Introduction to Random Signals and Noise

Introduction to Random Signals and Noise Wim C. van Etten
 ISBN: 0-470-02411-9

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John Wiley & Sons, Ltd

Telephone (+44) 1243 779777

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Other Wiley Editorial Offices

John Wiley & Sons Inc., 111 River Street, Hoboken, NJ 07030, USA

Jossey-Bass, 989 Market Street, San Francisco, CA 94103-1741, USA

Wiley-VCH Verlag GmbH, Boschstr. 12, D-69469 Weinheim, Germany

John Wiley & Sons Australia Ltd, 42 McDougall Street, Milton, Queensland 4064, Australia

John Wiley & Sons (Asia) Pte Ltd, 2 Clementi Loop #02-01, Jin Xing Distripark, Singapore 129809

John Wiley & Sons Canada Ltd, 22 Worcester Road, Etobicoke, Ontario, Canada M9W 1L1

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British Library Cataloguing in Publication Data

A catalogue record for this book is available from the British Library

ISBN-13 978-0-470-02411-9 (HB)
ISBN-10 0-470-02411-9 (HB)

Typeset in 10/12pt Times by Thomson Press (India) Limited, New Delhi
Printed and bound in Great Britain by Antony Rowe Ltd, Chippenham, Wiltshire
This book is printed on acid-free paper responsibly manufactured from sustainable forestry in which at least two trees are planted for each one used for paper production.

To Kitty,
to Sascha, Anne and Emmy,
to Björn and Esther

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Preface

Random signals and noise are present in several engineering systems. Practical signals seldom lend themselves to a nice mathematical deterministic description. It is partly a consequence of the chaos that is produced by nature. However, chaos can also be man-made, and one can even state that chaos is a *conditio sine qua non* to be able to transfer information. Signals that are not random in time but predictable contain no information, as was concluded by Shannon in his famous communication theory.

To deal with this randomness we have to nevertheless use a characterization in deterministic terms; i.e. we employ probability theory to determine characteristic descriptions such as mean, variance, correlation, etc. Whenever chaotic behaviour is time-dependent, as is often the case for random signals, the time parameter comes into the picture. This calls for an extension of probability theory, which is the theory of stochastic processes and random signals. With the involvement of time, the phenomenon of frequency also enters the picture. Consequently, random signal theory leans heavily on both probability and Fourier theories. Combining these subjects leads to a powerful tool for dealing with random signals and noise.

In practice, random signals may be encountered as a desired signal such as video or audio, or it may be an unwanted signal that is unintentionally added to a desired (information bearing) signal thereby disturbing the latter. One often calls this unwanted signal noise. Sometimes the undesired signal carries unwanted information and does not behave like noise in the classical sense. In such cases it is termed as interference. While it is usually difficult to distinguish (at least visually) between the desired signal and noise (or interference), by means of appropriate signal processing such a distinction can be made. For example, optimum receivers are able to enhance desired signals while suppressing noise and interference at the same time. In all cases a description of the signals is required in order to be able to analyse their impact on the performance of the system under consideration. In communication theory this situation often occurs. The random time-varying character of signals is usually difficult to describe, and this is also true for associated signal processing activities such as filtering. Nevertheless, there is a need to characterize these signals using a few deterministic parameters that allow a system user to assess system performance.

This book deals with stochastic processes and noise at an introductory level. Probability theory is assumed to be known. The same holds for mathematical background in differential and integral calculus, Fourier analysis and some basic knowledge of network and linear system theory. It introduces the subject in the form of theorems, properties and examples. Theorems and important properties are placed in frames, so that the student can easily summarize them. Examples are mostly taken from practical applications. Each chapter concludes with a summary and a set of problems that serves as practice material. The book is well suited for dealing with the subject at undergraduate level. A few subjects can be skipped if they do not fit into a certain curriculum. Besides, the book can also serve as a reference for the experienced engineer in his daily work.

In Chapter 1 the subject is introduced and the concept of a stochastic process is presented. Different types of processes are defined and elucidated by means of simple examples.

Chapter 2 gives the basic definitions of probability density functions and includes the time dependence of these functions. The approach is based on the 'ensemble' concept. Concepts such as stationarity, ergodicity, correlation functions and covariance functions are introduced. It is indicated how correlation functions can be measured. Physical interpretation of several stochastic concepts are discussed. Cyclo-stationary and Gaussian processes receive extra attention, as they are of practical importance and possess some interesting and convenient properties. Complex processes are defined analogously to complex variables. Finally, the different concepts are reconsidered for discrete-time processes.

