and Xue (2002) extend this notion and it is called EEBA. In their stability concept, both breaking up and merging into are allowed for coalitions. However in their definition, both breaking and merging occur at the same time, and this implies a lot of possibilities of changes in coalition structures at one step and makes its analysis complicated.

We extend the notion of EBA by a different way. In our definition, a domination between two coalition structures is defined by a sequence in which only two coalitions can merge into one coalition or one coalition can break into two coalitions. This gives a restriction to changes in coalition structures. Our definition of the domination is given as follows. The coalition structure z is said to sequentially dominate the coalition structure z' if there is a sequence of coalition structures starting from z to z' such that

(1) in each step, two coalitions may merge or one coalition may break into two coalitions, and

(2) in each step, the members in the merging coalitions or the breaking coalition prefer the payoffs of the final configuration z' to the present payoff.

A sequentially stable coalition structure is defined as a coalition structure which sequentially dominates all other coalition structures.

We compare these three notions and we show that these are characterized by vNM stable sets with respect to different domination relations.

Next we apply our stability concept to a common pool resource game, where each coalition structure corresponds to one coalition structure. We show that if the number of players is between 4 and 48, then for some concave production function, the coalition structure containing only the grand coalition is sequentially stable in the common pool resource game.

On the other hand, it is difficult to eliminate the stability of coalition structures containing singleton and $n-1$ person coalition because the singleton player gets the maximal payoff among the payoffs for all coalition structures. However we could show that the coalition structures containing singleton and $n-1$ person coalition are not sequentially stable under some concave production function for games with any number of players.

2 Dominations and Some Basic Concepts

Let $N = \{1, 2, ..., n\}$ be a set of players. A subset S of N is called a coalition. First we define a set of feasible payoff vectors under a coalition structure. We use the concept of a coalition structure how players form coalitions. Here a *coalition structure* $\mathcal{P}$ is a partition $\{S_1, S_2, ..., S_k\}$ of N, where $S_1, S_2, ..., S_k$ in $\mathcal{P}$ are disjoint and $\cup_{j=1}^{k} S_j = N$. The set of partitions of N is denoted by $\Pi(N)$.

We assume that given any coalition structure $\mathcal{P} \in \Pi(N)$, the feasible payoff vector under $\mathcal{P}$, $u(\mathcal{P}) = (u_1(\mathcal{P}), u_2(\mathcal{P}), ..., u_n(\mathcal{P})) \in \mathbb{R}^n$, is uniquely determined.

We give an example of a feasible payoff vector.

Example 1. A game in partition function form (N, v) is defined by a pair of a set of players N and a partition function v which assigns to each pair of a partition $\mathcal{P} \in \Pi(N)$ and a coalition $S \in \mathcal{P}$, a real value $v(S|\mathcal{P})$. Given a game in partition function form, the feasible payoff vector under $\mathcal{P}$ is given by $u_i(\mathcal{P}) = \frac{v(S|\mathcal{P})}{|S|}$ $\forall i \in S, \forall S \in \mathcal{P}$.

We introduce two special types of coalition structures. $\mathcal{P}^N = \{N\}$ is called a *grand* coalition structure, and $\mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\}$ is called a *singleton* coalition structure or *individual* coalition structure. We also say that $\mathcal{P}'$ is a *finer* coalition structure of $\mathcal{P}$ ($\mathcal{P}$ is a *coarser* coalition structure of $\mathcal{P}'$), if the coalition structure $\mathcal{P}'$ is given by re-dividing the coalition structure $\mathcal{P}$, that is, $\forall S' \in \mathcal{P}', \exists S \in \mathcal{P}$ such that $S' \subseteq S$ and $|\mathcal{P}'| > |\mathcal{P}|$.

We introduce several stability concepts for a set of coalition structures. This is an alternative way to define a core of a game with externalities. For this purpose, we define two simple concepts of dominations between two coalition structures.

Definition 1. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ is *dominated* by $\mathcal{P}'$ if

(1) $\mathcal{P}'$ is a finer coalition structure of $\mathcal{P}$, and

(2) there exists $T \in \mathcal{P}'$ such that $T \notin \mathcal{P}$ and $u_i(\mathcal{P}') > u_i(\mathcal{P})$ $\forall i \in T$.

Definition 2. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ is *directly dominated* by $\mathcal{P}'$ under $\mathcal{P}'$ if

(1) $\mathcal{P}'$ is a finer coalition structure of $\mathcal{P}$, and $|\mathcal{P}'| = |\mathcal{P}| + 1$,

(2) there exists $T \in \mathcal{P}'$ such that $T \notin \mathcal{P}$ and $u_i(\mathcal{P}') > u_i(\mathcal{P})$ $\forall i \in T$.

We can define stable coalition structures by these definitions of dominations.

The following definition is a natural extension of the credible core by Ray(1989) to games with externalities.

Definition 3. A *credible* coalition structure is given as follows:

(1) $\mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\}$ is credible.

(2) For $k \quad (k = n-1, n-2, ..., 1)$, $\mathcal{P}$ with $|\mathcal{P}| = k$ is credible if $\mathcal{P}$ is not directly dominated by any coalition structure $\mathcal{P}'$ where $\mathcal{P}'$ is credible and $|\mathcal{P}'| = k+1$.

We call this $\mathcal{P}$ a credible coalition structure. Moreover, the set of all credible coalition structures is called a *credible core*, and is denoted by CC.

This is a recursive definition. First, according to (1), $\mathcal{P}^I$ is credible. Second, we can check whether or not each of a coalition structure of $(n-1)$ coalitions is credible by using the fact $\mathcal{P}^I$ is credible. Third, we can check whether or not each of a coalition structure of $(n-2)$ coalitions is credible by using the fact obtained in the second step, and so on.

Ray and Vohra (1997) extends the credible core concept by a different way. Their concept is called "equilibrium binding agreement (EBA)". The following definition of an EBA coalition structure is the same as their concept properly, but it is expressed by a simpler way using a recursive definition.

Definition 4. An *EBA* coalition structure is given as follows:
(1) $\mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\}$ is an EBA.
(2') For $k \quad (k = n-1, n-2, ..., 1)$, $\mathcal{P}$ with $|\mathcal{P}| = k$ is an EBA if $\mathcal{P}$ is not dominated by any coalition structure $\mathcal{P}'$ where $\mathcal{P}'$ is an EBA and $|\mathcal{P}'| > k$.
The set of all EBA coalition structures is called an *EBA core*.

The difference between the two definitions is as follows: In a credible coalition structure, only the direct domination is considered, but in an EBA coalition structure, every any domination is considered.

We consider the stability concepts of coalition structures with respect to only deviation of coalitions but not for merge of coalitions in both Definitions 3 and 4.

3 Credible Cores in Common Pool Resource Games

Here we apply the above two cores to an economy with externalities. Consider the following game of an economy with a common pool resource. For any player $i \in N$, let $x_i \geq 0$ represent the amount of labor input of i. Clearly, the overall amount of labor is given by $\sum_{j \in N} x_j$. The technology that determines the amount of product is considered to be a joint production function of the overall amount of labor $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0, \lim_{x \to \infty} f'(x) = 0, f'(x) > 0$ and $f''(x) < 0$ for $x > 0$. The distribution of the product is supposed to be proportional to the amount of labor expended by players. In other words, the amount of the product assigned to player i is given by $\frac{x_i}{\sum_{j \in N} x_j} \cdot f(\sum_{j \in N} x_j)$. The price of the product is normalized to be one unit of money and let q be a cost of labor per unit, and we suppose $0 < q < f'(0)$.

Then individual i's income is denoted by

$$m_i(x_1, x_2, ..., x_n) = \frac{x_i}{x_N} f(x_N) - q x_i.$$

The total income of coalition S is denoted by

$$m_S \equiv \sum_{i \in S} m_i = \frac{x_S}{x_N} f(x_N) - q x_S,$$

where $x_S \equiv \sum_{i \in S} x_i$. We consider a game where each coalition is a player. It chooses its total labor input and its payoff is given by the sum of the income over its members. Naturally we can define a Nash equilibrium of that game.

Definition 5. $(x^*_{S_1}, x^*_{S_2}, ..., x^*_{S_k})$ is an *equilibrium under* $\mathcal{P} \iff$

$$m_{S_j}(x^*_{S_j}, x^*_{S_{-j}}) \geq m_{S_j}(x_{S_j}, x^*_{S_{-j}}), \quad \forall j, \quad \forall x_{S_j} \in \mathbb{R}_+.$$

Proposition 1 (Funaki and Yamato(1999)). *For any* $\mathcal{P} = \{S_1, S_2, ..., S_k\}$*, there exists a unique equilibrium* $(x^*_{S_1}, x^*_{S_2}, ..., x^*_{S_n})$ *under* $\mathcal{P}$ *which satisfies*

$$f'(x^*_N) + \frac{(k-1)f(x^*_N)}{x^*_N} = kq, \quad x^*_{S_j} = \frac{x^*_N}{k} \ \forall j, \quad x^*_{S_j} > 0 \ \forall j,$$

where $x^*_N = \sum_{j=1}^k x^*_{S_j}$.

Given a coalition structure $\mathcal{P} = \{S_1, ..., S_k\}$, let $(x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P}))$ be a unique equilibrium under $\mathcal{P}$ and let $x^*_N(\mathcal{P}) = \sum_{i=1}^k x^*_{S_i}(\mathcal{P})$. Moreover, let $m^*_{S_i}(\mathcal{P}) = m_{S_i}(x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P}))$ be the equilibrium income of coalition S_i for $i = 1, .., k$ and therefore
$m^*_N(\mathcal{P}) = \sum_{i=1}^k m_{S_i}(x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P}))$. The following result is given by Funaki and Yamato (1999).

Proposition 2 (Funaki and Yamato(1999)). *For two coalition structures* $\mathcal{P}_k = \{S_1, S_2, ..., S_k\}$ *and* $\mathcal{P}'_{k'} = \{S'_1, S'_2, ..., S'_{k'}\}$ *with* $k < k'$,

$$x^*_N(\mathcal{P}_k) < x^*_N(\mathcal{P}'_{k'}), \quad \frac{m^*_N(\mathcal{P}_k)}{n} > \frac{m^*_N(\mathcal{P}'_{k'})}{n},$$
$$S \in \mathcal{P}_k \ \text{ and } \ S \in \mathcal{P}'_{k'} \implies m^*_S(\mathcal{P}_k) > m^*_S(\mathcal{P}'_{k'}).$$

We assume that for a common pool resource game, the feasible payoff vector is given by $u_i(\mathcal{P}) = \frac{m^*_{S_j}(\mathcal{P})}{|S_j|} \ \forall i \in S_j, \forall S_j \in \mathcal{P}$. It is natural to consider this because of the symmetry of players.

The following is an important lemma to obtain Theorems 1 and 2.

Lemma 1. *In a common pool resource game, let a coalition structure* $\mathcal{P} \neq \mathcal{P}^I$ *be given. Without loss of generality, denote the coalition structure by* $\mathcal{P} = \{S_1, S_2, ..., S_k\}$*, where* $S_1 = \{1, 2, ..., r\}$*,* $2 \leq r \leq n$*, and* $1 \leq k \leq n-r+1$*. Suppose that the coalition* S_1 *is divided into two subcoalitions,* $S'_1 \equiv \{1, .., \ell\}$ *and* $S''_1 \equiv \{\ell+1, .., r\}$*, where* $1 \leq \ell \leq r/2$*. All other players do not change their behavior in coalition formation. Denote this coalition structure* $\mathcal{P}' = \{S'_1, S''_1, S_2, ..., S_k\}$*. Let* $m^*_1(\mathcal{P}) \equiv m^*_{S_1}(\mathcal{P})/r$ *and* $m^*_1(\mathcal{P}') \equiv m^*_{S'_1}(\mathcal{P}')/\ell$*. Then* $m^*_1(\mathcal{P}') > m^*_1(\mathcal{P})$ *if* $k^2/(k+1)^2 \geq \ell/r$*, in particular, if (i)* $r = n$ *and* $n/\ell \geq 4$*, (ii)* $3 \leq r \leq n-1$ *and* $\ell/r \leq 4/9$*, or (iii)* $r = 2$ *and* $k \geq 3$*.*

Proof. By Proposition 1,

$$m^*_1(\mathcal{P}) = m^*_{S_1}(\mathcal{P})/r = \frac{f(x^*_N(\mathcal{P})) - qx^*_N(\mathcal{P})}{rk} = \frac{f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})}{rk^2},$$

$$m^*_1(\mathcal{P}') = m^*_{S'_1}(\mathcal{P}')/\ell = \frac{f(x^*_N(\mathcal{P}')) - qx^*_N(\mathcal{P}')}{\ell(k+1)} = \frac{f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})}{\ell(k+1)^2}.$$

Therefore,

$$m_1^*(\mathcal{P}')-m_1^*(\mathcal{P}) =$$

$$\frac{rk^2\{f(x_N^*(\mathcal{P}')) - f'(x_N^*(\mathcal{P}'))x_N^*(\mathcal{P}')\} - \ell(k+1)^2\{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}}{r\ell k^2(k+1)^2}.$$

Here, $0 < f(x_N^*(\mathcal{P}))-x_N^*(\mathcal{P})f'(x_N^*(\mathcal{P})) < f(x_N^*(\mathcal{P}'))-x_N^*(\mathcal{P}')f'(x_N^*(\mathcal{P}'))$ holds because $f(x) - xf'(x)$ is increasing for $x > 0$, and $x_N^*(\mathcal{P}) < x_N^*(\mathcal{P}')$ by Proposition 2. Therefore, $m_1^*(\mathcal{P}') > m_1^*(\mathcal{P})$ if $A \equiv rk^2 - \ell(k+1)^2 \geq 0$, that is, $k^2/(k+1)^2 \geq \ell/r$. This condition is satisfied in the following cases.

$Case$ 1. $r = n$ and $n/\ell \geq 4$ Note that $r = n$ if and only if $k = 1$. Hence, $A = n - 4\ell \geq 0$ if $n/\ell \geq 4$.

$Case$ 2. $3 \leq r \leq n-1$ and $4/9 \geq \ell/r$: Since $r \neq n, k \geq 2$. Also, $k^2/(k+1)^2$ is increasing for $k > 0$. Therefore, $k^2/(k+1)^2 \geq 4/9$. Accordingly, if $4/9 \geq \ell/r$, then $A \geq 0$.

$Case$ 3. $r = 2$ and $k \geq 3$: Since $r = 2, \ell = 1$. Thus $A = (k-1)^2-2 \geq 2 > 0$.

Q.E.D.

We apply the stability concepts to this common pool resource game.

Example 2. In a common pool resource game, suppose a production function $f(x)$ is given by $f(x) = \sqrt{x}$.

(1) When $n = 4$, the singleton coalition structure $\mathcal{P}^I$ and all coalition structures consisting of two coalitions are both credible and EBA coalition structures.

(2) When $n = 5$, all coalition structures consisting of odd number of coalitions are credible. All coalition structures consisting of odd number of coalitions except for $\{\{i\}, \{j\}, T\}$ $(|T| = 3)$ are EBA coalition structures.

(3) When $n = 6$, all coalition structures containing even number of coalitions are credible. Only the grand coalition structure $\mathcal{P}^N$, the singleton coalition structure $\mathcal{P}^I$, $\{Q, R\}$ $\;(|Q| = |R| = 3)$ and $\{\{i\}, \{j\}, T, U\}$ $\;(|T| = |U| = 2)$ are EBA coalition structures.

The following theorem shows that if the number of players is odd, then coalition structures consisting of odd numbers of coalitions are credible, in particular, the grand coalition structure is credible and a credible core allocation exists. If the number of players is even, then coalition structures consisting of even numbers of coalitions are credible. In this case, although the grand coalition structure is not credible, coalition structures consisting of (*n-1*) -person coalition and one-person coalition are credible. This result is rather simple, but for the EBA coalition structures, it is not easy to get a general result.

Theorem 1. *In a common pool resource game, let $n \geq 4$. If n is odd, $\mathcal{P}$ consisting of odd number of coalitions is credible, and $CC(\mathcal{P}^N) \neq \emptyset$. If n is even, $\mathcal{P}$ consisting of even number of coalitions is credible, and $CC(\mathcal{P}^{N\setminus i}) \neq \emptyset$. Here* $\mathcal{P}^{N\setminus i} = \{N \setminus \{i\}, \{i\}\}$

Proof. Consider the case $n \geq 5$ first. According to the proof of Theorem 1, for the payoff vector z in $\mathcal{F}(\mathcal{P})$ with $\mathcal{P} \neq \mathcal{P}^I$, $\mathcal{P}$ is directly blocked by some T under some $\mathcal{P}$'. Consider any coalition structure $\mathcal{P}$ such that $|\mathcal{P}| - 1 = |\mathcal{P}^I|$ and $\mathcal{P}^I$ is finer than $\mathcal{P}$. Since $\mathcal{P}^I$ is credible by definition, the above result implies that $\mathcal{P}$ is directly blocked by finer credible coalition structure $\mathcal{P}^I$. This means that $\mathcal{P}$ is not credible.. The set of such $\mathcal{P}$ is denoted by $\mathbf{P}^2$. That is,

$$\mathbf{P}^2 = \{\mathcal{P} || \mathcal{P}| - 1 = |\mathcal{P}^I| \text{ and } \mathcal{P}^I \text{ is finer than } \mathcal{P}\}.$$

By a simple consideration, we have $\mathbf{P}^2 = \{\mathcal{P} || \mathcal{P}| = n - 1\}$. The above result directly implies that any $\mathcal{P}' \in \mathbf{P}^3$ is credible because any $\mathcal{P} \in \mathbf{P}^2$ is not credible, where

$$\mathbf{P}^3 = \{\mathcal{P}' || \mathcal{P}'| - 1 = |\mathcal{P}| \text{ for some } \mathcal{P} \in \mathbf{P}^2 \text{ and } \mathcal{P} \text{ is finer than } \mathcal{P}'\}$$

$$= \{\mathcal{P}' || \mathcal{P}'| = n - 2\}.$$

This consideration implies that any $\mathcal{P} \in \mathbf{P}^m$ is credible if $m = n - 2k(k = 0, 1, 2, ...)$, and not credible if $m = n - 2k - 1(k = 0, 1, 2, ...,)$. Since $m = n - 2k$ is odd if n is odd, $\mathcal{P}$ consisting of odd number of coalitions is credible, and $\mathcal{P}^N \in \mathbf{P}^n$ is credible, that is, $CC(\mathcal{P}^N) \neq \emptyset$. Since $m = n - 2k$ is even if n is even, $\mathcal{P}$ consisting of even number of coalitions is credible, and $\mathcal{P}^{N \setminus i} \in \mathbf{P}^{(n-1)}$ is credible, that is, $CC(\mathcal{P}^{N \setminus i}) \neq \emptyset$.

For the case $n = 4$, put $r = 2$ and $k = 3$ in Lemma 1. This implies $\mathcal{P} \in \mathbf{P}^2$ is not credible because $\mathcal{P}^I$ is credible. Then $\mathcal{P} \in \mathbf{P}^3$ is credible. Put $r = 3$ and $\ell = 1$ in Lemma 1. This implies $\mathcal{P} \in \mathbf{P}^4$ is not credible because $\mathcal{P} \in \mathbf{P}^3$ is credible.

Q.E.D.

Unfortunately we cannot find a general property of an EBA core of a common pool resource game as the following example illustrates.

Example 3. In a common pool resource game, let $f(x) = x^\alpha$, and let $n = 8$. When $\alpha = 0.2, 0.5, 0.8$, the grand coalition structure $\mathcal{P}^N$ is both credible and is an EBA. When $\alpha = 0.001, 0.9, 0.995$, the grand coalition structure $\mathcal{P}^N$ is not an EBA but credible.

In both definitions of credible cores and EBA cores, only breaking up is allowed for coalitions. In the next section, we propose another new stability concept of coalition structures such that coalitions can both break up and merge into.

4 Sequentially Stable Coalition Structures

In this section, we give our main stability concept called a "Sequentially Stable coalition structure". First we give a definition of sequential domination, and after that we give a definition of a sequentially stable coalition structure.

Definition 6. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ sequentially dominates $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^{T}$ such that

(1) $\mathcal{P}_T = \mathcal{P}$ and $\mathcal{P}_0 = \mathcal{P}'$,

(2) for all t $(0 \leq t \leq T-1)$, either $\mathcal{P}_{t+1}$ is a finer coalition structure of $\mathcal{P}_t$ with $|\mathcal{P}_{t+1}| = |\mathcal{P}_t| + 1$, or $\mathcal{P}_{t+1}$ is a coarser coalition structure of $\mathcal{P}_t$ with $|\mathcal{P}_{t+1}| = |\mathcal{P}_t| - 1$, and

(3) for all t $(0 \leq t \leq T-1)$, for some $S \in \mathcal{P}_{t+1}$ with $S \notin \mathcal{P}_t$,

$$u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \quad \forall i \in S.$$

We use the following notation for this sequence of coalition structures:

$$\mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \dots \to \mathcal{P}_T.$$

The condition (3) shows that if $\mathcal{P}_{t+1}$ is a finer coalition structure of $\mathcal{P}_t$, for any member i in one of the divided two coalitions S and T such that $S, T \in \mathcal{P}_{t+1}$ and $S \cup T \in \mathcal{P}_t$, his payoff $u_i(\mathcal{P}_t)$ is smaller than his terminal payoff $u_i(\mathcal{P}_T)$; and if $\mathcal{P}_{t+1}$ is a coarser coalition structure of $\mathcal{P}_t$, for any member i in two combining coalitions S and T such that $S, T \in \mathcal{P}_t$ and $S \cup T \in \mathcal{P}_{t+1}$, his payoff $u_i(\mathcal{P}_t)$ is smaller than his terminal payoff $u_i(\mathcal{P}_T)$.

Definition 7. We say that $\mathcal{P}^* \in \Pi(N)$ is a *sequentially stable coalition structure* if for all other coalition structures $\mathcal{P} \neq \mathcal{P}^*$, $\mathcal{P}^*$ sequentially dominates $\mathcal{P}$.

We will compare our domination notion with those of Ray and Vohra(1997) and Diamantoudi and Xue (2002). We have the domination due to Ray and Vohra called *RV-domination* by changing the condition (2) in Definition 6 into the following condition (2').

Definition 8. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ RV-dominates $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^{T}$ such that

(1) $\mathcal{P}_T = \mathcal{P}$ and $\mathcal{P}_0 = \mathcal{P}'$,

(2') for all t $(0 \leq t \leq T-1)$, $\mathcal{P}_{t+1}$ is a finer coalition structure of $\mathcal{P}_t$ with $|\mathcal{P}_{t+1}| = |\mathcal{P}_t| + 1$.

(3) for all t $(0 \leq t \leq T-1)$, for some $S \in \mathcal{P}_{t+1}$ with $S \notin \mathcal{P}_t$,

$$u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \quad \forall i \in S.$$

Note in condition (2'), only refinement of coalition structures is allowed. The set of EBA coalition structures is defined by the following set E of coalition structures such that

(a) $\mathcal{P}^I \in E$ and

(b) for any coalition structure $\mathcal{P}' \notin E$, there exists $\mathcal{P} \in E$ such that $\mathcal{P}$ RV-dominates $\mathcal{P}'$, and

(c) for any coalition structure $\mathcal{P}' \in E$, there is no $\mathcal{P} \in E$ such that $\mathcal{P}$ RV-dominates $\mathcal{P}'$.

Indeed the set E is the vNM-stable set via RV- domination (Diamantoudi and Xue (2002)) because condition (b) corresponds to the external stability of the vNM-stable set, and condition (c) corresponds to the internal stability of the vNM-stable set. For our notion of sequential domination, the singleton set consisting of any sequentially stable coalition structure is also the vNM-stable set via that domination.

If we change the conditions (2) and (3) in Definition 6 into the following conditions (2") and (3'), then we have a domination concept of Diamantoudi and Xue (2002) called *DX-domination*.

Definition 9. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ DX-dominates $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^{T}$ such that

(1) $\mathcal{P}_T = \mathcal{P}, \mathcal{P}_0 = \mathcal{P}'$, and

(2") for all t $(0 \leq t \leq T-1)$, $\mathcal{P}_{t+1}$ and $\mathcal{P}_t \equiv \{S_1, S_2, ..., S_k\}$ satisfy the following condition; there exists a coalition $Q(t) \subseteq N$ such that

(i) $Q(t) = Q_1 \cup Q_2 \cup ... \cup Q_l, Q_j \in \mathcal{P}_{t+1}\ \forall j = 1, 2, .., l$ and Q_js are disjoint,

(ii) $\forall j = 1, 2, ..., k, S_j \cap Q(t) \neq \emptyset \Rightarrow S_j \setminus Q(t) \in \mathcal{P}_{t+1}$,

(iii) $\forall j = 1, 2, ..., k, S_j \cap Q(t) = \emptyset \Rightarrow S_j \in \mathcal{P}_{t+1}$.

(3') for all t $(0 \leq t \leq T-1)$,

$$u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \quad \forall i \in Q(t).$$

In condition (2"), many possibility of refining and merging are allowed. Their definition and our definition of domination are both farsighted and coalition structures other than breaking or merging coalitions do not change. However the breaking or merging is step by step in our definition, but a jump is allowed in their definition. The vNM-stable set of the coalition structures using DX-domination is called the set of Extended EBA (EEBA) coalition structures. For two coalition structures $\mathcal{P}$ and $\mathcal{P}'$, $\mathcal{P}$ DX-dominates $\mathcal{P}'$ if $\mathcal{P}$ sequentially dominates $\mathcal{P}'$ because (2) in Definition 6 implies (2") in Definition 9. Hence the sequentially stable coalition structure is an EEBA structure.

The properties of EEBA coalition structures are examined in Diamantoudi and Xue (2002). In their paper, they give the following proposition:

Definition 10. The coalition structure $\mathcal{P} \in \Pi(N)$ is Pareto efficient if there does not exist $\mathcal{P}' \in \Pi(N)$ such that $u_i(\mathcal{P}') > u_i(\mathcal{P})$ for any $i \in N$.

Proposition 3 (Diamantoudi and Xue (2002)). Let $\mathcal{P}^* \in \Pi(N)$ be Pareto efficient. $\mathcal{P}^*$ is an EEBA if

(a) $u_i(\mathcal{P}^*) > u_i(\mathcal{P}^I)$ $\forall i \in N$, and

(b) for all $\mathcal{P} \in \Pi(N)$ such that $\mathcal{P} \neq \mathcal{P}^*$ and $\mathcal{P} \neq \mathcal{P}^I$, there is a coalition $S \in \mathcal{P}$ such that $|S| > 1$ and $u_i(\mathcal{P}^*) > u_i(\mathcal{P})$ for some $i \in S$.

The similar proposition holds for our notion of a sequential domination.

Proposition 4. Let $\mathcal{P}^* \in \Pi(N)$ be Pareto efficient. $\mathcal{P}^*$ is sequentially stable if

(a) $\mathcal{P}^*$ sequentially dominates $\mathcal{P}^I$, and

(b) for all $\mathcal{P} \in \Pi$ such that $\mathcal{P} \neq \mathcal{P}^*$ and $\mathcal{P} \neq \mathcal{P}^I$, there is a coalition $S \in \mathcal{P}$ such that $|S| > 1$ and for some member $i \in S$, $u_i(\mathcal{P}^*) > u_i(\mathcal{P})$.

Proof. Take any $\mathcal{P}$ such that $\mathcal{P} \neq \mathcal{P}^*$. We have to find a sequence of coalition structures from any $\mathcal{P}$ to $\mathcal{P}^*$ satisfying (1)(2)(3) in Definition 6. First we construct a sequence $\{\mathcal{P}_k\}_{k=0}^R$ of coalition structures from $\mathcal{P}$ to $\mathcal{P}^I$, where $\mathcal{P}_0 = \mathcal{P}$ to $\mathcal{P}_R = \mathcal{P}^I$ $(R \leq n)$. In the sequence $\{\mathcal{P}_k\}_{k=0}^R$, for any $\mathcal{P}_k$ such that $\mathcal{P}_k \neq \mathcal{P}^I$, one person deviates from one of the largest coalition in $\mathcal{P}_k$. In this step, the deviated person prefers $\mathcal{P}^*$ to $\mathcal{P}$ because of (b). Second, (a) implies that the existence of a sequence of coalition structures from $\mathcal{P}^I$ to $\mathcal{P}^*$. Combining these sequences, we obtain the desired sequence of coalition structures. This implies $\mathcal{P}^*$ sequentially dominates $\mathcal{P}$.

Q.E.D.

5 Sequentially Stable Coalition Structures in Common Pool Resource Game

We apply our stability concept, sequentially stable coalition structure, to a common pool resource game.

The following lemma gives a necessary and sufficient condition that the payoff configuration in the grand coalition structure is preferable to the coalition structure in another coalition structure for all players.

Lemma 2. In a common pool resource game, let a coalition structure $\mathcal{P}$ be given. Without loss of generality, denote the coalition structure by $\mathcal{P} = \{S_1, S_2, S_3, ..., S_k\}$, where $|S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq ... \leq |S_k| = r_k$. Let

$$B(k) \equiv \{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/[k^2\{f(x_N^*(\mathcal{P}^N))-$$

$$-f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\}], \qquad (1)$$

where $\mathcal{P}^N = \{1, 2, .., n\}$ is the grand coalition structure. Then for each $i \in N$, $m_i^*(\mathcal{P}) \gtreqless m_i^*(\mathcal{P}^N)$ if and only if $B(k) \gtreqless r_1/n$.

Proof. By Proposition 1,

$$m_i^*(\mathcal{P}) = m_{S_j}^*(\mathcal{P})/r_j = [f(x_N^*(\mathcal{P})) - qx_N^*(\mathcal{P})]/(r_j k) =$$
$$= \left[f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\right]/(r_j k^2),$$

for $i \in S_j$ and $j = 1, ..., k$. Notice that for the grand coalition structure $\mathcal{P}^N$, $k = 1$ and $r_1 = n$, so that $m_i^*(\mathcal{P}^N) = \left[f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\right]/n$ for $i \in N$. We also remark that a player belonging to the smallest coalition, S_1, obtains the highest payoff among all players, that is, the payoff of each player i, $m_i^*(\mathcal{P})$, is less than or equal to $m_{S_1}^*(\mathcal{P})/r_1$. Therefore, each $i \in N$, $m_i^*(\mathcal{P}) \gtreqless m_i^*(\mathcal{P}^N)$ if and only if $B(k) = \{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/[k^2\{f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\}] \gtreqless r_1/n$.

Q.E.D.

We check the sequential stability of the grand coalition structure. First consider a case $n = 2^m$ $(m \geq 2)$. We say $\mathcal{P}$ is *a k-th stage coalition structure* if $|\mathcal{P}| = k$.

Theorem 2. If $B(k) < 1/2^{k-1}$ for all $k(k = 2, ..., m, m+1)$, the grand coalition structure is sequentially stable.

Proof. We have to show that every coalition structure other than the grand coalition structure $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$. In the following, we denote a coalition structure $\mathcal{P} = \{S_1, S_2, S_3, ..., S_k\}$, where $|S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq ... \leq |S_k| = r_k$, by $\{r_1; r_2; r_3; ...; r_k\}$, because the payoff is determined by the sizes of all cotillions in a coalition structure.

Consider a coalition structure $\mathcal{P}^*$ consisting of the following $(m+1)$ coalitions: two 1-person coalitions, one 2-person coalition, one 4-person coalition, one 8-person coalition, ..., and one 2^{m-1}-person coalition. This coalition structure is denoted by $\{1; 1; 2; 4; 8; ....; 2^{m-1}\}$.

The proof consists of four steps.

(Step 1) $\mathcal{P}^*$ is sequentially dominated by $\mathcal{P}^N$:

Consider a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^m$ such that $\mathcal{P}_0 = \mathcal{P}^*, \mathcal{P}_m = \mathcal{P}^N$, and the two coalitions of the smallest size in $\mathcal{P}_t$ merge in $\mathcal{P}_{t+1}$ for $t = 0, 1, 2, ..., m-1$. This sequence is expressed by

$$\mathcal{P}_0 = \mathcal{P}^* = \{1; 1; 2; 4; 8; ...; 2^{m-2}; 2^{m-1}\} \to \mathcal{P}_1 = \{2; 2; 4; 8; ...; 2^{m-2}; 2^{m-1}\}$$
$$\to \mathcal{P}_2 = \{4; 4; 8; ...; 2^{m-2}; 2^{m-1}\} \to$$
$$.... \to .... \to \mathcal{P}_{m-2} = \{2^{m-2}; 2^{m-2}; 2^{m-1}\} \to$$

$$\mathcal{P}_{m-1} = \{2^{m-1}; 2^{m-1}\} \to \mathcal{P}_m = \mathcal{P}^N = \{2^m\}$$

First, it follows from Lemma 2 that the 2nd stage coalition structure $\mathcal{P}_{m-1} = \{2^{m-1}; 2^{m-1}\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-1}/2^m = 1/2 > B(2)$ by the hypothesis.

Next, it follows from Lemma 1 that the 3rd stage coalition structure $\mathcal{P}_{m-2} = \{2^{m-2}; 2^{m-2}; 2^{m-1}\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-2}/2^m = 1/4 > B(3)$ by the hypothesis.

In general, for $k = 2, ..., m, m+1$, it follows from Lemma 2 that the k-th stage coalition structure $\mathcal{P}_{m-k+1} = \{2^{m-k+1}; 2^{m-k+1}; 2^{m-k+2}; 2^{m-k+3}; ...; 2^{m-1}\}$ is sequentially dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-k+1}/2^m = 1/2^{k-1} > B(k)$ by the hypothesis.

Therefore, the $(m+1)$-th stage coalition structure $\mathcal{P}_0 = \mathcal{P}^* = \{1; 1; 2; 4; ...; 2^{m-1}\}$ is sequentially dominated by $\mathcal{P}^N$.

(Step 2) Every $(m+1)$-th stage coalition structure is sequentially dominated by $\mathcal{P}^N$:

Take any $(m+1)$-stage coalition structure $\mathcal{P}$.

First we consider a sequence $\{\mathcal{P}_t\}_{t=0}^T$ such that

1) $\mathcal{P}_0 = \mathcal{P} = \{r_1; r_2; r_3; ...; r_{m-1}; r_m; r_{m+1}\}$

2) $\mathcal{P}_T = \{1; 1; 1; ...; 1; 2^m - m\}$, where $|\mathcal{P}_T| = m+1$.

3) If t is zero or even, then the largest and the second largest coalitions in $\mathcal{P}_t$ merge in $\mathcal{P}_{t+1}$.

4) If t is odd, then one person belonging to the largest coalition in $\mathcal{P}_t$ deviates and forms one person coalition in $\mathcal{P}_{t+1}$.

Then the sequence $\{\mathcal{P}_t\}_{t=0}^T$ of coalition structures is given by:

$$\mathcal{P}_0 = \{r_1; r_2; r_3; ..., r_{m-1}; r_m; r_{m+1}\} \quad ((m+1)\text{-th stage})$$

$$\to \mathcal{P}_1 = \{r_1; r_2; r_3; ...; r_{m-1}; r_m + r_{m+1}\} \quad (m\text{-th stage})$$

$$\to \mathcal{P}_2 = \{1; r_1; r_2; r_3; ...; r_{m-1}; r_m + r_{m+1} - 1\} \quad ((m+1)\text{-th stage})$$

$$\to ... \to ...$$

$$\to \mathcal{P}_{T-2} = \{1; 1; 1; ...; 1; r_1; \sum_{k=2}^{m+1} r_k - m + 1\} \quad ((m+1)\text{-th stage})$$

$$\to \mathcal{P}_{T-1} = \{1; 1; 1; ...; 1; \sum_{k=1}^{m+1} r_k - m + 1\} \quad (m\text{th-stage})$$

$$\to \mathcal{P}_T = \{1; 1; 1; 1; ...; 1; \sum_{k=1}^{m+1} r_k - m\} = \{1; 1; 1; ...; 1; 2^m - m\} \quad ((m+1)\text{th-stage})$$

Next consider $\{\mathcal{P}_t\}_{t=T}^{T+T'}$ such that

1) $\mathcal{P}_T = \{1; 1; 1; ...; 1; 2^m - m\}$,

2) $\mathcal{P}_{T+T'} = \mathcal{P}^* = \{1; 1; 2; 4; 8; ...; 2^{m-2}; 2^{m-1}\}$,

3) If $t = T + \lambda$ and λ is zero or even ($\lambda \le T' - 2$), then the smallest coalition of more than one members and a 1-person coalition in $\mathcal{P}_{T+\lambda}$ merge in $\mathcal{P}_{T+\lambda+1}$.

4) If $t = T + \lambda$ and λ is odd ($\lambda \le T' - 2$), then $2^{m-\frac{\lambda+1}{2}}$ persons in the coalition of $2^{m-\frac{\lambda+1}{2}+1} - (m - \frac{\lambda+1}{2})$ persons in $\mathcal{P}_{T+\lambda}$ deviate and form a coalition in $\mathcal{P}_{T+\lambda+1}$. Note that $2^{m-\frac{\lambda+1}{2}+1} - (m - \frac{\lambda+1}{2}) \ge 1$.

5) If $t = T + T' - 1$, then two one-person coalitions in $\mathcal{P}_{T+T'-1}$ merge in $\mathcal{P}_{T+T'}$.

This sequence $\{\mathcal{P}_t\}_{t=T}^{T+T'}$ of coalition structures is given by:

$$\begin{aligned}
&\mathcal{P}_T = \{1;1;1;1;...;1;1;1;1;2^m - m\} \ \ ((m+1)\text{-th stage})\\
&\to \mathcal{P}_{T+1} = \{1;1;1;1;...;1;1;1;2^m - m + 1\} \ \ (m\text{-th stage})\\
&\to \mathcal{P}_{T+2} = \{1;1;1;1;...;1;1;1;2^m - m + 1 - 2^{m-1};2^{m-1}\}\\
&\qquad = \{1;1;1;1;...;1;1;1;2^{m-1} - m + 1;2^{m-1}\} \ \ ((m+1)\text{-th stage})\\
&\to \mathcal{P}_{T+3} = \{1;1;1;1;...;1;1;2^{m-1} - m + 2;2^{m-1}\} \ \ (m\text{-th stage})\\
&\to \mathcal{P}_{T+4} = \{1;1;1;1;...;1;1;2^{m-1} - m + 2 - 2^{m-2};2^{m-2};2^{m-1}\}\\
&\qquad = \{1;1;1;1;...;1;1;2^{m-2} - m + 2;2^{m-2};2^{m-1}\} \ \ ((m+1)\text{-th stage})\\
&\to \mathcal{P}_{T+5} = \{1;1;1;1;...;1;2^{m-2} - m + 3;2^{m-2};2^{m-1}\} \ \ (m\text{-th stage})\\
&\qquad \to ... \to ...\\
&\to \mathcal{P}_{T+T'-1} = \{1;1;1;1;4;8;...;2^{m-3};2^{m-2};2^{m-1}\} \ \ ((m+1)\text{-th stage})\\
&\to \mathcal{P}_{T+T'} = \{1;1;2;4;8;...;2^{m-3};2^{m-2};2^{m-1}\} \ \ (m\text{-th stage})
\end{aligned}$$

This sequence ends at the coalition structure $\mathcal{P}_{T'} = \mathcal{P}^*$.

Hence if we combine two sequences $\{\mathcal{P}_t\}_{t=0}^{T}$ and $\{\mathcal{P}_t\}_{t=T}^{T+T'}$, we can get a sequence $\{\mathcal{P}_t\}_{t=0}^{T+T'}$ from any $(m+1)$-th stage coalition structure $\mathcal{P}$ to $\mathcal{P}^*$. Note that only $(m+1)$-th stage and m-th stage coalition structures appear in this sequence.

Each member of any coalition in $(m+1)$-th stage coalition structure prefers the payoff under the grand coalition structure $\mathcal{P}^N$ to the payoff under the $(m+1)$-th stage coalition structure because of $B(m+1) < 1/2^m$. Moreover any deviating coalition in the process from m-th stage coalition structure to $(m+1)$-th stage coalition structure consists of at least two players. Each member of such a deviating coalition prefers the payoff in the grand coalition structure $\mathcal{P}^N$ to the payoff in the m-th stage coalition structure, because of $B(m) < 2/2^m = 1/2^{m-1}$ by Lemma 1.

Therefore if we combine this sequence $\{\mathcal{P}_t\}_{t=0}^{T'}$ and a sequence from $\mathcal{P}_{T+T'} = \mathcal{P}^*$ to $\mathcal{P}^N$, every coalition structure in the sequence $\{\mathcal{P}_t\}_{t=0}^{T+T'}$ is sequentially dominated by $\mathcal{P}^N$. And so is the $(m+1)$-th stage coalition structure $\mathcal{P}$. This completes the proof of Step 2.

(Step 3) Every coalition structure $\mathcal{P}$ of less than $m+1$ coalitions other than the grand coalition structure $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$.

First, we show that each member of a coalition of the maximal size in any coalition structure $\overline{\mathcal{P}}$ prefers her payoff under $\mathcal{P}^N$ to her payoff under $\overline{\mathcal{P}}$. Denote $\overline{\mathcal{P}}$ by $\overline{\mathcal{P}} = \{S_1, S_2, S_3, ..., S_k\}$, where $|S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq ... \leq |S_k| = r_k$. Because $r_k \geq r_i$ for all r_i, $kr_k \geq \sum_{i=1}^{k} r_i = n$, that is, $r_k/n \geq 1/k$. Since $B(k) < 1/2^{k-1}$, it follows that $r_k/n \geq 1/k \geq 1/2^{k-1} > B(k)$. By Lemma 1, we have the desired result.

Take any coalition structure $\mathcal{P}$ of less than $m+1$ coalitions other than $\mathcal{P}^N$. Consider the following sequence $\{\mathcal{P}_t\}$ starting from $\mathcal{P}$ to some $(m+1)$-stage coalition structure $\mathcal{P}'$: one person in a coalition of the maximal size in $\mathcal{P}_t$ deviates and forms a 1-person coalition in $\mathcal{P}_{t+1}$. Notice that such a person in $\mathcal{P}_t$ prefers her payoff under $\mathcal{P}^N$ to her payoff under $\mathcal{P}_t$, as shown above. Moreover, it is easy to construct a sequence of coalition structures from $\mathcal{P}$ to $\mathcal{P}^N$ by combining the above sequence from $\mathcal{P}$ to $\mathcal{P}'$ and the sequence from $\mathcal{P}'$ to $\mathcal{P}^N$ in Step 2. These imply that $\mathcal{P}$ is sequentially dominated by $\mathcal{P}^N$.

(Step 4) Every coalition structure $\mathcal{P}$ of more than $m+1$ coalitions is sequentially dominated by $\mathcal{P}^N$.

Take any k-th stage coalition structure $\mathcal{P}$ of more than $m+1$ coalitions. Since $B(k)$ is a decreasing function by Lemma 2, $B(k) < B(m+1) < 1/2^m = 1/n \leq r_i/n$ holds for any $r_i \geq 1$. This together with Lemma 1 imply that each member of any coalition in $\mathcal{P}$ prefers her payoff under the grand coalition structure $\mathcal{P}^N$ to her payoff under $\mathcal{P}$.

Consider a sequence $\{\mathcal{P}_t\}$ starting from $\mathcal{P}$ to some $(m+1)$-stage coalition structure $\mathcal{P}'$ such that two coalitions in $\mathcal{P}_t$ merge and form one coalition in $\mathcal{P}_{t+1}$. Notice that each member in these two coalitions in $\mathcal{P}_t$ prefers her payoff under $\mathcal{P}^N$ to her payoff under $\mathcal{P}_t$, as shown above. Moreover, it is easy to construct a sequence of coalition structures from $\mathcal{P}$ to $\mathcal{P}^N$ by combining the above sequence from $\mathcal{P}$ to $\mathcal{P}'$ and the sequence from $\mathcal{P}'$ to $\mathcal{P}^N$ in Step 2. These imply that $\mathcal{P}$ is sequentially dominated by $\mathcal{P}^N$. Q.E.D.

Next consider a case that $n = 2^m + l \ \ (m \geq 2, 1 \leq l \leq 2^m - 1)$.

Theorem 3. If $B(k) < \frac{2^{m-k+1}}{n} \ \ (k = 2, ..., m, m+1)$,the grand coalition structure is sequentially stable.

Proof. The basic idea of the proof is the same as that of the proof of Theorem 3. Consider a coalition structure $\mathcal{P}^* = \{1; 1; 2; 4; 8; ....; 2^{m-2}; 2^{m-1} + l\}$ consisting of $(m+1)$ coalitions.

(Step 1) $\mathcal{P}^*$ is sequentially dominated by $\mathcal{P}^N$.

Consider a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^{m}$ such that $\mathcal{P}_0 = \mathcal{P}^*, \mathcal{P}_m = \mathcal{P}^N$, and the two coalitions of the smallest size in $\mathcal{P}_t$ merge in $\mathcal{P}_{t+1}$ for $t = 0, 1, 2, ..., m-1$. This sequence is expressed by

$$\mathcal{P}^* = \{1; 1; 2; 4; 8; ...; 2^{m-2}; 2^{m-1} + l\} \rightarrow \{2; 2; 4; 8; ...; 2^{m-2}; 2^{m-1} + l\}$$

$$\to \{4;4;8;...;2^{m-2};2^{m-1}+l\} \to ....$$

$$\to \{2^{m-2};2^{m-2};2^{m-1}+l\} \to \{2^{m-1};2^{m-1}+l\} \to \{2^m+l\} = \mathcal{P}^N$$

First, it follows from Lemma 1 that the 2nd stage coalition structure $\mathcal{P}_{m-1} = \{2^{m-1};2^{m-1}+l\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-1}/n > B(2)$ by the hypothesis.

Next, it follows from Lemma 1 that the 3rd stage coalition structure $\mathcal{P}_{m-2} = \{2^{m-2};2^{m-2};2^{m-1}+l\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-2}/n > B(3)$ by the hypothesis.

In general, for $k = 2,...,m,m+1$, it follows from Lemma 1 that the k-th stage coalition structure $\mathcal{P}_{m-k+1} = \{2^{m-k+1};2^{m-k+1};2^{m-k+2};2^{m-k+3};...;2^{m-1}+l\}$ is sequentially dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-k+1}/n > B(k)$ by the hypothesis. Therefore the $(m+1)$-th stage coalition structure $\mathcal{P}_0 = \mathcal{P}^* = \{1;1;2;4;...;2^{m-1}+l\}$ is sequentially dominated by $\mathcal{P}^N$.

We omit the rest of the proof that is similar to that of Proposition 3. Q.E.D.

Theorem 4. If $B(k) < \frac{2^{m-k+1}}{n}$ $(k = 2,...,m), B(m+1) \geq \frac{1}{n}$ and $B(m+2) < \frac{1}{n}$ the grand coalition structure is sequentially stable.

Proof. The important difference from Theorem 4 is that one person coalition in $(m+1)$-th stage coalition structure does not like to move to $\mathcal{P}^N$, but members in coalitions with two or more players like to move to the destination coalition structure $\mathcal{P}^N$. We follow the procedures of the proof of Theorems 3 and 4.

Consider a coalition structure $\mathcal{P}^{**} = \{1;1;2;2;2;8;....;2^{m-2};2^{m-1}+l\}$ consisting of $(m+2)$ coalitions instead of $\mathcal{P}^* = \{1;1;2;4;8;...;2^{m-1}+l\}$ in Theorems 3 and 4.

(Step 1) $\mathcal{P}^{**}$ is sequentially dominated by $\mathcal{P}^N$. Consider a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^{m+1}$ such that $\mathcal{P}_0 = \mathcal{P}^{**}, \mathcal{P}_{m+1} = \mathcal{P}^N$,

$$\mathcal{P}^{**} = \{1;1;2;2;2;8;...;2^{m-2};2^{m-1}+l\}$$

$$\to \mathcal{P}_1 = \{2;2;2;2;8;...;2^{m-2};2^{m-1}+l\} \ \ ((m+1)\text{-th stage})$$

$$\to \mathcal{P}_2 = \{2;2;4;8;...;2^{m-2};2^{m-1}+l\} \ \ (m\text{-th stage})$$

$$\to \mathcal{P}_3 = \{4;4;8;...;2^{m-2};2^{m-1}+l\} \ \ ((m-1)\text{-th stage})$$

$$\to .... \to ....$$

$$\to \mathcal{P}_{m-1} = \{2^{m-2};2^{m-2};2^{m-1}+l\}$$

$$\to \mathcal{P}_m = \{2^{m-1};2^{m-1}+l\} \to \mathcal{P}_{m+1} = \{2^m+l\} = \mathcal{P}^N$$

We can prove that $\mathcal{P}_m = \{2^{m-1};2^{m-1}+l\}, \mathcal{P}_{m-1} = \{2^{m-2};2^{m-2};2^{m-1}+l\}$,..., and $\mathcal{P}_2 = \{2;2;4;8;...;2^{m-2};2^{m-1}+l\}$ are sequentially dominated by $\mathcal{P}^N$ by the same argument as Step 1 in the proof of Theorem 4.

Also, it follows from Lemma 1 that $\mathcal{P}_1 = \{2;2;2;2;8;...;2^{m-2};2^{m-1}+l\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 2/n > B(m+1)$ which is obtained from $2/n > B(m)$ by the hypothesis and $B(m) > B(m+1)$.
Moreover, it follows from Lemma 1 that $\mathcal{P}_0 = \mathcal{P}^{**} = \{1;1;2;2;2;8;...;2^{m-2};$ $2^{m-1}+l\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 1/n > B(m+2)$ by the hypothesis.
Therefore $\mathcal{P}^{**} = \{1;1;2;2;2;8;...;2^{m-1}+l\}$ is sequentially dominated by $\mathcal{P}^N$.

(Step 2) Every $(m+1)$-th stage coalition structure is sequentially dominated by $\mathcal{P}^N$:

Take any $(m+1)$-stage coalition structure $\mathcal{P}$.

First we consider a sequence $\{\mathcal{P}_t\}_{t=0}^{T}$ such that

1) $\mathcal{P}_0 = \mathcal{P} = \{r_1;r_2;r_3;...;r_{m-1};r_m;r_{m+1}\}$

2) $\mathcal{P}_T = \{1;1;1;...;1;2^m+l-m\}$, where $|\mathcal{P}_T| = m+1$.

3) If t is zero or even, then one person belonging to the largest coalition in $\mathcal{P}_t$ deviates and forms one person coalition in $\mathcal{P}_{t+1}$.

4) If t is odd, then the largest and the second largest coalitions in $\mathcal{P}_t$ merge in $\mathcal{P}_{t+1}$.

Then the sequence $\{\mathcal{P}_t\}_{t=0}^{T}$ of coalition structures is given by:

$$\mathcal{P}_0 = \{r_1;r_2;r_3;...,r_{m-1};r_m;r_{m+1}\} \quad ((m+1)\text{-th stage})$$

$$\rightarrow \mathcal{P}_1 = \{1;r_1;r_2;r_3;...;r_{m-1};r_m;r_{m+1}-1\} \quad ((m+2)\text{-th stage})$$

$$\rightarrow \mathcal{P}_2 = \{1;r_1;r_2;r_3;...;r_{m-1};r_m+r_{m+1}-1\} \quad ((m+1)\text{-th stage})$$

$$\rightarrow \mathcal{P}_3 = \{1;1;r_1;r_2;r_3;...;r_{m-1};r_m+r_{m+1}-2\} \quad ((m+2)\text{-th stage})$$

$$\rightarrow ... \rightarrow ...$$

$$\rightarrow \mathcal{P}_{T-2} = \{1;1;1;...;1;r_1;\sum_{k=2}^{m+1} r_k - m + 1\} \quad ((m+1)\text{-th stage})$$

$$\rightarrow \mathcal{P}_{T-1} = \{1;1;1;1;...;1;r_1;\sum_{k=2}^{m+1} r_k - m\} \quad ((m+2)\text{th-stage})$$

$$\rightarrow \mathcal{P}_T = \{1;1;1;1;...;1;\sum_{k=1}^{m+1} r_k - m\} = \{1;1;1;1;...;1;2^m+l-m\}$$

$$((m+1)\text{th-stage})$$

Next consider $\{\mathcal{P}_t\}_{t=T}^{T+T'}$ such that

1) $\mathcal{P}_T = \{1;1;1;...;1;2^m+l-m\}$,

2) $\mathcal{P}_{T+T'} = \mathcal{P}^{**} = \{1;1;2;2;2;8;...;2^{m-2};2^{m-1}+l\}$.

3) If $t=0$, $2^{m-1}-m$ persons in the coalition of 2^m+l-m persons in $\mathcal{P}_0$ deviate and form a coalition in $\mathcal{P}_1$.

4) If $t = T+\lambda$ and λ is odd ($\lambda \le T'-3$), then the smallest coalition of more than one members and a 1-person coalition in $\mathcal{P}_{T+\lambda}$ merge in $\mathcal{P}_{T+\lambda+1}$.

5) If $t = T + \lambda$ and λ is even ($\lambda \leq T' - 3$), then $2^{m-\frac{\lambda}{2}-1}$ persons in the coalition of $2^{m-\frac{\lambda}{2}} - (m - \frac{\lambda}{2})$ persons in $\mathcal{P}_{T+\lambda}$ deviate and form a coalition in $\mathcal{P}_{T+\lambda+1}$. Note that $2^{m-\frac{\lambda}{2}} - (m - \frac{\lambda}{2}) \geq 1$.

6) If $t = T + T' - 2$, two one person coalitions in $\mathcal{P}_{T+T'-2}$ merge in $\mathcal{P}_{T+T'-1}$.

7) If $t = T + T' - 1$, one 4 person coalition in $\mathcal{P}_{T+T'-1}$ is divided into two 2 person coalitions in $\mathcal{P}_{T+T'}$.

This sequence $\{\mathcal{P}_t\}_{t=T}^{T+T'}$ of coalition structures is given by:

$$\mathcal{P}_T = \{1;1;1;1;...;1;1;1;2^m + l - m\} \ \ ((m+1)\text{-th stage})$$
$$\to \mathcal{P}_{T+1} = \{1;1;1;1;...;1;1;1;2^{m-1} - m;2^{m-1} + l\} \ \ ((m+2)\text{-th stage})$$
$$\to \mathcal{P}_{T+2} = \{1;1;1;1;...;1;1;2^{m-1} - m + 1;2^{m-1} + l\} \ \ ((m+1)\text{-th stage})$$
$$\to \ \ \mathcal{P}_{T+3} = \{1;1;1;1;...;1;1;2^{m-2} - m+1;2^{m-2};2^{m-1}+l\} \ \ ((m+2)\text{-th stage})$$
$$\to \ \ \mathcal{P}_{T+4} = \{1;1;1;1;...;1;2^{m-2} - m + 2;2^{m-2};2^{m-1} + l\} \ \ ((m+1)\text{-th stage})$$
$$\to \ \ \mathcal{P}_{T+5} = \{1;1;1;1;...;1;2^{m-3} - m+2;2^{m-3};2^{m-2};2^{m-1}+l\} \ \ ((m+2)\text{-th stage})$$
$$\to ... \to ...$$
$$\to \mathcal{P}_{T+T'-3} = \{1;1;1;1;12;...;2^{m-3};2^{m-2};2^{m-1} + l\} \ \ ((m+1)\text{-th stage})$$
$$\to \mathcal{P}_{T+T'-2} = \{1;1;1;1;4;8;...;2^{m-3};2^{m-2};2^{m-1}\} \ \ (m+2)\text{-th stage})$$
$$\to \mathcal{P}_{T+T'-1} = \{1;1;2;4;8;...;2^{m-3};2^{m-2};2^{m-1}\} \ \ ((m+1)\text{-th stage})$$
$$\to \mathcal{P}_{T+T'} = \{1;1;2;2;2;8;...;2^{m-3};2^{m-2};2^{m-1} + l\} \ \ (m+2)\text{-th stage}) = \mathcal{P}^{**}$$

Hence if we combine two sequences $\{\mathcal{P}_t\}_{t=0}^{T}$ and $\{\mathcal{P}_t\}_{t=T}^{T+T'}$, we can get a sequence $\{\mathcal{P}_t\}_{t=0}^{T+T'}$ from any $(m+1)$-th stage coalition structure $\mathcal{P}$ to $\mathcal{P}^{**}$. Note that only deviation of a coalition with 2 or more members appears for all $(m+1)$-th coalition structures in this sequence.

The rest part of the proof is the same as the proof of Theorem 4. Q.E.D.

We now apply the above theorems when the production function is give by $f(x) = x^{\alpha}$ $(0 < \alpha < 1)$. First of all, by Proposition 1, it is easy to check that for any $\mathcal{P}$,

$$x_N^*(\mathcal{P}) = (\alpha + k - 1)(x_N^*(\mathcal{P}))^{\alpha-1}/(kq) = \left(\frac{\alpha - 1 + k}{kq}\right)^{1/(1-\alpha)},$$

$$m_1^*(\mathcal{P}) = m_{S_1}^*(\mathcal{P})/r_1 = [f(x_N^*(\mathcal{P})) - qx_N^*(\mathcal{P})]/(r_1 k)$$
$$= \left[f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\right]/(r_1 k^2) = (1-\alpha)(x_N^*(\mathcal{P}))^{\alpha}/(r_1 k^2).$$

Notice that if $\mathcal{P} = \mathcal{P}^N$, then $k = 1$ and $r_1 = n$, so that

$$x_N^*(\mathcal{P}^N) = \alpha(x_N^*(\mathcal{P}^N))^{\alpha-1}/q = \left(\frac{\alpha}{q}\right)^{1/(1-\alpha)}.$$

$$f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N) = (1-\alpha)(x_N^*(\mathcal{P}^N))^{\alpha}.$$

This implies

$$B(k) = \{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/[k^2\{f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\}]$$

$$= \frac{1}{k^2}\left(\frac{\alpha - 1 + k}{\alpha k}\right)^{\alpha/(1-\alpha)}. \qquad (*)$$

Corollary 1. If $f(x) = x^{\alpha}$ and $4 \leq n \leq 48$, then for some α, the grand coalition structure $\mathcal{P}^N$ is sequentially stable in the common pool game.

Proof. Note that $B(k)$ is an increasing function of α, and $\lim_{\alpha\to 0} B(k) = 1/k^2$ for any k. Hence for sufficiently small $\alpha > 0$, $B(k)$ is very close to $1/k^2$.

First, consider the case of $n = 2^m$. In this case, it follows from Theorem 2 that for $m \leq 5$, that is, for $n = 4, 8, 16, 32$, $\mathcal{P}^N$ is sequentially stable for a sufficiently small α, since $\lim_{\alpha\to 0} B(k) = 1/k^2 < 1/2^{k-1}$ for $k = 2, 3, 4, 5, 6$: $\lim_{\alpha\to 0} B(2) = 1/4 < 1/2$, $\lim_{\alpha\to 0} B(3) = 1/9 < 1/2^2 = 1/4$, $\lim_{\alpha\to 0} B(4) = 1/16 < 1/2^3 = 1/8$, $\lim_{\alpha\to 0} B(5) = 1/25 < 1/2^4 = 1/16$, and $\lim_{\alpha\to 0} B(6) = 1/36 < 1/2^5 = 1/32$.

Next consider the case of $n = 2^m + l \quad (1 \leq l \leq 2^m - 1)$. There are four subcases to examine:

1) $m = 2$. In this case, $n \in \{5, 6, 7\}$. Since $\lim_{\alpha\to 0} B(2) = 1/4 < \frac{2^{m-k+1}}{n} = 2/n$ and $\lim_{\alpha\to 0} B(3) = 1/9 < \frac{2^{m-k+1}}{n} = 1/n$, it follows from Theorem 3 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small α.

2) $m = 3$. In this case, $n \in \{9, 10, ..., 14, 15\}$. Since $\lim_{\alpha\to 0} B(2) = 1/4 < \frac{2^{m-k+1}}{n} = 4/n$, $\lim_{\alpha\to 0} B(3) = 1/9 < \frac{2^{m-k+1}}{n} = 2/n$, and $\lim_{\alpha\to 0} B(4) = 1/16 < \frac{2^{m-k+1}}{n} = 1/n$, it follows from Theorem 3 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small α.

3) $m = 4$. In this case, $n \in \{17, 18, ..., 30, 31\}$. Suppose that $n \leq 24$. Since $\lim_{\alpha\to 0} B(2) = 1/4 < \frac{2^{m-k+1}}{n} = 8/n$, $\lim_{\alpha\to 0} B(3) = 1/9 < \frac{2^{m-k+1}}{n} = 4/n$, $\lim_{\alpha\to 0} B(4) = 1/16 < \frac{2^{m-k+1}}{n} = 2/n$, and $\lim_{\alpha\to 0} B(5) = 1/25 < \frac{2^{m-k+1}}{n} = 1/n$, it follows from Theorem 3 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small α if $17 \leq n \leq 24$.

On the other hand, when $\lim_{\alpha\to 0} B(5) > 1/n$, that is, $n \in \{25, 26, ..., 31\}$, $\lim_{\alpha\to 0} B(6) = 1/36 < 1/n$. It follows from Theorem 4 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small α if $25 \leq n \leq 31$.

4) $m = 5$. In this case, $n \in \{32, 33, ..., 62, 63\}$. Suppose that $n \leq 35$. Since $\lim_{\alpha\to 0} B(2) = 1/4 < \frac{2^{m-k+1}}{n} = 16/n$, $\lim_{\alpha\to 0} B(3) = 1/9 < \frac{2^{m-k+1}}{n} = 8/n$,

$\lim_{\alpha\to 0} B(4) = 1/16 < \frac{2^{m-k+1}}{n} = 4/n, \lim_{\alpha\to 0} B(5) = 1/25 < \frac{2^{m-k+1}}{n} = 2/n$, and $\lim_{\alpha\to 0} B(6) = 1/36 < \frac{2^{m-k+1}}{n} = 1/n$, it follows from Theorem 3 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small α if $32 \leq n \leq 35$.

On the other hand, when $\lim_{\alpha\to 0} B(6) > 1/n$, if $\lim_{\alpha\to 0} B(7) = 1/49 < 1/n$, that is, $n \in \{36, 37, ..., 48\}$, it follows from Theorem 4 that $\mathcal{P}^N$ is sequentially stable for a sufficiently small α if $32 \leq n \leq 48$. Q.E.D.

This corollary says that if we apply our stability concept to a common pool resource game, the grand coalition structure is sequentially stable in the common pool resource game with between 4 and 48 players, for some concave production function. For more than 48 players, it might be possible to prove the similar theorem, but the proof needs much more steps and it will be very complicated. The important thing is if we succeed to prove the similar theorem for more players by following this proof, we also face an upper limit. Thus we need a very different proof to extend the corollary essentially.

Let $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N\setminus\{i\}\}, (\{i\} \in N)$ and for $\mathcal{P}$ such that $\#\mathcal{P} = k, C(k) \equiv \{f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})\}/\{f(x^*_N(\mathcal{P}^{N\setminus\{i\}})) - f'(x^*_N(\mathcal{P}^{N\setminus\{i\}}))x^*_N(\mathcal{P}^{N\setminus\{i\}})\}$.

Theorem 5. Let $n \geq 5$. If $C(3) \geq \frac{9}{8}$, then the coalition structures $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N\setminus\{i\}\}, (\{i\} \in N)$ are not sequentially stable in the common pool resource game.

Proof.

It suffices to show that there is a coalition structure which is not sequentially dominated by the coalition structure $\mathcal{P}^{N\setminus\{i\}}$.

We will show that any coalition structure containing three coalitions is not sequentially dominated by $\mathcal{P}^{N\setminus\{i\}}$ if $C(3) \geq \frac{9}{8}$.

Consider a coalition structure containing 3 coalitions $\mathcal{P} = \{S_1, S_2, S_3\}$ where $|S_1| \leq |S_2| \leq |S_3|$. We will show that every coalition structure with 3 coalitions is not sequentially dominated by $\mathcal{P}^{N\setminus\{i\}}$. Since in this sequential domination, we have to consider a merging of two coalitions, it is enough to show that the payoff of the player in one of two coalitions of the coalition structure with 3 coalitions is smaller than the payoff in the coalition $N\setminus\{i\}$ of coalition structure $\mathcal{P}^{N\setminus\{i\}}$. Hence if the largest payoff of a player in S_2 out of several coalition structures with 3 coalitions is smaller than the payoff of a player in $N\setminus\{i\}$, we can attain our purpose.

Then we have to compare the payoff $m^*_j(\mathcal{P})$ of player j in a coalition S_2 of the smallest size with the payoff $m^*_j(\mathcal{P}^{N\setminus\{i\}})$.

Remark that such a coalition structure is given by $|S_1| = 1, |S_2| = |S_3| = \frac{n-1}{2}$ if n is odd, and $|S_1| = 1, |S_2| = \frac{n-2}{2}, |S_3| = \frac{n+2}{2}$. if n is even.

By Proposition 1,

$$m^*_j(\mathcal{P}) = \left[f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})\right]/(9r_2),$$

for $j \in S_2$, and

$$m_j^*(\mathcal{P}^{N\backslash\{i\}}) = \left[f(x_N^*(\mathcal{P}^{N\backslash\{i\}})) - f'(x_N^*(\mathcal{P}^{N\backslash\{i\}}))x_N^*(\mathcal{P}^{N\backslash\{i\}})\right]/(4(n-1)),$$

for $j \in N\backslash\{i\}$.
Note that for $j \in S_2$, $m_j^*(\mathcal{P}) \geq m_j^*(\mathcal{P}^{N\backslash\{i\}})$ iff $[4(n-1)/(9r_2)]$ $\{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/\{f(x_N^*(\mathcal{P}^{N\backslash\{i\}})) - f'(x_N^*(\mathcal{P}^{N\backslash\{i\}}))x_N^*(\mathcal{P}^{N\backslash\{i\}})\}] = [4(n-1)/(9r_2)]C(3) \geq 1$. There are two cases to examine. First, if n is even, consider a coalition structure $\mathcal{P}$ with $r_2 = (n-2)/2$. In this case, $4(n-1)/(9r_2) = \frac{8(n-1)}{9(n-2)}$, so that if $C(3) \geq \frac{9}{8}$, then $m_j^*(\mathcal{P}) > m_j^*(\mathcal{P}^{N\backslash\{i\}})$. Second, if n is odd, consider a coalition structure $\mathcal{P}$ with $r_2 = (n-1)/2$. In this case, $4(n-1)/(9r_2) = \frac{8}{9}$, so that if $C(3) \geq \frac{9}{8}$, then $m_j^*(\mathcal{P}) \geq m_j^*(\mathcal{P}^{N\backslash\{i\}})$. Q.E.D.

By applying this theorem to the case in which the production function is give by $f(x) = x^\alpha$ $(0 < \alpha < 1)$, we have the following:

Corollary 2. Let $n \geq 5$. If $f(x) = x^\alpha$ and $\alpha \geq 0.583804$, then the coalition structures $\mathcal{P}^{N\backslash\{i\}} = \{\{i\}, N \setminus \{i\}\}$, $(\{i\} \in N)$ are not sequentially stable in the common pool resource game.

Proof. It is easy to see that

$$C(3) = \left(\frac{3(\alpha+1)}{2(\alpha+2)}\right)^{-\alpha/(1-\alpha)}.$$

Therefore, $C(3) > \frac{9}{8}$ iff $1/C(3) = \left(\frac{3(\alpha+1)}{2(\alpha+2)}\right)^{\alpha/(1-\alpha)} < \frac{8}{9}$. Figure 1 illustrates the function $1/C(3) - \frac{8}{9}$. It is not hard to check that if $1/C(3) < \frac{8}{9}$ if $\alpha \geq 0.5083804$. Q.E.D.

It is difficult to eliminate the stability of coalition structures containing singleton and $n-1$ person coalition because the singleton player gets the maximal payoff among the payoffs for all coalition structures. (See Diamantoudi and Xue (2002).) However we could show that the coalition structures containing singleton and $n-1$ person coalition are not sequentially stable under some concave production function for games with any number of players.

It is not true that all the coalition structures other than the grand coalition structure are not sequentially stable for the common pool resource game with $f(x) = x^\alpha$ for some α and $3 \leq n \leq 48$. In a 6-person game, we have an example that shows some coalition structure including two coalitions is also sequentially stable. It can be proved that, in these games, coalition structures with two or more coalitions are sequentially stable.

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On noncooperative games and minimax theory

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Abstract

In this note we review some known minimax theorems with applications in game theory and show that these results form an equivalent chain which includes the strong separation result in finite dimensional spaces between two disjoint closed convex sets of which one is compact. By simplifying the proofs we intend to make the results more accessible to researchers not familiar with minimax or noncooperative game theory.

1 Introduction.

In a two person noncooperative zero sum game one faces the following problem. Let X be the set of actions of player 1 and Y the set of actions of player 2. If player 1 chooses action $x \in X$ and player 2 chooses action $y \in Y$, then player 2 has to pay to player 1 an amount $f(x, y)$ with $f : X \times Y \to \mathbb{R}$ a given function. This function is called the payoff function of player 1. Since player 1 likes to gain as much profit as possible, but at the moment he does not know how to achieve this, he first decides to compute a lower bound on his profit. To do this, player 1 argues as follows : if he chooses action $x \in X$, then he wins at least $\inf_{y \in Y} f(x, y)$ irrespective of the action of player 2. Therefore a lower bound on the profit for player 1 is given by

$$r_* := \sup_{x \in X} \inf_{y \in Y} f(x, y). \tag{1}$$

Similarly player 2 likes to minimize his losses. Therefore, he also decides to compute first an upper bound on his losses. If he decides to choose action $y \in Y$ it follows that he loses at most $\sup_{x \in X} f(x, y)$ and this is independent of the action of player 1. Therefore an upper bound on his losses is given by

$$r^* := \inf_{y \in Y} \sup_{x \in X} f(x, y). \tag{2}$$

Since the profit of player 1 is at least r_* and the losses of player 2 is at most r^* and the losses of player 2 are the profits of player 1, it follows that $r_* \leq r^*$. If

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$r_* = r^*$, then this equality is called a **minimax result**. If additionally inf and sup are attained, an optimal action for both players can then be easily derived. However, in general $r_* < r^*$, as the following example shows.

Example 1 *Let* $f : [0,1] \times [0,1] \to \mathbb{R}$ *given by* $f(x,y) = (x-y)^2$. *For this function it holds* $0 = r_* < r^* = \frac{1}{4}$. *For this example it is not obvious which actions should be selected by the two players.*

By extending the sets of actions of each player, it is possible to show under certain conditions that the extended game satisfies a minimax result. In the next definition we introduce the set of **mixed strategies**.

Definition 2 *For a nonempty set* D *of actions and* $d \in D$ *let* ϵ_d *denote the one-point probability measure concentrated on the set* $\{d\}$ *and denote by* $\mathcal{F}(D)$ *the set of all probability measures on* D *with a finite support.*

Introducing the unit simplex $\Delta_k := \{\alpha : \sum_{i=1}^k \alpha_i = 1, \alpha_i \geq 0, 1 \leq i \leq k\}$, it follows by Definition 2 that λ belongs to the set $\mathcal{F}(D)$ if and only if there exist some $k \in \mathbb{N}$ and a set $\{d_1, ..., d_k\} \subseteq D$ such that

$$\lambda = \sum\nolimits_{i=1}^{k} \lambda_i \epsilon_{d_i}, (\lambda_1, ..., \lambda_k) \in \Delta_k \text{ and } \lambda_i \geq 0.$$

A game theoretic interpretation of a mixed strategy $\lambda \in \mathcal{F}(D)$ is now given by the following. If a player with action set D selects the mixed strategy $\lambda = \sum_{i=1}^k \lambda_i \epsilon_{d_i} \in \mathcal{F}(D)$, then with probability $\lambda_i, 1 \leq i \leq k$ this player will use action $d_i \in D$. By the above definition it is clear that the action set D of any player can be identified with the set of one-point probability measures; therefore the set D is often called the set of **pure strategies** for that player. Assume that player 1 uses the set $\mathcal{F}(X)$ of mixed strategies and the same holds for player 2 using the set $\mathcal{F}(Y)$. This means that the payoff function f should be extended to a function $f_e : \mathcal{F}(X) \times \mathcal{F}(Y) \to \mathbb{R}$ given by

$$f_e(\lambda, \mu) := \sum\nolimits_{i=1}^{m} \sum\nolimits_{j=1}^{n} \lambda_i \mu_j f(x_i, y_j) \tag{3}$$

with $\lambda = \sum_{i=1}^m \lambda_i \epsilon_{x_i} \in \mathcal{F}(X)$ and $\mu = \sum_{j=1}^n \mu_j \epsilon_{y_j} \in \mathcal{F}(Y)$. This extension represents the expected profit for player 1 or expected loss of player 2. In [3] the authors showed that several well known minimax theorems form an equivalent chain and this chain includes the strong separation result in finite dimensional spaces between two disjoint convex sets of which one is closed and the other compact. By reducing the number of results in this equivalent chain and by giving more transparent and simpler proofs, we intend to make the results more accessible to researchers not familiar with minimax or noncooperative game theory. The first minimax result was proved in a famous paper by von Neumann (cf.[6]) in 1928 for X and Y unit simplices in finite dimensional vector spaces and f affine in both variables. Later on, the conditions on the function f were weakened and more general sets X and Y were considered. These results turned out to be useful also in optimization theory (see for instance [2]) and were derived by means of short or long proofs using a version of the Hahn Banach theorem in either finite or infinite dimensional vector spaces. With von Neumann's result as a starting point, we will show that several of these so-called generalizations published in the literature can be derived from each other using only elementary observations. Before introducing this chain of equivalent minimax results we need the following notations. The set $\mathcal{F}_2(X) \subseteq \mathcal{F}(X)$ denotes the set of two-point probability measures on X. This means that λ belongs to $\mathcal{F}_2(X)$ if and only if

$$\lambda = \lambda_1 \epsilon_{x_1} + (1 - \lambda_1)\epsilon_{x_2} \tag{4}$$

with $x_i, 1 \leq i \leq 2$ different elements of X and $0 < \lambda_1 < 1$ arbitrarily chosen. Also, for each $0 < \alpha < 1$ the set $\mathcal{F}_{2,\alpha}(X)$ represents the set of two point probability measures with $\lambda_1 = \alpha$ in relation (4). On the set Y similar spaces of probability measures with finite support are introduced.

2 Equivalent minimax results

To start in a chronological order we first mention the famous von Neumann's minimax result (cf.[6]).

Theorem 3 *(von Neumann, 1928). If X and Y are finite sets, then it follows that*

$$\max_{\lambda \in \mathcal{F}(X)} \min_{\mu \in \mathcal{F}(Y)} f_e(\lambda, \mu) = \min_{\mu \in \mathcal{F}(Y)} \max_{\lambda \in \mathcal{F}(X)} f_e(\lambda, \mu).$$

A generalization of Theorem 3 due to Wald [7]) and published in 1945 is given by the next result. This result plays a fundamental role in the theory of statistical decision functions. While in case of Theorem 3 the action sets of players 1 and 2 are finite, this condition is relaxed in Wald's theorem claiming that only one set should be finite.

Theorem 4 *(Wald, 1945). If X is an arbitrary nonempty set and Y is a finite set, then it follows that*

$$\sup_{\lambda \in \mathcal{F}(X)} \min_{\mu \in \mathcal{F}(Y)} f_e(\lambda, \mu) = \min_{\mu \in \mathcal{F}(Y)} \sup_{\lambda \in \mathcal{F}(X)} f_e(\lambda, \mu).$$

In order to prove Wald's theorem by von Neumann's theorem, we first need the following elementary lemma. For its proof, see for instance [3]. Recall a function is upper semicontinuous if all its upper level sets are closed. For every set D let $< D >$ be the set of all finite subsets of D.

Lemma 5 *If the set X is compact and the function $h : X \times Y \to \mathbb{R}$ is upper semicontinuous on X for every $y \in Y$, then $\max_{x \in X} \inf_{y \in Y} h(x, y)$ is well defined and*

$$\max_{x \in X} \inf_{y \in Y} h(x, y) = \inf_{Y_0 \in <Y>} \max_{x \in X} \min_{y \in Y_0} h(x, y).$$

Since for every $\mu \in \mathcal{F}(Y)$ and $J \subseteq X$ it is easy to see that

$$\sup_{\lambda \in \mathcal{F}(J)} f_e(\lambda, \mu) = \sup_{x \in J} f_e(\epsilon_x, \mu), \tag{5}$$

we are now ready to derive Wald's minimax result from von Neumann's minimax result. Observe Wald (cf.[7]) uses in his paper von Neumann's minimax result and the Lebesgue dominated convergence theorem.

Theorem 6 *von Neumann's minimax result ⇒Wald's minimax result.*

Proof. If $\alpha := \sup_{\lambda \in \mathcal{F}(X)} \min_{\mu \in \mathcal{F}(Y)} f_e(\lambda, \mu)$ then clearly

$$\alpha = \sup_{J \in <X>} \max_{\lambda \in \mathcal{F}(J)} \min_{\mu \in \mathcal{F}(Y)} f_e(\lambda, \mu). \tag{6}$$

Since the set Y is finite we may apply von Neumann's minimax result in relation (6) and this implies in combination with relation (5) that

$$\begin{aligned} \alpha &= \sup_{J \in <X>} \min_{\mu \in \mathcal{F}(Y)} \max_{\lambda \in \mathcal{F}(J)} f_e(\lambda, \mu) \\ &= \sup_{J \in <X>} \min_{\mu \in \mathcal{F}(Y)} \max_{x \in J} f_e(\epsilon_x, \mu) \\ &= -\inf_{J \in <X>} \max_{\mu \in \mathcal{F}(Y)} \min_{x \in J} (-f_e(\epsilon_x, \mu)). \end{aligned} \tag{7}$$

The finiteness of the set Y also implies that the set $\mathcal{F}(Y)$ is compact and the function $\mu \to f_e(\epsilon_x, \mu)$ is continuous on $\mathcal{F}(Y)$ for every $x \in X$. This shows in relation (7) that we may apply Lemma 5 with the set X replaced by $\mathcal{F}(Y)$, Y by X and $h(x, y)$ by $-f_e(\epsilon_x, \mu)$ and so it follows that

$$\alpha = \min_{\mu \in \mathcal{F}(Y)} \sup_{x \in X} f_e(\epsilon_x, \mu). \tag{8}$$

Finally by relation (5) with J replaced by X the desired result follows from relation (8). □

In 1996 Kassay and Kolumbán (cf.[4]) introduced the following class of functions.

Definition 7 *The function $f : X \times Y \to \mathbb{R}$ is called weakly concavelike on X if for every I belonging to $< Y >$ it follows that*

$$\sup_{\lambda \in \mathcal{F}(X)} \min_{y \in I} f_e(\lambda, \epsilon_y) \leq \sup_{x \in X} \min_{y \in I} f(x, y).$$

Since ϵ_x belongs to $\mathcal{F}(X)$ it is easy to see that f is weakly concavelike on X if and only if for every $I \in < Y >$ it follows that

$$\sup_{\lambda \in \mathcal{F}(X)} \min_{y \in I} f_e(\lambda, \epsilon_y) = \sup_{x \in X} \min_{y \in I} f(x, y)$$

and this equality also has an obvious interpretation within game theory. The main result of Kassay and Kolumbán is given by the following theorem (cf.[4]).

Theorem 8 *(Kassay-Kolumbán, 1996). If X is a compact subset of a topological space and the function $f : X \times Y \to \mathbb{R}$ is weakly concavelike and upper semicontinuous on X for every $y \in Y$, then it follows that*

$$\inf_{\mu\in\mathcal{F}(Y)} \max_{x\in X} f_e(\epsilon_x, \mu) = \max_{x\in X} \inf_{y\in Y} f_e(x, y).$$

At first sight this result might not be recognized as a minimax result. However, it is easy to verify for every $x \in X$ that

$$\inf_{y\in Y} f(x, y) = \inf_{\mu\in\mathcal{F}(Y)} f_e(\epsilon_x, \mu). \tag{9}$$

By relation (9) an equivalent formulation of Theorem 8 is now given by

$$\inf_{\mu\in\mathcal{F}(Y)} \max_{x\in X} f_e(\epsilon_x, \mu) = \max_{x\in X} \inf_{\mu\in\mathcal{F}(Y)} f_e(\epsilon_x, \mu),$$

and so the result of Kassay and Kolumban is actually a minimax result. We now give an elementary proof of Theorem 8 using Wald's minimax theorem.

Proof. Let $\alpha = \inf_{\mu\in\mathcal{F}(Y)} \max_{x\in X} f_e(\epsilon_x, \mu)$, $\beta = \max_{x\in X} \inf_{\mu\in\mathcal{F}(Y)} f_e(\epsilon_x, \mu)$ and suppose by contradiction that $\alpha > \beta$. (The inequality $\beta \leq \alpha$ always holds.) Let γ so that $\alpha > \gamma > \beta$. Then by relation (9) and Lemma 5 we have

$$\gamma > \beta = \max_{x\in X} \inf_{y\in Y} f(x, y) = \inf_{Y_0\in<Y>} \max_{x\in X} \min_{y\in Y_0} f(x, y).$$

Therefore, there exists a finite subset $Y_0 \in< Y >$ such that

$$\max_{x\in X} \min_{y\in Y_0} f(x, y) < \gamma$$

and this implies using f is weakly concavelike on X that

$$\sup_{\lambda\in\mathcal{F}(X)} \min_{y\in Y_0} f_e(\lambda, \epsilon_y) < \gamma. \tag{10}$$

Similarly to relation (9), it is easy to see that for every $\lambda \in \mathcal{F}(X)$ and every $\mu \in \mathcal{F}(Y)$ the relations

$$\inf_{\mu\in\mathcal{F}(Y_0)} f_e(\lambda, \mu) = \min_{y\in Y_0} f_e(\lambda, \epsilon_y)$$

and

$$\sup_{\lambda\in\mathcal{F}(X)} f_e(\lambda, \mu) = \max_{x\in X} f_e(\epsilon_x, \mu)$$

hold, and these together with (10) and Wald's theorem imply

$$\begin{aligned}\alpha > \gamma &> \sup_{\lambda\in\mathcal{F}(X)} \inf_{\mu\in\mathcal{F}(Y_0)} f_e(\lambda, \mu) = \inf_{\mu\in\mathcal{F}(Y_0)} \sup_{\lambda\in\mathcal{F}(X)} f_e(\lambda, \mu) \\ &\geq \inf_{\mu\in\mathcal{F}(Y)} \sup_{\lambda\in\mathcal{F}(X)} f_e(\lambda, \mu) = \inf_{\mu\in\mathcal{F}(Y)} \max_{x\in X} f_e(\epsilon_x, \mu) = \alpha.\end{aligned}$$

This is clearly a contradiction and we have completed the proof. □

In 1952 Kneser (cf.[5]) proved a general minimax result useful in game theory. Its proof is ingenious and very elementary and uses only some simple computations and the well-known result that any upper semicontinuous function attains its maximum on a compact set.

Theorem 9 *(Kneser, 1952). If X is a nonempty convex compact subset of a topological vector space and Y is a nonempty convex subset of a vector space and the function $f : X \times Y \to \mathbb{R}$ is affine in both variables and upper semicontinuous on X for every $y \in Y$, then it follows that*

$$\max_{x\in X} \inf_{y\in Y} f(x,y) = \inf_{y\in Y} \max_{x\in X} f(x,y). \tag{11}$$

One year later, generalizing the proof and result of Kneser, Ky Fan (cf.[1]) published his celebrated minimax result. To show his result Ky Fan introduced the following class of functions which we call Ky Fan convex (Ky Fan concave) functions.

Definition 10 *The function $f : X \times Y \to \mathbb{R}$ is called Ky Fan concave on X if for every $\lambda \in \mathcal{F}_2(X)$ there exists some $x_0 \in X$ satisfying*

$$f_e(\lambda, \epsilon_y) \leq f(x_0, y)$$

for every $y \in Y$. The function $f : X \times Y \to \mathbb{R}$ is called Ky Fan convex on Y if for every $\mu \in \mathcal{F}_2(Y)$ there exists some $y_0 \in Y$ satisfying

$$f_e(\epsilon_x, \mu) \geq f(x, y_0)$$

for every $x \in X$. Finally, the function $f : X \times Y \to \mathbb{R}$ is called Ky Fan concave-convex on $X \times Y$ if f is Ky Fan concave on X and Ky Fan convex on Y.

By induction it is easy to show that one can replace in the above definition $\mathcal{F}_2(X)$ and $\mathcal{F}_2(Y)$ by $\mathcal{F}(X)$ and $\mathcal{F}(Y)$. Although rather technical, the above concept has a clear interpretation in game theory. It means that the payoff function f has the property that any arbitrary mixed strategy is dominated by a pure strategy. Eliminating the linear structure in Kneser's proof Ky Fan (cf.[1]) showed the following result.

Theorem 11 *(Ky Fan, 1953). If X is a compact subset of a topological space and the function $f : X \times Y \to \mathbb{R}$ is Ky Fan concave-convex on $X \times Y$ and upper semicontinuous on X for every $y \in Y$, then it follows that*

$$\max_{x\in X} \inf_{y\in Y} f(x,y) = \inf_{y\in Y} \max_{x\in X} f(x,y).$$

Proof. In what follows we show that Ky Fan's minimax theorem can easily be proved by Kassay-Kolumbán's result. Indeed, it is easy to see that every Ky Fan concave function on X is also weakly concavelike on X. By Theorem 8 it follows that

$$\max_{x\in X} \inf_{y\in Y} f(x,y) = \inf_{\mu\in\mathcal{F}(Y)} \max_{x\in X} f_e(\epsilon_x, \mu). \tag{12}$$

Also, since f is Ky Fan convex on Y, there exists for every $\mu \in \mathcal{F}(Y)$ some $y_0 \in Y$ such that $f_e(\epsilon_x, \mu) \geq f(x, y_0)$ for every $x \in X$. Thus,

$$\max_{x\in X} f_e(\epsilon_x, \mu) \geq \max_{x\in X} f(x, y_0) \geq \inf_{y\in Y} \max_{x\in X} f(x,y)$$

implying that

$$\inf_{\mu\in\mathcal{F}(Y)} \max_{x\in X} f_e(\epsilon_x, \mu) \geq \inf_{y\in Y} \max_{x\in X} f(x, y)$$

and this, together with (12) leads to

$$\max_{x\in X} \inf_{y\in Y} f(x, y) \geq \inf_{y\in Y} \max_{x\in X} f(x, y).$$

Since the reverse inequality always holds, we have equality in the last relation and the proof is complete. $\square$

We show now that the following well-known strong separation result in convex analysis can easily be proved by Kneser's minimax theorem.