In Chapter 3 a description of stochastic processes in the frequency domain is given. This results in the concept of power spectral density. The bandwidth of a stochastic process is defined. Such an important subject as modulation of stochastic processes is presented, as well as the synchronous demodulation. In order to be able to define and describe the spectrum of discrete-time processes, a sampling theorem for these processes is derived.

After the basic concepts and definitions treated in the first three chapters, Chapter 4 starts with applications. Filtering of stochastic processes is the main subject of this chapter. We confine ourselves to linear, time-invariant filtering and derive both the correlation functions and spectra of a two-port system. The concept of equivalent noise bandwidth has been defined in order to arrive at an even more simple description of noise filtering in the frequency domain. Next, the calculation of the spectrum of random data signals is presented. A brief resumé of the principles of discrete-time signals and systems is dealt with using the z-transform and discrete Fourier transform, based on which the filtering of discrete-time processes is described both in time and frequency domains.

Chapter 5 is devoted to bandpass processes. The description of bandpass signals and systems in terms of quadrature components is introduced. The probability density functions of envelope and phase are derived. The measurement of spectra and operation of the spectrum analyser is discussed. Finally, sampling and conversion to baseband of bandpass processes is discussed.

Thermal noise and its impact on systems is the subject of Chapter 6. After presenting the spectral densities we consider the role of thermal noise in passive networks. System noise is considered based on the thermal noise contribution of amplifiers, the noise figure and the influence of cascading of systems on noise performance.

Chapter 7 is devoted to detection and optimal filtering. The chapter starts by considering hypothesis testing, which is applied to the detection of a binary signal disturbed by white Gaussian noise. The matched filter emerges as the optimum filter for optimum detection performance. Finally, filters that minimize the mean squared error (Wiener filters) are derived. They can be used for smoothing stored data or portions of a random signal that arrived in the past. Filters that produce an optimal prediction of future signal values can also be designed.

Finally, Chapter 8 is of a more advanced nature. It presents the basics of random point processes, of which the Poisson process is the most well known. The characteristic function plays a crucial role in analysing these processes. Starting from that process several shot noise processes are introduced: the homogeneous Poisson process, the inhomogeneous Poisson process, the Poisson impulse process and the random-pulse process. Campbell's theorem is derived. A few application areas of random point processes are indicated.

The appendices contain a few subjects that are necessary for the main material. They are: signal space representation and definitions of attenuation, phase shift and decibels. The rest of the appendices comprises basic mathematical relations, a summary of probability theory, definitions of special functions, a list and properties of Fourier transform pairs, and a few mathematical and physical constants.

Finally, I would like to thank those people who contributed in one way or another to this text. My friend Rajan Srinivasan provided me with several suggestions to improve the content. Also, Arjan Meijerink carefully read the draft and made suggestions for improvement.

Last but certainly not least, I thank my wife Kitty, who allowed me to spend so many hours of our free time to write this text.

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Enschede, The Netherlands

1
Introduction

1.1 RANDOM SIGNALS AND NOISE

In (electrical) engineering one often encounters signals that do not have a precise mathematical description, since they develop as random functions of time. Sometimes this random development is caused by a single random variable, but often it is a consequence of many random variables. In other cases the causes of randomness are not clear and a description is not possible, but the signal is characterized by means of measurements only.

A random time function may be a desired signal, such as an audio or video signal, or it may be an unwanted signal that is unintentionally added to a desired (information) signal and disturbs the desired signal. We call the desired signal a random signal and the unwanted signal noise. However, the latter often does not behave like noise in the classical sense, but it is more like interference. Then it is an information bearing signal as well, but undesired. A desired signal and noise (or interference) can, in general, not be distinguished completely; by means of well-defined signal processing in a receiver, the desired signal may be favoured in a maximal way whereas the disturbance is suppressed as much as possible. In all cases a description of the signals is required in order to be able to analyse its impact on the performance of the system under consideration. Especially in communication theory this situation often occurs. The random character as a function of time makes the signals difficult to describe and the same holds for signal processing or filtering. Nevertheless, there is a need to characterize these signals by a few deterministic parameters that enable the system user to assess the performance of the system. The tool to deal with both random signals and noise is the concept of the stochastic process, which is introduced in Section 1.3.

This book gives an elementary introduction to the methods used to describe random signals and noise. For that purpose use is made of the laws of probability, which are extensively described in textbooks [1–5].

1.2 MODELLING

When studying and analysing random signals one is mainly committed to theory, which however, can be of good predictive value. Actually, the main activity in the field of random signals is modelling of processes and systems. Many scientists and engineers have

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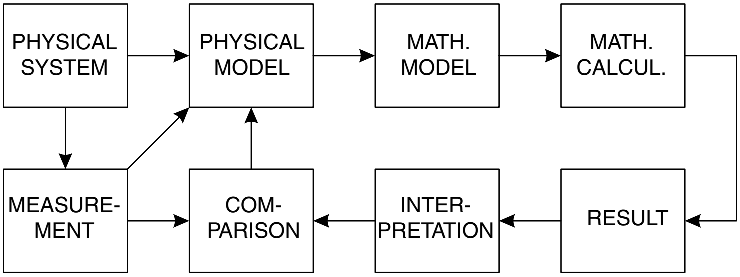

Figure 1.1 The process of modelling

contributed to that activity in the past and their results have been checked in practice. When a certain result agrees (at least to a larger extent) with practical measurements, then there is confidence in and acceptance of the result for practical application. This process of modelling has schematically been depicted in Figure 1.1.

In the upper left box of this scheme there is the important physical process. Based on our knowledge of the physics of this process we make a physical model of it. This physical model is converted into a mathematical model. Both modelling activities are typical engineer tasks. In this mathematical model the physics is no longer formally recognized, but the laws of physics will be included with their mathematical description. Once the mathematical model has been completed and the questions are clear we can forget about the physics for the time being and concentrate on doing the mathematical calculations, which may help us to find the answers to our questions. In this phase the mathematicians can help the engineer a lot. Let us suppose that the mathematical calculations give a certain outcome, or maybe several outcomes. These outcomes would then need to be interpreted in order to discover what they mean from a physical point of view. This ends the role of the mathematician, since this phase is maybe the most difficult engineering part of the process. It may happen that certain mathematical solutions have to be discarded since they contradict physical laws. Once the interpretation has been completed there is a return to the physical process, as the practical applicability of the results needs to be checked. In order to check these the quantities or functions that have been calculated are measured. The measurement is compared to the calculated result and in this way the physical model is validated. This validation may result in an adjustment of the physical model and another cycle in the loop is made. In this way the model is refined iteratively until we are satisfied about the validation. If there is a shortage of insight into the physical system, so that the physical model is not quite clear, measurements of the physical system may improve the physical model.

In the courses that are taught to students, models that have mainly been validated in this way are presented. However, it is important that students are aware of this process and the fact that the models that are presented may be a result of a difficult struggle for many years by several physicists, engineers and mathematicians. Sometimes students are given the opportunity to be involved in this process during research assignments.

1.3 THE CONCEPT OF A STOCHASTIC PROCESS

In probability theory a random variable is a rule that assigns a number to every outcome of an experiment, such as, for example, rolling a die. This random variable X is associated with a sample space S, such that according to a well-defined procedure to each event s in the