Theorem 12 *If $X \subseteq \mathbb{R}^n$ is a closed convex set and $Y \subseteq \mathbb{R}^n$ a compact convex set and the intersection of X and Y is empty, then there exists some $s_0 \in \mathbb{R}^n$ satisfying*

$$\sup\{s_0^\top x : x \in X\} < \inf\{s_0^\top y : y \in Y\}.$$

Proof. Since $X \subseteq \mathbb{R}^n$ is a closed convex set and $Y \subseteq \mathbb{R}^n$ is a compact convex set we obtain that $H := X - Y$ is a closed convex set. It is now easy to see that the strong separation result as given in Theorem 12 holds if and only if there exists some $s_0 \in \mathbb{R}^n$ satisfying $\sigma_H(s_0) := \sup\{s_0^\top x : x \in H\} < 0$. To verify this, we assume by contradiction that $\sigma_H(s) \geq 0$ for every $s \in \mathbb{R}^n$. This clearly implies $\sigma_H(s) \geq 0$ for every s belonging to the compact Euclidean unit ball E and applying Kneser's minimax result we obtain

$$\sup_{h\in H} \inf_{s\in E} s^\top h = \inf_{s\in E} \sup_{h\in H} s^\top h \geq 0. \tag{13}$$

Since by assumption the intersection of X and Y is nonempty, we obtain that 0 does not belong to $H := X - Y$ and this implies using H is closed that $\inf_{h\in H} \|h\| > 0$. By this observation we obtain for every $h \in H$ that $-h\|h\|^{-1}$ belongs to E and so for every $h \in H$ it follows that $\inf_{s\in E} s^\top h \leq -\|h\|$. This implies that

$$\sup_{h\in H} \inf_{s\in E} s^\top h \leq \sup_{h\in H} -\|h\| = -\inf_{h\in H} \|h\| < 0$$

and we obtain a contradiction with relation (13). Hence there must exist some $s_0 \in \mathbb{R}^n$ satisfying $\sigma_H(s_0) < 0$ and we are done. $\square$

Observe that without loss of generality one may suppose that the vector s_0 in Theorem 12 belongs to Δ_n (the unit simplex in $\mathbb{R}^n$). An easy consequence of Theorem 12 is the following result.

Lemma 13 *If $C \subseteq \mathbb{R}^n$ is a convex compact set, then it follows that*

$$\inf_{u\in C} \max_{\alpha\in\Delta_n} \alpha^\top u = \max_{\alpha\in\Delta_n} \inf_{u\in C} \alpha^\top u.$$

Proof. It is obvious that

$$\inf_{u\in C}\max_{\alpha\in\Delta_n}\alpha^\top u \geq \max_{\alpha\in\Delta_n}\inf_{u\in C}\alpha^\top u. \tag{14}$$

To show the reverse inequality, we assume by contradiction that

$$\inf_{u\in C}\max_{\alpha\in\Delta_n}\alpha^\top u > \max_{\alpha\in\Delta_n}\inf_{u\in C}\alpha^\top u := \gamma. \tag{15}$$

Let $\mathbf{e}$ be the vector $(1, ..., 1)$ in $\mathbb{R}^n$ and introduce the mapping $H : C \to \mathbb{R}^n$ given by $H(u) := u - \beta\mathbf{e}$ with β satisfying

$$\inf_{u\in C}\max_{\alpha\in\Delta_n}\alpha^\top u > \beta > \gamma \tag{16}$$

If we assume that $H(C)\cap\mathbb{R}^n_-$ is nonempty, then there exists some $u_0 \in C$ satisfying $u_0 - \beta\mathbf{e} \leq 0$. This implies $\max_{\alpha\in\Delta_n}\alpha^\top u_0 \leq \beta$ and we obtain a contradiction with relation (16). Therefore $H(C)\cap\mathbb{R}^n_-$ is empty. Since $H(C)$ is convex and compact and $\mathbb{R}^n_-$ is closed and convex, we may apply Theorem 12. Hence one can find some $\alpha_0 \in \Delta_n$ satisfying $\alpha_0^\top u - \beta \geq 0$ for every $u \in C$ and using also the definition of γ listed in relation (15) this implies that

$$\gamma \geq \inf_{u\in C}\alpha_0^\top u \geq \beta.$$

Hence we obtain a contradiction with relation (16) and the desired result is proved. □

Finally we show that von Neumann's minimax theorem (Theorem 3) is an easy consequence of Lemma 13. In this way we close the equivalent chain of results considered in this note.

Proof. Indeed, let $m := card(X)$ and introduce the mapping $L : \mathcal{F}(Y) \to \mathbb{R}^m$ given by

$$L(\mu) := (f_e(\epsilon_x, \mu))_{x\in X}.$$

It is easy to see that the range $L(\mathcal{F}(Y)) \subseteq \mathbb{R}^m$ is a convex compact set. Applying now Lemma 13 yields

$$\begin{aligned}\inf_{\mu\in\mathcal{F}(Y)}\max_{\lambda\in\mathcal{F}(X)} f_e(\lambda,\mu) &= \inf_{u\in L(\mathcal{F}(Y))}\max_{\alpha\in\Delta_m}\alpha^\top u\\ &= \max_{\alpha\in\Delta_m}\inf_{u\in L(\mathcal{F}(Y))}\alpha^\top u\\ &= \max_{\lambda\in\mathcal{F}(X)}\inf_{\mu\in\mathcal{F}(Y)} f_e(\lambda,\mu),\end{aligned}$$

which completes the proof. □

As we have seen, the equivalent minimax results presented here corresponds to different zero-sum games with different action sets. From our technique it follows that finite pure action sets and compact pure action sets are not really "far apart".

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Stable Profit Sharing in Patent Licensing: General Bargaining Outcomes

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Abstract

In a generalized framework of oligopolistic markets, we study how many potential licensees an external licensor of a patented innovation should negotiate with on the license issue and how much profit sharing the licensor can gain through the negotiation, from a viewpoint of the stability of coalition structures. The core with coalition structure is empty, unless the grand coalition forms under some condition. The bargaining set with coalition structure is a singleton, if the number of licensees optimal for the licensor is larger than that of non-licensees. The bargaining set coincides with the core, if the core is nonempty.

Keywords : licensing, oligopolistic markets, stable profit sharing, bargaining set with coalition structure, core
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1 Introduction

Patent licensing problems in oligopolistic markets have been studied only with noncooperative licensing policies; fixed fees or royalty in Kamien and Tauman (1984, 1986), and auction in Katz and Shapiro (1985, 1986). Kamien, Oren and Tauman (1992) compared those three policies for general demand functions: in the Cournot competition, it is never optimal for an external patent holder to license a cost-reducing innovation by means of the royalty. Muto (1993) studied the optimal licensing in the Bertrand duopoly with differentiated commodities: there is a case where it is optimal for an external patent holder to license by means of the royalty.

On the other hand, licensing through negotiations are also observed in reality. If there are many potential licensees, it may be best for the licensor to adopt simpler noncooperative policies in order to reduce large "transaction costs" due to persistent negotiations. However, such costs will not be so large if the licensor negotiate with a limited number of potential licensees.

Hence, we study bargaining outcomes in the patent licensing problems, applying cooperative solutions with coalition structures considered by Thrall and Lucas (1963), Aumann and Maschler (1964), and Aumann and Drèze (1974). Our questions are: (1) how many potential licensees should an external licensor of a patented innovation negotiate with? (2) how much profit sharing can the licensor gain through the negotiation?

A key problem with us is how to define the worth of each coalition of players. Driessen, Muto and Nakayama (1992) applied a classical definition to an information trading: the information is shared in a most efficient way among the seller and the potential buyers. However, not all the potential buyers are provided with the information, although they can share their total profits. We define the worth of each coalition in a more natural way.

Watanabe and Tauman (2003) proposed an alternative definition that reflects a sophisticated nature of events under a subtle mixture of conflict and cooperation: licensees can form a cartel S to enhance their oligopolistic power, whereas non-licensees may react also by forming some cartels. Then, the licensees in S might not merge into a single entity, but gather as smaller subcartels in S forming the headquarter-subsidiaries relationship.

In this paper, we assume that any forms of cooperations among firms are prohibited (by law) for the sake of a fair comparison of our results with noncooperative ones in the literature. A group of the licensor and the potential licensees forms only for the negotiation on the license issue. However, such a group formation appears also in Watanabe and Tauman's setting under some conditions as described in section 3; even if firms are allowed to cooperate in the market, firms in any groups will decide not to do so by themselves.

Another key point in this paper is to study the bargaining outcomes in a generalized framework of patent licensing models that have ever been studied. In the classical literature, functional forms, market structures, and characteristics of the innovation are specified, e.g., linear demand and cost functions, Cournot oligopoly, a non-drastic cost-reducing innovation under the "perfect" patent protection, and so there is no possibility of relicensing and spillover of the innovation to non-licensees. However, the stable profit sharing of the licensor can be characterized in a much less specified model.

Our main results are: (1) the core with coalition structure is empty, if the grand coalition does not form under some condition. it is always empty in the classical linear model followed by the Cournot competition. (2) the upper and lower bounds of the bargaining set with coalition structure is characterized. it is a singleton, if the number of licensees optimal for the licensor is larger than that of non-licensees. (3) the bargaining set coincides with the core, if the core is nonempty. even in a linear model, it can be nonempty, if commodities are differentiated.

We hereafter refer to the "stable profit sharing" as the set of the licensor's payoffs that belong to the bargaining set with coalition structure, since the core is empty under almost every coalition structure as stated above.

The outline of this paper is as follows. For better understanding our generalization, Section 2 gives a classical linear model of patent licensing. Section 3 formalizes our general cooperative licensing game. The core and the bargaining set with coalition structure are the solution concepts we study. Section 4 and 5 provide the results. Some discussions and remarks on related literature are stated in the last section.

2 A Linear Model

There are n firms operating in the market, where $2 \leq n < \infty$. Each firm i produces $q_i(\geq 0)$ units of an identical commodity with the same unit cost $c(> 0)$ of production. The market price p of the commodity is determined by $p = \max(a - \sum_{i \in N} q_i, 0)$, where $a \in (c, \infty)$ is a constant. An external licensor has a patent of an innovation which reduces the unit cost of production from c to $c - \epsilon$, where $c - \epsilon > 0$.

The profit of firm i is $u_i(q) = pq_i - C(q_i)$. $C(q_i) = (c - \epsilon)q_i$ if i is a licensee of the patented innovation, and $C(q_i) = cq_i$ otherwise. Without any production facilities, the licensor takes no action in the market but shares some of the profits of licensees in return for licensing his innovation.

Below is the game where the licensor sells the license to firms by means of fixed fees only. (1) The licensor first shows the prices of the patented innovation to firms. (2) Each firm next decides whether or not to purchase it at each price shown by the licensor simultaneously and independently of the other firms. (3) Finally they compete *à la* Cournot in the market, knowing that which firms are licensed or not. (The game with licensing by means of royalties only or auctions is played in a similar manner. See Kamien and Tauman (1984, 1986) and Katz and Shapiro (1985, 1986).)

The game is analyzed backwardly in the spirit of the subgame perfection. Given that s firms are licensed, let $W(s)$ and $L(s)$ denote the equilibrium profit of each licensee and that of each non-licensee, respectively. Let $\hat{s} := (a - c)/\epsilon$. If $a - c - \epsilon \geq 0$ (non-drastic innovation), then

$$W(s) = \begin{cases} ((a - c + (n - s + 1)\epsilon)/(n + 1))^2 & \text{if } s \leq \hat{s} \\ ((a - c + \epsilon)/(\hat{s} + 1))^2 & \text{if } s > \hat{s} \end{cases}$$

$$L(s) = \begin{cases} ((a - c - s\epsilon)/(n + 1))^2 & \text{if } s \leq \hat{s} \\ 0 & \text{if } s > \hat{s} \end{cases}$$

Regardless of the innovation being non-drastic or drastic ($a - c < \epsilon$), the equilibrium profits of licensees and non-licensees are summarized as

$$\begin{aligned} W(1) > \cdots > W(s) > \cdots > W(n) > L(0) > \cdots > L(s) > \cdots > L(\hat{s} - 1) \\ \geq L(\hat{s}) = \cdots = L(n - 1) = 0. \end{aligned}$$

3 A Licensing Game with Coalition Structures

We generalize the linear model described in section 2. There are n firms with the same cost functions and an external licensor of a patented innovation. $N = \{1, \ldots, n\}$ is the set of all the firms and the external licensor is denoted as player 0. The market can be either the Cournot or the Bertrand oligopoly for homogeneous or differentiated commodities. The innovation can be a quality-improving technology as well as a drastic or non-drastic cost-reducing one. Let $\{0\} \cup S$ $(S \subseteq N)$ denote the set of the licensor and all the potential licensees. No firm outside $\{0\} \cup S$ is licensed.

The game has two stages. It starts with negotiations among the licensor and the firms in S. The patented innovation is licensed to the firms in S, while how much each firm pays to the licensor is determined in negotiations. Next, firms compete in the market, knowing that which firms are licensed or not. No cooperation among firms is allowed in the market for a fair comparison of our results with noncooperative ones.

The equilibrium profit of each firm $i \in N$ in the market is determined for general (symmetric) demand and cost functions. Given that s firms hold the license, $W(s)$ and $L(s)$ denote the equilibrium profit of each licensee and that of each non-licensee, respectively. We require only the following:

$$W(s) > L(0) > L(s) \quad {}^{\forall}s.$$

$W(1) < W(2)$ can happen in the Bertrand duopoly with substitutive commodities at a sufficiently small rate. Muto and Watanabe (2004) showed that the prices raised by two licensees may cover the decreased demand for the commodity of the formerly single licensee. Our model contains this case. The "spillover" of the patented innovation to non-licensees is also included in this model, since the magnitudes of equilibrium profits are not concerned. Suppose that non-licensees can also utilize the patented innovation with some probability due to the spillover. Then, $W(s)$ and $L(s)$ are interpreted as the expected equilibrium profits when s firms are officially licensed.

We hereafter formalize this situation as a cooperative game with sidepayments. Denote by s^* the number of licensees such that $s^*(W(s^*) - L(0)) \geq s(W(s) - L(0))$ for any s.

Let $s = |S|$. The worth of each set of players is then characterized by

$$v(\{0\}) = v(\emptyset) = 0 \qquad v(\{0\} \cup S) = sW(s)$$
$$v(S) = sL(n - s).$$

The licensor 0 can gain nothing without selling the innovation, since he or she has no production facilities. $v(\{0\} \cup S) = sW(s)$ is the total equilibrium profits of licensees in S. $v(S)$ is the total equilibrium profits that the firms in S can guarantee for themselves even in the worst anticipation that all the other $n - s$ firms are licensed when firms in S jointly break off the negotiations. In the linear model described in section 2, our v-function is the same as in Watanabe and Tauman (2003), if $s \le (n + 1)/2$ and $s \le \hat{s}$. That implies that even if firms are allowed to cooperate in the market, firms in any coalitions will decide not to do so by themselves. Any firms can equally be a member of S, since every firm is identical before licensed. Hence, we can apply $v(S)$ to a group S of non-licensees.

Given a set $S \subseteq N$ of firms, negotiations are done only within $\{0\} \cup S$, and so the permissible coalition structure is $P^S = (\{0\} \cup S, \{\{i\}\}_{i \in N \setminus S})$. Since no cooperation is allowed in the market, coalition $\{0\} \cup S$ forms only for negotiations.

The set of imputations under a coalition structure P^S is then defined as

$$X^S = \{x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} | x_0 + \textstyle\sum_{i \in S} x_i = sW(s),$$
$$x_0 \ge 0,\ x_i \ge L(n - 1)\ {}^\forall i \in S, \quad x_i = L(s)\ {}^\forall i \in N \setminus S\}.$$

The core with coalition structure P^S is defined as

$$C^S = \{x \in X^S | \sum_{i \in T} x_i \ge v(T)\ {}^\forall T \subseteq \{0\} \cup N,\ T \cap (\{0\} \cup S) \ne \emptyset\}.$$

It can be shown that $C^S = \{x \in X^S | \sum_{i \in T} x_i \ge v(T)\ {}^\forall T \subseteq \{0\} \cup N\}$.

Let $i, j \in \{0\} \cup S$ and $x \in X^S$. We say that i has an objection (y, T) against j in x if $i \in T$, $j \notin T$, $T \subseteq \{0\} \cup N$, $y_k > x_k\ {}^\forall k \in T$, and $\sum_{k \in T} y_k \le v(T)$, and that j has a counter objection (z, R) to i's objection (y, T) if $j \in R$, $i \notin R$, $R \subseteq \{0\} \cup N$, $z_k \ge x_k\ {}^\forall k \in R$, $z_k \ge y_k\ {}^\forall k \in R \cap T$, and $\sum_{k \in R} z_k \le v(R)$. We say that i has a *valid* objection (y, T) against j in x if (y, T) is not countered.

The bargaining set with coalition structure P^S is defined as

$$M^S = \{x \in X^S | \text{no player in } \{0\} \cup S \text{ has a valid objection in } x\}.$$

The bargaining set contains other several cooperative solutions. It is clear that $C^S \subset M^S$ under any coalition structure P^S by the definitions.

Let $i, j \in N$. We say that i and j are *substitutes* in game v if

$$v(S \cup \{i\}) = v(S \cup \{j\}) \ {}^{\forall} S \subset (N \setminus \{i, j\}).$$

Since all the firms in S are substitutes in game v, the following symmetric sets facilitate our analysis:

$$\begin{array}{l} \tilde{X}^S = \{x \in X^S | x_i = x_j \ {}^{\forall} i, j \in S\} \\ \tilde{C}^S = C^S \cap \tilde{X}^S, \quad \tilde{M}^S = M^S \cap \tilde{X}^S \end{array}$$

4 The Core with P^S

Lemma 1 *If $C^S \neq \emptyset$, then there exists an $x \in C^S$ such that $x_i = \bar{x} \ \ {}^{\forall} i \in S$.*

Proof : Let $y = (y_0, y_1, \ldots, y_n) \in C^S$. Define $x = (x_0, x_1, \ldots, x_n) \in X^S$ by $x_j = y_j$ if $j \notin S$ and $x_i = \bar{x} = (1/s) \sum_{i \in S} y_i = (1/s) y(S)$ if $i \in S$. Fix a coalition $T \subseteq \{0\} \cup N$ such that $T \cap S \neq \emptyset$. Let $l = |T \cap S|$. Then $\min_{U \subset S, \ |U|=l} y(U) \leq (l/s) y(S) = x(T \cap S)$. Hence,

$$\begin{aligned} x(T) = \ & x(T \setminus S) + x(T \cap S) \ \geq \ y(T \setminus S) + \min\nolimits_{U \subset S, \ |U|=l} y(U) \\ \geq \ & \min\nolimits_{U \subset S, \ |U|=l} y((T \setminus S) \cup U) \ \geq \ v((T \setminus S) \cup U) = v(T), \end{aligned}$$

since $y((T \setminus S) \cup U) \geq v((T \setminus S) \cup U)$ and $v((T \setminus S) \cup U) = v(T)$ by the fact that all the firms in S are substitutes. *Q.E.D.*

Proposition 1 $C^S = \emptyset$ *if* $S \neq N$

Proof : We first show that $\tilde{C}^S = \emptyset$ if $S \neq N$. Suppose $\tilde{C}^S \neq \emptyset$ and take $x \in \tilde{C}^S$. Let $x_i = \bar{x} \ {}^{\forall} i \in S$. Then, we have $\bar{x} > L(0)$. Otherwise, we would have $s\bar{x} + (n - s)L(s) < nL(0)$ since $L(s) < L(0)$, which would imply that coalition N could block x.

Take a coalition $\{0\} \cup T$ with $|T| = |S|$ where $T \subseteq N \setminus S$ if $|S| \leq n/2$ and $T \supseteq N \setminus S$ if $|S| > n/2$. Let $t = |T|$. Then, we have $x_0 + \sum_{i \in T} x_i < sW(s) = tW(t)$, since $x_0 + s\bar{x} = sW(s)$ and $\bar{x} > L(0) > L(s) (= x_i \ ^\forall i \in N \setminus S)$. Hence, $\tilde{C}^S = \emptyset$ if $S \neq N$.

Now we know $\tilde{C}^S = \emptyset$, which implies that $C^S = \emptyset$ because of Lemma 1. *Q.E.D.*

Proposition 2 *$\tilde{C}^N \neq \emptyset$ if and only if $s^* = n$.*

Proof: ($\Rightarrow$) Suppose $s^* < n$. If $x \in \tilde{C}^N \neq \emptyset$, then we have

$$\bar{x} \geq L(0) \tag{1}$$

$$x_0 + s\bar{x} \geq sW(s), \ \ s = 0, 1, \cdots, n-1, \tag{2}$$

where $x_i = \bar{x} \ ^\forall i \in S$ and $x_0 = nW(n) - n\bar{x}$. Letting $s = s^*$ in (2), we obtain $nW(n) - n\bar{x} + s^*\bar{x} \geq s^*W(s^*)$ or $(n - s^*)\bar{x} \leq nW(n) - s^*W(s^*)$. By (1), we get $(n - s^*)L(0) \leq nW(n) - s^*W(s^*)$ or

$$s^*(W(s^*) - L(0)) \leq n(W(n) - L(0)),$$

contradicting the uniqueness of s^*.

($\Leftarrow$) Take x such that

$$x_i = \begin{cases} n(W(n) - L(0)) & \text{if } i = 0 \\ L(0) & \text{if } i \in N. \end{cases}$$

Since $s^* = n$, it is easily shown that $x \in \tilde{C}^N$. *Q.E.D.*

Remark 1: It is easily confirmed that $s^* < n$ in the linear model described in section 2. Watanabe and Tauman (2003) showed that the core of the linear model is empty as the number of licensees tends to infinity. We could show the same result even in the case of a finite number of players, although our v-function is slightly different from theirs.

We could know that $C^S = \emptyset$ unless $s = s^* = n$ by Proposition 1 and 2. Let us next consider the bargainig set with P^S.

5 The Symmetric Bargaining Set with P^S

It suffices to examine objections and counter objections of the licensor 0 and a licensee $i \in S$, because of the licensees' symmetric payoffs $\bar{x}$.

Lemma 2 *Suppose* $n/2 \le s < n$. *If* $x \in \tilde{M}^S$, *then* $\bar{x} \le L(0)$.

Proof : Suppose $\bar{x} > L(0)$ and take the licensor 0's objection $(y, \{0\} \cup T)$ against firm $i \in S$ such that $|T| = s$, $T \supseteq N \setminus S$ and

$$y_k = \begin{cases} x_0 + \epsilon & \text{if } k = 0 \\ \bar{x} + \epsilon & \text{if } k \in T \cap S \\ L(0) + \epsilon & \text{if } k \in T \cap (N \setminus S), \end{cases}$$

where $\epsilon = (n-s)(\bar{x} - L(0))/(s+1) > 0$.
Note that

$$\begin{aligned} y_0 + \textstyle\sum_{k \in T} y_k &= x_0 + (2s-n)\bar{x} + (n-s)L(0) + (s+1)\epsilon \\ &= x_0 + (2s-n)\bar{x} + (n-s)L(0) + (n-s)(\bar{x} - L(0)) \\ &= x_0 + s\bar{x} = sW(s). \end{aligned}$$

Since $y_k > L(0)\ \forall k \in T$ and $x_k = \bar{x} > L(0)\ \forall k \in N \setminus T$, any firm $i \in S$ has no counter objection against $(y, \{0\} \cup T)$. Contradiction. *Q.E.D.*

Lemma 3 *Suppose* $1 \le s \le n/2$. *If* $x \in \tilde{M}^S$ *and if* $s(W(s) - L(0)) \le (n-s)(W(n-s) - L(0))$, *then* $\bar{x} \le L(0)$.

Proof: Suppose $\bar{x} > L(0)$. Then $x_0 < s(W(s) - L(0))$. Take the licensor 0's objection $(y, \{0\} \cup (N \setminus S))$ against firm $i \in S$ with

$$y_k = \begin{cases} (n-s)(W(n-s) - L(0)) & \text{if } k = 0 \\ L(0) & \text{if } k \in N \setminus S, \end{cases}$$

Since $y_k > L(0)\ \forall k \in N \setminus S$ and $x_k = \bar{x} > L(0)\ \forall k \in S$, any firm $i \in S$ has no counter objection against $(y, \{0\} \cup (N \setminus S))$. Contradiction. *Q.E.D.*

Lemma 4 *Suppose* $1 \le s \le n/2$. *If* $x \in \tilde{M}^S$ *and if* $0 < s(W(s) - W(t)) \le (n-s)(W(n-s) - L(0))$, *then* $\bar{x} \le W(t)$.

Proof: Suppose $\bar{x} > W(t)$. Then $x_0 < s(W(s) - W(t))$. Since $W(t) > L(0)$, the same argument as in the proof of Lemma 2 applies. *Q.E.D.*

Lemma 5 *If* $x \in \tilde{M}^S$*, then* $x_0 \leq s^*(W(s^*) - L(0))$*.*

Proof : Suppose $x_0 > s^*(W(s^*) - L(0))$. Then, $\bar{x} = (sW(s) - x_0)/s < (sW(s) - s^*(W(s^*) - L(0)))/s \leq L(0)$ by s^*. Take an objection (y, N) of $i \in S$ against 0 such that $y_k = L(0)$ $^\forall k \in N$. If 0 had a counter objection $(z, \{0\} \cup T)$, then we would have $z_0 \geq x_0 > s^*(W(s^*) - L(0))$ and $z_k \geq y_k = L(0)$ $^\forall k \in T$, and thus we would reach a contradiction $z_0 + \sum_{k \in T} z_k > s^*(W(s^*) - L(0)) + tL(0) \geq tW(t)$. *Q.E.D.*

Proposition 3 *Let* $x \in M^S$*. Then, we have the following.*
(a) *If* $1 \leq s \leq n/2$ *and* $s(W(s) - W(t)) \leq (n - s)(W(n - s) - L(0))$*, then*

$$s(W(s) - W(t)) \leq x_0 \leq s^*(W(s^*) - L(0)),$$

where $W(t)$ *is the lowest one satisfying the above condition.*
(b) *If* $n/2 \leq s < n$ *or* $s(W(s) - L(0)) \leq (n - s)(W(n - s) - L(0))$*, then*

$$s(W(s) - L(0)) \leq x_0 \leq s^*(W(s^*) - L(0)).$$

(c) *If* $s^* < s = n$*, then* $n(W(n) - L(0)) \leq x_0 \leq s^*(W(s^*) - L(0))$*.*

Proof : Lemma 2 to Lemma 5 jointly implies (a) and (b). Consider the case (c). Let $\bar{x} = L(0) + z$ where $z > 0$. If $x_0 = n(W(n) - \bar{x})$, then 0 can make an objection $(y, \{0\} \cup S^*)$ where $y_i > x_i$ for any $i \in \{0\} \cup S^*$, since $n(W(n) - \bar{x}) = n(W(n) - L(0)) - nz < s^*(W(s^*) - L(0)) - s^*z = s^*(W(s^*) - \bar{x})$. Any counter objection cannot be made by $i \in N \setminus S^*$, since $\bar{x} > L(0)$. Lemma 5 completes (c). *Q.E.D.*

Let S^* be a set $S \subseteq N$ with $|S| = s^*$. Proposition 3 (b) directly implies the next corollary.

Corollary 1 *If* $n/2 \leq s^* < n$*, then* $\tilde{M}^{S^*} = \{x^*\}$ *where*

$$x_i = \begin{cases} s^*(W(s^*) - L(0)) & \text{if } i = 0 \\ L(0) & \text{if } i \in S^* \\ L(s^*) & \text{if } i \in N \setminus S^*. \end{cases}$$

In the linear model described in section 2, it is easily shown that there exists a threshold $\hat{\epsilon}$ of the cost reduction such that $n/2 \leq s^*$ if $\epsilon \leq \hat{\epsilon}$.

Note: The bargaining set merely suggests the set of payoffs reachable by a series of objections and counter objections in a real negotiation. In that sense, it has no criterion for the value judgement on a payoff distribution. On the other hand, it generally comtains the "nucleolus" that is uniquely determined in any game v by repeatedly applying the principle of minimizing the maximum complaint on a payoff distribution. Corollary 1 implies that the bargaining set coincides with the nucleolus under a coalition structure P^{S^*} with $n/2 \leq |S^*| < n$ and then reflects a "fairness" notion that the nucleolus has.

Proposition 4 *Let x^* be such that*

$$x_i^* = \begin{cases} s^*(W(s^*) - L(0)) & \textit{if } i = 0 \\ L(0) & \textit{if } i \in S^* \\ L(s^*) & \textit{if } i \in N \setminus S^*, \end{cases}$$

where $1 \leq s^* \leq n$. Then $x^* \in \tilde{M}^{S^*}$.

Proof: Take any objection $(y, \{0\} \cup T)$ of 0 against $i \in S^*$ in x^*. Then, we have $\sum_{k \in T} y_k < tL(0)$. Otherwise, we would obtain $tW(t) \geq y_0 + \sum_{y \in T} y_k > s^*(W(s^*) - L(0)) + tL(0)$, contradicting the definition of s^*. Hence, i has a counter objection (z, N) against $(y, \{0\} \cup T)$ with

$$z_i = \begin{cases} L(0) & \text{if } k \in S^* \setminus T \\ y_k + \epsilon & \text{if } k \in T \\ L(0) & \text{if } k \in (N \setminus S^*) \setminus T. \end{cases}$$

where $\epsilon = (tL(0) - \sum_{k \in T} y_k)/t > 0$. In fact, $\sum_{k \in N} z_k = nL(0)$, $z_k \geq x_k \ ^\forall k \in N$ and $z_k > y_k \ ^\forall k \in T$.

Next take any objection (u, R) of $i \in S^*$ against 0 in x^*. Let

$$u_k' = \begin{cases} u_k & \text{if } k \in R \\ x_k & \text{if } k \in N \setminus R. \end{cases}$$

Order all the n firms according to their payoffs in the non-decreasing order, and take the first s^* firms. Then, we have $\sum_{k \in Q} u'_k < s^* L(0)$ where Q is the set of the first s^* firms. Hence 0 has a counter objection against (u, R). *Q.E.D.*

Lemma 6 *If $x \in \tilde{M}^{S^*}$, then $\bar{x} \geq L(0)$.*

Proof: Suppose $\bar{x} < L(0)$. Then, a licensee $i \in S^*$ has an objection (y, N) against the licensor 0 in x, where $y_k = L(0)\ ^\forall k \in N$. Suppose that 0 had a counter objection $(z, \{0\} \cup R)$ to i's objection (y, N). Then we would have

$$rW(r) \geq z_0 + \textstyle\sum_{k \in R} z_k$$
$$z_0 \geq x_0, \text{ and } z_k \geq y_k = L(0)\ ^\forall k \in R$$

Since $\bar{x} < L(0)$, it must be that $x_0 = s^* W(s^*) - s^* \bar{x} > s^* W(s^*) - s^* L(0)$. We would then obtain

$$rW(r) \geq z_0 + \sum_{k \in R} z_k > s^* W(s^*) - s^* L(0) + rL(0),$$

contradicting the definition of s^*. Thus, i's objection (y, N) could not be countered by 0. Contradiction. *Q.E.D.*

Proposition 5 *If $s^* = n$, then $\tilde{M}^N = \tilde{C}^N$. $x \in \tilde{C}^N$ is characterized as $x_0 = n(W(n) - \bar{x})$ and $L(0) \leq \bar{x} \leq \min_{s \neq n}(nW(n) - sW(s))/(n - s)$.*

Proof: ($\supseteq$) It is clear by the definitions of C^N and M^N.

($\subseteq$) Suppose that there exists $x \in \tilde{M}^N$ with $x \notin \tilde{C}^N$. Since $x \in \tilde{M}^N$, we must have $\bar{x} \geq L(0)$ by Lemma 6. Since $x \notin \tilde{C}^N$, there must exist $\{0\} \cup T$ such that

$$x_0 + \sum_{i \in T} x_i < tW(t), \text{ where } t < n.$$

Let $(y, \{0\} \cup T)$ be 0's objection against any $i \in N \setminus T$ in x, where $y_k = x_k + \epsilon\ ^\forall k \in \{0\} \cup T$ and $(t+1)\epsilon = tW(t) - (x_0 + \sum_{i \in T} x_i) > 0$. Since $\bar{x} \geq L(0)$, i has no counter objection, contradicting that $x \in \tilde{M}^S$.

The system of inequalities to characterize $\tilde{C}^N$ yields $L(n - s) \leq \bar{x} \leq (nW(n) - sW(s))/(n - s)$ for any s. By Lemma 6, $\bar{x} \geq L(0)$. *Q.E.D.*

Even with the linear demand and cost functions, it can be that $\tilde{C}^N \neq \emptyset$ if the commodities are differentiated. See Muto and Watanabe (2004).

Remark 2: Proposition 3, 4 and 5 jointly imply that $s^*(W(s^*)-L(0))$ is the "stable profit sharing" always, but that he or she cannnot gain more than that amount and may not obtain it unless $s = s^*$ and $n/2 \leq s^* < n$. Hence, the licensor should invite $s^*(< n)$ firms to the negotiation if $n/2 \leq s^* < n$, which is the most favorable coalition size for him. When $s^* = n$, there are some cases where it is better for him not to invite all the n firms to the negotiation, if (collective) bargaining power of the firms is quite large. For example, it is better for the licensor to invite $n-1$ firms to the negotiations if $(n-1)(W(n-1)-L(0)) \geq n(W(n)-\bar{x})$ where $\bar{x} \geq L(0)$.

6 Concluding Remarks

Negotiations versus Fees

Recall the linear model in section 2. Kamien and Tauman (1986) showed that it is better for the licensor to license the innovation by means of fixed fees only than by means of royalties only. In the same model, we can find some cases where negotiations are superior to fixed fees.

Let ϵ denote the magnitude of the cost reduction, and let π^*_{PH} denote the profit of the patent holder who licenses using the fixed fees only. In the case of moderate cost reduction such that $2(a-c)/(3n-2) \leq \epsilon \leq 1 + (a-c)/n$, the stable profit sharing $s^*(W(s^*)-L(0))$ is lower than π^*_{PH}, regardless of any s^*. On the other hand, if $\epsilon \leq 2(a-c)/(3n-2)$ or if $1+(a-c)/n \leq \epsilon$, it can then be that $s^*(W(s^*)-L(0)) > \pi^*_{PH}$.

When $\epsilon \leq 2(a-c)/(3n-2)$ (or $1+(a-c)/n \leq \epsilon$), n (or $\hat{s}$) firms are licensed by means of fixed fees to obtain π^*_{PH}. In the former case, we know that $s^* < n$ by Remark 1, and obviously $s^* \leq \hat{s}$ by the definition of $\hat{s}$ in the latter case. Negotiations could be superior to fixed fees in such cases that the innovation is trivial or very nice.

Muto and Watanabe (2004) showed that it can be optimal for the licensor to sell the innovation by means of negotiations in the Bertrand duopoly with differentiated commodities. The interpretation of such cases is not so easy, since it depends also on the rate of substitution (complementarity) between the commodities.

Let $\mathrm{Sh}_0(v)$ denote the Shapley value of the licensor[1] and let $x \in M^S$. Lemma 5 shows that $x_0 \leq s^*(W(s^*) - L(0))$ if x_0 is a stable profit sharing. Watanabe and Tauman (2003) also showed in the linear model that $\mathrm{Sh}_0(v) > s^*(W(s^*) - L(0))$ as the number of licensees tends to infinity. It happens in our model if

$$\begin{aligned}&(1/n+1)\textstyle\sum_{s=1}^{n-\hat{s}} sL(0)) + (1/n+1)\sum_{s=n-\hat{s}+1}^{n-1} s(L(0) - L(n-s))\\ &> s^*(W(s^*) - L(0)) - (1/n+1)\textstyle\sum_{s=1}^{n} s(W(s) - L(0)).\end{aligned}$$

It is well known that the Shapley value is not necessarily in the core, but its relationship to the bargaining set has not been studied comprehensively. In this paper, $\hat{s}$ did not play any important role. With more specified assumptions on it, we could have proceed further on that topic.

Limitation of Sidepayments

We could have studied an alternative model where no sidepayments are allowed except payments of fees to the licensor: in $\{0\} \cup S$, each $i \in S$ pays p_i to the licensor 0, ${}^\forall S \subseteq N$, and there is no money transfer in S. Assume the uniform pricing scheme: $p_i = p\ {}^\forall i \in S$. We can regain almost the same results even in this setup. Hence, the assumption on the sidepayments does not play any important role for our propositions. Below is the addendum to extend our model.

The permissible coalition structure is

$$P^S = (\{0\} \cup S, \{\{i\}\}_{i \in N \setminus S}),\ {}^\forall S \subseteq N,$$

and so the characteristic function is given by

$$\begin{aligned}&V(\{0\} \cup S) = \{(x_i)_{i \in \{0\} \cup S} | x_0 \leq sp,\ x_i \leq W(s) - p,\ 0 \leq p \leq W(s)\}\\ &V(\{0\}) = 0, \quad V(S) = \{(x_i)_{i \in S} | x_i \leq L(n-s)\}.\end{aligned}$$

The imputations under a coalition structure P^S is defined by

$$\begin{aligned}X^S = \{x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} | x_0 = sp,\ x_i = W(s) - p,\ {}^\forall i \in S\\ 0 \leq p \leq W(s) - L(n-1),\ x_i \geq L(n-s),\ {}^\forall i \in N \setminus S\}.\end{aligned}$$

The core C^S is defined by

$$\begin{aligned}C^S = \{x \in X^S | \text{ for any } T \subseteq \{0\} \cup N \text{ with } T \cap (\{0\} \cup S) \neq \emptyset,\\ \text{there exists no } y \in V(T) \text{ such that } y_k > x_k,\ {}^\forall k \in T\}.\end{aligned}$$

Let $i, j \in \{0\} \cup S$ and $x \in X^S$. i has an objection (y, T) against j in x if $i \in T$, $j \notin T$, $T \subseteq \{0\} \cup N$, $y_k > x_k$ $^\forall k \in T$, and $y \in V(T)$. j has a counter objection (z, R) to i's objection (y, T) if $j \in R$, $i \notin R$, $R \subseteq \{0\} \cup N$, $z_k \geq x_k$ $^\forall k \in R$, $z_k \geq y_k$ $^\forall k \in R \cap T$, and $z \in V(R)$. i has a *valid* objection (y, T) against j in x if (y, T) is not countered. The bargaining set M^S is defined by

$$M^S = \{x \in X^S | \text{no player in } \{0\} \cup S \text{ has a valid objection in } x\}.$$

Final Remarks

Similar results are obtained with other solution concepts such as the strong equilibrium and the coalition-proof Nash equilibrium. We will show them precisely in another paper. For reference, see Muto (1987, 1990) and Nakayama and Quintas (1991).

Since our primary purpose was to show the general bargaining outcomes, it was difficult to analyze the welfare issues, especially in the consumer's surplus. The welfare analysis with more specified models are left for our future research.

In this paper, we assumed that no cooperation is allowed in the market. However, cooperative actions are observed in reality. Watanabe and Tauman (2003) defined the worth of each coalition under a subtle mixture of conflict and cooperation. Their main result is an asymptotic equivalence: with many small firms in the Cournot oligopolistic market, the Shapley value of the licensor of a patented cost-reducing innovation approximates the payoff he or she obtains in the linear non-cooperative licensing games (described in section 2) traditionally studied in the literature. Extending their idea, Watanabe (2004) argues how to represent strategic-form games in coalitional form without sidepayments.

Footnotes

(1) Let $\mathcal{R}(s+1)$ be an ordering of $n+1$ players where the licensor 0 is at the $(s+1)$-st place. $j\mathcal{R}(s+1)0$ means that firm j precedes the licensor 0 in the ordering $\mathcal{R}(s+1)$. Denote by $\mathcal{P}_0^{\mathcal{R}(s+1)} = \{j \in N | j\mathcal{R}(s+1)0\}$ the set of firms that precedes the licensor in $\mathcal{R}(s+1)$. Since every firm is identical before licensed, there are $n!$ such orderings that have the same marginal contribution $v(\mathcal{P}_0^{\mathcal{R}(s+1)} \cup \{0\}) - v(\mathcal{P}_0^{\mathcal{R}(s+1)})$ of the licensor to coalition $S^0 = \mathcal{P}_0^{\mathcal{R}(s+1)} \cup \{0\}$. The Shapley value of the licensor in our licensing game is

$$\begin{aligned}\mathrm{Sh}_0(v) &= (1/(n+1)!) \textstyle\sum_{s=1,\cdots,n+1} n!\{v(\mathcal{P}_0^{\mathcal{R}(s+1)} \cup \{0\}) - v(\mathcal{P}_0^{\mathcal{R}(s+1)})\} \\ &= (1/n+1) \textstyle\sum_{s=1}^{n} s(W(s) - L(n-s)).\end{aligned}$$

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On Bargaining Sets and Voting Games*

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Abstract

Using α-effectiveness we define the NTU games corresponding to simple majority voting, plurality voting, and approval voting. The Aumann-Davis-Maschler bargaining set of a simple majority voting game is nonempty if there are at most three alternatives and it may be empty for four or more alternatives, whereas the Mas-Colell bargaining set may be empty only for more than five alternatives. However, if the number of players tends to infinity, then the bargaining sets of simple majority voting games are likely to be nonempty. The emptiness of an upper hemicontinuous extension of the Mas-Colell bargaining set for a simple majority voting game with four persons is used to conclude that the Mas-Colell bargaining set of a non-levelled superadditive NTU game may be empty.

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1 Introduction

The Voting Paradox prevents us from applying the majority voting rule to choice problems with more than two alternatives. The standard way to avoid the paradox is to assume that the preferences of the voters are

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restricted so that the method of decision by majority yields no cycles (see Gaertner (2001) for a recent comprehensive survey). In this paper we follow a different path. It is well-known that the Voting Paradox is equivalent to the emptiness of the core of the corresponding cooperative majority voting game. We have chosen to investigate two bargaining sets which include the core: The Aumann-Davis-Maschler bargaining set and the Mas-Colell bargaining set. While it is well-known that the Aumann-Davis-Maschler bargaining set may be empty for superadditive NTU games, the problem of the non-emptiness of the Mas-Colell bargaining set (for the same class of games) was open when we started our investigation. Indeed, Example 4.5 provides for the first time a superadditive NTU game with an empty Mas-Colell bargaining set. Although the foregoing two bargaining sets may be empty, they perform much better than the core; for example, both are nonempty for the Voting Paradox and satisfy interesting asymptotic results.

We shall now review our results. At the end of the review we shall present our main conclusions.

In Section 2 we derive the exact form of the cooperative NTU games which correspond to simple majority voting, plurality voting, and approval voting (see Brams and Fishburn (1983)). We also recall the definitions of the Aumann-Davis-Maschler and Mas-Colell bargaining sets of cooperative NTU games. Throughout our study we focus, almost exclusively, on the foregoing two bargaining sets of simple majority voting games.

Section 3 deals with the Aumann-Davis-Maschler bargaining set. We report that it is nonempty for three alternatives. We show by means of an example that it may be empty (for a simple majority voting game), when there are four or more alternatives. Nevertheless, in a simple probabilistic model, if the number of alternatives is fixed, then the probability that the Aumann-Davis-Maschler bargaining set is nonempty tends to one as the number of voters tends to infinity.

Our main existence theorem is presented in Section 4: The Mas-Colell bargaining set of a simple majority voting game is nonempty for five (or less) alternatives. For six alternatives Example 4.5 shows that the Mas-Colell bargaining set (of a simple majority voting game) may be empty. Finally, we report in Section 4 the following result: If R^N is a profile of preferences of the n members of the set N of voters and if $k \geq n+2$, then the Mas-Colell bargaining set of any simple majority voting game that is derived from the k-th replication of R^N is nonempty.

In Section 5 we introduce an extension of the Mas-Colell bargaining set which is upper hemicontinuous. The emptiness of this extension for a four-person simple majority voting game with ten alternatives can be used to show the existence of a four-person non-levelled superadditive NTU game with an empty Mas-Colell bargaining set. This result solves an open problem of Vohra (1991).

Now we present our conclusions. Let (N, V) be a simple majority voting game and let x be an individually rational payoff vector. Then x is in a bargaining set if: (i) x is (weakly) Pareto optimal; and (ii) for every objection (in the sense of the bargaining set) there is a counter objection. Our study proves that the tension between (i) and (ii) is so strong that for six or more alternatives all bargaining sets may be empty. This is our first conclusion. Our second conclusion is more vague: If the number of players tends to infinity and the number of alternatives is held fixed, then the bargaining sets of (simple majority) voting games are likely to be non-empty.

Proofs of the results are contained in Peleg and Sudhölter (2004, 2005).

2 Preliminaries

Let $N = \{1, \ldots, n\}$, $n \geq 3$, be a set of voters, also called players, and let $A = \{a_1, \ldots, a_m\}$, $m \geq 3$, be a set of m alternatives. For $S \subseteq N$ we denote by $\mathbb{R}^S$ the set of all real functions on S. So $\mathbb{R}^S$ is the $|S|$-dimensional Euclidean space. (Here and in the sequel, if D is a finite set, then $|D|$ denotes the cardinality of D.) If $x, y \in \mathbb{R}^S$, then we write $x \geq y$ if $x^i \geq y^i$ for all $i \in S$. Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$ and we write

$x \gg y$ if $x^i > y^i$ for all $i \in S$. Denote $\mathbb{R}^S_+ = \{x \in \mathbb{R}^S \mid x \geq 0\}$. A set $C \subseteq \mathbb{R}^S$ is comprehensive if $x \in C$, $y \in \mathbb{R}^S$, and $y \leq x$ imply that $y \in C$. An *NTU game* with the player set N is a pair (N, V) where V is a function which associates with every *coalition* S (that is, $S \subseteq N$ and $S \neq \emptyset$) a set $V(S) \subseteq \mathbb{R}^S$, $V(S) \neq \emptyset$, such that $V(S)$ is closed and comprehensive and $V(S) \cap (x + \mathbb{R}^S_+)$ is bounded for every $x \in \mathbb{R}^S$.

We shall focus on choice by simple majority voting, by plurality voting, and by approval voting. The corresponding three strategic game forms leading to three kinds of NTU voting games may be described as follows. The first game form consists of the voters selecting an element of A. If a strict majority of voters agrees on $\alpha \in A$, then the outcome is α; otherwise no alternative is selected. The second game form is a multi-valued game form which differs from the first game form only inasmuch as the set of all alternatives that are announced by a maximal number of voters is selected. In the third game form each voter has to announce a nonempty subset – a ballot – of alternatives. The outcome is the set of alternatives that are members of a maximal number of ballots.

We shall now assume that each $i \in N$ has a linear preference R^i on A. Thus, for every $i \in N$, R^i is a complete, transitive, and antisymmetric binary relation on A. Moreover, let u^i, $i \in N$, be a utility function that represents R^i. With the exception of Section 5 we shall always assume that

$$\min_{\alpha \in A} u^i(\alpha) = 0 \text{ for all } i \in N. \tag{2.1}$$

As we are going to break ties by even-chance lotteries, we shall further assume that the utilities are *weakly cardinal*, that is, they satisfy the expected utility hypothesis for even-chance lotteries (see Fishburn (1972)). For each of the three strategic game forms any utility profile $u^N = (u^i)_{i \in N}$ that satisfies the foregoing assumptions determines its corresponding strategic game. These considerations motivate us to define the cooperative NTU *voting games* that are associated (via α-effectiveness) with our strategic games. Indeed, let u^N be a utility profile that satisfies (2.1). The NTU game (N, V_{u^N}) associated with choice by simple majority voting and called *simple majority voting game* (see Aumann (1967)) is defined by

$$V_{u^N}(S) = \{x \in \mathbb{R}^S \mid x \leq 0\} \text{ if } S \subseteq N, 1 \leq |S| \leq \frac{n}{2}; \tag{2.2}$$

$$V_{u^N}(S) = \{x \in \mathbb{R}^S \mid \exists \alpha \in A \text{ such that } x \leq u^S(\alpha)\} \text{ if } S \subseteq N, |S| > \frac{n}{2}. \tag{2.3}$$

The coalition function of the *plurality voting game*, that is, the NTU game associated with choice by plurality voting, is denoted by $V^{pl}_{u^N}$ and it may differ from V_{u^N} only for coalitions $S \subseteq N$ such that $|S| = n/2$ and for the grand coalition N. Indeed, we define

$$V^{pl}_{u^N}(S) = \left\{ x \in \mathbb{R}^S \,\middle|\, \exists\ \alpha \in A \text{ such that } x \leq \frac{1}{2} u^S(\alpha) \right\} \text{ for all } S \subseteq N, |S| = \frac{n}{2}, \tag{2.4}$$

and

$$V^{pl}_{u^N}(N) = \left\{ x \in \mathbb{R}^N \,\middle|\, \begin{array}{c} \exists\ B \subseteq A \text{ such that } 1 \leq |B| \leq n, \\ \left(\left[\frac{n}{|B|}\right] - 1\right)|A| + |B| \geq n, \text{ and } x \leq \frac{\sum_{\beta \in B} u^N(\beta)}{|B|} \end{array} \right\}, \tag{2.5}$$

where $[r]$ denotes the largest integer less than or equal to r. Indeed, if $|S| = n/2$ and all members of S select the same alternative α, then a player $i \in S$ cannot be prevented from the utility $u^i(\alpha)/2$ even if all members of $N \setminus S$ select i's worst alternative (see (2.1)). Moreover, if B is the set of alternatives that are announced by a maximal number t of voters, then $0 \leq n - t|B| \leq (t-1)(|A| - |B|)$ and, hence, $t \leq [n/|B|]$ and

$$n - |B| \leq ([n/|B|] - 1)|A|. \tag{2.6}$$

If $B \subseteq A$ satisfies (2.6), then there exists a profile of strategies that results in the outcome B.

Now, if approval voting is employed, if $S \subseteq N$ satisfies $|S| = n/2$, and if each member j of S selects a ballot B^j, then the strategies of the players in $N \setminus S$ may induce the following sets of outcomes: (1) Any subset of $\bigcup_{j \in S} B^j$ and (2) any superset of $\bigcap_{j \in S} B^j$. Hence, if $i \in S$, then $N \setminus S$ may prevent i from receiving more than the utility

$$\min \left\{ \min_{\beta \in \bigcup_{j \in S} B^j} u^i(\beta), \min_{C \supseteq \bigcap_{j \in S} B^j} \sum_{\gamma \in C} \frac{u^i(\gamma)}{|C|} \right\} \leq$$

$$\leq \min\left\{\min_{\beta\in B^j} u^i(\beta), \min_{C\supseteq B^j} \sum_{\gamma\in C} \frac{u^i(\gamma)}{|C|}\right\} \;\forall j \in S.$$

Also, if all members of the grand coalition select $B \subseteq A$, then the resulting utility profile is $\sum_{\beta\in B} u^N(\beta)/|B|$. Hence, the NTU game associated with choice by approval voting, $(N, V^{ap}_{u^N})$, called *approval voting game*, differs from (N, V_{u^N}) only inasmuch as for any $S \subseteq N, |S| = \frac{n}{2}$,

$V^{ap}_{u^N}(S) =$

$$\left\{x \in \mathbb{R}^S \middle| \exists\, \emptyset \neq B \subsetneq A \quad \text{such that} \right.$$
$$\left. x^i \leq \min\left\{\min_{\beta\in B} u^i(\beta), \min_{\emptyset\neq C\subseteq A\setminus B} \frac{\sum_{\beta\in B\cup C} u^i(\beta)}{|B|+|C|}\right\} \forall\, i \in S\right\}, \tag{2.7}$$

and

$$V^{ap}_{u^N}(N) = \left\{x \in \mathbb{R}^N \middle| \exists\, \emptyset \neq B \subseteq A \text{ such that } x \leq \frac{\sum_{\beta\in B} u^N(\beta)}{|B|}\right\}. \tag{2.8}$$

Hence, for each coalition S, $V_{u^N}(S)$ (or $V^{pl}_{u^N}(S), V^{ap}_{u^N}(S)$, respectively) consists of all vectors $x \in \mathbb{R}^S$ that S can get, regardless of the strategies chosen by the members of $N\setminus S$, with respect to choice by simple majority voting (or plurality voting, approval voting, respectively). Note that the selection of no alternative in the context of choice by simple majority voting is assumed to result in the utility 0 for each voter.

Notation 2.1 In the sequel let $L = L(A)$ denote the set of linear preferences on A. If $R^N \in L^N$, then denote

$$\mathcal{U}^{R^N} = \{(u^i)_{i\in N} \mid u^i \text{ is a representation of } R^i \text{ satisfying (2.1) } \forall i \in N\}.$$

Remark 2.2 Let $R^N \in L^N$. Then the associated simple majority voting games are derived from each other by ordinal transformations. The associated plurality voting games and the associated approval voting games may not be derived from each other by an ordinal transformation, because weakly cardinal utilities may not be covariant under monotone transformations.

Let (N, V) be an NTU game. The pair (N, V) is *zero-normalized* if $V(\{i\}) = -\mathbb{R}_+^{\{i\}} (= \{x \in \mathbb{R}^i \mid x \leq 0\})$ for all $i \in N$. Also, (N, V) is *superadditive* if for every pair of disjoint coalitions S, T, $V(S) \times V(T) \subseteq V(S \cup T)$. It should be remarked that the three foregoing NTU games are zero-normalized and superadditive.

Now we shall recall the definitions of two bargaining sets introduced by Davis and Maschler (1967) and by Mas-Colell (1989). Let (N, V) be a zero-normalized NTU game and $x \in \mathbb{R}^N$. We say that x is

- *individually rational* if $x \geq 0$;
- *Pareto optimal* (in $V(N)$) if $x \in V(N)$ and if $y \in V(N)$ and $y \geq x$ imply $x = y$;
- *weakly Pareto optimal* (in $V(N)$) if $x \in V(N)$ and if for every $y \in V(N)$ there exists $i \in N$ such that $x^i \geq y^i$;
- a *preimputation* if x is weakly Pareto optimal in $V(N)$;
- an *imputation* if x is an individually rational preimputation.

A pair (P, y) is an *objection* at x if $\emptyset \neq P \subseteq N$, y is Pareto optimal in $V(P)$, and $y > x^P$. An objection (P, y) is *strong* if $y \gg x^P$. The pair (Q, z) is a *weak counter objection* to the objection (P, y) if $Q \subseteq N$, $Q \neq \emptyset, P$, if $z \in V(Q)$, and if $z \geq (y^{P \cap Q}, x^{Q \setminus P})$. A weak counter objection (Q, z) is a *counter objection* to the objection (P, y) if $z > (y^{P \cap Q}, x^{Q \setminus P})$. A strong objection (P, y) is *justified in the sense of the bargaining set* if there exist players $k \in P$ and $\ell \in N \setminus P$ such that there does not exist any weak counter objection (Q, z) to (P, y) satisfying $\ell \in Q$ and $k \notin Q$. The *bargaining set* of (N, V), $\mathcal{M}(N, V)$, is the set of all imputations x that do not have strong justified objections at x in the sense of the bargaining set (see Davis and Maschler (1967)). An objection (P, y) is *justified in the sense of the Mas-Colell bargaining set* if there does not exist any counter objection to (P, y). The *Mas-Colell bargaining set* of (N, V), $\mathcal{MB}(N, V)$, is the set of all imputations x that do not have a justified objection at x in the sense of the Mas-Colell bargaining set (see Mas-Colell (1989)).

Notation 2.3 If $R^N \in L^N$ and $\alpha, \beta \in A$, $\alpha \neq \beta$, then α *dominates* β (abbreviated $\alpha \succ_{R^N} \beta$) if $|\{i \in N \mid \alpha\ R^i\ \beta\}| > \frac{n}{2}$. For $R \in L$ and for $k \in \{1, \ldots, m\}$, let $t_k(R)$ denote the k-th alternative in the order R. Also, for $B \subseteq A$ let $R_{|B}$ denote the restriction of R to B.

Remark 2.4 Let $u^N \in \mathcal{U}^{R^N}$, let $B \subsetneq A$, let $i \in N$, and let

$$(t_1({R^i}_{|A \setminus B}), \ldots, t_{m-|B|}({R^i}_{|A \setminus B})) = (\alpha_1, \ldots, \alpha_{m-|B|})$$

be the vector of alternatives in $A \setminus B$ ordered by R^i. For $j = 1, \ldots, m - |B|$, define

$$z_j = \frac{1}{m-j+1}\left(\sum_{\beta \in B} u^i(\beta) + \sum_{k=j}^{m-|B|} u^i(\alpha_k)\right).$$

It can be deduced that the sequence $(z_j)_{j=1}^{m-|B|}$ is unimodal, i.e., there exists $t \in \{1, \ldots, m - |B|\}$ such that $z_k > z_{k+1}$ for $k \leq t-1$, $z_k < z_{k+1}$ for $k > t$, and $z_t \leq z_{t+1}$ if $t < m - |B|$. We conclude that

$$\min_{\emptyset \neq C \subseteq A \setminus B} \sum_{\beta \in B \cup C} \frac{u^i(\beta)}{|B| + |C|} = \min_{j=1,\ldots,m-|B|} z_j = z_t.$$

This remark enables us to easily compute (2.7), taking (2.1) into account, that is,

$$t_m(R^i) \in B \quad \Rightarrow \quad \min_{\beta \in B} u^i(\beta) = 0 \leq z_t, \tag{2.9}$$

$$t_m(R^i) \notin B \quad \Rightarrow \quad u^i(\alpha_{m-|B|}) = u^i(t_m(R^i)) = 0. \tag{2.10}$$

We shall say that an alternative $\alpha \in A$ is a *weak Condorcet winner* (with respect to R^N) if $\beta \not\succ_{R^N} \alpha$ for all $\beta \in A$.

3 The Aumann-Davis-Maschler Bargaining Set

Throughout this section and Section 4 let $R^N \in L(A)^N$, $u^N \in \mathcal{U}^{R^N}$ (see Notation 2.1), $V = V_{u^N}$ (see (2.2) and (2.3)) and let $\succ \; = \; \succ_{R^N}$ (see Notation 2.3).

Theorem 3.1 *If* $|A| = 3$, *then* $\mathcal{M}(N, V_{u^N}) \neq \emptyset$.

In order to partially characterize the bargaining set, for $\alpha, \beta \in A$, $\alpha \neq \beta$, let

$$D_{\alpha\beta}(R^N) = D_{\alpha\beta} = \{i \in N \mid \alpha \ R^i \ \beta\}.$$

Theorem 3.2 *Let* $A = \{a, b, c\}$. *Assume that* $a \succ b$, $b \succ c$, $c \succ a$, *and that*

$$|D_{\alpha\beta}| > \frac{n}{2} + 1 \text{ for all } (\alpha, \beta) \in \{(a, b), (b, c), (c, a)\}.$$

If $x \in \mathbb{R}^N$ *satisfies*

$$0 \leq x \leq u^N(\alpha) \text{ for some } \alpha \in A \tag{3.1}$$

and

$$x^i \leq u^i(t_2(R^i)) \text{ for all } i \in N, \tag{3.2}$$

then $x \in \mathcal{M}(N, V)$.

Remark 3.3 In fact $|D_{ca}| > \frac{n}{2} + 1$ is not used when $x \leq u^N(a)$. Thus, the following stronger result may be deduced.

Corollary 3.4 *Let* $A = \{a, b, c\}$. *Assume that* $x \in \mathbb{R}^N$ *satisfies* $0 \leq x \leq (u^i(t_2(R^i)))_{i \in N}$ *and assume that* $a \succ b$, $b \succ c$, *and* $c \succ a$. *Then* $x \in \mathcal{M}(N, V)$ *in each of the following three cases:*

$$\left(x \leq u^N(a) \text{ and } |D_{ab}|, |D_{bc}| > \tfrac{n}{2} + 1\right), \quad \text{or}$$
$$\left(x \leq u^N(b) \text{ and } |D_{bc}|, |D_{ca}| > \tfrac{n}{2} + 1\right), \quad \text{or}$$
$$\left(x \leq u^N(c) \text{ and } |D_{ca}|, |D_{ab}| > \tfrac{n}{2} + 1\right).$$

By means of an example we shall show that $\mathcal{M}(N, V_{u^N})$ may be empty for any $u^N \in \mathcal{U}^{R^N}$, provided $|A| \geq 4$.

Example 3.5 Let $A = \{a, b, c, d\}$, let $n = 3$, let R^N be given by Table 3.1, let $u^N \in \mathcal{U}^{R^N}$, and let $V = V_{u^N}$. Then $\mathcal{M}(N, V) = \emptyset$. Example 3.5 shows that the tension between (weak) Pareto optimality and stability (à la Aumann and Maschler (1964)) may result in an empty bargaining set.

Table 3.1: Preference Profile of a 4-Alternative Voting Problem

R^1	R^2	R^3
a	c	b
b	a	c
d	d	d
c	b	a

Example 3.5 may be generalized to any number $m \geq 4$ of alternatives. Indeed, let $A = \{a, b, c, d_1, \ldots, d_k\}$, where $k = m - 3$, and define R^N by

$$\begin{aligned} R^1 &= (a, b, d_1, \ldots, d_k, c), \\ R^2 &= (c, a, d_1, \ldots, d_k, b), \\ R^3 &= (b, c, d_1, \ldots, d_k, a), \end{aligned}$$

and note that $\mathcal{M}(N, V_{u^N}) = \emptyset$ for any $u^N \in \mathcal{U}^{R^N}$. More interestingly, Example 3.5 can be generalized to yield an empty bargaining set for simple majority voting games on four alternatives with infinitely many numbers of voters.

Example 3.6 (Example 3.5 generalized) Let

$$\begin{aligned} &R_1 = (a, b, d, c), \quad R_2 = (a, c, d, b), \quad R_3 = (b, a, d, c), \\ &R_4 = (b, c, d, a), \quad R_5 = (c, a, d, b), \quad R_6 = (c, b, d, a), \end{aligned}$$

and let $k \in \mathbb{N}$. Let $N = \{1, \ldots, 6k - 3\}$ and let $R^N \in L^N$ satisfy

$$|\{j \in N \mid R^j = R_i\}| = \begin{cases} \quad k & , \text{if } i = 1, 4, 5, \\ k - 1 & , \text{if } i = 2, 3, 6. \end{cases}$$

Then $\mathcal{M}(N, V_{u^N}) = \emptyset$ for any $u^N \in \mathcal{U}^{R^N}$. Indeed, $k = 1$ coincides with Example 3.5.

Notwithstanding Example 3.5, there is a simple probabilistic model in which most preference profiles lead to a nonempty bargaining set $\mathcal{M}$ as the number of players becomes large. Let $|A| = m \geq 4$ and let $L(A) = L$. Assume that each $R \in L$ appears with positive probability $p_R > 0$ in the population of potential voters, where $\sum_{R \in L} p_R = 1$. Now let $(\mathcal{R}^i)_{i \in \mathbb{N}}$ be a sequence of independent and identically distributed random variables such that $Pr(\{\mathcal{R}^i = R\}) = p_R$ for all $i \in \mathbb{N}$, $R \in L$. Call $R^N \in L^N$ *good* if for all $\alpha \in A$ there exists $i \in N$ such that $\alpha = t_m(R^i)$. If R^N is good, then $\left(u^i(t_m(R^i))\right)_{i \in N} \in \mathcal{M}(N, V_{u^N})$ for any $u^N \in \mathcal{U}^{R^N}$. By the law of large numbers, $\lim_{n \to \infty} Pr\left(\left\{\mathcal{R}^N \text{ is good}\right\}\right) = 1$, where $\mathcal{R}^N = \left(\mathcal{R}^1, \ldots, \mathcal{R}^n\right)$. Hence, $\lim_{n \to \infty} Pr\left(\left\{\mathcal{M}\left(N, V\left(\mathcal{R}^N\right)\right) \neq \emptyset\right\}\right) = 1$, where $\left(N, V\left(\mathcal{R}^N\right)\right)$ is a random NTU game which is a simple majority voting game V_{u^N}, $u^N \in \mathcal{U}^{R^N}$, for any realization R^N of $\mathcal{R}^N$.

4 The Mas-Colell Bargaining Set

Remark 4.1 If there exists a weak Condorcet winner with respect to R^N, then $\mathcal{MB}(N, V_{u^N})$ contains the set of the utility profiles of all weak Condorcet winners.

In the case of three alternatives we may deduce the following results.

Theorem 4.2 *If $|A| = 3$ and if there is no weak Condorcet winner with respect to R^N and if $x \in \mathbb{R}^N$ satisfies*

$$0 \leq x^i \leq u^i(t_2(R^i)) \text{ for all } i \in N; \tag{4.1}$$

$$\text{there exists } \alpha \in A \text{ such that } x \leq u^N(\alpha), \tag{4.2}$$

then $x \in \mathcal{MB}(N, V)$.

Corollary 4.3 *If $|A| = 3$ and there is no weak Condorcet winner with respect to R^N, then $\mathcal{M}(N, V) \subseteq \mathcal{MB}(N, V)$.*

Examples show that the inclusion in the foregoing corollary may be strict.

Theorem 4.4 *If $m \leq 5$, then $\mathcal{MB}(N, V_{u^N}) \neq \emptyset$ for all $u^N \in \mathcal{U}^{R^N}$.*

Table 4.1: Preference Profile leading to an empty $\mathcal{MB}$

R^1	R^2	R^3	R^4
a_1	a_4	a_3	a_2
a_2	a_1	a_4	a_3
c	c	c	b
b	b	b	a_4
a_3	a_2	a_1	c
a_4	a_3	a_2	a_1

We shall now present an example of a simple majority voting game on six alternatives whose Mas-Colell bargaining set is empty.

Example 4.5 Let $n = 4$, $A = \{a_1, \ldots, a_4, b, c\}$, let $R^N \in L^N$ be given by Table 4.1 and let $u^N \in \mathcal{U}^{R^N}$. It may be verified that $\mathcal{MB}(N, V_{u^N}) = \emptyset$.

Example 4.5 may be generalized to any number $m \geq 6$ of alternatives. Also, if $R_i = R^i$ for $i = 1, \ldots, 4$, if

$$R_5 = (a_2, a_1, c, b, a_3, a_4), R_6 = (a_4, a_3, c, b, a_1, a_2),$$

if $n = 4 + 2k$ for some $k \in \mathbb{N}$, if $\widetilde{R}^N \in L^N$ such that

$$|\{j \in N \mid \widetilde{R}^j = R_i\}| = \begin{cases} k & , \text{if } i = 5, 6, \\ 1 & , \text{if } i = 1, 2, 3, 4, \end{cases}$$

then $\mathcal{MB}(N, V_{u^N}) = \emptyset$ for all $u^N \in \mathcal{U}^{\widetilde{R}^N}$.

In what follows we shall show that a suitable choice of utilities in Example 4.5 shows that the Mas-Colell bargaining set of a plurality or of a approval voting game on six alternatives may be empty.

Example 4.6 (Example 4.5 continued) We now specify a utility representation $u^N \in \mathcal{U}^{R^N}$ by

$$u^i(t_j(R^i)) = 6^5 - 6^{j-1} \text{ for all } i \in N \text{ and } j = 1, \ldots, 6.$$

Let (N, V) the corresponding plurality or approval voting game, that is, $V \in \{V^{pl}_{u^N}, V^{ap}_{u^N}\}$.

Then $\mathcal{MB}(N, V) = \emptyset$.

Remark 4.7 It is possible to modify the utility profile u^N of the foregoing example in such a way that the Mas-Colell bargaining sets of the approval or the plurality voting game are nonempty. Indeed, if we just replace u^i, $i = 1, 2$, by $\widetilde{u}^i$ which differs from u^i only inasmuch as $\widetilde{u}^i(t_j(R^i)) = 12 - 2j$ for $j = 4, 5$, then

$$x = \left(\frac{\widetilde{u}^1(a_3) + \widetilde{u}^1(a_4)}{2}, \frac{u^2(a_1)}{2}, u^{\{3,4\}}(a_4)\right) = (1, 3885, 7770, 7560) \in \mathcal{MB}(N, V).$$

In order to replicate the simple majority voting game (N, V_{u^N}), let $k \in \mathbb{N}$ and denote

$$kN = \{(j, i) \mid i \in N,\ j = 1, \ldots, k\}.$$

Furthermore, let $R^{(j,i)} = R^i$ and $u^{(j,i)} = u^i$ for all $i \in N$ and $j = 1, \ldots, k$. Then $(kN, V_{u^{kN}})$ is the *k-fold replication* of (N, V_{u^N}).

Remark 4.8 *If α is a weak Condorcet winner with respect to R^N, then $u^{kN}(\alpha) \in \mathcal{MB}(kN, V_{u^{kN}})$.*

Theorem 4.9 *If* $k \geq \left\{ \begin{array}{ll} n+2 & \text{, if } n \text{ is odd,} \\ \frac{n}{2}+2 & \text{, if } n \text{ is even,} \end{array} \right\}$ *then* $\mathcal{MB}(kN, V_{u^{kN}}) \neq \emptyset$.

It should be remarked that the foregoing theorem remains valid for any $u^{kN} \in \mathcal{U}^{R^{kN}}$.

5 A Non-Levelled Superadditive Game with an Empty $\mathcal{MB}$

In this section we show that there exists a **non-levelled**[1] superadditive NTU game whose Mas-Colell bargaining set is empty. Note that simple majority

[1]A zero-normalized NTU game (N, V) is *non-levelled* if for every coalition S every weakly Pareto optimal element of $V(S) \cap \mathbb{R}^S_+$ is Pareto optimal.

voting games are levelled. We shall now extend $\mathcal{MB}$ and modify the game of Example 4.5 suitably. Let (N, V) be a zero-normalized superadditive NTU game and let x be an imputation. A strong objection at x is *strongly justified* if it has no weak counter objection. The *extended bargaining set* $\mathcal{MB}^*(N, V)$ is the set of all imputations that do not have strongly justified strong objections. Clearly, $\mathcal{MB}(N, V) \subseteq \mathcal{MB}^*(N, V)$.

If (N, V) is the game of Example 4.5, then $(u^{\{1,2,3\}}(b), u^4(a_4)) \in \mathcal{MB}^*(N, V)$. However, the following example presents a game whose extended bargaining set is empty.

Example 5.1 Let $n = 4$, $A = \{a_1, \ldots, a_4, a_1^*, \ldots, a_4^*, b, c\}$, let $R^N \in L^N$ be given by Table 5.1, let u^N represent R^N such that $\min_{\alpha \in A} u^i(\alpha) > 0$ for all

Figure 5.1: A Preference Profile

R^1	R^2	R^3	R^4
a_1	a_4	a_3	a_2
a_2	a_1	a_4	a_3
a_2^*	a_1^*	a_4^*	a_3^*
a_1^*	c	a_3^*	a_2^*
c	a_4^*	c	b
b	b	b	a_4^*
a_3^*	a_2^*	a_1^*	a_4
a_3	a_2	a_1	c
a_4^*	a_3^*	a_2^*	a_1^*
a_4	a_3	a_2	a_1

$i \in N$, and let $V = V_{u^N}$. It may be verified that $\mathcal{MB}^*(N, V) = \emptyset$.

Let N be a finite nonempty set and let Γ denote the set of all superadditive zero-normalized NTU games (N, V).

The following lemma may be shown directly.

Lemma 5.2 *$\mathcal{M}B^*$ is an upper hemicontinuous correspondence*[2] *on* Γ.

With the help of Theorem 4 of Wooders (1983) it is possible to deduce the following desired result.

Theorem 5.3 *There exists a superadditive and non-levelled four-person game U such that $\mathcal{M}B(U) = \emptyset$.*

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[2] Here we identify any game $(N, V) \in \Gamma$ with V^+, defined by $V^+(S) = V(S) \cap \mathbb{R}^S_+$ for all $S \subseteq N$, $S \neq \emptyset$. The *distance* between two games V_1^+ and V_2^+ is the number $\delta(V_1^+, V_2^+) = \max_{\emptyset \neq S \subseteq N} d_S(V_1^+(S), V_2^+(S))$, where $d_S(\cdot, \cdot)$ is the Hausdorff distance between nonempty compact subsets of $\mathbb{R}^S$.

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Core-based solutions for assignment markets

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Abstract

This survey reviews some contributions to the literature that show that most cooperative solutions to the assignment market are determined by the core of the game. Different assignment markets with the same core have the same τ-value, the same kernel and the same nucleolus.

Key words: Assignment games, core, τ-value, kernel, nucleolus.

1 Introduction

In an assignment market, two disjoint sets of agents exist, let us say buyers and sellers, and one good is present in indivisible units. Each seller owns a unit of the indivisible good and each buyers needs exactly one unit. Differentiation in the units is allowed and therefore a buyer might place different valuations on the units of different sellers. When the difference between what an object on sale is worth to the buyer and the minimum that would be accepted by the seller is nonnegative, the trading between this pair of agents is possible and this difference is the joint profit that this mixed-pair will obtain if they trade. If the reservation price of a seller exceeds the worth that a buyer places in the object, no profit can be made by this mixed-pair of agents. In this way, a nonnegative assignment matrix is obtained with the profits of all pairings.

The assignment problem is then an operations research problem which looks for a matching from buyers to sellers that maximizes the total profit.

Under the assumption that side payments among agents are allowed, and identifying utility with money, Shapley and Shubik (1972) introduce a

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cooperative model for this two–sided market. A coalitional game is then defined where the characteristic function assigns to each coalition of agents the profit of an optimal matching in the corresponding submarket. They prove that the core of the assignment game is nonempty and, since it consists of the set of solutions to the dual assignment problem, it can be described just in terms of the assignment matrix. As a consequence of the lattice structure of the core of the assignment game, for each side of the market there is an extreme core allocation where all agents on this side simultaneously maximize their core payoff. Demange (1982) and Leonard (1983) show that, for an assignment market, the maximum core payoff of an agent is his or her marginal contribution to the grand coalition.

Several papers analyze the core of the assignment market. Balinsky and Gale (1987) give upper and lower bounds for the number of extreme core allocations, Hamers et al. (2002) show that every extreme core allocation is a marginal worth vector, although these are nonconvex games, and Núñez and Rafels (2003) characterize the set of extreme core allocations by means of the reduced marginal worth vectors.

Other works relate the core of the assignment game with other cooperative solutions such as the stable sets, the bargaining set, the kernel and the nucleolus. Solymosi and Raghavan (2001) determine when the core of an assignment game is a stable set, that is to say, a von Neumann and Morgenstern (1944) solution. To be more precise, the core is proved to be stable if and only if the minimum core payoff for each agent in the market is zero. Moreover, in Solymosi (1999) the core of the assignment game is proved to coincide with the bargaining set defined by Aumann and Maschler (1964). This implies the coincidence of the bargaining sets of any two assignment markets with the same core.

In Rochford (1984) some core allocations are selected by means of classical cooperative bargaining theory: if the optimal matching is assumed to be given exogenously, matched pairs engage in a pairwise bargaining process *à la* Nash which is solved symmetrically, after defining threats based on the outside opportunities given the current payoff to other pairs. In this way, a set of equilibria is defined (the symmetrically pairwise-bargained allocations or SPB allocations) which are stable under rebargaining. This set is proved to coincide with the intersection of the kernel (a well-known set solution for transferable utility cooperative games defined by Davis and Maschler, 1965) and the core of the assignment game. After Driessen (1998), we know that the kernel of an assignment game is a subset of the core and thus the set of SPB allocations coincides with the kernel of the assignment game. Moreover, Granot and Granot (1992) characterize those assignment markets where the kernel coincides with the whole core.

A well known single-valued core selection is the nucleolus (Schmeidler, 1969), and Solymosi and Raghavan (1994) provide an algorithm to compute the nucleolus of an assignment game. Another single-valued solution is the

τ-value (Tijs, 1981). For arbitrary cooperative games this solution may lie outside the core but in the case of the assignment game it turns out to be also a core selection (Núñez and Rafels, 2002a).

In this survey paper we review some of our contributions to this literature to emphasize the fact that most of the aforementioned solutions to the assignment game are determined by its core. This means that, as it happens with the bargaining set, different assignment markets with the same core also have the same τ-value, the same kernel and the same nucleolus.

The starting point is the characterization of the core of the assignment game given in Quint (1991): any "45^o lattice" can be associated with the core of an appropriately defined assignment game. However this assignment game might not be unique, and Quint asks for an assignment matrix with the entries "as high as possible". After presenting the assignment model in Section 2, in Section 3 we will define a representative matrix with the entries as high as possible among those defining games with the same core. In Section 4 and Section 5 we show that the τ-value, the kernel or the nucleolus of an assignment market ar those of the aformentioned representative.

2 The assignment model

Let $M = \{1, 2, \ldots, m\}$ be a set of buyers, $M' = \{1, 2, \ldots, m'\}$ a set of sellers and let $A = (a_{ij})_{(i,j)\in M\times M}$ be a nonnegative matrix where a_{ij} represents the profit obtained by the mixed–pair (i, j) if they trade. Let $n = m + m'$ denote the cardinality of $M \cup M'$. The assignment problem (M, M', A) consists in looking for an optimal matching between the two sides of the market. A matching for A is a subset μ of $M \times M'$ such that each $k \in M \cup M'$ belongs at most to one pair in μ. We will denote the set of matchings of A by $\mathcal{M}(A)$ or $\mathcal{M}(M, M')$. We say a matching μ is optimal if for all $\mu' \in \mathcal{M}(M, M')$, $\sum_{(i,j')\in\mu} a_{ij'} \geq \sum_{(i,j')\in\mu'} a_{ij'}$, and will denote the set of optimal matchings by $\mathcal{M}^*(A)$.

Assignment games were introduced by Shapley and Shubik (1972) as a cooperative model for a two–sided market with transferable utility. Given an assignment problem (M, M', A), the player set is $M \cup M'$, and the matrix A determines the characteristic function w_A. Given $S \subseteq M$ and $T \subseteq M'$, $w_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S, T)\}$, $\mathcal{M}(S, T)$ being the set of matchings between S and T. It will be assumed as usual that a coalition formed only by sellers or only by buyers has worth zero. For all $i \in M$ optimally matched by μ, we will denote by $\mu(i)$ the agent $j \in M'$ such that $(i, j) \in \mu$. Similarly, i could be denoted by $\mu^{-1}(j)$. Moreover, we say a buyer $i \in M$ is not assigned by μ if $(i, j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

Shapley and Shubik proved that the core of the assignment game $(M \cup M', w_A)$ is nonempty and can be represented in terms of any optimal matching μ of $M \cup M'$ by

$$C(w_A) = \left\{ (u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'} \;\middle|\; \begin{array}{l} u_i \geq 0, \text{ for all } i \in M\,;\; v_j \geq 0, \text{ for all } j \in M' \\ u_i + v_j = a_{ij} \text{ if } (i,j) \in \mu \\ u_i + v_j \geq a_{ij} \text{ if } (i,j) \notin \mu \\ u_i = 0 \text{ if } i \text{ not assigned by } \mu \\ v_j = 0 \text{ if } j \text{ not assigned by } \mu\,. \end{array} \right\} \tag{1}$$

Moreover, the core has a lattice structure with two special extreme points: the buyers–optimal core allocation, $(\overline{u}, \underline{v})$, where each buyer attains his maximum core payoff, and the sellers–optimal core allocation, $(\underline{u}, \overline{v})$, where each seller does.

¿From Demange (1982) and Leonard (1983) we know that the maximum core payoff of any player coincides with his marginal contribution:

$$\overline{u}_i = w_A(M \cup M') - w_A((M \cup M') \setminus \{i\}) \text{ and } \overline{v}_j = w_A(M \cup M') - w_A((M \cup M') \setminus \{j\})\,. \tag{2}$$

From (2) and the description of the core (1) the minimum core payoff of buyer i is

$$\underline{u}_i = a_{i\mu(i)} - w_A(M \cup M') + w_A((M \cup M') \setminus \{\mu(i)\}) \text{ for all } \mu \in \mathcal{M}^*(A)\,, \tag{3}$$

while the minimum core payoff of seller j is

$$\underline{v}_j = a_{\mu^{-1}(j)j} - w_A(M \cup M') + w_A((M \cup M') \setminus \{\mu^{-1}(j)\}) \text{ for all } \mu \in \mathcal{M}^*(A)\,. \tag{4}$$

Example 1 (Shapley and Shubik, 1972) *Let $M = \{1,2,3\}$ be the set of buyers, $M' = \{1',2',3'\}$ be the set of sellers and let the assignment matrix A be*

	1'	2'	3'
1	5	8	2
2	7	9	6
3	2	3	0

Notice there exists only one optimal matching $\mu = \{(1,2'),(2,3'),(3,1')\}$. Following Shapley and Shubik, to describe the core of the assignment game you do not need to compute the complete characteristic function since only the mixed-pair coalitions are relevant. Thus, the core of this game is:

$$C(w_A) = \left\{ (u,v) \in \mathbb{R}^6_+ \;\middle|\; \begin{array}{lll} u_1 + v_{1'} \geq 5, & u_1 + v_{2'} = 8, & u_1 + v_{3'} \geq 2, \\ u_2 + v_{1'} \geq 7, & u_2 + v_{2'} \geq 9, & u_2 + v_{3'} = 6, \\ u_3 + v_{1'} = 2, & u_3 + v_{2'} \geq 3, & u_3 + v_{3'} \geq 0\,. \end{array} \right\}$$

This set is the convex hull of its extreme points: (3,5,0;2,5,1), (3,6,0;2,5,0), (4,6,1;1,4,0), (5,6,1;1,3,0), (5,6,0;2,3,0) and (4,5,0;2,4,1). Although this core is a subset of $\mathbb{R}^6$, taking into account the equality constraints $u_1 + v_{2'} = 8$, $u_2 + v_{3'} = 6$, $u_3 + v_{1'} = 2$, the core is completely determined by its projection to the space of payoffs to the buyers and this projection is depicted in figure 1.

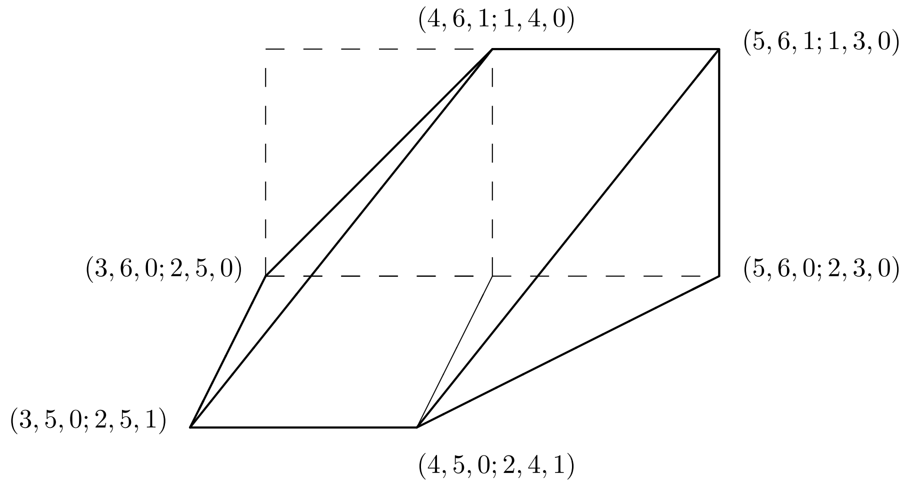

Figure 1: The core of the assignment game in Example 1

Among these extreme points we point out the buyers-optimal core allocation, $(\overline{u}, \underline{v}) = (5, 6, 1; 1, 3, 0)$, and the sellers-optimal core allocation, $(\underline{u}, \overline{v}) = (3, 5, 0; 2, 5, 1)$, which are the two more distant extreme points.

3 Buyer-seller exactness

Notice that in the above example $u_1 + v_{3'} \geq 3 > 2 = a_{13'}$ for all $(u, v) \in C(w_A)$. As a consequence, if we raise $a_{13'}$ in one unit, the resulting assignment game will have the same core. In fact, all matrices

$$A(\alpha) = \begin{pmatrix} 5 & 8 & \alpha \\ 7 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix}$$

with $0 \leq \alpha \leq 3$ define assignment games with the same core as $(M \cup M', w_A)$.

In Núñez and Rafels (2002b) an assignment game is defined to be buyer-seller exact if no matrix entry can be raised without modifying the core of the game.

Definition 2 *An assignment game $(M \cup M', w_A)$ is* **buyer-seller exact** *if and only if for all $i \in M$ and all $j \in M'$ there exists $(u, v) \in C(w_A)$ such that $u_i + v_j = a_{ij}$.*

It is then proved that for all assignment game $(M\cup M', w_A)$ there exists a unique buyer-seller assignment game with its same core. This is denoted by $(M\cup M', w_{A^r})$ and is the **buyer-seller exact representative** of the initial game. Notice that A^r is maximal among all matrices defining assignment games with the same core as $(M\cup M', w_A)$. Moreover, both matrices have at least one optimal matching in common.

Of course, given the core of the game, the buyer-seller exact representative is easily computed, as we have done in our example. But, how to obtain A^r directly from the assignment problem?

To answer this question we analyze how the payoff to a mixed-pair coalition is bounded in the core. From now on, without loss of generality we will assume that A is square by adding null rows or columns, and we will denote the j-th seller by j' to distinguish it from the j-th buyer. Then, an optimal matching can be assumed to be placed in the diagonal: $\mu = \{(i,i') \mid i \in M\} \in \mathcal{M}^*(A)$.

We define, for all $(i,j') \in M \times M'$, $K_{ij'}$ to be the upper core bound for the mixed-pair coalition $\{i,j'\}$, $K_{ij'} = \max_{(u,v)\in C(w_A)} u_i + v_{j'}$, and $k_{ij'}$ the lower core bound for the same coalition, $k_{ij'} = \min_{(u,v)\in C(w_A)} u_i + v_{j'}$. These bounds can be expressed in terms of the characteristic function.

Proposition 3 *Let $(M\cup M', w_A)$ be an assignment game. Then, $K_{ij'} = w_A(M\cup M') - w_A(M\cup M' \setminus \{i,j'\})$ and $k_{ij'} = a_{ii'} + a_{jj'} + w_A(M\cup M' \setminus \{j,i'\}) - w_A(M\cup M')$.*

This shows that, as it happens with the marginal contributions of one-player coalitions, all marginal contributions of mixed pair coalitions are attained in the core of the assignment game.

Then $(M\cup M', w_A)$ is buyer–seller exact if and only if $k_{ij'} = a_{ij'}$ for all $(i,j') \in M \times M'$.

All the same, we would like to be able to determine, just in terms of the matrix entries, whether an assignment game $(M\cup M', w_A)$ is buyer-seller exact and, if it is not, to compute the buyer-seller exact representative $(M\cup M', w_{A^r})$.

To this end, we recall a definition due to Solymosi and Raghavan (2001). An assignment game is **doubly dominant diagonal** if and only if $a_{ij'} + a_{kk'} \geq a_{ik'} + a_{kj'}$ for all $i,j,k \in M$ and different. This property characterizes those matrices with the property of buyer-seller exactness.

Theorem 4 (Núñez and Rafels, 2002b) *An assignment game $(M\cup M', w_A)$ is buyer–seller exact if and only if A is doubly dominant diagonal.*

Moreover, the representative matrix A^r can be computed from matrix A in the following way: $a^r_{ij'} = \max\{a_{ij'}, \tilde{a}_{ij'}\}$, where, for all $(i,j') \in M \times M'$,

$$\tilde{a}_{ij'} = \max_{\substack{k_1,k_2,\ldots,k_r \in M\setminus\{i,j\} \\ different}} \{a_{ik'_1} + a_{k_1k'_2} + \cdots + a_{k_rj'} - (a_{k_1k'_1} + \cdots + a_{k_rk'_r})\}. \quad (5)$$

To conclude this section, notice that, from the definition of the buyer-seller representative matrix A^r, two assignment games with the same core have the same buyer–seller exact representative. In some sense, the coincidence of the cores provides a classification of assignment games. This fact makes us question how the main cooperative solutions behave with respect to this classification.