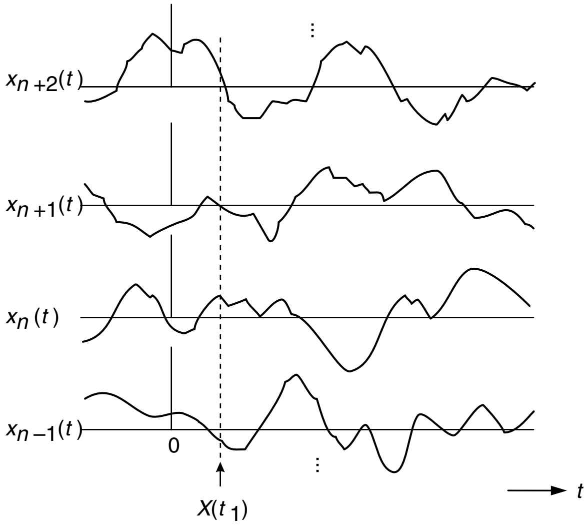

Figure 1.2 A few sample functions of a stochastic process

sample space a number is assigned to X and is denoted by $X(s)$. For stochastic processes, on the other hand, a time function $x(t,s)$ is assigned to every outcome in the sample space. Within the framework of the experiment the family (or ensemble) of all possible functions that can be realized is called the stochastic process and is denoted by $X(t,s)$. A specific waveform out of this family is denoted by $x_n(t)$ and is called a sample function or a realization of the stochastic process. When a realization in general is indicated the subscript n is omitted. Figure 1.2 shows a few sample functions that are supposed to constitute an ensemble. The figure gives an example of a finite number of possible realizations, but the ensemble may consist of an infinite number of realizations. The realizations may even be uncountable. A realization itself is sometimes called a stochastic process as well. Moreover, a stochastic process produces a random variable that arises from giving t a fixed value with s being variable. In this sense the random variable $X(t_1,s) = X(t_1)$ is found by considering the family of realizations at the fixed point in time t_1 (see Figure 1.2). Instead of $X(t_1)$ we will also use the notation X_1. The random variable X_1 describes the statistical properties of the process at the instant of time t_1. The expectation of X_1 is called the ensemble mean or the expected value or the mean of the stochastic process (at the instant of time t_1). Since t_1 may be arbitrarily chosen, the mean of the process will in general not be constant, i.e. it may have different values for different values of t. Finally, a stochastic process may represent a single number by giving both t and s fixed values. The phrase ‘stochastic process’ may therefore have four different interpretations. They are:

1. A family (or ensemble) of time functions. Both t and s are variables.
2. A single time function called a sample function or a realization of the stochastic process. Then t is a variable and s is fixed.
3. A random variable; t is fixed and s is variable.
4. A single number; both t and s are fixed.

Which of these four interpretations holds in a specific case should follow from the context.

Different classes of stochastic processes may be distinguished. They are classified on the basis of the characteristics of the realization values of the process x and the time parameter t. Both can be either continuous or discrete, in any combination. Based on this we have the following classes:

- Both the values of $X(t)$ and the time parameter t are continuous. Such a process is called a continuous stochastic process.
- The values of $X(t)$ are continuous, whereas time t is discrete. These processes are called discrete-time processes or continuous random sequences. In the remainder of the book we will use the term discrete-time process.
- If the values of $X(t)$ are discrete but the time axis is continuous, we call the process a discrete stochastic process.
- Finally, if both the process values and the time scale are discrete, we say that the process is a discrete random sequence.

In Table 1.1 an overview of the different classes of processes is presented. In order to get some feeling for stochastic processes we will consider a few examples.

Table 1.1 Summary of names of different processes

$X(t)$	Time	
	Continuous	Discrete
Continuous	Continuous stochastic process	Discrete-time process
Discrete	Discrete stochastic process	Discrete random sequence

1.3.1 Continuous Stochastic Processes

For this class of processes it is assumed that in principle the following holds:

$$-\infty < x(t) < \infty \quad \text{and} \quad -\infty < t < \infty \tag{1.1}$$

An example of this class was already given by Figure 1.2. This could be an ensemble of realizations of a thermal noise process as is, for instance, produced by a resistor, the characteristics of which are to be dealt with in Chapter 6. The underlying experiment is selecting a specific resistor from a collection of, let us say, 100 Ω resistors. The voltage across every selected resistor corresponds to one of the realizations in the figure.

Another example is given below.

Example 1.1:

The process we consider now is described by the equation

$$X(t) = \cos(\omega_0 t - \Theta) \tag{1.2}$$

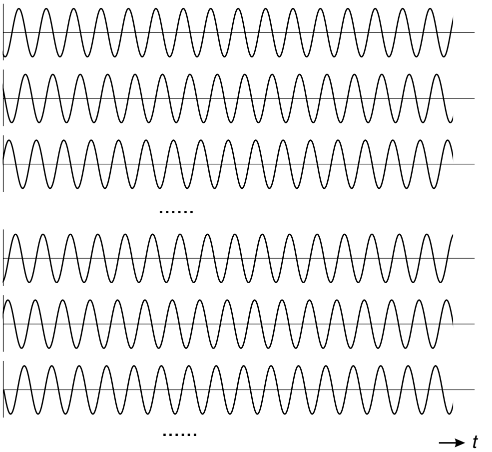

Figure 1.3 Ensemble of sample functions of the stochastic process $\cos(\omega_0 t - \Theta)$, with Θ uniformly distributed on the interval $(0, 2\pi]$

with ω_0 a constant and Θ a random variable with a uniform probability density function on the interval $(0, 2\pi]$. In this example the set of realizations is in fact uncountable, as Θ assumes continuous values. The ensemble of sample functions is depicted in Figure 1.3.

Thus each sample function consists of a cosine function with unity amplitude, but the phase of each sample function differs randomly from others. For each sample function a drawing is taken from the uniform phase distribution. We can imagine this process as follows. Consider a production process of crystal oscillators, all producing the same amplitude unity and the same radial frequency ω_0. When all those oscillators are switched on, their phases will be mutually independent. The family of all measured output waveforms can be considered as the ensemble that has been presented in Figure 1.3.