4 The kernel and the τ-value

In the case of the assignment game, it is quite straightforward to realize that these two cooperative solutions are completely determined by the core of the game.

The τ-value is a single-valued solution for coalitional games that was introduced by Tijs (1981) as a compromise value between a utopia vector and a minimal rights vector. In some games the τ-value does not lie in the core. This is not the case for the assignment game, since in Núñez and Rafels (2002a) its τ-value is proved to coincide with the midpoint of the segment determined by the buyers-optimal and the sellers-optimal core allocations:

$$\tau(w_A) = \frac{1}{2}(\overline{u}, \underline{v}) + \frac{1}{2}(\underline{u}, \overline{v}) \in C(w_A)\,.$$

Thus, two assignment games with the same core have the same τ-value. In Example 1, $\tau(w_{A(\alpha)}) = (4, 5.5, 0.5; 1.5, 4, 0.5)$ for all $0 \le \alpha \le 3$.

The case of the kernel is similar. Let us denote by $I(v)$ the set of imputations (efficient allocations that are individually rational) of a game (N, v). For zero–monotonic games ($v(S) \ge v(T) + \sum_{i \in S\setminus T} v(i)$, for all $T \subseteq S$), as it is the case of assignment games, the **kernel** is given by

$$\mathcal{K}(v) = \{z \in I(v) \mid s^v_{ij}(z) = s^v_{ji}(z),\ \forall i, j \in N,\ i \ne j, \}$$

where the maximum surplus $s^v_{ij}(z)$ of player i over another player j with respect to the allocation $z \in \mathbf{R}^N$ is defined by

$$s^v_{ij}(z) = \max\{v(S) - z(S) \mid S \subseteq N\,,\ i \in S\,,\ j \notin S\}\,.$$

It is known from Maschler, Peleg and Shapley (1979) that given two coalitional games with the same core, the intersections of the kernel and the core also coincide. As a consequence, if $(M \cup M', w_A)$ and $(M \cup M', w_B)$ are two assignment games with the same core, then $\mathcal{K}(w_A) \cap C(w_A) = \mathcal{K}(w_B) \cap C(w_B)$. Since the kernel of an assignment game is always included in the core (Driessen, 1998), the above equality is equivalent to $\mathcal{K}(w_A) = \mathcal{K}(w_B)$.

As it is shown in Núñez (2004), the kernel of Example 1 reduces to only one point, thus being the nucleolus.

5 All assignment games with the same core have the same nucleolus

The nucleolus is a single-valued solution for coalitional games that was introduced by Schmeidler (1969) as the imputation that lexicographically minimizes the vector formed by the excesses of all nontrivial coalitions in non-increasing order. This minimum always exists and reduces to only one point. Moreover, if the core is nonempty, the nucleolus lies in the core.

It is known from Huberman (1980) that only essential coalitions are to be considered in the computation of the nucleolus of a coalitional game. From his definition, it is easy to check that, for assignment games, only one-player coalitions and mixed-pair coalitions are essential.

Given an assignment game $(M \cup M', w_A)$, for all $x \in I(w_A)$ and all $S \subseteq M \cup M'$, the *excess of coalition* S at x is

$$e(S,x) = v(S) - \sum_{i \in S} x_i = v(S) - x(S).$$

Then, for all $x \in I(w_A)$ define the *vector* $\theta(x) \in \mathbf{R}^{m \times m' + m + m'}$ *of excesses* of all non-trivial essential coalitions at x in non-increasing order: $\theta(x)_k = e(S_k, x)$, where $e(S_k, x) \geq e(S_{k+1}, x)$ and S_k is either a one-player coalition or a mixed-pair coalition.

The **nucleolus** of $(M \cup M', w_A)$ is the imputation $\nu(w_A)$ that minimizes $\theta(x)$, with respect to the lexicographic order, over the set of imputations:

$$\theta(\nu(v)) \leq_{Lex} \theta(x)\,, \quad \text{for all } x \in I(w_A)\,.$$

Solymosi and Raghavan (1994) adapt the definition of lexicographic center due to Maschler, Peleg and Shapley (1979) to the case of the assignment game. With some small changes, their definition of lexicographic center of an assignment game is used in Núñez (2004).

Given $(M \cup M', w_A)$, we take an optimal matching $\mu \in \mathcal{M}^*(A)$, and consider the set of coalitions

$$\mathcal{P} = \{\{k\} \mid k \in M \cup M'\} \cup \{\{i, j'\} \mid i \in M\,,\ j' \in M'\}\,.$$

We iteratively construct
• $\Sigma^0 \supseteq \Sigma^1 \supseteq \cdots \supseteq \Sigma^{s+1}$ and $\Delta^0 \subseteq \Delta^1 \subseteq \cdots \subseteq \Delta^{s+1}$ sets of coalitions in $\mathcal{P}$ such that for all $0 \leq r \leq s+1$, (Δ^r, Σ^r) is a partition of $\mathcal{P}$, and
• $X^0 \supseteq X^1 \supseteq \cdots \supseteq X^{s+1}$ a sequence of payoff sets, such that:

Initially $\Delta^0 = \{\{i, j'\} \mid (i, j') \in \mu\} \cup \{\{k\} \mid k \in M \cup M' \text{ not matched by } \mu\}$, $\Sigma^0 = \mathcal{P} \setminus \Delta^0$, and $X^0 = C(w_A)$.

For $r \in \{0, 1, \ldots, s\}$ define recursively

1. $\alpha^{r+1} = \min_{(u,v) \in X^r} \max_{S \in \Sigma^r} e(S, (u, v))$

2. $X^{r+1} = \{(u,v) \in X^r \mid \max_{S\in\Sigma^r} e(S,(u,v)) = \alpha^{r+1}\}$

3. $\Sigma_{r+1} = \{S \in \Sigma^r \mid e(S,(u,v))$ is constant on $X^{r+1}\}$

4. $\Sigma^{r+1} = \Sigma^r \setminus \Sigma_{r+1}$, $\Delta^{r+1} = \Delta^r \cup \Sigma_{r+1}$

where s is the last index for which $\Sigma^r \neq \emptyset$. The set X^{s+1} is the **lexicographic center** of $(M \cup M', w_A)$.

It is then proved that the lexicographic center is well defined and it reduces to only one point which coincides with the nucleolus.

In our Example 1, the process begins with a linear program with 6 variables and 15 constraints, and you obtain:

$$\nu(w_A) = (4,\ 5.667,\ 0.333;\ 1.667,\ 4,\ 0.333)\,.$$

In Maschler, Peleg and Shapley (1979) an example is given of two cooperative games with the same core but different nucleoli. Next theorem shows that this cannot happen with two assignment games.

Theorem 5 (Núñez, 2004) *Let $(M \cup M', w_A)$ be an assignment game with the same number of agents on each side of the market and $(M \cup M', w_{A^r})$ its buyer-seller exact representative. Then,*

$$\nu(w_A) = \nu(w_{A^r})\,.$$

To prove that, take $(\Delta^0, \Sigma^0), \ldots, (\Delta^{s+1}, \Sigma^{s+1})$ and $X^0, \ldots, X^{s+1}$ the partitions and payoff sets in the definition of lexicographic center of $(M \cup M', w_A)$, and consider also $(\tilde{\Delta}^0, \tilde{\Sigma}^0), \ldots, (\tilde{\Delta}^{s'+1}, \tilde{\Sigma}^{s'+1})$ and $\tilde{X}^0, \ldots, \tilde{X}^{s'+1}$ the partitions and payoff sets in the definition of lexicographic center of $(M \cup M', w_{A^r})$. Then it can be proved (by induction on r) that for all $0 \leq r \leq s+1$, $\Delta^r = \tilde{\Delta}^r$, $\Sigma^r = \tilde{\Sigma}^r$ and $X^r = \tilde{X}^r$, and consequently $\nu(w_A) = \nu(w_{A^r})$.

As a consequence of the above theorem, if $(M \cup M', w_A)$ and $(M \cup M', w_B)$ are two assignment games with the same core and the same number of agents on each side of the market, then $\nu(w_A) = \nu(w_{A^r}) = \nu(w_{B^r}) = \nu(w_B)$. The result is easily extended to assignment games where one side of the market has more agents than the opposite side.

Thus, all the markets $(M \cup M', w_{A_\alpha})$, with $\alpha \in [0,3]$, in our previous example have the same nucleolus:

$$\nu(w_{A(\alpha)}) = (4,\ 5.667,\ 0.333;\ 1.667,\ 4,\ 0.333)\,.$$

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Convex Geometry and Bargaining

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Abstract

We discuss the construction of a superadditive bargaining solution in the spirit of Maschler–Perles. The family of polyhedra admitting such a solution is provided by the "cephoids", i.e., finite sums of "prisms". The geometrical shape of these polyhedra, the partially ordered set of their maximal faces, and the combinatorial structure describing this poset provide the foundation for the construction of a bargaining solution.

1 Superadditivity of Solutions

The Shapley value ([15]) is a mapping defined on TU games (on real valued set functions) with values in Euclidean space (distributions of wealth) respecting anonymity, Pareto efficiency, and a dummy property. The solution concept is uniquely defined by the additional requirement that it should be *additive*. More precisely, if $\boldsymbol{I}$ is a finite set and $\mathcal{P} = \mathcal{P}(\boldsymbol{I})$ the power set (the set of coalitions), then a function $\boldsymbol{v} : \mathcal{P} \to \mathbb{R}$ satisfaying $\boldsymbol{v}(\emptyset) = 0$ is a ***game***. A mapping $\Phi : \{\boldsymbol{v} \,|\, \boldsymbol{v} \text{ is a game }\} \to \mathbb{R}^{\{\boldsymbol{I}\}}$ is additive if, for any two games $\boldsymbol{v}, \boldsymbol{w}$ it satisfies

$$\Phi(\boldsymbol{v}) + \Phi(\boldsymbol{w}) = \Phi(\boldsymbol{v} + \boldsymbol{w}) .$$

The SHAPLEY VALUE for a game $\boldsymbol{v}$ is given by

$$\Phi_i(\boldsymbol{v}) := \sum_{S \in \mathcal{P}} \frac{(s-1)!(n-s)!}{n!} (v(S) \;-\; v(S \setminus \{i\})).$$

It can be computed in various other ways and enjoys a host of nice properties; the amount of work dealing with this concept is huge.

Additivity of a solution concept is traditionally justified by risk neutrality. Players facing a lottery of two games do not distinguish between the value of the expected game and the expected value of the games. Yet, one can as well think that players act independently in two remote games and evaluate the expectations either in both games independently or else by establishing a package deal.

If side payments are not permitted, than additivity of a solution concept cannot be obtained. Yet, the sum of two games (or in particular of two bargaining problems) is well defined. This is the algebraic sum which is obtained by summing all pairs of utility vectors available to a coalition in the two games.

One can at best hope for superadditivity. This means that by striking a package deal both players improve their situation with regard to the solution concept. Or else, when facing a lottery of two games, it is an advantage for both players to contract *ex ante* and evaluate the expected game according to the solution concept.

We shall first of all necessarily restrict the discussion on bargaining problems. The NASH bargaining solution [8] is not superadditive as is seen by simple examples for two persons. The MASCHLER–PERLES solution [7] on the other hand is uniquely defined by superadditivity – given that one takes the usual requirements regarding symmetry etc. for granted. The serious problem is that the MASCHLER–PERLES solution exists for two person games only. In fact, PERLES [12] proved in 1982 that a superadditive solution generally does not exist for three and more persons.

Yet, when generalizing the SHAPLEY value to the NTU context the general approach assumes coincidence with the Nash solution for the n person bargaining problem (see SHAPLEY [16] AUMANN [1], HARSHANYI [4], [5], KERN [6]). This idea is justified on the overwhelming acceptance of the NASH–solution. It may also recognize the fact that there is no superadditive solution in higher dimensions and *a fortiori* there will be no superadditive SHAPLEY VALUE. Yet, one might argue that this is not a particulaly consistent approach.

We establish a solution concept that is superadditive on a certain sub–class of games. As it turns out to establish this class requires quite an effort. The nature of sums of convex polyhedra is discussed within the framework of convex geometry,this task may, depending on the context, in itself be quite involved. The particular class we will exhibit is the family of "cephoids" which are sums of hyperplane games (bargaining situations) with variing normal vector.

Let us recall some basic facts. The MASCHLER–PERLES bargaining solution (MASCHLER–PERLES [7], see also [13] for a textbook presentation) is a mapping defined on 2–dimensional bargaining problems respecting anonymity, Pareto efficiency, and affine transformations of utility. Moreover, this mapping is *superadditive* by which property it is uniquely characterized. We want to be more precise.

A ***bargaining problem*** is a pair $(\mathbf{0}, U)$ with a compact, convex, and comprehensive subset $\emptyset \neq U \subseteq \mathbb{R}^2_+$. $\mathbf{0} \in U$ is the ***status quo point*** and U the ***feasible set***. Players may reach agreement on some feasible utility vector. Or else end up at the *status quo point*, which we assume to be $\mathbf{0}$. It suffices to mention U.

A ***solution*** is a mapping $\boldsymbol{\varphi}$ that, based on some axiomatic justification, assignes to each bargaining problem U a Pareto efficient vector $\boldsymbol{\varphi}(U)$.

Suppose two players are engaged in two "remote" problems U and U' simultaneously. In the beginning, they considered these to be different affairs, thus wanted to settle for $\boldsymbol{\varphi}(U)$ and $\boldsymbol{\varphi}(U')$ separately. Later on they realized that one should consider giving in with respect to one contract in favor of receiving concessions with respect to the other one. That is, they decided to consider this to be one problem. The utilities available are now $\{x + x' | x \in U, x' \in U'\} =: U + U'$. If a solution is superadditive, i.e., satisfies $\boldsymbol{\varphi}(U + U') \geq \boldsymbol{\varphi}(U) + \boldsymbol{\varphi}(U')$, it turnes out that both players profit from a *quid quo pro*.

Od course the interpretation that players face a lottery involving two bargaining problems applies as well. Superaddivity is then seen to consistently favor contracting *ex ante*, thereby increasing expected utility (see [7] or [13], *p.562*, for a detailed discussion).

We focus on *polyhedral* bargaining problems. A bargaining problem (in $\mathbb{R}^2$) is ***polyhedral*** if the Pareto surface consists of line segments only. The MASCHLER–PERLES solution $\boldsymbol{\mu}$ is based on the observation that every polyhedral bargaining solution in $\mathbb{R}^2$ is an sum of "elementary" bargaining problems that are generated by a line segment (thus reflect constant transfer of utility). More precisely, let $\boldsymbol{a} = (a_1, a_2) > 0 \in \mathbb{R}^n$. We introduce the unit vectors $\boldsymbol{e}^i$, the vectors $\boldsymbol{a}^i := a_i\boldsymbol{e}^i$ $(i \in \boldsymbol{I})$, and associate with $\boldsymbol{a}$ the ***triangle***

$$\Pi^{\boldsymbol{a}} := \operatorname{convH}\left(\left\{\mathbf{0}, \boldsymbol{a}^1, \boldsymbol{a}^2, \right\}\right). \tag{1.1}$$

The Pareto curve of this triangle is the ***line segment*** $\Delta^{\boldsymbol{a}}$ which is given by

$$\Delta^{\boldsymbol{a}} := \operatorname{convH}\left(\left\{\boldsymbol{a}^1, \boldsymbol{a}^2\right\}\right). \tag{1.2}$$

Now, a bargaining problem is seen to be polyhedral if and only if the feasible set is given by

$$\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}} \tag{1.3}$$

with a suitable family of (positive) vectors

$$\left(\boldsymbol{a}^{(k)}\right)_{k \in \boldsymbol{K}}, \quad \boldsymbol{K} := \{1, \ldots, K\}$$

To any triangle $\Pi^{\boldsymbol{a}}$ we associate the ***volume*** $V(\boldsymbol{a}) := \frac{1}{2}a_1a_2 =$ area $(\Pi^{\boldsymbol{a}})$.

Consider the case that all triangles involved in a representation (1.3) have equal volume. The bargaining problems having this property form a dense subset of the set of all bargaining problems (employing the Hausdorff metric). Similarly, whenever we deal with the sum of two bargaining problems, we may assume that the summands as well as the sum are dyadic with the same basis.

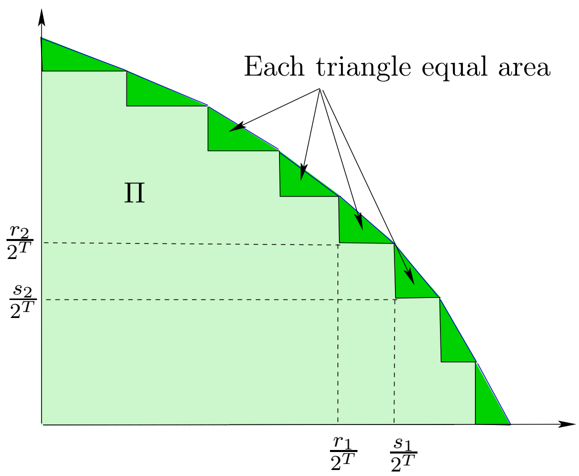

Figure 1.1: A standard dyadic bargaining problem

Definition 1.1. *A bargaining problem* ***standard dyadic*** *if the feasible set is a polyhedron represented as in* (1.3) *with dyadic vectors all generating equal volume.*

We assume the enumeration of the triangles to be such that the *tangents* (i.e., the quotients $\frac{a_2^{(k)}}{a_1^{(k)}}$) are *decreasing* with the index k. The MASCHLER–PERLES solution for a standard dyadic bargaining problem is then defined inductively as follows: For $K = 1$ it is the midpoint of the line segment (the Pareto curve). For $K = 2$ (and assuming that the two triangles are not homothetic) it is the unique vertex of $\Pi = \Pi^{(1)} + \Pi^{(2)}$. For $K \geq 3$ it is defined by the *recursive formula*

$$\begin{aligned} \boldsymbol{\mu}(\Pi) &= \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{K}} \Pi^{a^{(k)}}\right) \\ &:= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)}\right) + \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{K}-\{1,K\}} \Pi^{a^{(k)}}\right). \end{aligned} \tag{1.4}$$

This formula in fact implies uniquenes of the solution on standard dyadic bargaining problems. For, every superadditive solution $\boldsymbol{\mu}$ is necessarily additive whenever the solutions of the two summands admit of a joint normal.

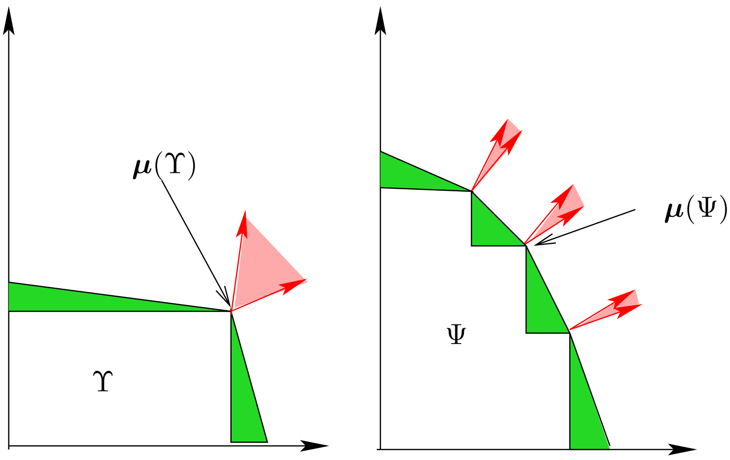

Figure 1.2: Additivity of the solution

To see this more clearly, consider Figure 1.2. Note that the sum of two Pareto efficient vectors is Pareto efficient if and only if both admit of a joint normal (equivalently: a joint tangency). In Figure 1.2, the vertex of Υ admits of a joint normal with *each* Pareto efficient point of Ψ (some normal cones are indicated). If the volumes of the two triangles involved in Υ are equal, then the solution yields this cornerpoint, denoted by $\boldsymbol{\mu}(\Upsilon)$, hence $\boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi)$ is Pareto efficient. As the solution is superadditive, we must necessarily have $\boldsymbol{\mu}(\Upsilon + \Psi) = \boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi)$.

Given our enumeration, the first polyhedron $\Pi^{(1)} + \Pi^{(K)}$ plays the role of Υ, hence its solution admits of a joint normal with *every* Pareto efficient point of the second polyhedron.

We present two stories concerning the supperadditive solution. The first one is due to MASCHLER–PERLES: two travelers (it may help to think of donkey carts) move along the Pareto curve in a way that the product of the speed components in axis directions is equal at any moment.When they start out at the extremal points of the Pareto surface, then they meet eventually at the solution point.

The second story is a reframed version of the above one: instead of traveling along the Pareto surface with a certain speed, we may define a *surface measure* on it such the density (w.r. to the local Lebesgue mesure) correponds to the travelling speed as mentioned above.

To every line segment $\Delta^{(k)}$ (and every translate of such line segment) we assign a length measure $\boldsymbol{\iota}_\Delta$ (i.e. a dilated version of Lebesgue measure) by setting

$$\boldsymbol{\iota}_\Delta(\Delta^{(k)}) \;=\; \sqrt{a_1^{(k)} a_2^{(k)}} \;:=\; \alpha_k. \tag{1.5}$$

Then, for every $k \in \boldsymbol{K}$ we take a corresponding multiple of the unit simplex, i.e., we put

$$\widehat{\Pi}^{(k)} \;:=\; \sqrt{\alpha_k}\Pi^{\boldsymbol{e}} \tag{1.6}$$

such that the surface has length $\lambda(\widehat{\Delta}^{(k)}) = \alpha_k$. Summing these we obtain a multiple of the unit simplex $\widehat{\Pi} = \sum_{k\in\boldsymbol{K}} \widehat{\Pi}^{(k)}$ the surface $\widehat{\Delta}$ of which has length

$$\lambda(\widehat{\Delta}) = \sum_{k\in\boldsymbol{K}} \alpha_k \;, \tag{1.7}$$

which is the same as the total length of the Pareto curve of Π in terms of the surface measure.

Therefore, we now construct a bijective and "locally" affine linear mapping, say $\boldsymbol{\kappa} \;:\; \Delta \to \widehat{\Delta}$ by mapping the translates of the various $\Delta^{(k)}$ on copies of the $\widehat{\Delta}^{(k)}$ in the order dictated by the slopes $|\frac{a_2^{(k)}}{a_1^{(k)}}|$. That is, if $\Delta^{(1)}$ has the smallest slope $|\frac{a_2^{(1)}}{a_1^{(2)}}|$, then a copy of $\Delta^{(1)}$ is the line segment in the uppermost left corner of Δ and this is mapped on a line segment of length α_1 in the uppermost left corner of $\widehat{\Delta}$ etc. Figure 1.3 indicates the procedure. Now, if $\bar{\boldsymbol{x}}$ is the midpoint of $\widehat{\Delta}$, then the MASCHLER–PERLES solution is given by

$$\boldsymbol{\mu}(\Pi) \;=\; \boldsymbol{\kappa}^{-1}(\bar{\boldsymbol{x}}). \tag{1.8}$$

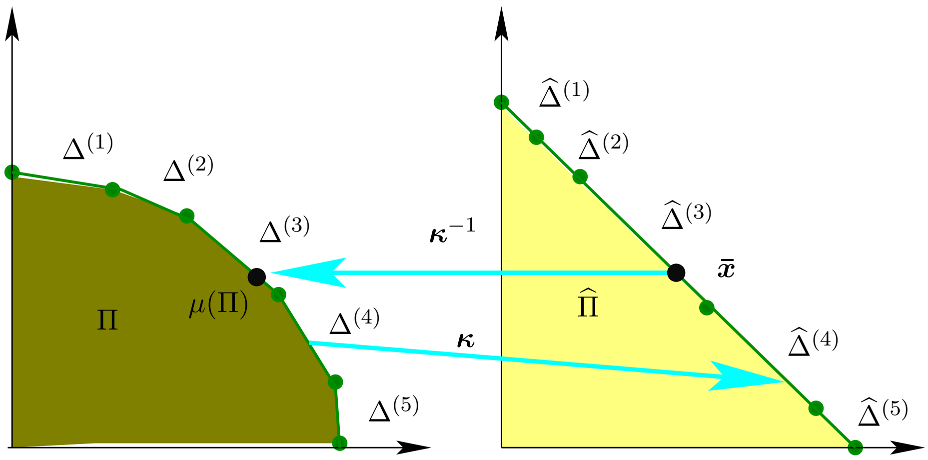

Figure 1.3: Mapping Δ on $\widehat{\Delta}$

2 Sums of Prisms

In order to generalize the two dimensional results, we consider the class of polytopes in $\mathbb{R}^n_+$ that are sums of prisms. It is our aim to exhibit the structure of the surface of these polyhedra and, based on this structure, to define a surface measure that resembles the one presented earlier. Given a sum of prisms, there appears a the shape of a cephalopod on the surface. Therefore, we call the polytopes of our family "cephoids". A cephoid is formally describes as follows.

We denote by $\boldsymbol{I} := \{1, \ldots, n\}$ the set of coordinates of $\mathbb{R}^n$ and by $\boldsymbol{e}^i$ the i^{th} unit vector of $\mathbb{R}^n$ $(i \in \boldsymbol{I})$. Also write $\boldsymbol{e} = (1, \ldots, 1)$. Let $\boldsymbol{a} = (a_1, \ldots, a_n) > 0 \in \mathbb{R}^n_+$. Put $\boldsymbol{a}^i := a_i \boldsymbol{e}^i$ $(i \in \boldsymbol{I})$ and associate with $\boldsymbol{a}$ the ***prism*** $\Pi^{\boldsymbol{a}}$ which is given by

$$\Pi^{\boldsymbol{a}} := \operatorname{conv}\left\{\mathbf{0}, \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right\}. \tag{2.1}$$

The (outward) face of this prism is the ***simplex*** $\Delta^{\boldsymbol{a}}$ which is given by

$$\Delta^{\boldsymbol{a}} := \operatorname{conv}\left\{\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right\}. \tag{2.2}$$

For any $\boldsymbol{J} \subseteq \boldsymbol{I}$ we write $\mathbb{R}^n_{\boldsymbol{J}} = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, x_i = 0 \ (i \notin \boldsymbol{J})\}$. Accordingly, we obtain the ***subprism*** of $\Pi^{\boldsymbol{a}}$ given by

$$\Pi^{\boldsymbol{a}}_{\boldsymbol{J}} := \{x \in \Pi^{\boldsymbol{a}} \,|\, x_i = 0 \ (i \notin \boldsymbol{J})\} = \Pi^{\boldsymbol{a}} \cap \mathbb{R}^n_{\boldsymbol{J}}, \tag{2.3}$$

a similar notation is used for the simplex, $\Delta^{\boldsymbol{a}}$ we write for the subface generated by the coordinates $i \in \boldsymbol{J}$

$$\Delta^{\boldsymbol{a}}_{\boldsymbol{J}} := \{x \in \Delta^{\boldsymbol{a}} \,|\, x_i = 0 \ (i \notin \boldsymbol{J})\} = \Delta^{\boldsymbol{a}} \cap \mathbb{R}^n_{\boldsymbol{J}}. \tag{2.4}$$

Now we consider the Minkowski sum of prisms.

Definition 2.1. *Let* $\boldsymbol{a}^{\bullet} := (\boldsymbol{a}^{(k)})_{k=1}^{K}$ *denote a family of positive vectors and let*

$$\Pi = \sum_{k=1}^{K} \Pi^{\boldsymbol{a}^{(k)}} \tag{2.5}$$

be the (algebraic) sum. Then Π *is called a* ***cephoid****.*

Note that the representation of a cephoid by a family of prisms is in general not unique. Some few examples may serve to motivate a condition that ensures uniqueness.

Example 2.2. *A prism may be represented as a cephoid in various ways. E.g., let* $\Pi = \Pi^{\boldsymbol{e}}$ *be the unit prism and let* Π *be represented as the sum* $\Pi = \Pi^{\alpha\boldsymbol{e}} + \Pi^{\beta\boldsymbol{e}}$ *with* $\alpha, \beta \geq 0, \quad \alpha + \beta = 1$*. The outer surface, i.e., the unit simplex* $\Delta = \Delta^{\boldsymbol{e}} = \Delta^{\alpha\boldsymbol{e}} + \Delta^{\beta\boldsymbol{e}}$ *is the union of the two translates* $\alpha\boldsymbol{e}^1 + \Delta^{\beta\boldsymbol{e}}$*,* $\beta\boldsymbol{e}^2 + \Delta^{\alpha\boldsymbol{e}}$ *and a "diamond"* $\Delta_{13}^{\alpha\boldsymbol{e}} + \Delta_{23}^{\beta\boldsymbol{e}}$*. (cf. Figure 2.1)*

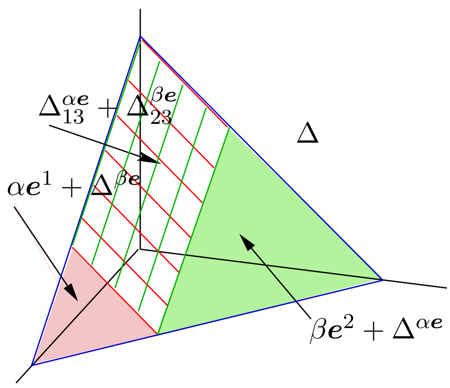

Figure 2.1: The unit simplex as a cephoid

However, the representation is not unique. As all prisms involved are homothetic, the vector used to translate a prism is rather arbitrary. Now consider two nonhomothetic prisms. The sum is indicated in Figure 2.2. Again there are the translates of the two prisms involved, i.e. $\Delta^{\boldsymbol{a}} + \boldsymbol{b}^1$ *and* $\Delta^{\boldsymbol{b}} + \boldsymbol{a}^1$*. The "diamond" is the sum* $\Delta_{23}^{\boldsymbol{a}} + \Delta_{13}^{\boldsymbol{b}}$*.*

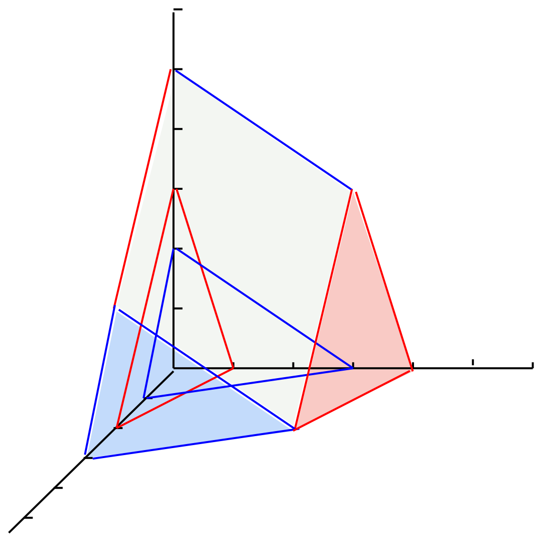

Figure 2.2: Adding two non–homothetic prisms

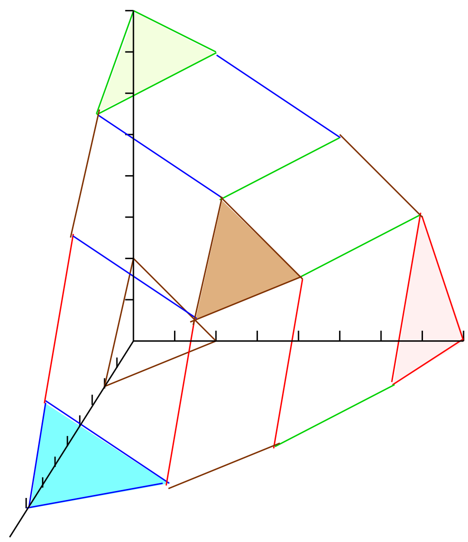

Figure 2.3: A sum of four prisms

The sum of four prisms is depicted in Figure 2.3.

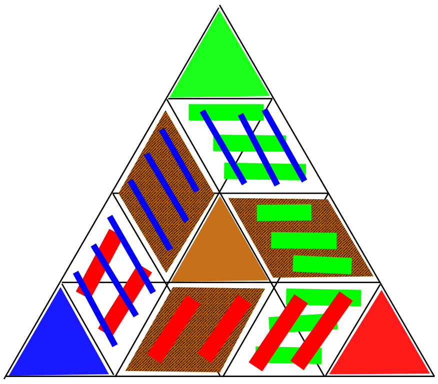

Figure 2.4: $\Delta^{4\boldsymbol{e}}$ as the sum of four prisms

Compare this with Figure 2.4, which is the sum of four copies of the unit simplex. The common structure is obvious, but of course the representation is not unique. The planar case is in some way "degenerate". However it serves to represent the surface structure of the cephoid in Figure 2.3.

The exact definition of a nondegenerate family is omitted, see [9]. However, it is worthwhile to note:

Theorem 2.3. *A nondegenerate cephoid is uniquely represented as a sum of nonhomothetic prisms.*

The proof follows from general theorems of convex geometry. (see [14]) Henceforth we shall, therefore, attach the term "nondegenerate" to a cephoid (i.e., the sum) as well as the generating family.

As it turns out, the general structure of a cephoidal surface is at best represented on (a positive multiple of) the unit simplex: there is a "canonical" mapping between the two surfaces preserving the partially ordered set of faces. (E.g. Figure 2.4 is the "canonical representation" of Figure 2.3) This geometric structure is accompanied by a combinatorial structure correponding to the poset (partially ordered set) of maximal faces.

3 The Canonical Representation

Recall the similar structure exhibited in Figures 2.3 and 2.4. There is a mapping of the surface structure of a cephoid on a suitable positive multiple of the unit simplex such that both structures are "combinatorically equivalent", i.e., the posets (partially ordered sets) of subfaces are isomorphic (see [3]).

In order to simplify the notation, we use $\boldsymbol{K} := \{1, \ldots, K\}$ for the index set of a family of prisms. We consider a family $(\boldsymbol{a}^{(k)})_{k \in \boldsymbol{K}}$ of vectors in general position; the prism $\Pi := \sum_{k \in \boldsymbol{K}} \Pi^{(k)}$ and its surface $\Delta := \sum_{k \in \boldsymbol{K}} \Delta^{(k)}$ are defined as previously.

We take K copies of $\boldsymbol{e}$ which we denote by $\boldsymbol{a}^{0(1)}, \ldots, \boldsymbol{a}^{0(K)}$. As in SECTION 2 we write $\boldsymbol{a}^{0(k)i} := a_i^{0(k)} \boldsymbol{e}^i$, where $a_i^{0(k)}$ denotes the $i'th$ coordinate of $\boldsymbol{a}^{0(k)}$.

For every $k \in \boldsymbol{K}$ let $\Pi^{0(k)} := \Pi^e$ and $\Delta^{0(k)} := \Delta^e$ be a copy of the unit prism and simplex respectively. The (homothetic) sums generated are denoted by

$$\Pi^0 := \sum_{k \in \boldsymbol{K}} \Pi^{0(k)} = \Pi^{K\boldsymbol{e}} = K\Pi^{\boldsymbol{e}}$$

and

$$\Delta^0 := \sum_{k \in \boldsymbol{K}} \Delta^{0(k)} = \Delta^{K\boldsymbol{e}} = K\Delta^{\boldsymbol{e}}$$

respectively. As all prisms involved are homothetic, the simplex Δ^0 has the (trivial) face poset of the unit simplex.

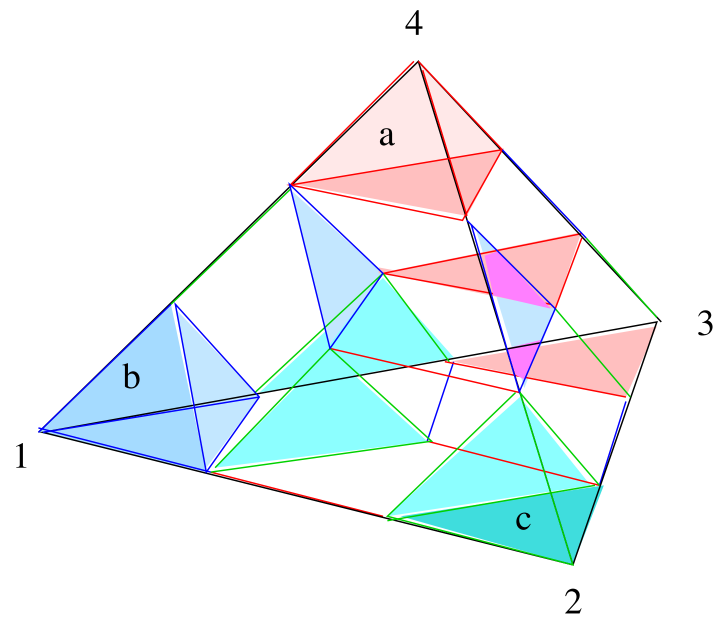

Figure 3.1: A sum of 3 prisms in 4 dimensions.

The ***canonical representation*** of Δ is the suitable projection of the outer surface Δ of a cephoid Π on Δ^0 in a way which preserves the poset of faces. For example, the canonical representation of the cephoid represented in Figure 2.3 is indicated by Figure 2.4. Here, both Δ and Δ^0 are two dimensional while the prisms $\Pi^{(k)} = \Pi^{(\boldsymbol{a}^{(k)})}$ as well as the resulting cephoid Π are three dimensional.

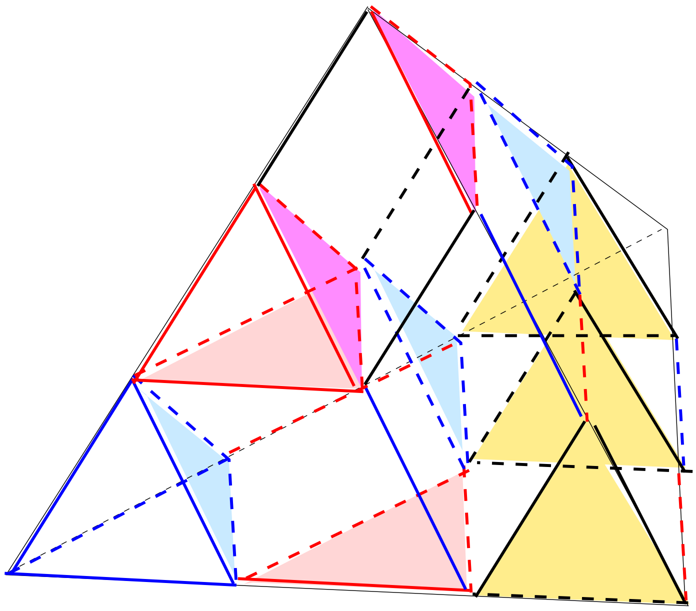

Figure 3.2: A sum of 3 prisms – not all prisms at a vertex

Analogously, for $n = 4$ dimensions and $K = 3$ prisms, we represent the sum of the prisms canononically on a suitable multiple of the unit simplex of $\mathbb{R}^4$, which is a three dimensional tetrahedron. It turns out, that there are three translates of simplices on Δ. Each of these generates 'tentacles' consisting of 2 cylinders. Thus, we find immediately nine maximal faces that involve a vertex. However, in addition to these nine faces, there is exactly 1 *block*, i.e., a maximal face that is the sum of three edges, each one taken from. one of the prisms involved.

The canonical representation of a sum of three prisms in $\mathbb{R}^4$ has a surface Δ which is presented in Figure 3.1.

The translates of the simplices are located at the vertices of Δ. Each simplex generates two cylinders which together form a "tentacle" issued from that simplex. Finally, there is the "block", which is the representation of a part of Δ which can be written

$$\Delta_{12}^{(\boldsymbol{a})} + \Delta_{23}^{(\boldsymbol{b})} + \Delta_{34}^{(\boldsymbol{c})}. \tag{3.1}$$

In Figure 3.2 we perceive a variant – not all translates of the three prisma involved are licated at some ertex.

Given nondegeneracy, the number of maximal faces of a cephoid is actually depending on the dimension n and the number of prisms involved K only.

E.g., four prisms in $\mathbb{R}^4$ yield a cephoid with 20 maximal faces; a canonical

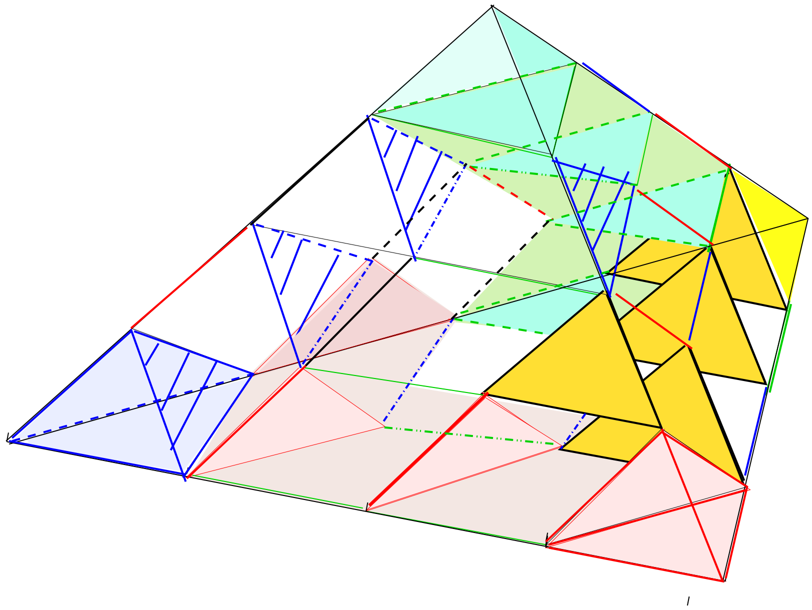

Figure 3.3: Summing 4 prisms in $\mathbb{R}^4$.

representation is depicted in Figure 3.3.

4 The Surface Measure

In order to illustrate the procedure for the surface measure, we start out with a three dimensional cephoid. Recall, that the surface measure in the two dimensional case involves the *area* of the prisms involved. It is, therefore, appropriate to use the *volume* in order to generate the ***surface measure.***

Let $\boldsymbol{a} = (a_1, a_2, a_3) > 0$ be a positive vector and let $\Pi^{\boldsymbol{a}}$ be the prism associated, the surface is the simplex $\Delta^{\boldsymbol{a}}$. The volume of $\Pi^{\boldsymbol{a}}$ is $V(\Pi^{\boldsymbol{a}}) = \frac{a_1 a_2 a_3}{6}$. We use the volume in order to define a measure on the surface, as follows. First of all, assign an area to $\Delta^{\boldsymbol{a}}$ which is given by

$$\boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{a}}) = \sqrt[3]{[6V(\Pi^{\boldsymbol{a}})]^2} = \sqrt[3]{[a_1 a_2 a_3]^2} \tag{4.1}$$

The same area is associated to any translate of $\Delta^{\boldsymbol{a}}$. Then we obtain in particular for $\boldsymbol{d} \in \mathbb{R}^3_+$ and $\varepsilon > 0$

$$\boldsymbol{\iota}_{\Delta}(\boldsymbol{d} + \Delta^{\varepsilon \boldsymbol{a}}) = \varepsilon^2 \boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{a}}). \tag{4.2}$$

Now we observe that this definition generates a σ–additive set function on $\Delta^{\boldsymbol{a}}$. For, let us decompose $\Delta^{\boldsymbol{a}}$ canonically into 4 similar simplices as indicated by Figure 4.1. We define

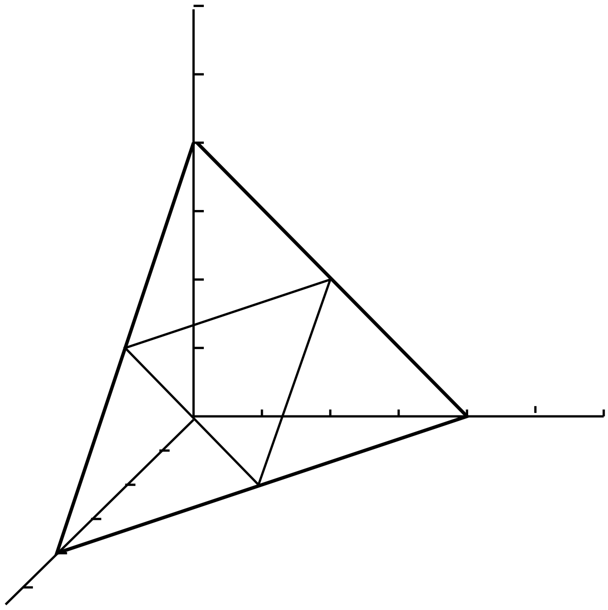

Figure 4.1: Canonical Decomposition of a Simplex

$$(4.3) \qquad \Delta^{a*} \ := \ \operatorname{convH}\left(\left\{\frac{\boldsymbol{a}^1+\boldsymbol{a}^2}{2}, \frac{\boldsymbol{a}^1+\boldsymbol{a}^2}{3}, \frac{\boldsymbol{a}^2+\boldsymbol{a}^3}{3}, \right\}\right)$$

in order to obtain

$$(4.4) \qquad \Delta^{\boldsymbol{a}} = \bigcup_{i=1}^{3}\left(\frac{1}{2}\boldsymbol{e}^i + \frac{1}{2}\Delta^{\boldsymbol{a}}\right) \cup \left(\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \frac{1}{2}\Delta^{a*}\right).$$

Each of the 4 triangles involved has measure $\boldsymbol{\iota}_\Delta(\frac{1}{2}\boldsymbol{d} + \frac{1}{2}\Delta^{\boldsymbol{a}})$ and because of (4.2) we have

$$(4.5) \qquad 4\boldsymbol{\iota}_\Delta(\frac{1}{2}\boldsymbol{d} + \frac{1}{2}\Delta^{\boldsymbol{a}}) = \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}).$$

The decomposition of a simplex into 4 equal subsimplices may be continued and we obtain an additive set function on the field generated by these simplices on $\Delta^{\boldsymbol{a}}$. By the usual extension theorems, we obtain the surface measure $\boldsymbol{\iota}_\Delta$ on the surface $\boldsymbol{d} + \Delta^{\boldsymbol{a}}$ of every translate $\boldsymbol{d} + \Pi^{\boldsymbol{a}}$ of some prism $\Pi^{\boldsymbol{a}}$ (the σ–algebra is generated by the relative topology).

Now we turn to the sum of two prisms. Let $\boldsymbol{a}, \boldsymbol{b} > 0$ and consider the polyhedron $\Pi^{\boldsymbol{ab}} \ := \ \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$. Figure 4.2 shows the situation in which we assume

$$(4.6) \qquad \begin{array}{c} a_2 > a_2 > a_1 \ ; \quad b_3 < b_1 < b_2; \\ a_2 > b_2, \ b_3 > a_3, \ b_1 > a_1. \end{array}$$

The two vectors are nondegenerate.

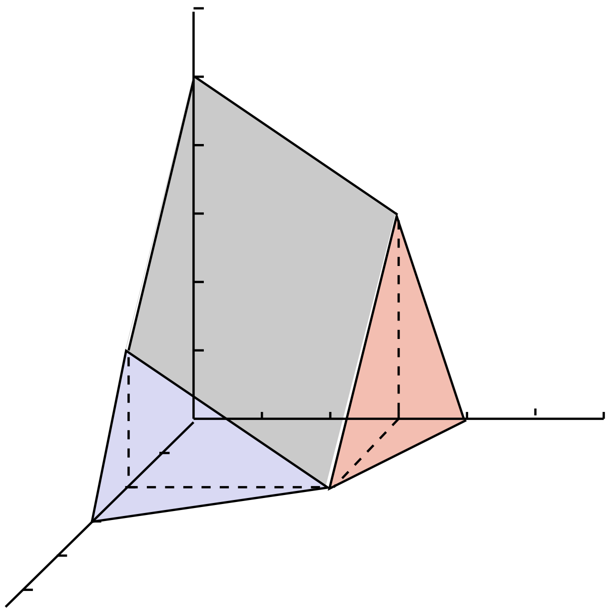

Figure 4.2: The sum of two prisms

The surface consists of the translates $\boldsymbol{b}^1 + \Pi^{\boldsymbol{a}}$ and $\boldsymbol{a}^2 + \Pi^{\boldsymbol{b}}$ and the diamond

$$\Lambda^{\boldsymbol{ab}} \;=\; \Lambda^{\boldsymbol{ab}}_{23\ 13} \;=\; \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{13} \tag{4.7}$$

which is the sum of the subsimplices of $\Delta^{\boldsymbol{a}}$ and $\Delta^{\boldsymbol{b}}$ indicated.

Now we define a measure on the surface $\Lambda^{\boldsymbol{ab}}$ consistently to the one on the surface of the simplices. There is a marked difference to the two dimensional case,as the diamond is the first new type of a maximal face that appears in three dimensions. (The next new type is the *block* in four dimensions). $\Lambda^{\boldsymbol{ab}}_{23\ 13}$ now receives a measure that depends on the volumes of the two prisms invoved (consider Figure 4.2). We define

$$\boldsymbol{\iota}_{\Delta}(\Lambda^{\boldsymbol{ab}}) \;:=\; 2\sqrt[3]{6V(\Pi^{\boldsymbol{a}})6V(\Pi^{\boldsymbol{b}})}\ . \tag{4.8}$$

The generalization to several dimensions is now at hand. Again, we use the *volume* in order to define a measure on the *surface* of a cephoid. We start out with a prism. Let $\boldsymbol{a} = (a_1, \ldots, a_n) > 0$ be a positive vector and let $\Pi^{\boldsymbol{a}}$ be the prism associated, the surface is the simplex $\Delta^{\boldsymbol{a}}$.

The volume of $\Pi^{\boldsymbol{a}}$ is

$$V(\Pi^{\boldsymbol{a}}) = \frac{\prod_{i\in\boldsymbol{I}} a_i}{n!}.$$

We associate a ***surface measure*** of

$$(4.9)\qquad \sqrt[n]{(n!)^{n-1}(V(\Pi^{\boldsymbol{a}}))^{n-1}} \ =: \ \sqrt[n]{v_n(V(\Pi^{\boldsymbol{a}}))^{n-1}}$$

to any translate of the surface $\Delta^{\boldsymbol{a}}$. In particular, the simplex $\Delta^{\boldsymbol{e}}$ (the surface of the unit prism $\Pi^{\boldsymbol{e}}$) receives surface measure 1. Next we we turn to more general types of maximal faces of a cephoid (diamonds, blocks,...).

Such a maximal face is given by a system of index sets $\boldsymbol{J} = (\boldsymbol{J}^{(1)}, \ldots, \boldsymbol{J}^{(K)})$ which is called the ***reference system***. A maximal face is the sum of certain subfaces of the prisms involved. That is, such a face may be written

$$(4.10)\qquad \boldsymbol{F} = \Delta^{(1)}_{\boldsymbol{J}^{(1)}} + \ldots + \Delta^{(K)}_{\boldsymbol{J}^{(K)}}$$

The numbers $j_l \ := \ |\boldsymbol{J}^{(l)}|$ satisfy

$$(4.11)\qquad (j_1 - 1) + \ldots + (j_K - 1) = n - 1 \ , \ \ j_1 + \ldots + j_K = n + K - 1 \ .$$

This is a consequence of the nondegeneracy assumption (see [9]). Consider the Minkowski sum

$$(4.12)\qquad \Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(1)}} + \ldots + \Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(K)}} \ .$$

The (Lebesgue) surface measure of this convex compact polyhedron is a multiple of the surface of the unit simplex, this multiple is denoted by $c_{\boldsymbol{J}}$. Of course the number depends on $j_1, \ldots, j_K$ only and not on the ordering of these indices. Thus we write

$$(4.13)\qquad c_{\boldsymbol{J}} \ = \ c_{j_1,\ldots,j_K} \ := \ \frac{\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(1)}} + \ldots + \Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(K)}})}{\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}})},$$

where $\boldsymbol{\lambda}$ denotes the Lebesgue measure. E.g., for $n = 3$ two triangles will fit into a diamond, hence $c_{13} = 1, c_{22} = 2$. For $n = 4$ three tetrahedra just fill a cylinder and two cylinders fill a cube, hence $c_{114} = c_{141} = c_{411} = 1$, $c_{123} = ... = 3$, and $c_{222} = 6$, etc.

Having obtained the above defined "normalizing coefficients" we can now proceed by defining a surface measure on any face of a cephoid.

Definition 4.1. *Let $\boldsymbol{a}^{\bullet}$ be a positive family of vectors and let $\boldsymbol{F}$ be a maximale face represented via a family of index sets $\boldsymbol{J}$ by*

$$(4.14)\qquad \boldsymbol{F} \ = \ \Delta^{(1)}_{\boldsymbol{J}^{(1)}} + \ldots + \Delta^{(K)}_{\boldsymbol{J}^{(K)}}.$$

*Then the **surface measure** associated with $\boldsymbol{F}$ is given by*

$$(4.15)\qquad \boldsymbol{\iota}_{\Delta}(\boldsymbol{F}) = c_{\boldsymbol{J}} \sqrt[n]{(v_n) \left[V(\Pi^{(1)})\right]^{j_1 - 1} \cdot \ldots \cdot \left[V(\Pi^{(K)})\right]^{j_K - 1}}.$$

Within the following lemma we list some obvious properties of the surface measure. This shows that the surface measure exhibits the "appropriate behaviour".

Lemma 4.2.

1. *For* $\boldsymbol{t} = (t_1, \ldots, t_K) > 0$ *and* $\boldsymbol{t}\boldsymbol{a}^{(\bullet)} = (t_k \boldsymbol{a}^{(k)})_{k \in \boldsymbol{K}}$ *let* $\boldsymbol{tF}$ *denote the face corresponding to a face* $\boldsymbol{F}$*. Then*

$$\boldsymbol{\iota}_\Delta(\boldsymbol{tF}) \;=\; t_1^{j_1-1} \cdot \ldots \cdot t_K^{j_K-1} \boldsymbol{\iota}_\Delta(\boldsymbol{F}). \tag{4.16}$$

2. *In particular, for* $\boldsymbol{t} = (\varepsilon, \ldots, \varepsilon)$*, we obtain from* (4.11)

$$\boldsymbol{\iota}_\Delta(\varepsilon \boldsymbol{F}) \;=\; \varepsilon^{n-1} \boldsymbol{\iota}_\Delta(\boldsymbol{F}). \tag{4.17}$$

   Equations (4.16) *and* (4.17) *show that* $\boldsymbol{\iota}_\Delta(\bullet)$ *behaves like the Lebesgue measure of the surface.*

3. *If, for some family* $\boldsymbol{a}^\bullet$*, we have* $\boldsymbol{a}^{(1)} = \ldots = \boldsymbol{a}^{(K)}$*, then it follows that a face* $\boldsymbol{F}$ *represented by* (4.10) *satisfies*

$$\boldsymbol{\iota}_\Delta(\boldsymbol{F}) \;=\; c_{\boldsymbol{J}} \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}^{(1)}}) \;. \tag{4.18}$$

4. *More generally, if for some family* $\boldsymbol{a}^\bullet$ *the volumes satisfy*

$$V(\Pi^{\boldsymbol{a}^{(1)}}) = \ldots = V(\Pi^{\boldsymbol{a}^{(K)}}) \;,$$

   then it follows that a face $\boldsymbol{F}$ *represented by* (4.10) *satisfies*

$$\boldsymbol{\iota}_\Delta(\boldsymbol{F}) \;=\; c_{\boldsymbol{J}} \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}^{(1)}}). \tag{4.19}$$

Corollary 4.3. *Let* $\boldsymbol{a}^\bullet$ *be a family of vectors and let* Π, Δ *be the cephoid generated and its surface. Let* $\boldsymbol{F}$ *be a maximal face of* Δ *represented by* $\boldsymbol{J}$ *as in* (4.10)*. Then there is a measure* $\boldsymbol{\iota}_\Delta$ *defined on* $\boldsymbol{F}$ *which satisfies* (4.15)*, has the properties stated in Lemma 4.2, and is continuous as a function on families* $\boldsymbol{a}^\bullet$*.*

Definition 4.4. *As was the case for 3 dimensions, we call the measure* $\boldsymbol{\iota}_\Delta$ *the* ***surface measure****.*

Remark 4.5. 1. *Let* $\boldsymbol{e} := (1, \ldots, 1)$ *The measure* $\boldsymbol{\iota}_\Delta$ *on* $\Delta^{\boldsymbol{e}}$ *is the Lebesgue measure* $\boldsymbol{\lambda}$ *normalized to* $\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{e}}) = 1$.

2. *A sum of homothetic prism is a multiple of one of those prisms. While the surface structure is not unique, the surface measure is seen to be a multiple of Lebesgue measure – independently on a homothetic decomposition and the surface structure. The measure* $\boldsymbol{\iota}_\Delta$ *behaves consistently with any surface structure.*

Now we create a second mapping (apart from the canonical one) which carries the surface of a cephoid onto the one of a suitable multiple of the unit simplex such that *the surface measure is carried into Lebesgue measure*. The procedure is quite similar. We arrange the surface structure of $\widehat{\Delta}$ in a way such that the surface structure (the poset of maximal faces) of Δ is preserved. This is achieved by mapping the extremals of the faces of Δ bijectively onto certain corresponding vectors of $\widehat{\Delta}$ such that the *surface measure is transported into the Lebesgue measure.*

In a well defined sense, the mapping $\boldsymbol{\kappa}$ defined this way constitutes a piecewise linear isomorphism between Δ and $\widehat{\Delta}$.

Geometrically, the difference between the canonical representation and the measure preserving representation consists just in a different size/volume/surface of the images of the various faces.

E.g. Figure 4.3 is a relative of Figure 3.1 inasmuch as it represents the sum of three cephoids in $\mathbb{R}^4$. The location of the cylinders is however slightly different (observe the cylinders generated by prism $\boldsymbol{b}$ in 3.1). And the surface measure of the various maximal faces (or rather their representations) (which is the *volume* of the polyhedra in Figure 4.3 as our surface is three dimensional), is not normalized but varies depending on the surface measure.

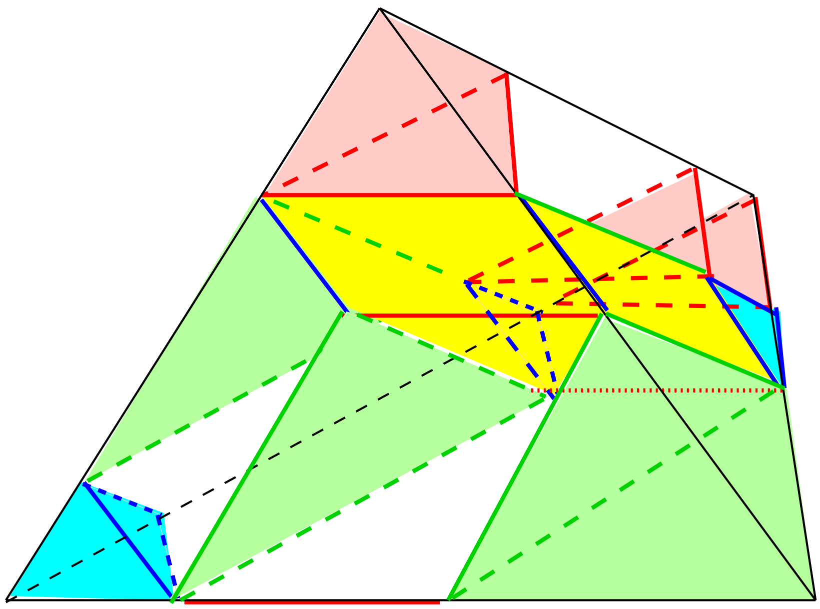

Figure 4.3: The measure preserving representation of a cephoid

5 The Superadditive Solution

Finally we describe an n–dimensional version of the Maschler–Perles superadditive solution ([7]). It is well known that we cannot expect such a solution to behave superadditively for all bargaining problems (see Perles([12]). Yet, there is a class of bargaining solutions on which a superadditive solutions exist.

The appropriate definition generalizing the two–dimensional version makes use of the measure preserving representation.

Definition 5.1. *Let Π be a cephoid. Let $\widehat{\Delta} = \Delta^{\widehat{\alpha}\boldsymbol{e}}$ be the appropriate multiple of the unit simplex of $\mathbb{R}^n$ to carry the measure preserving representation of Π . That is, there is the bijective and locally affine linear mapping*

$$\boldsymbol{\kappa} \;:\; \Pi \rightarrow \widehat{\Delta}$$

that preserves the poset of maximal faces and carries the surface measure into the Lebesgue measure. Let

$$\boldsymbol{\mu}(\Delta^{\widehat{\alpha}\boldsymbol{e}}) \;:=\; \frac{\widehat{\alpha}}{n}\boldsymbol{e}$$

denote the barycenter of $\Delta^{\widehat{\alpha}\boldsymbol{e}}$. Then

$$\boldsymbol{\mu}(\Pi) \;:=\; \boldsymbol{\kappa}^{-1}(\widehat{\alpha}\boldsymbol{e}) \tag{5.1}$$

*is the **solution** of Π.*

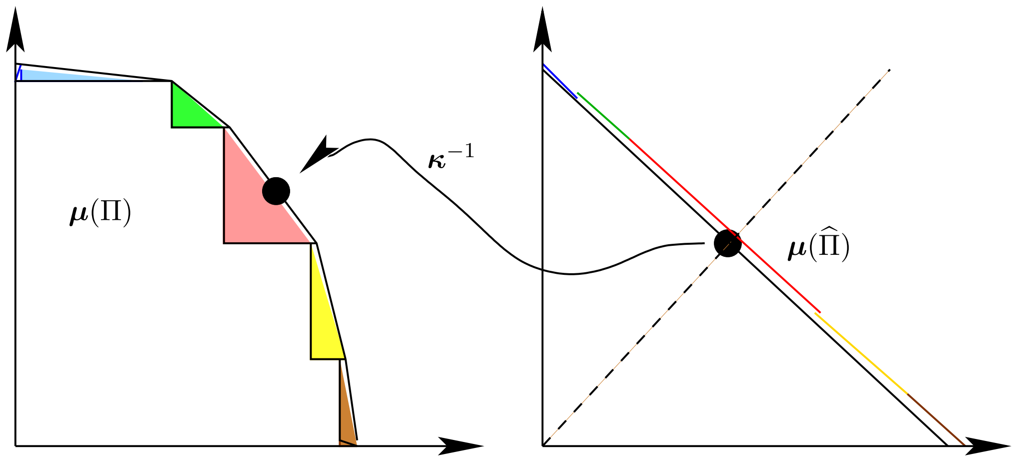

Figure 5.1: The mapping $\boldsymbol{\kappa}^{-1}$ for a two person problem

In two dimension a polyhedral bargaining problem is bijectively mapped on a suitable multiple of the two dimensional unit simplex and Maschler–Perles solution is the inverse of the barycenter under the mesure preserving mapping $\boldsymbol{\kappa}$. Obviously we imitate this procedure for the general case of n dimensions. Figure 5.1 indicates the geometrical setup and figures 5.2 and 5.3 show the analogous geometrical setup for a 3 dimensional bargaining problem.

The inverse image of the barycenter of the measure preserving representation defines the generalizes Maschler–Perles solution.

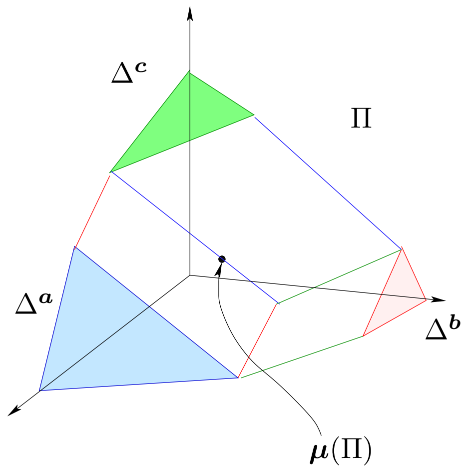

Figure 5.2: A 3–dimensional bargaining problem

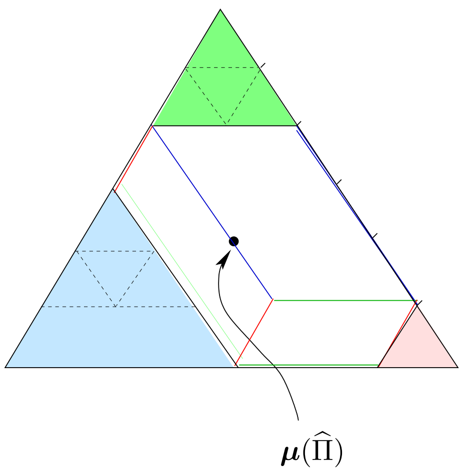

Figure 5.3: The measure preserving representation of 5.2

Now there should be a short discussion concerning the family of bargaining problems admitting the solution to be superadditive. In two dimensions, a polyhedral bargaining problem is bounded by a Pareto curve which consists of line segments. The slopes of these line segments can be *ordered*. The essential requirement for the general case is that some kind of ordering can be imposed on the normals of the prisms involved in a general cephoid.

Example 5.2. *For $n = 3$, Concider a **cyclic** case in which the coordinates of the vectors are ordered as follows:*

$$
\begin{array}{ccccccccc}
a_1^{(1)} & \geq & \dots & \geq & a_1^{(K)} & & & & \\
a_2^{(\frac{K}{3}+1)} & \geq & \dots & \geq & a_2^{(K)} \geq & a_2^{(1)} & \geq & \dots \geq & a_2^{(\frac{K}{3})} \\
a_3^{(\frac{2K}{3}+1)} & \geq & \dots & \geq & a_3^{(K)} \geq & a_3^{(1)} & \geq & \dots \geq & a_3^{(\frac{2K}{3})}
\end{array}
\tag{5.2}
$$

E.g.t Figure 5.4 represents a cyclic polyhedron (assuming that the volume of the prisms is equal). However, the orientation in the setup suggested by this figure is clockwise, i.e., mathematically negative. The orientation above is mathematically positive. Yet, both versions are, in a sense, well ordered.

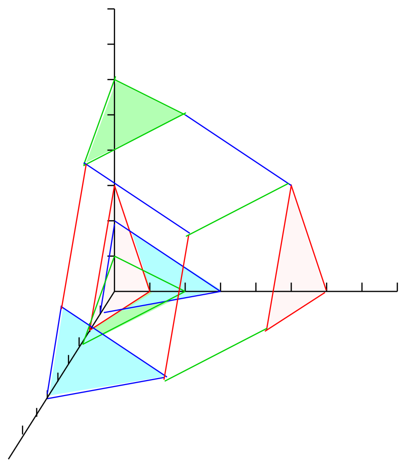

Figure 5.4: a cyclic bargaining problem

The above example requires that the total number K of prisms involved is a multiple of the dimension n. This condition is actually not too restrictive. It can be achieved by replacing each prism by a sum of three homothetic $\frac{1}{n}$–copies of itself. Also, the requirement that all the prisms involved have equal volume is not as strong as it might appear on a first glance. The details of the discussion can be found in [9] and [10].

Thus, we come up with

Definition 5.3. *A family* $\boldsymbol{a}^{\bullet}$ *of positive vectors as well as the cephoid* Π *generated is called* ***well ordered*** *if the following conditions are satisfied.*

1. $\boldsymbol{a}^{\bullet}$ *is weakly nondegenerate in the sense that it is obtained by a nondegenerate family via a possible repetition of some of its members, thus admitting of homothetic prisms.*
2. *The* K *prisms involved have equal volume.*
3. n *is a divisor of* K*.*
4. *There is a decomposition of* $\boldsymbol{K}$ *into* n *disjoint subsets* $\boldsymbol{K}_i$ $(i \in \boldsymbol{I})$ *such that* $a_i^{(k)} \geq a_i^{(l)}$ $(k \in \boldsymbol{K}_i, l \notin \boldsymbol{K}_i)$ *holds true.*

We are now in the position to state the version of superadditivity that holds true for our solution.

Theorem 5.4 (see [10]). $\boldsymbol{\mu}$ *behaves superadditively along decompositions of a well ordered polyhedron. More precisely, if* Π *is a well ordered polyhedron and* Π *is the sum of two bargaining problems, say* $\Pi = \Upsilon + \Psi$*, then*

1. Υ, Ψ *are cephoids.*
2. $\boldsymbol{\mu}$ *is superadditive, i.e.,*

$$\boldsymbol{\mu}(\Pi) \geq \boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi) \tag{5.3}$$

holds true.

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Games and Geometry

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1 Introduction

The main aim of this paper is to look with a geometric eye to the theory of cooperative games with transferable utility. We observe that several concepts from Euclidean and convex geometry such as hyperplanes, hypercubes, simplices, (polyhedral) cones, extreme points and directions, generating sets, polytopes, halfspaces, orthants, centers of gravity or barycenters have been used in cooperative game theory to obtain insight in the structure of games and in the description of solution concepts.

The outline of the paper is as follows. Section 2 recalls basic cooperative game theoretic notions and game properties which are used throughout the paper. In Section 3 we briefly present three geometric representations of cooperative games with transferable utility that have been extensively used in the analysis of such games. Section 4 deals with polyhedral cones of cooperative games and Section 5 is devoted to the geometry of solution concepts for such games. We conclude the paper in Section 6 with some remarks on geometry in non-cooperative game theory.

2 Preliminaries

A cooperative n-person game is a pair $< N, v >$ where N is the set of players, usually of the form $\{1, 2, \ldots, n\}$, and $v : 2^N \to \mathbb{R}$ with $v(\phi) = 0$ is the characteristic function with domain the family 2^N of subsets of N. For each coalition S, $v(S)$ indicates the worth generated by cooperation of players in S. Normally a game is identified with its characteristic function.

The subgame of a game v relative to a non-empty coalition S is the game $< S, v_{|S} >$ where $v_{|S}$ is defined to be the restriction of v to 2^S. The unanimity game u_S based on $\phi \neq S \subset N$ is defined by $u_S(T) = 1$ if $T \supset S$ and $u_S(T) = 0$ otherwise. The dual game v^* of v is defined by $v^*(S) = v(N) - v(N \backslash S)$ for each $\phi \neq S \subset N$.

The value $v(\{k\})$ is called the individual value of player k, and $i(v) = (v(\{1\}), v(\{2\}), \ldots, v(\{n\}))$ is the individual rational point of v. The real number $v^*(\{k\}) = v(N) - v(N \backslash \{k\})$ is the marginal contribution of k

to the grand coalition N, called also the utopia payoff for player k, and $u(v) = (v^*(\{1\}), v^*(\{2\}), \ldots, v^*(\{n\}))$ is the utopia point of v. The maximal remainder point $a(v)$ of v, called also the minimum right vector, is defined by $a_k(v) = \max_{S:S\ni k}(v(S) - \sum_{i\in S\setminus\{k\}} u^i(v))$ for each $k \in N$.

A game v is called quasi-balanced if $\sum_{i=1}^{n} a_i(v) \leq v(N) \leq \sum_{i=1}^{n} u_i(v)$ and $a_i(v) \leq u_i(v)$ for all $i \in N$. A game v is said to be monotonic if $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subset T$.

Given an ordering $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ of players in N, the σ-marginal vector $m^\sigma(v) \in \mathbb{R}^n$ of v is the vector which has for each $k \in N$ as its $\sigma(k)$-th coordinate the real number
$v(\sigma(1), \sigma(2), \ldots, \sigma(k)) - v(\sigma(1), \sigma(2), \ldots, \sigma(k-1))$.

For a coalition $S \in 2^N$, e^S denotes the characteristic vector of S with $(e^S)_i = 1$ if $i \in S$ and $(e^S)_i = 0$ otherwise. For $S \in 2^N\setminus\{\phi\}$, the per capita value of S is $v(S)/|S|$, where $|S|$ denotes the number of elements of S.

To solve the problem of how to divide $v(N)$ among the players in N when all of them agree to cooperate, several solution concepts have been proposed. Sometimes subsets of payoff distributions of $v(N)$ are assigned to games as solutions. The following three such subsets will play a role in this paper:

- the imputation set $I(v)$ of v defined by

$$I(v) = \{x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = v(N),\ x_i \geq v(\{i\}) \text{ for each } i \in N\};$$

- the dual imputation set $I^*(v)$ of v defined by

$$I^*(v) = \{x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = v(N),\ x_i \leq v^*(\{i\}) \text{ for each } i \in N\};$$

- the core $C(v)$ of v introduced by Gillies (1953) is defined by

$$C(v) = \{x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = v(N), \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in 2^N\}.$$

A game with a non-empty core is called balanced. A balanced game whose subgames are also balanced is called totally balanced. A game is called exact if for each $S \in 2^N\setminus\{\phi\}$ there exists $x \in C(v)$ such that $x(S) = v(S)$.

Of the one-point solution concepts for cooperative games we mention here only the Shapley value (Shapley, 1953), the nucleolus (Schmeidler, 1969) and the τ-value (Tijs, 1981).

- The Shapley value $\phi(v)$ of v is equal to the average $\frac{1}{n!}\sum\limits_{\sigma\in\Pi(N)} m^\sigma(v)$ of marginal vectors; here we denote by $\Pi(N)$ the set of $n!$ orderings of N.

- The nucleolus $N(v,X)$ of a game v w.r.t. a non-empty closed set $X \subset \mathbb{R}^n$ is defined by $N(v,X) := \{x \in X|\theta(x) \preceq \theta(y) \text{ for all } y \in X\}$, where $\theta(x)$ is a (2^n-1)-dimensional vector of the excesses of coalitions $S \in 2^N\backslash\{\phi\}$ w.r.t. x in decreasing order and $\preceq$ denotes the lexicographical order.

- The τ-value $\tau(v)$ of a quasi-balanced game v is defined as the feasible compromise between $u(v)$ and $a(v)$; in formula $\tau(v) = \alpha u(v) + (1-\alpha)a(v)$, where α is the unique real number such that $\sum\limits_{i=1}^{n}\tau_i(v) = v(N)$.

It turns out that for various classes of games the nucleolus and the τ-value coincide and then $\tau(v)$ has a computational advantage over the nucleolus.

A game v is called superadditive if

$$v(S\cup T) \geq v(S) + v(T) \text{ for all } S,T \in 2^N,\ \ S\cap T = \phi$$

An interesting and important class of cooperative games is the class of convex games introduced by Shapley (1971). A game v is called convex if

$$v(S\cup T) + v(S\cap T) \geq v(S) + v(T) \text{ for all } S,T \in 2^N.$$

A game v is called a big boss game with n as big boss if the following properties hold:

(i) $v(S) = 0$ if $n \notin S$ (big boss property);

(ii) v is monotonic;

(iii) $v(N) - v(N\backslash S) \geq \sum\limits_{i\in S} M_i(v)$ for each $S \in 2^{N\backslash\{n\}}$ (union property).

For an introduction to cooperative game theory the reader is referred to Driessen (1988), Tijs (2003) and Branzei, Dimitrov and Tijs (2005).

3 Geometric representations of cooperative games

A first geometric representation identifies a game with a vector in $\mathbb{R}^{2^n-1}$. This can be done by ordering in some way all non-empty coalitions $(S_1, S_2, \ldots, S_{2^n-1})$ and identifying the game v with the (2^n-1)-dimensional vector whose i-th coordinate is $v(S_i)$. The game space G^N is treated as a Euclidean vector space with dimension 2^n-1 in which many of the considered classes of games may be seen as polyhedral cones, i.e. as a finite intersection of closed halfspaces. It is possible to find a set of simple games, for example the unanimity games, that form a basis for the linear space G^N.

The second geometric approach is to represent any n-person game v as a real-valued function w defined on the set of extreme points of the unit hypercube in R^n by $w(e^S) = v(S)$ for each $S \in 2^N$. This geometric representation is based on the idea that there is a one-to-one correspondence between 2^N and $ext[0,1]^N$ which associates with each coalition $S \in 2^N$ its characteristic vector e^S. The multi-linear extension of v, introduced by Owen (1972), extends the function w to the entire hypercube. Further, Aubin (1974, 1981) introduces the notion of fuzzy game, specifically a cooperative game with fuzzy coalitions, as a function from $[0,1]^N$ to N. In Muto et al. (2005) generalized cores of fuzzy games are introduced based on looking with a geometrical eye to the (Aubin) core.

The third geometric representation of an n-person game v is to convert it to a real-valued function $u : \left\{ \frac{e^S}{|S|} | S \in 2^N \backslash \{\phi\} \right\} \cup \{0\} \to \mathbb{R}$ defined by $u\left(\frac{e^S}{|S|}\right) = \frac{v(S)}{|S|}$ for each $S \in 2^N \backslash \{\phi\}$, and $u(0) = 0$. With each subsimplex's barycenter $\frac{e^S}{|S|}$ the function u associates the per capita value of S. Such representation is used by Branzei and Tijs (2001a) to reformulate the Bondareva-Shapley result in geometric terms. Azrieli and Lehrer (2004) consider the concavification of u, denoted *cav* u, which is a function defined on the entire simplex as the minimum of all concave functions that are larger than or equal to u. They show that *cav* u induces a game which is the totally balanced cover of v and use *cav* u to characterize well-known classes of cooperative games like balanced, totally balanced, exact and convex games. A similar approach is developed by Tijs et al. (2005) for games with restricted cooperation and fuzzy games.

4 Polyhedral cones of cooperative games

A primary task of cooperative game theory is to obtain insight in the structure of several types of games considered as subsets in the game space G^N. Most of the corresponding classes of games may be seen as polyhedral cones.

A cone K in G^N is a set of games with the property that $\alpha v + \beta w \in K$ if $v, w \in K$ and α, β are non-negative real numbers. Note that $I^N = \{v \in G^N | I(v) \neq \phi\}$ and $I_*^N = \{v \in G^N | I^*(v) \neq \phi\}$ are cones of games as well as the related classes of games with a simplicial core $SI^N = \{v \in G^N | \phi \neq C(v) = I(v)\}$ and $SI_*^N = \{v \in G^N | \phi \neq C(v) = I^*(v)\}$ which are called simplex games and dual simplex games, respectively (Branzei and Tijs (2001b)). A cone K is said to be polyhedral if it is the intersection of a finite family of closed halfspaces. Examples of polyhedral cones of cooperative games are the cones of: superadditive games, convex games, (totally) balanced games, big boss games, quasi-balanced games, information market games (Muto, Potters and Tijs (1989); Branzei, Tijs and Timmer (2000)).

A polyhedral cone is characterized by the existence of a finite generating set, which is a finite subset of the cone with the property that each element in the cone can be described as a non-negative weighted sum of the elements of this set. Each polyhedral cone can be described by its extreme directions. For a systematic study of polyhedral cones of cooperative games we refer to Derks (1991). See also Tijs and Branzei (2002) for perfect cones of cooperative games.

The characterization of the extreme directions of a cone of games and the construction of a (finite) generating set are also useful for describing the behavior of solution concepts defined on the underlying cone. An exhaustive study of the cone of superadditive games and the cone of convex games is done by Rosenmüller (1977) from the viewpoint that the extremality of a game in the geometrical sense should correspond to a notion of extremality in the social behavior sense.

5 Geometry and solution concepts for cooperative games

We start this section by looking from a geometric point of view to the imputation set $I(v)$ and the dual imputation set $I^*(v)$ of a game $v \in G^N$.

The imputation set $I(v)$ of v is equal to the intersection of the efficiency hyperplane $\{x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = v(N)\}$ and the orthant $\{x \in \mathbb{R}^n | x \geq i(v)\}$ of individual rational vectors. $I(v)$ is non-empty iff $v(N) \geq \sum_{i=1}^{n} v(\{i\})$. If $v(N) > \sum_{i=1}^{n} v(\{i\})$, i.e. the game v is N-essential, $I(v)$ is an $(n-1)$-dimensional simplex with extreme points $f^1(v), f^2(v), \ldots, f^n(v)$, where $(f^i(v))_k = v(\{k\})$ if $k \neq i$, and $(f^i(v))_i = v(N) - \sum_{k \in N \setminus \{i\}} v(\{k\})$.

The dual imputation set $I^*(v)$ of v is equal to the intersection of the efficiency hyperplane and the orthant $\{x \in \mathbb{R}^n | x \leq u(v)\}$ of subutopic vectors. $I^*(v)$ is non-empty iff $\sum_{i=1}^{n} v^*\{i\} \geq v(N)$. In case of strict inequality $I^*(v)$ is an $(n-1)$-dimensional simplex with extreme points $g^1(v), g^2(v), \ldots, g^n(v)$, where $(g^i(v))_k = v^*(\{k\})$ if $k \neq i$, and $(g^i(v))_i = v(N) - \sum_{k \in N \setminus \{i\}} v^*(\{k\})$.

The core $C(v)$ of v is a subset of $I(v) \cap I^*(v)$. The core $C(v)$ is the bounded solution set of a set of linear inequalities, which means that the core is a polytope, i.e. the convex hull of a finite set of vectors in $\mathbb{R}^n$.

The core $C(v)$ is non-empty iff $v(N) \geq \sum_{S \in 2^N \setminus \{\phi\}} \lambda_S v(S)$ in case $\lambda_S \geq 0$ for all $S \in 2^N \setminus \{\phi\}$, and $\sum_{S \in 2^N \setminus \{\phi\}} \lambda_S e^S = e^N$. These balancedness conditions (Bondareva (1963) and Shapley (1967)) have been reformulated by Branzei and Tijs (2001a) in geometric terms: $C(v)$ is non-empty iff for each way of writing the barycenter of the imputation set $I(v)$ as a convex combination of barycenters of subsimplices of $I(v)$, the per capita value of N is at least as large as the corresponding convex combination of per capita values of the coalitions $S \in 2^N \setminus \{\phi\}$.

The core of a convex game and the core of a big boss game are special polytopes with a beautiful geometric structure. The paper by Shapley (1971) basically describes the shape of the core of an convex game. The paper by Muto et al. (1989) describes the shape of the core of a big boss game.

The core is a useful solution concept and it is extensively studied. However, as we already noted, not all games possess a non-empty core. Moreover, in case $C(v) \neq \phi$, often the core is large and then the problem to single out a unique allocation in the core arises. From a geometric viewpoint the "center of gravity" of the core could be a candidate.

Now we briefly discuss three well-known single-valued (or one-point) solution concepts for cooperative games focusing on their geometric facet.

The Shapley value of v is the average of marginal vectors and each marginal vector $m^\sigma(v)$, $\sigma \in \Pi(N)$, corresponds to a walk according to the ordering σ along the edges of the unit hypercube $[0,1]^N$ starting from the vertex 0 and ending at the vertex e^N. For a convex game the Shapley value is essentially the center of gravity of the core. However, there are games with a non-empty core whose Shapley value does not lie in the core.

In this respect, the nucleolus has an advantage over the Shapley value. The nice properties that the nucleolus possesses and its geometric construction motivate its choice as a unique outcome in the core. In the paper by Maschler, Peleg and Shapley (1979) a sequence of geometric constructions leading to the nucleolus point is given, which is based on "pushing" at equal speed hyperplanes of the form $x(S) = v(S) + k$, where k is a constant.

The τ-value of a quasi-balanced game v is the intersection point between the efficiency hyperplane and the line segment with endpoints the utopia point $u(v)$ and the maximal remainder point $a(v)$ of v. This corresponds to the situation for quasi-balanced games where the utopia point and the minimum right vector are at different sides of the efficiency hyperplane. For a big boss game the τ-value is the center of the core and coincides with the nucleolus (see Muto et al. (1988)). Two single-valued solution concepts discussed by Branzei and Tijs (2001b), namely CIS (the center of the imputation set) and ENSR (the equal split of the non-separable rewards solution) have the property $\text{CIS}(v) = \text{ENSR}(v) = \tau(v)$ for each superadditive simplex game v and $\text{ENSR}(v) = \tau(v)$ for each dual simplex game v. Recall that

CIS : $I^N \to \mathbb{R}^n$ is defined by $\mathrm{CIS}(v) = \frac{1}{n}\sum_{i=1}^{n} f^i(v)$ for each $v \in I^N$, and ENSR : $I_*^N \to \mathbb{R}^*$ is defined by $\mathrm{ENSR}(v) = \frac{1}{n}\sum_{i=1}^{n} g^i(v)$ for each $v \in I_*^N$. For the geometry of the τ-value for games on matroids we refer to the paper of Bilbao et al. (2002).

For additional information on the geometry of solution concepts for n-person cooperative games the reader is referred to Spinetto (1971, 1974).

For (super)additivity properties of solution concepts for cooperative games have been sudied on different cones of games; we refer here to Dragan, Potters and Tijs (1989), Branzei and Tijs (2001b) and Tijs and Branzei (2002).

6 Final remarks

In the foregoing sections we have concentrated on geometric issues in cooperative game theory. The next few sentences deal with geometry and non-cooperative game theory. The interesting proof of the minimax theorem of J. von Neumann (1928) by Ville (1938) uses a separation theorem of two disjoint convex sets. For a finite matrix game the sets of optimal mixed strategies are polytopes in the simplices of mixed strategies for which interesting dimension relations exist (cf. Bohnenblust, Karlin and Shapley (1950) and Gale and Sherman (1950)). The extreme points of these optimal strategy spaces correspond to submatrices (Shapley and Snow (1950)). Geometric methods to solve $2 \times n$- and $m \times 2$-matrix games are well-known (see Tijs (2003), chapter 3). For each finite bimatrix game (Nash (1950)) the set of Nash equilibria is a finite union of polytopes (cf. Jansen, Jurg and Vermeulen (2002)). For $m \times \infty$-matrix games (Tijs (1975)) the strategy spaces of the players are convex sets but not necessarily polytopes. In fact for each compact convex subset C of the mixed strategy space Δ^m of player 1 there exists an $m \times \infty$-matrix such that C is the optimal strategy space of player 1.

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Comparing selfishness and versions of cooperation as the voting strategies in a stochastic environment

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Sociologists have been discussing for a long time the matter of whether altruistic behavior is socially advantageous or not. Evidently, a single “altruist” can hardly expect any advantage among “egoists” except moral gain; most probably he/she will be “swallowed up”. The initial idea of this work was as follows. A certain version of altruism can be approximated by “group egoism” with respect to a very large group. Suppose that, in some community, a group consisting of group egoists competes with individual egoists. In case the group wins and it is possible to freely join it (and leave it as well), the inflow of new members will expand the group (possibly, even all people will enter it), and as a result, the group-egoistic behavior will become more and more altruistic. This idea reminds the idea of competition between the capitalist and radical socialist development trends, but some very important peculiarities distinguish it from the well-known historical implementations.

The present paper investigates the dynamics of communities that consecutively vote for external motions distributed in accordance with a stochastic law. More specifically, the current state of the system is characterized by the vector of actors’ welfare. At every stage, the body of voters can preserve *status quo* or accept a new external motion. The motion is a vector of algebraic increments $(d_1,\dots,d_n)$ of actors’ welfare, where n is the number of voters and $(d_1,\dots,d_n)$ a sample from a normal distribution $N(\mu,\sigma^2)$. Each actor votes for a motion or against it, being guided by his/her own selfish or cooperative voting principle. A selfish person i votes for a motion if and only if $d_i > 0$. The number of selfish persons (=egoists) is ℓ; $n-\ell = g$ is the number of group members. We consider a variety of group voting principles. The simplest ones are as follows:

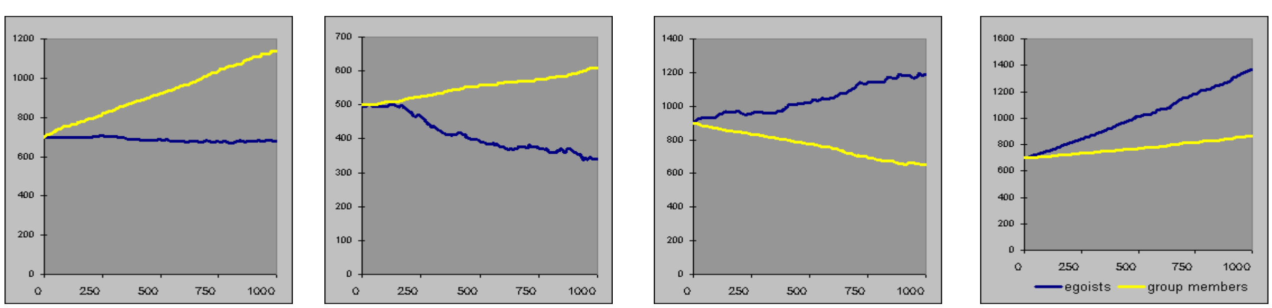
Principle A$_1$. *The group, $\mathcal{G}$, votes for a motion $(d_1,\dots,d_n)$ if and only if* $\#\{i\in\mathcal{G}\mid d_i>0\} > g/2$.

Principle B. *The group, $\mathcal{G}$, votes for a motion $(d_1,\dots,d_n)$ if and only if* $\sum_{i\in\mathcal{G}} d_i > 0$.

Thus, all members of $\mathcal{G}$ vote similarly. The collective decisions are made by means of threshold majority voting with some threshold $\alpha\in[0,1]$: a motion passes if and only if the number of voters that support it exceeds αn. In this case, then $(d_1,\dots,d_n)$ is added to the vector of actors’ welfare; otherwise this vector remains the same. One more natural principle of group voting is as follows:

Principle A$_2$. *The group, $\mathcal{G}$, votes for a motion $(d_1,\dots,d_n)$ if and only if* $\#\{i\in\mathcal{G}\mid d_i>0\} > \alpha g$.

Let $2\beta = \ell/n$. The following diverse examples of social dynamics are typical of the corresponding parameters. In all of them, $n=200$, $\sigma=10$, and the group follows the utilitarian principle B.



a. $\mu=0$, $2\beta=0.5$, $\alpha=0.5$ b. $\mu=-1$, $2\beta=0.08$, $\alpha=0.07$ c. $\mu=-1$, $2\beta=0.08$, $\alpha=0.04$ d. $\mu=0.5$, $2\beta=0.08$, $\alpha=0.97$

Figure 1: Examples of social dynamics: the average welfare of egoists and group members vs step number.

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These figures have been obtained by simulation, as well as the following plots which represent the average welfare of egoists and group members after 500 steps versus the decision threshold α.

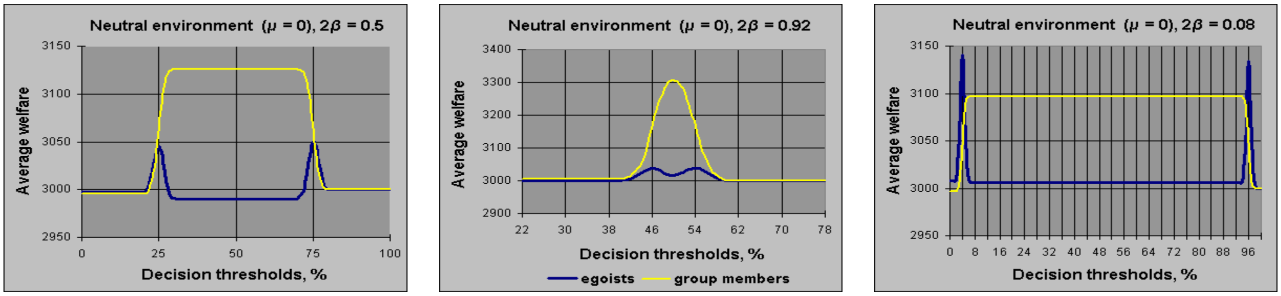


Figure 2: Average welfare in one simulated implementation with $n = 450$, $\sigma = 10$, principle B, and $s = 500$.

The first author has found the mathematical expectations of actors' one-step welfare increments: $M(\widetilde{d}_{\mathcal{E}})$ for egoists and $M(\widetilde{d}_{\mathcal{G}})$ for group members. Their approximations in the matrix notation are as follows:

$$M(\widetilde{d}^A_{\mathcal{E}}) \approx [P^A_G\ Q^A_G]\left(\mu\begin{bmatrix}F_\gamma\\ F_\alpha\end{bmatrix} + \frac{\sigma f}{\sqrt{pq\ell}}\begin{bmatrix}f_\gamma\\ f_\alpha\end{bmatrix}\right),\quad M(\widetilde{d}^A_{\mathcal{G}}) \approx [F_\gamma\ F_\alpha]\left(\mu\begin{bmatrix}P^A_G\\ Q^A_G\end{bmatrix} + \frac{\sigma f f^A_G}{\sqrt{pqg}}\begin{bmatrix}1\\ -1\end{bmatrix}\right), \tag{1}$$

$$M(\widetilde{d}^B_{\mathcal{E}}) \approx [P^B_G\ Q^B_G]\left(\mu\begin{bmatrix}F_\gamma\\ F_\alpha\end{bmatrix} + \frac{\sigma f}{\sqrt{pq\ell}}\begin{bmatrix}f_\gamma\\ f_\alpha\end{bmatrix}\right),\quad M(\widetilde{d}^B_{\mathcal{G}}) \approx [F_\gamma\ F_\alpha]\left(\mu\begin{bmatrix}P^B_G\\ Q^B_G\end{bmatrix} + \frac{\sigma f^B_G}{\sqrt{g}}\begin{bmatrix}1\\ -1\end{bmatrix}\right), \tag{2}$$

where

$$P^A_G \approx F\left(\frac{pg - 0.5 - [\alpha' g]}{\sqrt{pqg}}\right),\quad Q^A_G \approx 1 - F\left(\frac{pg - 0.5 - [\alpha' g]}{\sqrt{pqg}}\right),\quad f^A_G = f\left(\frac{pg - 0.5 - [\alpha' g]}{\sqrt{pqg}}\right) \tag{3}$$

with $\alpha' = g/2$ in the case of principle A_1 and $\alpha' = \alpha$ in the case of principle A_2, $[\alpha' g]$ the integer part of $\alpha' g$,

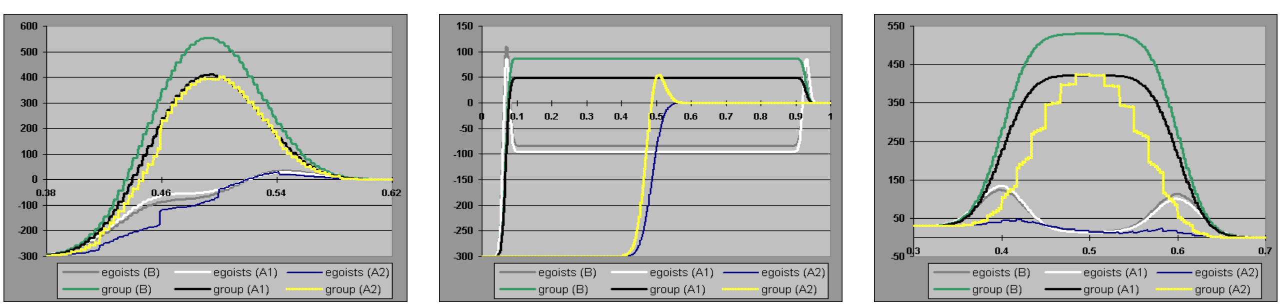
$$P^B_G = F\left(\frac{\mu\sqrt{g}}{\sigma}\right),\quad Q^B_G = 1 - P^B_G,\quad f^B_G = f\left(\frac{\mu\sqrt{g}}{\sigma}\right), \tag{4}$$

$$F_\theta = F\left(\frac{p\ell - 0.5 - [\theta n]}{\sqrt{pq\ell}}\right),\quad f_\theta = f\left(\frac{p\ell - 0.5 - [\theta n]}{\sqrt{pq\ell}}\right),\quad \gamma = \alpha - (1 - 2\beta), \tag{5}$$

$$p = F\left(\frac{\mu}{\sigma}\right),\quad q = 1 - p,\quad f = f\left(\frac{\mu}{\sigma}\right),\quad \text{the superscripts A and B denote the voting principles,} \tag{6}$$

$F(\cdot)$ and $f(\cdot)$ being the cumulative distribution function and the density of the standard normal distribution.

Three examples of the expected welfare dynamics obtained with the above expressions are given in Fig. 3.



a. $n = 300$, $\mu = -0.3$, $2\beta = 0.92$ b. $n = 450$, $\mu = -0.3$, $2\beta = 0.14$ c. $n = 300$, $\mu = 0.03$, $2\beta = 0.8$

Figure 3: Expected welfare after a series of $s = 1000$ steps with $\sigma = 10$ and principles A_1, A_2 and B.

Owing to the preferable welfare dynamics of the group, the prospects of group expansion through the entry of egoists are quite natural. As another conclusion, the group is challenged by the tradeoff between maximizing the absolute values of its welfare and relative advantages over egoists.

Cooperative models of Joint Implementation ⋆

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In this work we construct multistage cooperative model of the Kyoto Protocol realization and suggest time-consistent solutions to numerical examples with three country groups.

Without a doubt, climate change is the first among the global environmental threats to civilization at the beginning of the XXI Century. The importance of this problem is demonstrated by the adaptation costs the global community pays to protect itself from a growing number of natural disasters. The United Nations Framework Convention on Climate Change was signed at the World Summit on the Environment and Development in Rio de Janeiro in 1992, and the Kyoto Protocol to the Convention was adopted in 1997 [20]. The Kyoto Protocol proposes six innovative "mechanisms:" joint implementation, clean development, emission trading, joint fulfilment, banking, and sinks. The mechanisms aim to reduce the costs of curbing emissions by allowing Parties (Party is a term of Kyoto Protocol and means a country, or group of countries, that has ratified the Kyoto Protocol) to pursue opportunities to cut emissions more cheaply abroad than at home. The cost of curbing emissions varies considerably from region as a result of differences in, for example, energy sources, energy efficiency and waste management. It makes economic sense to cut emissions where it is cheapest to do so, given that the impact on the atmosphere is the same.

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The Kyoto protocol defines six flexibility mechanisms and three of them have the following sense: "joint implementation" provides for Annex B Parties (mostly highly developed industry countries) to implement projects that reduce emission, or remove carbon from the air, in other Annex B Parties, in return for emission reduction units (ERUs); the "clean development" mechanism provides for Annex B Parties to implement projects that reduce emissions in non-Annex B Parties, in return for certified emission reductions (CERs), and assist the host Parties in achieving sustainable development and contributing to the ultimate objective of the Convention; "emission trading" provides for Annex B Parties to acquire units from other Annex B Parties. The emission reduction units and certified emission reductions generated by the flexibility mechanisms can be used by Annex B Parties to help meet their emission targets.

That flexibility mechanisms are the base of the co-operation behavior because joint implementation, clean development, and emission trading comprehend that Parties work together and receive common "benefit" (emission reduction and economy of total cost). which should be allocated fairly. Joint implementation projects demand to develop distribution principles to allocate the benefit. Cooperative games with transferable utility (TU-games) are the mathematical models of such conflicts. Cooperative theory takes into account only income of cooperation but does not consider various coalitional attitudes towards extra payoffs due to common actions. In the problem of extra payoff allocation analyst should pay attention to desires and ambitions of individual players and coalitions to propose realizability of a solution. Cooperative game theory treats many optimality concepts (core, Shapley value, nucleolus, etc.) It is possible to choose an appropriate solution by a number of axioms [12]. We offer a tool to compare different solutions via their attraction for every coalition. The main idea of the proposed method is based on multicriteria methodology and the ASPID technique [10]. Vector Shapley and nucleolus are compared under different available information about excess preferences.

It is natural to use the dynamic cooperative theory to model the Kyoto Protocol realization [8]. An important problem in a dynamic cooperative theory is the time-consistency of a solution [13]. As in the theory of non zero-sum differential games [1,3,9], the use of optimality principles from the static theory in dynamic TU-games leads to contradictions arising from loss of time-consistency. Time-consistency of the optimality principle means that any segment of an optimal trajectory determines the optimal motion with respect to relevant initial states of the process. This property holds for the overwhelming majority of classical optimal control problems and follows from the Bellman optimality principle [2].

The absence of time-consistency in the optimality principle involves the possibility that the previous "optimal" decision are abandoned at some current moment of time, thereby making meaningless the problem of seeking an optimal control as such. This is why particular emphasis is placed on the construction of time-consistent optimality principles. This problem has attracted much attention [4,5,11,14–19].

The problem of time-consistency of a solution in a differential TU-cooperative game was investigated for the first time in [13]. It is directly relevant to regularization methods of cooperative games [13,21,22]. We suggest constructing time-consistent optimality principles for multistage cooperative games on the basis of "regularization" of optimality principles from the differential cooperative game theory. The idea of regularization is based on constructing delays of the payoffs to the players along optimal trajectory of the game.

We study time-consistency of the subcore selectors and propose two imputation distribution procedures that provide non-negative payoffs at every moment of the game. Both algorithms are based on delays of total payoff at a current moment of the game to avoid debtors at the following steps. We formulate necessary and sufficient conditions for the time-consistency of an imputation from the subcore in a multistage cooperative game. The results of this part were partially presented in [23,24].

Then we formulate a new problem of minimal reduction, and apply it to the regularization of dynamic TU-games. We apply a reduced game due to Davis and Maschler [6] and a modified Davis-Maschler reduced game to get the appropriate IDPs. This approach we can use even in the case of no time-consistent imputation in the core of a balanced game [7] .

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Repeated games with lack of information on one side and multistage auctions

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Models of multistage counter-auctions with asymmetric information, as introduced by De Meyer and Saley (2002), are considered. At such auctions two stockbrokers carry on repeated bidding with risky assets (shares). Before the bidding chance move determines the final value of one share once for all. This value is 1 with probability p and 0 with probability $1-p$. Player 1 is informed on the final value, Player 2 is not. Both players know p. At each subsequent step $t = 1, 2, ..., n$ both players propose simultaneously their prices. The maximal bid wins and one share is transacted at this price. Each player aims to maximize the value of his final portfolio (money plus shares).

At each step both players are supposed to remind all previous bids including these of their opponent. This allows uninformed player to draw conclusions on the proper final value of risky asset from the opponent's actions. The informed player should take it into account and try to reveal as few as possible information.

In game theory this kind of problems is modelled with repeated games with lack of information at one side (Aumann, Maschler (1995)). In such games players have to play a matrix game n times. Payoffs depend on the state of nature s, chosen at the beginning of the game from the finite set S by a chance move. Player 1 is informed on the result of the chance move while Player 2 knows only its prior probability distribution. During the game both players learn the opponent's choice of actions and make statistical inference based on these observations. At the end of the game Player 2 pays to Player 1 the average of intermediate payoffs. Such game may be regarded as a stochastic game with a set of probability distributions over S as a state space.

At each stage of the game Player 2 should reestimate his prior information on the base of Player 1's actions. Player 1 should take into account the opportunity of such reestimation and try to reveal as less as possible information to the opponent.

De Meyer and Saley constructed a class of zero-sum repeated games $G_n(p)$ with lack of information at one side modelling multistage counter-auctions. The sets of strategies (the sets of bids) are intervals [0,1]. Two payoffs are defined over the unit square according to two possible outcomes of chance move. One step gain of player buying the asset (and the opponent's loss) is equal to the difference between the final value of asset and his bid.

Before the start of the game chance move determines the payoff function with probabilities p and $1-p$. Players play this game n times. Player 1 is informed on the outcome of chance move. Both players know p. After each step players learn opponent's move. At the end of the game player 2 pays player 1 the sum of one step gains.

They obtain solutions for these n-step games and their asymptotics. The values $V_n(p)$ of n-step games infinitely grow up as $\sqrt{n}$. $V_n(p)/\sqrt{n}$ converges to a limit expressed through normal distribution. Thus the asymptotics of values for these games turns to be similar to the asymptotics of values for the games studied in the works Mertens, Zamir (1995), while the structure of games essentially differs.

It seems more realistic to assume that players may assign only threshold auction bids. We consider games $G_n^m(p)$ modelling auctions with the admissible bids being multiples of $1/m$.

The game $G_n^m(p)$ is an "eventually revealing" game. It means that in the long run the information advantage of Player 1 annihilates and the loss of Player 2 becomes equal to his loss in the case if he knows the state of nature a priori.

The asymptotics of values for these games essentially differs from such asymptotics for the games studied by De Meyer and Saley. We show that as the number of steps n infinitely increase the values of the games $G_n^m(p)$ converge to a piecewize-linear function of p. For $p \in [k/m, (k+1)/m], k = 0, \ldots, m-1$

$$\lim_{n\to\infty} \mathrm{val} G_n^m(p) = p(m-2k-1)/2 + k(k+1)/2m.$$

For p from this interval the optimal first move of player 2 converges to the bid k/m.

Thus, in the game with m admissible bids player 2 in the long run learns almost for sure the final value of risky asset. On the other hand, optimal behavior of player 1 requires permanent randomization. To determine his bids he employs lotteries depending on his private information. Therefore, in this model it is just the randomized behavior of player 1 that generates stochastic fluctuations of stock prices.

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War and Peace in Veto Voting *

Vladimir Gurvich †

Abstract

Let $I = \{i_1, \dots, i_n\}$ be a set of voters (players) and $A = \{a_1, \dots, a_p\}$ be a set of candidates (outcomes). Each voter $i \in I$ has a preference P_i over the candidates. We assume that P_i is a complete order on A. The preference profile $P = \{P_i, i \in I\}$ is called a *situation.* A situation is called *war* if the set of all voters I is partitioned in two coalitions K_1 and K_2 such that all voters of K_i have the same preference, $i = 1, 2$, and these two preferences are opposite. For a simple class of veto voting schemes we prove that the results of elections in all war situations uniquely define the results for all other (*peace*) situations.

Key words: veto, voting scheme, voting by veto, veto power, veto resistance, voter, candidate, player, outcome, coalition, block, effectivity function, veto function, social choice function, social choice correspondence

1 Main Theorem

We follow standard concepts and notation of veto voting theory; see e.g. [1, 2]. Let $I = \{i_1, \dots, i_n\}$ be a set of voters (players) and $A = \{a_1, \dots, a_p\}$ be a set of candidates (outcomes). Each voter $i \in I$ has a preference (a complete order) P_i over all candidates. The set of all preferences $P = \{P_i, i \in I\}$ is called a *preference profile* or a *situation.* A situation is called *war* if the set of voters I is partitioned in two coalitions K_1 and K_2 such that all voters of K_i have the same preference, $i = 1, 2$, and these two preferences are opposite.

Further, each voter $i \in I$ has μ_i veto cards and each candidate $a \in A$ has λ_a counter-veto cards. Positive integers μ_i and λ_a are called the *veto power* of $i \in I$ and *veto resistance* of $a \in A$, respectively. The corresponding integral-valued functions. $\mu : I \to \mathbb{Z}_+$ and $\lambda : I \to \mathbb{Z}_+$ are called veto power and veto resistance distributions.

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Let us define the *veto order* σ_μ as a word in the alphabet $I = \{i_1, \dots, i_n\}$ in which every letter $i \in I$ appears exactly μ_i times and hence each word σ_μ has the same length $\sum_{i \in I} \mu_i$. The triplet $(\lambda, \mu, \sigma_\mu)$ is called *veto voting scheme (VVS)*. It is realized as follows. In the given order σ_μ the voters put their veto cards against the candidates until all veto cards are finished. The voters have perfect information. It is forbidden to over-veto, that is as soon as a candidate a has got λ_a veto cards he is eliminated and no more veto cards can be used against him. All non-eliminated candidates are elected. Obviously, this set will be empty unless total veto power is strictly less than total veto resistance, that is

$$\sum_{i \in I} \mu_i < \sum_{a \in A} \lambda_a \tag{1.1}$$

If we assume further that

$$\sum_{a \in A} \lambda_a - \sum_{i \in I} \mu_i = 1. \tag{1.2}$$

then exactly one candidate is elected in each situation. However, unlike (1.1), this assumption is not mandatory.

The voters may behave in many different, sometimes rather sophisticated, ways, see [1, 2]. In this paper we consider only the simplest concept of their so-called *sincere* behavior. This means that each voter $i \in I$ always put each veto card against the worst (with respect to the preference P_i) not yet eliminated candidate. Hence, given a VVS $(\lambda, \mu, \sigma_\mu)$, a set of elected candidates $B = B(P) \subseteq A$ is uniquely defined for every situation $P = \{P_i, i \in I\}$.

In general, a mapping $S : P \to 2^A$ which assigns a set of candidates to every preference profile is called a *social choice correspondence (SCC)*, and it is called a *social choice function (SCF)* if only one candidate is elected, that is $|S(P)| = 1$ for each situation P. Thus, every veto voting scheme $(\lambda, \mu, \sigma_\mu)$ defines a SCC $S_{\lambda,\mu,\sigma_\mu}$ which is an SCF whenever (1.2) holds. The SCC or SCF generated by a veto voting scheme are called *veto SCC and SCF*, respectively.

A veto order σ_μ is called *simple* if the voters do not alternate, or more precisely, if there exists a permutation τ of I such that first the voter $\tau^{-1}(i_1)$ put all veto cards, followed by $\tau^{-1}(i_2)$, etc. Obviously, a simple veto order σ_μ is uniquely determined by μ and τ. The corresponding veto voting scheme and SCC we will call *simple* and denote by (λ, μ, τ) and $S_{\lambda,\mu,\tau}$, respectively.

In this paper we prove that each simple veto SCC is uniquely defined by the values it takes in the war situations. More precisely, the following statement holds.

Theorem 1 *Given two simple veto voting schemes $VVS' = (\lambda', \mu', \tau)$ and $VVS'' = (\lambda'', \mu'', \tau)$ which generate social choice correspondences $S' = S_{\lambda',\mu',\tau}(P)$ and $S'' = S_{\lambda'',\mu'',\tau}(P)$, respectively, if $S'(P) = S''(P)$ for each war situation P then $S'(P) = S''(P)$ for all P.*

Note, however, that Theorem 1 can not be promoted to the rank of a "general law of diplomacy". For example, it is not general enough just because it only holds when the two involved veto orders coincide, moreover, it must be a simple order; see Example 1 below.

Further, let us remark that in a war situation the veto order (simple or not) does not matter at all. In this case all candidates are uniquely ordered and all voters are split in two coalitions which veto candidates "from two opposite ends of this order". Some "moderate centrist" candidates will be elected and the set of these candidates does not depend on the order in which the voters act. More accurately these arguments are summarized as follows.

Lemma 1 *Given distributions λ, μ and two veto orders $\sigma'_\mu, \sigma''_\mu$, the equality $S_{\lambda,\mu,\sigma'_\mu}(P) = S_{\lambda,\mu,\sigma''_\mu}(P)$ holds for each war situation P.*

Yet, for other (peace) situations the result can depend on the veto order.

Example 1 *Let us consider two voters of veto power 3 and 1 and three candidates of veto resistance 1,2, and 2, that is $I = \{i_1, i_2\}, A = \{a_1, a_2, a_3\}$, $\mu_1 = 3, \mu_2 = 1$, $\lambda_1 = 1, \lambda_2 = \lambda_3 = 2$. Note that (1.2) holds and hence this voting scheme generates an SCF. Let the preferences be $a_1 > a_2 > a_3$ and $a_2 > a_1 > a_3$ for i_1 and i_2 respectively. This profile defines a peace situation P.*

First, let us consider two simple veto orders. If i_1 votes first then i_1 eliminates a_3 and puts one remaining veto card against a_2. Still a_2 is not eliminated, yet. Moreover, a_2 will be elected, since i_2 vetoes a_1. If i_2 votes first i_2 puts the veto card against a_3. This allows i_1 to eliminate both a_3 and a_2. Hence. in this case a_1 is elected.

Now let us consider two veto orders i_1, i_1, i_2, i_1 and i_1, i_2, i_1, i_1. These orders are not simple and they have similar pattern: first i_1, then i_2, and then i_1 again. Yet, these two orders result in electing different candidates. In the first case i_1 eliminates a_3, then i_2 eliminates a_1, and a_2 is elected. In the second case i_1 puts just one veto card against a_3, then i_2 eliminates a_3, and now i_1 can eliminate a_2 by the two remaining veto cards, hence, a_1 is elected.

Finally, let us remark that, according to Lemma 1, all four veto orders considered above would give the same result in each war situation.

2 An equivalent statement

The theorem can be equivalently reformulated in a less emotional way.

The *veto function* is defined as a mapping $V : 2^I \times 2^A \to \{0,1\}$, that is V has two arguments: a coalition of voters $K \subseteq I$ and a block of candidates $B \subseteq A$. The equalities $V(K,B) = 1$ and $V(K,B) = 0$ mean that K can, and respectively can not, veto B. The complementary function $E(K,B) = V(K, A \setminus B)$ is called *effectivity function*; see [1, 2].

Each pair of distributions $\mu : I \to \mathbb{Z}_+$ and $\lambda : I \to \mathbb{Z}_+$, generates a veto function $V = V_{\mu,\lambda}$

$$V(K,B) = 1 \text{ iff } \sum_{i \in K} \mu_i \geq \sum_{a \in B} \lambda_a. \tag{2.3}$$

In other words, K can veto B if the voters from K has sufficiently many veto-cards to eliminate all candidates from B. In these terms we can reformulate Theorem 1 as follows.

Theorem 2 *Let $VVS' = (\lambda', \mu', \tau)$ and $VVS'' = (\lambda'', \mu'', \tau)$ be two simple veto voting schemes such that they have the same simple veto order τ and their veto functions $V' = V_{\mu',\lambda'}$ and $V'' = V_{\mu'',\lambda''}$ are equal, that is $V'(K,B) = V''(K,B)$ for all $K \subseteq I, B \subseteq A$. Then the SCCs $\mathcal{S}' = \mathcal{S}_{\mu',\lambda',\tau}$ and $S'' = \mathcal{S}_{\mu'',\lambda'',\tau}$ are equal, too, that is $S'(P) = S''(P)$ for every situation P.*

To prove that Theorems 1 and 2 are equivalent we only need to show that Theorem 2 becomes trivial if we restrict ourselves by the war situations only. In other words, given a veto function, the results of elections in all war situations are uniquely defined, and vice versa. Due to Lemma 1, this is true for all (not only simple) veto orders.

Lemma 2 *Given two veto voting schemes $VVS' = (\lambda', \mu', \sigma'_{\mu'})$ and $VVS'' = (\lambda'', \mu'', \sigma''_{\mu''})$ which generate veto functions $V' = V_{\lambda',\mu',\sigma'_{\mu'}}$, $V'' = V_{\lambda'',\mu'',\sigma''_{\mu''}}$ and SCCs $S' = S_{\lambda',\mu',\sigma'_{\mu'}}$, $S'' = S_{\lambda'',\mu'',\sigma''_{\mu''}}$, the following claims are equivalent:*

(i) $V' = V''$, that is $V'(K,B) = V''(K,B)$ for all $K \subseteq I, B \subseteq A$,

(ii) $S'(P) = S''(P)$ for every war situation P.

Proof . Suppose that $V' \neq V''$, say $1 = V'(K,B) \neq V''(K,B) = 0$ for some $K \subseteq I, B \subseteq A$, that is in VVS' coalition K can veto block B but in VVS'' it can not. Consider a complete order P_0 over A such that each candidate from B precedes each candidate from $A \setminus B$. Let a_0 be the last candidate from B in this order. Define a war situation P as follows. All voters from K prefer candidates according to P_0, (that is for them $A \setminus B$ is better than B) and all voters from $I \setminus K$ have the opposite preference. Then obviously,

$a_0 \notin S'(P)$, since $V'(K, B) = 1$ and in VVS' coalition K can veto the whole block B including a_0. Yet $a_0 \in S''(P)$ since $V''(K, B) = 0$, that is in VVS'' coalition K has not enough veto power to eliminate B and hence a_0 will remain unvetoed. Thus $S'(P) \neq S''(P)$.

Vice versa, suppose that $S'(P) \neq S''(P)$ for a war situation P defined by a complete order P_0 over A and a partition $K, I \setminus K$. Without loss of generality, we can assume that $a_0 \in S''(P) \setminus S'(P)$, that is $a_0 \notin S'(P)$ and $a_0 \in S''(P)$. Let B consists of a_0 and all candidates preceding a_0 in order P_0. Then obviously, $V'(K, B) = 1$, otherwise a_0 would be elected in VVS', and $V''(K, B) = 0$, otherwise a_0 would be vetoed in VVS''.

Let us underline again that all above arguments are based on Lemma 1.

□

3 Proof of Theorems 2

Now without loss of generality we can assume that permutation τ is identical, that is i_1 distributes all veto cards first, followed by i_2, etc. In this case the argument τ becomes irrelevant and we will omit it in all formulas.

Given a voting scheme (λ, μ), let us fix a voter $i \in I$ and a candidate $a \in A$. We say that i *kills* a if a is eliminated and the last veto card against him is put by i. We say that i *wounds* a if i puts at least one veto card against a but does not eliminate a, that is either a is elected or eliminated (killed) later by some other voter. Finally, we say that i ignores a if i puts no veto card against a.

Lemma 3 *A voter can ignore and/or kill several candidates but wound at most one.*

Proof . Indeed, if i wounds a then i can not switch to another candidate a' before a is killed. This follows from our two basic assumptions: (i) the veto order is simple and (ii) the voting is sincere. Let us remark that both assumptions are important. □

Given a veto voting scheme (λ, μ), we will divide the voting procedure in $|I| = n$ steps. The m-th one begins after $m - 1$ voters are finished. In other words, we will consider voting schemes $VVS_m = (\lambda, \mu, m)$ with a truncated set of voters $I_m = \{i_1, \ldots, i_m\}$, where $m = 1, \ldots, n$.

Given m and a candidate $a \in A$, let $s_m(a)$ be the sequence of those voters from I_m who put at least one veto card against a. The length of this sequence is a non-decreasing function of m. We will also add a special symbol 0 at the end of $s_m(a)$ if after m steps a is eliminated. This symbol 0 will remain the last one, since it is not allowed to over-veto.

Now let us compare two veto voting schemes $VVS' = (\lambda', \mu', \tau) = (\lambda', \mu')$ and $VVS'' = (\lambda'', \mu'', \tau) = (\lambda'', \mu'')$. Theorem 2 claims that if two veto functions are equal $V'(K,B) = V''(K,B)$ for all $K \subseteq I$ and $B \subseteq A$ then the result of elections is always the same, that is $S'(P) = S''(P)$ for every situation P.

We would like to prove (by induction on m) a stronger claim: $s'_m(a) = s''_m(a)$ for each m and $a \in A$. Unfortunately, this is not true. Yet, it becomes true if we slightly modify the voting procedure. Let us introduce one additional rule: the last veto card of a voter must wound, not kill. We will refer to this rule as the DNK-rule. Suppose that a voter $i \in I$ runs out of the veto cards exactly after killing a candidate $a' \in A$. In this case we will give one extra veto card to i and one extra anti-veto card to the candidate a who follows a' in the order P_i. Anyway, these two cards will immediately annihilate each other since i will put the extra card against a by the next (and the last) move. Hence, this modification preserves the SCC $S(P)$ in every situation P. Though the veto function $V(K,B)$ changes, we will see that this does not create a problem.

We will prove by induction on m that $s'_m(a) = s''_m(a)$ for all $a \in A$. Let $m = 1$ and suppose that $s'_1(a) \neq s''_1(a)$. There are two possibilities.

Case 1. The candidate a is eliminated in one voting scheme, say VVS', but not in the other, that is $\mu'(i_1) \geq \lambda'(B) = \sum_{b \in B} \lambda'(b)$, $\quad \mu''(i_1) < \lambda''(B) = \sum_{b \in B} \lambda''(b)$, where block B consists of a and all candidates worse than a with respect to P_{i_1}. In other words, i_1 has enough veto power to eliminate B in VVS' but not in VVS''. Indeed, in this case $s'_1(a) \neq s''_1(a)$, more precisely, $s'_1(a) = (s''_1(a), 0)$. Then it is clear that veto functions V' and V'' are not equal either, because $1 = V'(\{i_1\}, B) \neq V''(\{i_1\}, B) = 0$, and we get a contradiction.

Case 2. In VVS' and VVS'' voter i_1 eliminates the same set of candidates. Yet, in VVS' i_1 runs out of the veto cards exactly after killing a candidate, while in VVS'' i_1 still have some extra veto cards to wound a new candidate a but not enough for killing a. Though in this case $s'_1(a)$ and $s''_1(a)$ differ, yet, we must apply the DNK-rule and they become equal.

Analogous (just a little more sophisticated) arguments work for the general inductive step from $m-1$ to m. We assume that $s'_{m-1}(a) = s''_{m-1}(a)$ for all $a \in A$ and will prove indirectly that $s'_m(a) = s''_m(a)$ for all $a \in A$. Assume that $s'_m(a) \neq s''_m(a)$ for some $a \in A$ and consider again the same two cases. In case 2 our arguments do not change, except we substitute i_m for i_1.

Case 2. In VVS' and VVS'' voter i_m eliminates the same set of candidates. Yet, in VVS' i_m runs out of the veto cards exactly after killing a candidate a', while in VVS'' i_m still have some extra veto cards to wound a new candidate a, though not enough to kill a. In this case $s'_m(a) \neq s''_m(a)$ but we apply the DNK-rule and they become equal.

Case 1. Candidate a is killed by i_m in one voting scheme, say in VVS'', but not in the other. Perhaps, in VVS' voter i_m did not even wound a. Yet obviously, there is a candidate a^0 wounded but not killed by i_m. By induction hypothesis, in VVS'' the voter i_m kills a^0 before a. In this case $s'_1(a^0) \neq s''_1(a^0)$ (more precisely, $(s'_m(a^0), 0) = s''_m(a^0)$) and we will prove that $V' \neq V''$. Consider VVS', set $B_1 = \{a^0\}$, and denote by K_1 the coalition of all voters who wound a^0. In particular, $i_m \in K_1$. By Lemma 3, the voters of K_1 wound no other candidate but perhaps they kill some other. Denote by B_2 the set of all candidates killed by K_1 (if any) and by K_2 the set of all voters who wound a candidate from B_2. By construction, $K_1 \cap K_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. By Lemma 3, the voters of K_2 wound no other candidates, except the candidates from B_2, but perhaps they kill some other. Denote by B_3 the set of all candidates killed by K_2 (if any) and by K_3 the set of all voters who wound a candidate from B_3, etc. Obviously, this process will finish and we get a family of coalitions $K_1, \ldots, K_p$ and blocks $B_1, \ldots, B_q$ such that

(i) $p = q$ or $p = q + 1$;

(ii) the obtained coalitions and blocks are pairwise disjoint;

(iii) the voters from K_j wound only candidates from B_j and kill only candidates from B_{j+1}, and vice versa, the candidates from B_j are wounded only by the voters from K_j and are killed only by the candidates from K_{j-1}.

Let $K = \cup_{j=1}^{p} K_j$ and $B = \cup_{j=1}^{q} B_j$. By (iii), K put all veto cards only against B and vice versa, each veto card put against B belongs to K. Yet, K can not eliminate B in VVS', hence $V'(K, B) = 0$. Now let us consider VVS'' and the same pair (K, B). By induction hypothesis, (iii) "almost holds" again, except the voter i_m after eliminating a^0 can (and must, according to the DNK-rule) spend the remaining veto cards out of B. Hence, $V''(K, B) = 1$, since in VVS'', unlike VVS', a^0 is killed by i_m. Thus $1 = V''(K, B) \neq V'(K, B) = 0$.

Finally, we have to recall that in m steps the original distributions $\lambda', \mu', \lambda'', \mu''$ can change. Yet, $S'(P)$ and $S''(P)$ can not. Let us show that $V'(K, B)$ and $V''(K, B)$ (for the considered pair (K, B)) can not change either. Indeed, in both VVS' and VVS'', the voters from $K \setminus \{i_m\}$ put all veto cards only against B and the candidates from B receive all veto cards only from the voters of K. Hence, the numbers of extra cards given to K and to B are equal. Thus the modified veto functions take in (K, B) the same values as the original ones. In particular, $1 = V''(K, B) \neq V'(K, B) = 0$ and we get a contradiction.

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Owen coalitional value without additivity axiom*

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Abstract

We show that the Owen value for TU games with coalition structure can be characterized without additivity axiom similarly as it was done by Young for the Shapley value for general TU games. Our axiomatization via four axioms of efficiency, marginality, symmetry across coalitions, and symmetry within coalitions is obtained from the original Owen's one by replacement of the additivity and null-player axioms via marginality. We show that the alike axiomatization for the generalization of the Owen value suggested by Winter for games with level structure is valid as well.

Keywords: cooperative TU game, coalitional structure, Owen value, axiomatic characterization, marginality.

Mathematics Subject Classification 2000: 91A12

1 Introduction

We consider the Owen value for TU games with coalition structure that can be regarded as an expansion of the Shapley value for the situation when a coalition structure is involved. The Owen value was introduced in [2] via a set of axioms it determining. These axioms were vastly inspired by original Shapley's axiomatization that in turn exploits the additivity axiom. However, the additivity axiom that being a very beautiful mathematical statement does not express any fairness property. Another axiomatization of the Shapley value proposed by Young [5] via marginality, efficiency, and symmetry appears to be more attractive since all the axioms present different reasonable properties of fair division. The goal of this paper is to evolve the Young's approach to the case of the Owen value for games with coalition structure. We provide a new axiomatization for the Owen value

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without additivity axiom that is obtained from the original Owen's one by the replacement of additivity and null-player via marginality. We show that the similar axiomatization can be also obtained for the generalization of the Owen value suggested by Winter in [4] for games with level structure.

Sect. 2 introduces basic definitions and notation. In Sect. 3, we present an axiomatization for the Owen value for games with coalition structure and for the Winter's generalization for games with level structure on the basis of marginality axiom.

2 Definitions and notation

First recall some definitions and notation. A *cooperative game with transferable utility (TU game)* is a pair $\langle N, v\rangle$, where $N = \{1, \ldots, n\}$ is a finite set of $n \geq 2$ players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ (or $S \in 2^N$) of s players is called a *coalition*, and the associated real number $v(S)$ presents the *worth* of the coalition S. For simplicity of notation and if no ambiguity appears, we write v instead of $\langle N, v\rangle$ when refer to a game, and also omit the braces when writing one-player coalitions such as $\{i\}$. The set of all games with a fixed player set N we denote $\mathcal{G}_N$. For any set of games $\mathcal{G} \subseteq \mathcal{G}_N$, a *value on* $\mathcal{G}$ is a mapping $\psi\colon \mathcal{G} \to \mathbb{R}^n$ that associates with each game $v \in \mathcal{G}$ a vector $\psi(v) \in \mathbb{R}^n$, where the real number $\psi_i(v)$ represents the *payoff* to the player i in the game v.

We consider games with coalition structure. A coalition structure $\mathcal{B} = \{B_1, ..., B_m\}$ on a player set N is a partition of the player set N, i.e., $B_1 \cup ... \cup B_m = N$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. Denote by $\mathfrak{B}_N$ a set of all coalition structures on N. In this context a value is an operator that assigns a vector of payoffs to any pair $(v, \mathcal{B})$ of a game and a coalitional structure on N. More precisely, for any set of games $\mathcal{G} \subseteq \mathcal{G}_N$ and any set of coalition structures $\mathfrak{B} \subseteq \mathfrak{B}_N$, a *coalitional value on* $\mathcal{G}$ *with a coalition structure from* $\mathfrak{B}$ is a mapping $\xi\colon \mathcal{G} \times \mathfrak{B} \to \mathbb{R}^n$ that associates with each pair $\langle v, \mathcal{B}\rangle$ of a game $v \in \mathcal{G}$ and a coalition structure $\mathcal{B} \in \mathfrak{B}$ a vector $\xi(v, \mathcal{B}) \in \mathbb{R}^n$, where the real number $\xi_i(v, \mathcal{B})$ represents the *payoff* to the player i in the game v with the coalition structure $\mathcal{B}$.

We say players $i, j \in N$ are *symmetric* with respect to the game $v \in \mathcal{G}$ if they make the same marginal contribution to any coalition, i.e., for any $S \subseteq N \backslash \{i, j\}$, $v(S \cup i) = v(S \cup j)$. A player i is a *null-player* in the game $v \in \mathcal{G}$ if he adds nothing to any coalition non-containing him, i.e., $v(S \cup i) = v(S)$, for every $S \subseteq N \backslash i$.

In what follows we denote the cardinality of any set A by $|A|$.

A coalitional value ξ is *efficient* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$,

$$\sum_{i \in N} \xi_i(v, \mathcal{B}) = v(N).$$

A coalitional value ξ is *marginalist* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, for every $i \in N$, $\xi_i(v, \mathcal{B})$ depends only upon the ith marginal utility vector $\{v(S \cup i) - v(S)\}_{S \subseteq N \setminus i}$, i.e.,

$$\xi_i(v, \mathcal{B}) = \phi_i(\{v(S \cup i) - v(S)\}_{S \subseteq N \setminus i}),$$

where $\phi_i \colon \mathbb{R}^{2^{n-1}} \to \mathbb{R}^1$.

A coalitional value ξ possesses the *null-player property* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, every null-player i in game v gets nothing, i.e., $\xi_i(v, \mathcal{B}) = 0$.

A coalitional value ξ is *additive* if, for any two $v, w \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$,

$$\xi_i(v + w, \mathcal{B}) = \xi_i(v, \mathcal{B}) + \xi_i(w, \mathcal{B}),$$

where $(v + w)(S) = v(S) + w(S)$, for all $S \subseteq N$.

We consider two symmetry axioms. First note that for a given game $v \in \mathcal{G}$ and coalition structure $\mathcal{B} = \{B_1, ..., B_m\} \in \mathfrak{B}$, we can define a game between coalitions or in other terms a *quotient game* $\langle M, v^{\mathcal{B}} \rangle$ with $M = \{1, \ldots, m\}$ in which each coalition B_i acts as a player. We define the quotient game $v^{\mathcal{B}}$ as:

$$v^{\mathcal{B}}(Q) = v(\bigcup_{i \in Q} B_i), \qquad \text{for all } Q \subseteq M.$$

A coalitional value ξ is *symmetric across coalitions* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, for any two symmetric in $v^{\mathcal{B}}$ players $i, j \in M$, the total payoffs for coalitions B_i, B_j are equal, i.e.,

$$\sum_{k \in B_i} \xi_k(v, \mathcal{B}) = \sum_{k \in B_j} \xi_k(v, \mathcal{B}).$$

A coalitional value ξ is *symmetric within coalitions* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, any two players who are symmetric in v and belong to the same coalition in B get the same payoffs, i.e., for any $i, j \in B_k \in \mathcal{B}$ that are symmetric in v,

$$\xi_i(v, \mathcal{B}) = \xi_j(v, \mathcal{B}).$$

The Owen value was introduced in Owen [2] as the unique efficient, additive, symmetric across coalitions, and symmetric within coalitions coalitional value that possesses the null-player property.[1] In the sequel the Owen value in a game v with a coalition structure $\mathcal{B}$ we denote $Ow(v, \mathcal{B})$. For any $v \in \mathcal{G}_N$ and any $\mathcal{B} \in \mathfrak{B}_N$, for all $i \in N$, $Ow_i(v, \mathcal{B})$ can be given by the following formula

$$\begin{aligned} Ow_i(v, \mathcal{B}) \quad = \quad & \sum_{\substack{Q \subseteq M \\ Q \not\ni k}} \sum_{\substack{S \subseteq B_k \\ S \not\ni i}} \frac{q!\,(m-q-1)!\,s!\,(b_k - s - 1)!}{m!\,b_k!} \cdot \\ & \qquad (v(\bigcup_{j \in Q} B_j \cup S \cup i) - v(\bigcup_{j \in Q} B_j \cup S)), \end{aligned} \tag{1}$$

[1] We present above the original Owen's axioms in the formulation of Winter [4].

where k is such that $i \in B_k \in \mathcal{B}$.

3 Axiomatization of the Owen value via marginality

We prove below that the Owen value defined on entire set of games $\mathcal{G}_N$ with any possible coalition structure from $\mathfrak{B}_N$ can be characterized by four axioms of efficiency, marginality, symmetry across coalitions, and symmetry within coalitions. Our proof strategy by induction is similar to that in Young [5].

Theorem 1 *The only efficient, marginalist, symmetric across coalitions, and symmetric within coalitions coalitional value defined on $\mathcal{G}_N \times \mathfrak{B}_N$ is the Owen value.*

Remark 2 Notice that similar as in Young for the Shapley value every efficient, marginalist, symmetric across coalitions, and symmetric within coalitions coalitional value defined on $\mathcal{G}_N \times \mathfrak{B}_N$ possesses the null-player property.

Remark 3 It is reasonable to note that for some subclasses of games $\mathcal{G} \subset \mathcal{G}_N$, for example for the subclass $\mathcal{G}_N^{sa}$ of superadditive games or for the subclass $\mathcal{G}_N^{c}$ of constant-sum games, if it is desired to stay entirely within one of these subclasses and not in the entire set of games $\mathcal{G}_N$, the same axiomatization for the Owen value via efficiency, marginality, symmetry across coalitions, and symmetry within coalitions is still valid. It can be proved similarly to the case of the Shapley value (see [5], [1]) adapting the ideas applied in the proof of Theorem 1.

Winter [4] introduced a generalization of the Owen value for games with level structure. A *level structure* is a finite sequence of partitions $\mathcal{L} = (\mathcal{B}_1, ..., \mathcal{B}_p)$ such that every $\mathcal{B}_i$ is a refinement of $\mathcal{B}_{i+1}$. Denote by $\mathfrak{L}_N$ the set of all level structures on N. In this context, for any set of games $\mathcal{G} \subseteq \mathcal{G}_N$ and any set of level structures $\mathfrak{L} \subseteq \mathfrak{L}_N$, a *level structure value on $\mathcal{G}$ with a level structure from* $\mathfrak{L}$ is an operator defined on $\mathcal{G} \times \mathfrak{L}$ that assigns a vector of payoffs to any pair $(v, \mathcal{L})$ of a game $v \in \mathcal{G}$ and a level structure $\mathcal{L} \in \mathfrak{L}$. It is not difficult to see that the Winter's extension of the Owen value for games with level structure admits the similar axiomatization with the replacement of two above mentioned symmetry axioms by the following two captured from [4].

A level structure value ξ is *coalitionally symmetric* if, for all $v \in \mathcal{G}$ and any level structure $\mathcal{L} = (\mathcal{B}_1, ..., \mathcal{B}_p)$, for each level $1 \leq k \leq p$ for any two symmetric in $v^{\mathcal{B}_k}$ players $i, j \in M_k$ such that $B_i, B_j \in \mathcal{B}_k$ are subsets of the same component in $\mathcal{B}_t$ for all $t > k$, the total payoffs for coalitions B_i, B_j are equal, i.e.,

$$\sum_{r \in B_i} \xi_r(v, \mathcal{L}) = \sum_{r \in B_j} \xi_r(v, \mathcal{L}).$$

A level structure value ξ is *symmetric within coalitions* if, for all $v \in \mathcal{G}$ and any level structure $\mathcal{L} = (\mathcal{B}_1, ..., \mathcal{B}_p)$, any two players i, j who are symmetric in v and for every level $1 \leq k \leq p$ simultaneously belong or not to the same non-singleton coalition in $\mathcal{B}_k$, get the same payoffs, i.e., $\xi_i(v, \mathcal{L}) = \xi_j(v, \mathcal{L})$.

Theorem 4 *The only efficient, marginalist, coalitionally symmetric, and symmetric within coalitions level structure value defined on $\mathcal{G}_N \times \mathfrak{L}_N$ is the Winter value for games with level structure.*

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Pareto choice functions and elimination of dominated strategies

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Abstract

Regarded demand functions as a special case of choice functions, Uzawa and Arrow referred analysis of rational demand behavior to an abstract framework of studying rational choice functions. The foundation for the rationalizability results in tradition choice theory is formed by properties of the Weak Axiom of Revealed Preference for a binary choice and Path Independence for a general choice. Plott considered the concept of "path independence" of a choice function C (i.e. for all A, $B \subset P$, $C(A \cup B) = C(C(A) \cup C(B)))$ as a means of weakening the condition of rationality in a manner which preserves one of the key properties of rational choice, namely that choice over any subset should be independent of the way the alternatives were initially divided up to consideration.

The aim of the talk is to apply theory of choice functions to analyze rationality of strategic behavior in noncooperative game theory. The idea is as follows. Let $\mathcal{U}$ be a set of n-persons games in the normal form. Given a game $G = (S_1, \dots, S_n; u_1, \dots, u_n)$, S_i denotes the set of strategies of the i-th agent and $u_i : \times_{i \in N} S_i \to \mathbb{R}$ denotes its payoff function, a solution concept $Sol(G) = (Sol(G)_i \subset S_i,\ i \in N)$ is a tuple of strategies of each players which are used in the solution.

Given a solution concept Sol, define a choice function for each player, say the first, of the form

$$C_{Sol}(A) := Sol(G|_A)_1 \subset A,$$

where we denote by $G|_A := (A, S_2, \dots, S_n; u_1, \dots, u_n)$ a sub-game of G under shrinking the strategy set of the first player from S_1 to A. So that, we call the strategic behavior of the first player *rational* under the solution concept Sol if the corresponding choice function is *rational*.

We demonstrate that a solution concept of the form elimination of purely dominated strategies lead to rational choice function being ordinally rationalizable and, moreover, any ordinally rationalizable choice function is of such a form. Elimination of weakly dominated strategies as the solution concept leads to path independent choice functions, and Plott choice function might be implemented of this form. The Nash equilibrium as a solution concept does not reveal a rationale strategic behavior in the framework of our approach.

Generalized kernels and bargaining sets for coalition systems.

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We consider a generalization of the theory of bargaining sets for cooperative TU–games (see [1], [4], [8]). Objections and counter-objections are permited only between elements of a fixed collection $\mathcal{A}$ of subsets of the player set. For $\mathcal{A}$, we define a generalization of the kernel $\mathcal{K}_{\mathcal{A}}$, contained in the corresponding bargaining set $\mathcal{M}^i_{\mathcal{A}}$. We give sufficient condition on $\mathcal{A}$ for existence of $\mathcal{K}_{\mathcal{A}}$. This condition is also necessary for existence of $\mathcal{K}_{\mathcal{A}}$ for all games but is not necessary for existence of the corresponding bargaining set. For a set of coalitions $\mathcal{A}$ we also define a generalization of the nucleolus $\mathcal{N}_{\mathcal{A}}$ but we have an example when it does not belong to nonempty $\mathcal{K}_{\mathcal{A}}$.

Let (N, v) be a cooperative TU-game, $K, L \subset N$, x be an imputation for (N, v). A *vector-objection of* K *against* L *in* x is a vector $y_C = \{y_i\}_{i\in C}$ such that $K \subset C \subset N$, $L \cap C = \emptyset$, $\sum_{i\in C} y_i = v(C)$, $y_i > x_i$ for all $i \in K$, and $y_i \geq x_i$ for all $i \in C$. A *vector counter–objection of* L *against* K *to the objection* y_C *in* x is a vector z_D such that $L \subset D \subset N$, $K \not\subset D$, $\sum_{i\in D} z_i = v(D)$, $z_i \geq x_i$ for all $i \in D$, $z_i \geq y_i$ for all $i \in C \cap D$. A vector objection is *justified* if there is no vector counter–objection to it.

For superadditive v, K has a justified vector objection against L in x iff K has a justified objection against L in x (w.r.t. the definitions in [1]).

Let $\mathcal{A}$ be a set of subsets of N. An individually rational imputation x of (N, v) belongs to the *bargaining set* $\mathcal{M}^i_{\mathcal{A}}(N, v)$ if for all $K, L \in \mathcal{A}$ there are no justified objections of K against L in x.

For a set $\mathcal{A}$ of subsets of N consider the following generalization of the kernel. Let $K, L \subset N$ and x be an imputation of (N, v). K *overweights* L in x if $K \cap L = \emptyset$, $\sum_{i\in L} x_i > v(L)$, and $s_{K,L}(x) > s_{L,K}(x)$, where

$$s_{P,Q}(x) = \max\{v(S) - \sum_{i\in S} x_i : \quad S \subset N, P \subset S, Q \not\subset S\}.$$

The set $\mathcal{K}_{\mathcal{A}}(N, v)$ is a set of all individually rational imputations x of (N, v) such that no $K \in \mathcal{A}$ overweights any $L \in \mathcal{A}$.

It was proved in [5] that $\mathcal{K}_{\mathcal{A}}(N, v) \subset \mathcal{M}^i_{\mathcal{A}}(N, v)$.

In what follows we suppose that $v(S) \geq \sum_{i\in S} v(\{i\})$ for all $S \subset N$. Then we can assume without loss of generality that $v(\{i\}) = 0$ for all $i \in N$ and $v(S) \geq 0$ for all $S \subset N$.

If $\mathcal{A}$ is a set of all singletons, then $\mathcal{M}^i_{\mathcal{A}} = \mathcal{M}^i_1$, $\mathcal{K}_{\mathcal{A}}$ is the kernel and the existence theorems for these sets are well known (see [2], [3], [7], [4]). Here we describe conditions on $\mathcal{A}$ that ensure the existence of $\mathcal{M}^i_{\mathcal{A}}(N, v)$ and $\mathcal{K}_{\mathcal{A}}(N, v)$ for all v.

A set of coalitions $\mathcal{A}$ generates the undirected graph $G = G(\mathcal{A})$, where $\mathcal{A}$ is the set of vertices and $K, L \in \mathcal{A}$ are adjacent iff $K \cap L = \emptyset$.

Theorem 1. *Let $\mathcal{A}$ be a set of subsets of N satisfying the following condition : if a single vertex is taken in each connected component of $G(\mathcal{A})$, then the union of the remaining elements of $\mathcal{A}$ does not contain N. Then $\mathcal{K}_{\mathcal{A}}(N,v) \neq \emptyset$ for all (N,v).*

The proof is based on Peleg's lemma [7].

If $\mathcal{A}$ is a collection of subsets of N, then a player $i \in S$ is called a *fanatic* for $\mathcal{A}$ if it belongs to precisely one element of $\mathcal{A}$.

Corollary. *If each element of $\mathcal{A}$ contains a fanatic of $\mathcal{A}$, then $\mathcal{K}_{\mathcal{A}}(N,v) \neq \emptyset$ for all (N,v).*

Let us consider the collections of subsets $\mathcal{A}$ that satisfy the condition of Theorem 1. If N is not covered by the elements of $\mathcal{A}$, then this condition is obviously fulfilled. Let $\mathcal{A}^0$ be the set of isolated vertices of graph $G(\mathcal{A})$, $\bar{\mathcal{A}} = \mathcal{A} \setminus \mathcal{A}^0$. Then the collections of coalitions $\mathcal{A}$ and $\bar{\mathcal{A}}$ satisfy the condition of Theorem 1 simultaneously.

Let us describe for some n all collections of coalitions $\bar{\mathcal{A}}$ that cover N.

If $n = 3$ then either $\bar{\mathcal{A}} = \{\{1\},\{2\},\{3\}\}$ or $\bar{\mathcal{A}} = \{\{i,j\},\{k\}\}$.

If $n = 4$ then either each element of $\bar{\mathcal{A}}$ contains a fanatic of $\bar{\mathcal{A}}$ or $\bar{\mathcal{A}} = \{\{i,j\},\{k,l\},\{i,k\},\{j,l\}\}$.

If $n = 5$ then either each element of $\bar{\mathcal{A}}$ contains a fanatic of $\bar{\mathcal{A}}$ or
$\bar{\mathcal{A}} = \{\{i,j\},\{k,l,m\},\{i,k\},\{j,l\}\}$ or
$\bar{\mathcal{A}} = \{\{i,j\},\{k,l,m\},\{i,k\},\{j,l,m\}\}$.

Note that for $\mathcal{A}$ satisfying the condition of Theorem 1 the case $|\bar{\mathcal{A}}| > n$ is possible.

Example 1. $n = 11$, $\mathcal{A} = \{S \cup \{11\},\ (\{1,2,3,4\} \setminus S) \cup \{i(S)\} :$
$|S| = 2,\ S \subset \{1,2,3,4\},\ i(S) \in \{5,6,7,8,9,10\},\ i(S) \neq i(T)$ for $S \neq T\}$. Here $\bar{\mathcal{A}} = \mathcal{A}$ and $|\mathcal{A}| = 12$.

Theorem 2. *If a collection $\mathcal{A}$ of subsets of N does not satisfy the condition of Theorem 1, then there exits (N,v) such that $\mathcal{K}_{\mathcal{A}}(N,v) = \emptyset$.*

For nonemptyness of $\mathcal{M}^i_{\mathcal{A}}$ the condition of Theorem 1 is not necessary.

Example 2. Let $\mathcal{A} = \{K, L, M\}$, where $K \subset L$, $K \neq L$, $M \cap L = \emptyset$, $M \cup L = N$. Then $\mathcal{A}$ does not satisfy the condition of Theorem 1 but it can be proved (as in Theorem 1) that $\mathcal{M}^i_{\mathcal{A}}(N,v) \neq \emptyset$.

An open problem is to find selectors of the set $\mathcal{K}_{\mathcal{A}}$. A generalization of the nucleolus is the set $\mathcal{N}_{\mathcal{A}}(N,v)$, where we consider only excesses of coalitions containing elements of $\mathcal{A}$. Differently from the case when $\mathcal{A}$ is the collection of singletons, it is possible that $\mathcal{N}_{\mathcal{A}}(N,v) \cap \mathcal{K}_{\mathcal{A}}(N,v) = \emptyset$ and $\mathcal{A}$ satisfies the condition of Theorem 1.

Example 3. Let $N = \{1,2,3,4\}$, $\mathcal{A} = \{\{1\},\{2\},\{3,4\}\}$, $v(\{i\}) = 0$ for all $i \in N$, $v(N) = v(S) = 1$ for $|S| = 3$, $v(1,2) = v(3,4) = 0$, $v(1,3) = v(2,3) = 1$, $2/3 < v(1,4) = v(2,4) < 1$.

Here $\mathcal{N}_{\mathcal{A}} = \{z\}$, where $z_1 = z_2 = z_3 = 1/2 - v(2,4)/4$, $z_4 = 3v(2,4)/4 - 1/2$. But $z \notin \mathcal{K}_{\mathcal{A}}(N, v)$ because $s_{\{1\},\{3,4\}}(z) = v(2,4)/2 > s_{\{3,4\},\{1\}}(z) = 1/2 - v(2,4)/4$.

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Information equilibrium: Existence and core equivalence*

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The paper deals with the existence and fuzzy core equivalence problems for so-called information equilibrium and its modifications in pure exchange economy with externalities. Some existence results similar to that obtained for the classic Walrasian equilibrium are established, and coalitional stability with respect to several types of fuzzy blocking is investigated. In particular, the concept of representative fuzzy core is introduced, and the coincidence of this core and the set of modified (dual) information equilibria is demonstrated under rather general assumptions. A notion of replica for the exchange economy with externalities is elaborated, and a generalization of the famous Debreu-Scarf Theorem on the shrinkability of the cores is proposed. Besides the existence and core equivalence problems, some Pareto-optimality and individual rationality properties of equilibrium in the models with nonautonomous preferences are discussed. The main tool we apply to investigate the Lindahlian type equilibria proposed in the paper is the information extension of the original market with externalities, introduced by V.L.Makarov (1982), *Modern Problems of Mathematics*, v.19, 23 - 59, Moscow: VINITI (in Russian).

Key words: Information equilibrium, representative fuzzy core, information extension.

§1. Modified information equilibrium

It is common knowledge (see, e.g., [1], [2]), that the Pareto-optimality of Walrasian equilibria in the models of economic exchange is guaranteed only in the case of autonomous preferences when the utility function of each agent depends only on the agent's own consumption. At the same time, many important problems of equilibrium analysis do not fit within the framework of the classical autonomy requirement. The most typical example is the problem of choosing the production level of public goods which is to be one and the same for the whole economic system and various modifications of the problem (see, e.g., [7], [8], [9] and references therein). Possible ways to eliminate the inefficiency of the standard market mechanism in the models with externalities (nonautonomous preferences) might consist in some kind

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of extension of the original market. Such an extension is obtained, for example, at the expense of introducing additional commodities for exchanging the information on the consumption structure of the economic system as a whole which would be appropriate for the agents.

In this section we introduce a rather general concept of equilibrium under the presence of externality, based on the information extension that was first proposed in [2] and in more general setting in [3]. First, remind that a model of economic exchange with externalities (nonautonomous preferences) is an economy of the form

$$\mathcal{E} =< N, L, \{X_i, w^i, u_i, \}_{i\in N} >,$$

where $N = \{1, \ldots, n\}$ is the set of agents and $X_i \subseteq \mathcal{R}^l,\ w^i \in \mathcal{R}^l$, and $u_i : \prod_{j\in N} X_j \to \mathcal{R}$ are respectively the consumption set, the initial endowments, and the utility function of an agent $i \in N$.

The distinction of the model $\mathcal{E}$ from the standard exchange model is that, in the first case, the values of the utility function of each agent are defined not only by the agent's individual consumption but also by the consumption of the other economic agents.

As it was already mentioned, the traditional Walrasian equilibria are, typically, not Pareto optimal in the presence of externalities. The generalized information equilibria presented in the paper not only retains the most important features of the classical equilibrium but also (according to the results concerning the fuzzy core, given in the paper) are localized in a sufficiently narrow section of the Pareto boundary of the model under consideration.

To give the main definitions we start with some technical notations. Put $A = (\mathcal{R}^l)^N$, $A_0 = \{(p^1, \ldots, p^n) \in (\mathcal{R}^l_+)^N \mid p^1 = \ldots = p^n\}$ and denote by $\mathcal{P}$ the set of individual prices of the model $\mathcal{E}$, defined by the formula

$$\mathcal{P} = \{(p_1, \ldots, p_n) \in A^N \setminus \{0\} \mid \exists p_0 \in A_0[\sum_{i\in N} p_i = p_0]\}.$$

From the definition of $\mathcal{P}$ it follows, that for any $p = (p_1, \ldots, p_n) \in \mathcal{P}$ there exists the vector $p^0 = p^0(p) \in \mathcal{R}^l$ such that $\sum_{i\in N} p_i = (p^0, \ldots, p^0)$. This vector $p^0(p)$ is said to be the public prices, corresponding to the individual prices $p = (p_1, \ldots, p_n)$. To introduce the income functions we deal with denote by $\mathcal{A}$ the set of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ of functions $\alpha_i : \mathcal{P} \to \mathcal{R},\ i \in N$, satisfying requirement similar to the Walras law for the standard exchange models

$$\sum_{i\in N} \alpha_i(p) = p^0 \cdot \sum_{i\in N} w^i, \quad p \in \mathcal{P}. \tag{1}$$

Further, put $X = \prod_{i\in N} X_i$ and denote by $X(N)$ the set of attainable allocations

$$X(N) = \{x = (x^i)_{i\in N} \in X \mid \sum_{i\in N} x^i \leq \sum_{i\in N} w^i\}.$$

Finally, as usual, for any $x \in X$ we put

$$P_i(x) = \{z \in X \mid u_i(z) > u_i(x)\}, \quad i \in N.$$

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be any n-tuple of income functions from $\mathcal{A}$.

Definition 1.1 *An allocation $\bar{x} \in X(N)$ is said to be an α-information equilibrium for the model $\mathcal{E}$ if there exist individual prices $\bar{p} = (\bar{p}_1, \dots, \bar{p}_n) \in \mathcal{P}$ such that for any $i \in N$ it holds*

$$\bar{p}_i \cdot \bar{x} = \alpha_i(\bar{p}), \quad \text{and } \bar{p}_i \cdot x > \alpha_i(\bar{p}) \text{ for any } x \in P_i(\bar{x}). \tag{2}$$

Denote by $W_I^\alpha(\mathcal{E})$ the set of the α-information equilibria for the model $\mathcal{E}$. From (1) and directly from the definition of α-information equilibria it follows that any allocation $\bar{x}$ belonging to $W_I^\alpha(\mathcal{E})$ is weakly Pareto-optimal. Namely, for any economy $\mathcal{E}$ with nonautonomous preferences it holds

$$W_I^\alpha(\mathcal{E}) \subseteq P(\mathcal{E}), \tag{3}$$

where $P(\mathcal{E})$ is the set of weak Pareto-optimal allocations of the economy $\mathcal{E}$.

The most interesting concrete forms of the income functions α_i seem to be two polar ones. Historically, the first one was introduced in the seminal work by Makarov V.L. [2] (see, also, [3]) as follows

$$\alpha_i(p) = p^0(p) \cdot w^i, \quad i \in N.$$

Some results, concerning the existence and fuzzy core equivalence problems for this type of income function can be found in [3], [4], [5] and [8]. Here we pay more attention to the polar case

$$\alpha_i(p) = p_i \cdot w, \quad i \in N,$$

with w to be equal to the allocation of initial endowments of the agents: $w = (w^1, \dots, w^n)$. It should be noted that α-information corresponding to this type of income functions are individually rational (which may not be the case for the first type mentioned above).

In the paper, we discuss rather natural (and typical in some cases for the models with externalities only) conditions providing quite general equilibrium existence results for several types of income functions, including those indicated above.

§2. Information fuzzy core

One of the main results of the paper is the core equivalence theorem that demonstrates the coincidence of the properly defined fuzzy core and the equilibrium set $W_I^\alpha(\mathcal{E})$ of the exchange model with externalities. As in

[3] and [8], the equivalence theorem is heavily relied upon a characterization of equilibrium allocations in terms of an appropriate, rather subtle fuzzy domination relation.

To show this fuzzy domination in case $\alpha_i(p) = p_i \cdot w,\ i \in N$, denote by $\mathcal{T}_N$ the set of so-called representative fuzzy coalitions, defined as follows

$$\mathcal{T}_N = \{(\tau_1, \ldots, \tau_n) \in \mathcal{R}^n \mid \tau_i \in (0, 1],\ i \in N\}.$$

Definition 2.1 *A fuzzy coalition $\tau = (\tau_1, \ldots, \tau_n) \in \mathcal{T}_N$ dominates (blocks) an $x \in X(N)$ if there exist an allocations $x_0 = (x_0^1, \ldots, x_0^n) \in X$ and $z_i = (z_i^1, \ldots, z_i^n) \in X,\ i \in N,$ such that*
(C$_\mathrm{I}$1) $z_i \in P_i(x)$, *for any* $i \in N$,
(C$_\mathrm{I}$2) $\tau_i(z^i - w) = (\tau_1(x_0^1 - w^1), \ldots, \tau_n(x_0^n - w^n))$ *for any* $i \in N$,
(C$_\mathrm{I}$3) $\sum_{i \in N} \tau_i x_0^i \leq \sum_{i \in N} \tau_i w^i$.

The set of allocations $x \in X(N)$ which are not blocked by any fuzzy coalition $\tau \in \mathcal{T}_N$ is denoted by $IC_F^w(\mathcal{E})$ and called the representative fuzzy w-core (w-core, shortly) for the model $\mathcal{E}$.
It is not very hard to prove the following amplification of the inclusion (3).

Proposition 2.1 *For any economy $\mathcal{E}$ with $\alpha_i(p) = p_i \cdot w,\ i \in N$, it holds*

$$W_I^\alpha(\mathcal{E}) \subseteq IC_F^w(\mathcal{E}).$$

In the paper, some rather general core equivalence results are given which demonstrate that the equality

$$W_I^\alpha(\mathcal{E}) = IC_F^w(\mathcal{E})$$

holds, provided that quite standard continuity and convexity assumptions on the parameters of the model $\mathcal{E}$ are fulfilled. In addition, the notions of replica and blocking in replica of the exchange model with externalities are proposed. It turns out, that one of the most interesting feature of domination in replicas is the dependence of utility levels obtained by the members of a blocking coalition M on the structure of the coalition: when evaluating the utility of an allocation suggested by this coalition, each economic agent consider both the average consumption of the agent's partners and the number of members of M which are identical to the agent. For more details on this feature of domination relations in replicas in case $\alpha_i(p) = p^0(p) \cdot w^i,\ i \in N$, see [3] and [8].

§3. Information extension of the model $\mathcal{E}$

It should be stressed that the key tool we use to investigate the models with externalities is the so-called information extension of the original economy $\mathcal{E}$. More or less detailed description of this extension can be found in [2],

[3], [5], and [8]. Here we restrict ourselves to the case $\alpha_i(p) = p_i \cdot w,\ i \in N$. To give a brief account on the standard model $\mathcal{E}^{\triangle}$ without any externalities which is associated with the original model $\mathcal{E} =< N, L, \{X_i, w^i, u_i, \}_{i\in N} >$, put $\mathcal{D} = \{(i,j) \mid i,j \in N,\ i \neq j\},\ \mathcal{D}_0 = \mathcal{D} \cup \{0\}$ and define linear operators $\Gamma_i : (\mathcal{R}^l)^N \to (\mathcal{R}^l)^{\mathcal{D}_0},\ i \in N$, by the formulas

$$\Gamma_i(x^1, \cdots, x^n)^d = \begin{cases} x^i, & d = 0; \\ -x^i, & d = (k,i),\ k \in N \setminus \{i\}; \\ x^k, & d = (i,k),\ k \in N \setminus \{i\}; \\ 0 & \text{otherwise.} \end{cases}$$

Put $N^{\triangle} = N,\ L^{\triangle} = L \times \mathcal{D}_0$, and denote by $w^i_{\triangle}$ and $X_i^{\triangle}$ the "information extensions" of the initial endowments and consumption sets of the original economy $\mathcal{E}$, defined by

$$w^i_{\triangle} = \Gamma_i(w),\ X_i^{\triangle} = \Gamma_i(X), \quad i \in N. \tag{4}$$

Further, for any $i \in N$ denote by $u_i^{\triangle}$ the function, defined on $X_i^{\triangle}$ by the formula

$$u_i^{\triangle}(\Gamma_i(x)) = u_i(x), \quad x \in X. \tag{5}$$

Finally, denote by $Y^{\triangle}$ the aggregated production set of the economy $\mathcal{E}^{\triangle}$, defined as follows

$$Y^{\triangle} = \{y = (y^d)_{d\in\mathcal{D}_0} \in (-\mathcal{R}^l_+) \times (\mathcal{R}^l)^{\mathcal{D}} \mid y^{(i,j)} = 0,\ (i,j) \in \mathcal{D}\}. \tag{6}$$

Now we are in position to give a formal description of so-called information extension in terms of a standard exchange model with aggregated production set.

Definition 3.1 *The standard exchange model (with aggregated production set), defined by*

$$\mathcal{E}^{\triangle} =< N, L^{\triangle}, \{X_i^{\triangle}, w^i_{\triangle}, u_i^{\triangle}\}_{i\in N}, Y^{\triangle} >,$$

with consumption sets, initial endowments, utility functions, and aggregated production set given by the formulas (4) - (6) is said to be the information extension (or information market) of the exchange model $\mathcal{E}$ with externalities.

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Values for TU games with linear cooperation structures

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The study of cooperative TU games with coalitional structures, given exogenously, are dated from the analysis of games in partition function form (Thrall, Lucas (1963)). The authors considered TU games generated by the vectors of coalitional values defined for each partition of the player set.

Later TU games with a coalitional strtucture, i.e. a partition of the player set have been studied. Auman and Drèze (1975) were the first to introduce such a structure into a study of value. Their outcomes were the payoff vectors whose sums of components for each coalition of the partition was equal to the corresponding characteristic function value. Thus, the values for such games, in general, were not efficient.

Owen (1977) extended the Shapley value (1953) to games with a coalitional structure by a modification of symmetry: in such games the symmetry is admitted between players from the same coalition and between the whole coalitions. Winter (1991) extended this approach to the games with an hierarchy of a finite sequence of partitions. Both values are expressed by the similar formulas: for each player the value is the expectation of its marginal contributions w.r.t. equally probable random orders of players consistent with the coalitional or level structure respectively.

Myerson (1977) defined an extension of the Shapley value to the case of a more general cooperation structure. This structure is defined by a graph whose vertices are players and a link between two vertices means the possibility of communication. Thus, the complete graph corresponds to the definition of the usual cooperative game, and the Myerson value coincide with the Shapley value. For cooperation graphs that are not connected the Myerson value is, in general, not efficient. Moreover, the definition of cooperation structure by graphs, i.e. by bilateral links between the players is not general, because the complete subgraph for a coalition S can express both pairwise communications and the complete communication of the players from S.

In this paper we extend the Owen and the Winter values to TU games with cooperation structures whose coalitions may intersect.

Let N be a finite set of players. A *linear cooperation structure (LCS)* on N is a collection $\mathcal{B} = (B_1, \ldots, B_k)$ of coalitions $B_l \subset N, l = 1, \ldots, k$ such that there is an ordering $\succ$ of N such that for each $l = 1, \dots, k$, $i, j \in B_l$ the relations $i \succ m \succ j$ imply $m \in B_l$.

Evidently, each partition of N is a linear cooperation structure. If $N = \{1, 2, 3\}$, then besides the partitions the linear cooperation structures consists of the following collections: $\{(1, 2), (2, 3)\}$, $\{(1, 3), (2, 3)\}$, $\{(1, 2), (1, 3)\}$. The ordering $\succ$, participating in the definition of LCS, is in some sense similar to the ordering defining single-peakened profiles of preferences: if two players are permitted to cooperate, all intermediate players w.r.t. the ordering may adjoin to them.

Therefore, the class of TU games $\langle N, v, \mathcal{B}\rangle$ with linear cooperation structures is an extenstion of the class of TU games with coalitional structures (CS) being partitions of the player sets.

In the sequel the notation $\mathcal{B}$ will be used for the LCSs, and the partitions will be denoted by $\mathcal{P}$.

Definition 1 Given a LCS $\mathcal{B}$ on a set N we call a permutation $\pi : N \to N$ *consistent* with the LCP $\mathcal{B}$ if $\pi\mathcal{B}$ is also a LCS.

Note that consistent permutations always exist. For example, let $(1, 2, \dots, n)$ be an ordering defining a LCS $\mathcal{B}$. Then the opposite ordering $(n, n-1, \dots, 1)$ defines the LCS $\pi\mathcal{B}$ for $\pi : (1, 2, \dots, n) \to (n, n-1, \dots, 1)$. It is easy to check that this definition extends permutations consistent with partitions. In fact, if $\mathcal{B}$ is a partition then any permutation consistent with $\mathcal{B}$ in the sense of Hart–Kurz (1984) is consistent with $\mathcal{B}$ in the sense of Definition 1 and vice versa.

Consider the class $\mathcal{G}_N^{lcs}$ of TU games with linear cooperation structures and a finite player set N. Let Φ be a value for this class. First, we are going to reformulate the Owen axioms for games with LCS. For this purpose we should define more precisely linear cooperative structures.

We call a linear cooperation structure $\mathcal{B}$ *decomposable*, if there is a partition $\mathcal{P} = (P_1, \dots, P_l)$ $l > 1$ of N such that

$$B \in \mathcal{B} \implies B \subset P_h \text{ for some } h = 1, \dots, l$$

and for each $h = 1, \dots, l$ the restriction $\mathcal{B}_h$ of $\mathcal{B}$ on P_h :

$$\mathcal{B}_h = \{B \in \mathcal{B}, B \subset P_h\} \tag{1}$$

is a LCS on P_h. Otherwise, a LCS is called *non decomposable*.

Thus, if a LCS $\mathcal{B}$ is decomposable, then there is a partition $\mathcal{P} = (P_1, \dots, P_l)$ of the set N such that for each $h = 1, \dots, l$ the collection $\mathcal{B}_h$ defined in (1) is a LCS which may be decomposable or not. In particular, a partition is a decomposable LCS. An hierarchical level structure due to Winter (1991), i.e. a finite sequence of partitions $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m)$ such that $\mathcal{P}_i$ is a refinement of $\mathcal{P}_{i+1}$, is also a decomposable LCS.

The general LCS $\mathcal{B}$ on N can be represented by the following way:

$$\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p), \tag{2}$$

where $\mathcal{B}_k, k = 1, \ldots, p$ is a non decomposable LCS defined on the corresponding coalition P_k of the partition $\mathcal{P} = (P_1, \ldots, P_p)$ of N, and an hierarchial level structure $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m)$ can be given on the partition $\mathcal{P}$ such that $\mathcal{P} = \mathcal{P}_1$, $\mathcal{P}_k$ is a partition of N for all $k = 1, \ldots, p$, and

$$\begin{array}{rl} T \in \mathcal{P}_{k+1} & \Longrightarrow\ T = \bigcup_{j \in J(T)} S_j, S_j \in \mathcal{P}_k \text{ for some index set} \\ & \quad J(T) \subset \{1, 2, \ldots, , |\mathcal{P}_k|\} \text{ and } k = 1, \ldots, m-1. \end{array} \tag{3}$$

In this case we call the partition $\mathcal{P}_k$ a *refinement* of $\mathcal{P}_{k+1}$. If all LCS $\mathcal{B}_k$, $k = 1, \ldots, p$ consist of a single coalition P_k, then such a decomposable LCS $\mathcal{B}$ is a *(hierarchical) level structure* considered by Winter (1991).

We begin with games with non decomposable LCS. Let $\mathcal{G}_N^{nd}$ be the class of TU games $\Gamma = \langle N, v, \mathcal{B} \rangle$, with the player set N, where $\mathcal{B}$ is either a non decomposable LCS, or a partition of N. Let Φ be a value for this class. Before giving modifications of Owen's axioms for the value Φ let us note that the restriction $\mathcal{B}_S$ of non decomposable LCS $\mathcal{B}$ on a subset $S \subset N$ is a LCS which can be decomposable or not. Thus, the Carrier axiom is formulated for the class $\mathcal{G}_N^{nd}$ as follows:

Carrier. If $R \subset N$ is the carrier of v, and the restriction $\mathcal{B}_R$ of $\mathcal{B}$ on R is a non decomposable LCS or a partition of R, then

$$\begin{array}{rl} \Phi_j(N, v, \mathcal{B}) = v(\{j\}) \text{ for all } j \in N \setminus R, & \sum_{i \in R} \Phi(N, v, \mathcal{B}) = v(R), \\ \mathcal{B}_R = \mathcal{B}'_R,\ R - \text{ the carrier of } \mathcal{B}' & \Longrightarrow \Phi(N, v, \mathcal{B}) = \Phi(N, v, \mathcal{B}'). \end{array}$$

Additivity. If $\langle N, v, \mathcal{B} \rangle$, $\langle N, v', \mathcal{B} \rangle \in \mathcal{G}_N^{nd}$, then

$$\Phi(N, v, \mathcal{B}) + \Phi(N, v', \mathcal{B}) = \Phi(N, v + v', \mathcal{B}).$$

Anonymity. Let $\langle N, v, \mathcal{B} \rangle$ be a game with a nondecomposable LCS $\mathcal{B}$, π be a permutation of N, consistent with $\mathcal{B}$. Then

$$\Phi(N, \pi v, \pi \mathcal{B}) = \pi \Phi(N, v, \mathcal{B}). \tag{4}$$

If $\langle N, v, \mathcal{P} \rangle$ is a game with coalitional structure $\mathcal{P}$, then equality (4) holds for every permutation $\pi : N \to N$.

Anonymity of the value for games with LCS states invariance of the value w.r.t. permutations consistent with the LCS. Indeed, other permutations transform a LCS into cooperation structures which are not linear.

Associate with each game $\langle N, v, \mathcal{P} \rangle$ with a coalitional structure $\mathcal{P} = (P_1, \ldots, P_m)$ a TU game $\langle M, v \rangle$ *between coalitions*, where $M = \{1, 2, \ldots, m\}$, $v(S) = v\left(\bigcup_{j \in S} P_j\right)$ for each $S \subset M$.

> **Inessential game**. If the game $\langle M, v\rangle$ between coalitions is inessential, then $\sum_{i\in P_j} \phi_i(N, v, \mathcal{P}) = v(P_j)$ for each $j = 1, \ldots, m$.

Let $\mathcal{B}$ be an arbitrary non decomposable LCS on the set N, $\succ$ be the ordering of N, corresponding to $\mathcal{B}$. If for some coalitions $S, T \subset N$ it holds $i \succ j$ for all $i \in S, j \in T$, then we write $S \succ T$.

The LCS $\mathcal{B}$ generates an ordered partition $\mathcal{S} = \{S_1, \ldots, S_l\}$ of N such that $S_p \succ S_q$ for $p > q$ onto coalitions of players belonging to the same coalitions from $\mathcal{B}$:

$$i, j \in S_k, k = 1, \ldots, l, \iff \{i \in B \in \mathcal{B} \longleftrightarrow j \in B\}. \tag{5}$$

For every coalition S from partition $\mathcal{S}$ (5) denote by $N_1(S), N_2(S)$ the coalitions of all predecessors and followers w.r.t. the ordering $\succ$, defining the LCS $\mathcal{B}$ respectively.

For LCS differing from partitions we should state one more axiom stating dependence of the value Φ only on characteristic function values of coalitions, "consistent" with the LCS:

> **Independence of coalitions not supporting the LCS**. Let $S_k \in \mathcal{S}$ be an arbitrary coalition from the partition $\mathcal{S}$ (5). Then $\Phi_i(N, v, \mathcal{B}), i \in S_k, k = 1, \ldots, l(i \in S_k, k = 1, \ldots, l$ does not depend on $v(T)$, for all $T \subset N$ such that
>
> $$S_k \not\subset T, \;\; T \cap N_1(S_k) \neq \emptyset, \quad T \cap N_2(S_k) \neq \emptyset,$$
>
> and not equal to the carrier of v.

This means that the values $\Phi_i(N, v, \mathcal{B})$ for players belonging to some S_k, do not depend on the characteristic function values of coalitions having "holes" on the coalition S_k.

Now we are ready to characterize the values of games for the whole class $\mathcal{G}_N^{nd}$.

Theorem 1 *The unique value Φ for the class $\mathcal{G}_N^{nd}$ satisfying axioms Carrier, Additivity, Anonymity, Inessential games, and Independence of coalitions not supporting the LCS, is defined by*

$$\Phi(N, v, \mathcal{B}) = E[v(\mathcal{P}^i \cup \{i\}) - v(\mathcal{P}^i)], \tag{6}$$

where the expectation E is over all equally probable random orders on a carrier of v consistent with $\mathcal{B}$. Here $\mathcal{B}$ is either a partition, or a non decomposable LCS.

A decomposable LCS differs from a level structure by the highest level where each of coalition of the partition $\mathcal{P}_1$ (in the level structure) is replaced by a non decomposable LCS. Therefore, it is not difficult to unify the axioms given in this and the previous section to the class of games with decomposable LCS and to characterize the value for this class.

Assume as earlier that the orderings defining the LCS $\mathcal{B}_k, k = 1, \dots, t$ coincides with the natural ordering of the numbers. Denote by Π the set of all permutations consistent with the level structure $\mathcal{P}$, and by $\mathcal{S}^k = (S_1^k, \dots, S_{q_k}^k)$ – the partition of $P_k^1, k = 1, \dots, |M^k|$ defined as in (5). Then we have

$$\begin{array}{ll} S_j^{k_1} > S_l^{k_2} & \text{for all } k_1 > k_2 \text{ and all } j, l, \\ S_{j_1}^k > S_{j_2}^k & \text{for all } j_1 > j_2 \text{ and all } k = 1, \dots, |M|. \end{array}$$

We call a permutation π *consistent* with the LCS $\mathcal{B}$, if $\pi \in \Pi$, and

$$i, j \in \pi S_l^k,\ i \succ_\pi \succ_\pi j \implies (\pi)^{-1} r \in S_l^k \text{ for all } k = 1, \dots, |M|, l = 1, \dots, q_k,$$

where the ordering $\succ_\pi$ corresponds to the LCS $\pi\mathcal{B}$. Thus, this definition is the composition of those of permutations consistent with level structures and with non decomposable LCS.

Let $\mathcal{G}_N^d$ be the class of TU games with decomposable LCS and the player set N. Give generalizations of the axioms given above for the characterization of the value for games from the class $\mathcal{G}_N^d$. The axioms Carrier and Additivity are immediately reformulated for this class of games.

> **Anonymity**. Let $\langle N, v, \mathcal{B}\rangle \in \mathcal{G}_N^d$. Then for each permutation π consistent with $\mathcal{B}$
> $$\pi\Phi(N, v, \mathcal{B}) = \Phi(N, \pi v, \pi\mathcal{B}).$$

In the same way as for games from the class $\mathcal{G}_N^{ls}$, with any game $\langle N, v, \mathcal{B}\rangle \in \mathcal{G}_N^d$ we can associate m games $\langle M^1, v^1\rangle, \dots, \langle M^m, v^m\rangle$ between coalitions, one for each hierarchy level (partition). The number of players $|M^k|$ is equal to the number of coalitions on the partition $\mathcal{P}_k$, and the characteristic function v^j is defined by

$$v^j(S) = v\left(\bigcup_{i \in S} P_i^j\right),$$

where $\mathcal{P}_j = (P_1^j, \dots, P_{|M^j|}^j)$, $j = 1, \dots, m$. The next axioms are the straightforward generalizations of the same axioms for games with level structures and non decomposable LCS respectively:

> **Inessential game**. If for some $k = 1, \dots, m$ the game $\langle M^k, v^k\rangle$ is inessential, then
> $$\sum_{i \in P_j^k} \Phi_i(N, v, \mathcal{B}) = v(P_k^j) \text{ for each } k = 1, \dots, M^j.$$

Independence of coalitions not supporting the LCS. Let $S_j^k \in \mathcal{S}$ be an arbitrary coalition from the partition $\mathcal{S}$. Then $\Phi_i(N, v, \mathcal{B}), i \in S_j^k$, $j = 1, \ldots, M^k$, $k = 1, \ldots, m$ does not depend on $v(T)$, for all $T \subset N$ such that

$$S_j^k \not\subset T, \;\; T \cap N_1(S_j^k) \neq \emptyset, \quad T \cap N_2(S_j^k) \neq \emptyset,$$

and not equal to the carrier of v.

The next theorem is the generalization of Theorem 1 and the Winter Theorem:

Theorem 2 *If a value Φ for the class $\mathcal{G}_N^d$ of games with decomposable LCS satisfies axioms Carrier, Additivity, Anonymity, Inessential games, and Independence of coalitions not supporting the LCS, then for each $\langle N, v, \mathcal{B} \rangle \in \mathcal{G}_N^d$*

$$\Phi(N, v, \mathcal{B}) = E(v(\mathcal{P}^i \cup \{i\}) - v(\mathcal{P}^i)),$$

where the expectation is over all equally probable random orders consistent with the decomposable LCS $\mathcal{B}$.

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