This process will get further attention in different chapters that follow.

□

1.3.2 Discrete-Time Processes (Continuous Random Sequences)

The description of this class of processes becomes more and more important due to the increasing use of modern digital signal processors which offer flexibility and increasing speed and computing power. As an example of a discrete-time process we can imagine sampling the process that was given in Figure 1.2. Let us suppose that to this process ideal sampling is applied at equidistant points in time with sampling period T_s; with ideal sampling we mean the sampling method where the values at T_s are replaced by delta functions of amplitude $X(nT_s)$ [6]. However, to indicate that it is now a discrete-time process we denote it by $X[n]$, where n is an integer running in principle from $-\infty$ to $+\infty$. We know from the sampling theorem (see Section 3.5.1 or, for instance, references [1] and [7]) that the original signal can perfectly be recovered from its samples, provided that the signals are band-limited. The process that is produced in this way is given in Figure 1.4, where the sample values are presented by means of the length of the arrows.

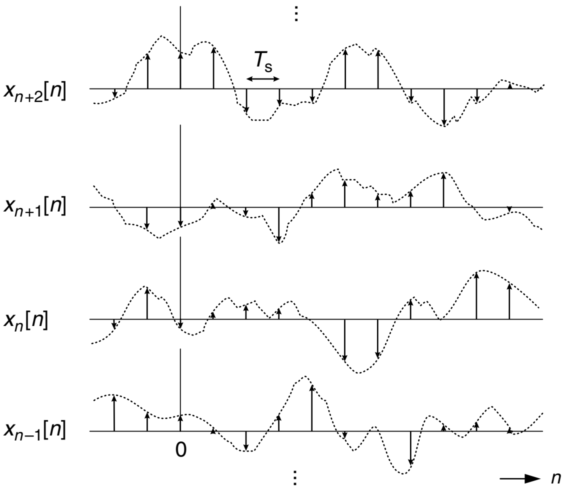

Figure 1.4 Example of a discrete-time stochastic process

Another important example of the discrete-time process is the so-called Poisson process, where there are no equidistant samples in time but the process produces ‘samples’ at random points in time. This process is an adequate model for shot noise and it is dealt with in Chapter 8.

1.3.3 Discrete Stochastic Processes

In this case the time is continuous and the values discrete. We present two examples of this class. The second one, the random data signal, is of great practical importance and we will consider it in further detail in Chapter 4.

Example 1.2:

This example is a very simple one. The ensemble of realizations consists of a set of constant time functions. According to the outcome of an experiment one of these constants may be chosen. This experiment can be, for example, the rolling of a die. In that case the number of realizations can be six $(n = 6)$, equal to the usual number of faces of a die. Each of the outcomes $s \in \{1, 2, 3, 4, 5, 6\}$ has a one-to-one correspondence to one of these numbered constant functions of time. The ensemble is depicted in Figure 1.5.

□

Example 1.3:

Another important stochastic process is the random data signal. It is a signal that is produced by many data sources and is described by

$$X(t) = \sum_n A_n p(t - nT - \Theta) \tag{1.3}$$

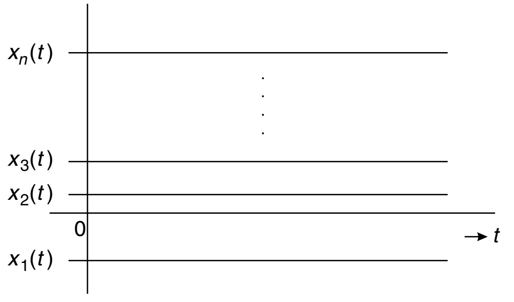

Figure 1.5 Ensemble of sample functions of the stochastic process constituted by a number of constant time functions

where $\{A_n\}$ are the data bits that are randomly chosen from the set $A_n \in \{+1, -1\}$. The rectangular pulse $p(t)$ of width T serves as the carrier of the information. Now Θ is supposed to be uniformly distributed on the bit interval $(0, T]$, so that all data sources of the family have the same bit period, but these periods are not synchronized. The ensemble is given in Figure 1.6.

□

1.3.4 Discrete Random Sequences

The discrete random sequence can be imagined to result from sampling a discrete stochastic process. Figure 1.7 shows the result of sampling the random data signal from Example 1.3.

We will base the further development of the concept, description and properties of stochastic processes on the continuous stochastic process. Then we will show how these are extended to discrete-time processes. The two other classes do not get special attention, but

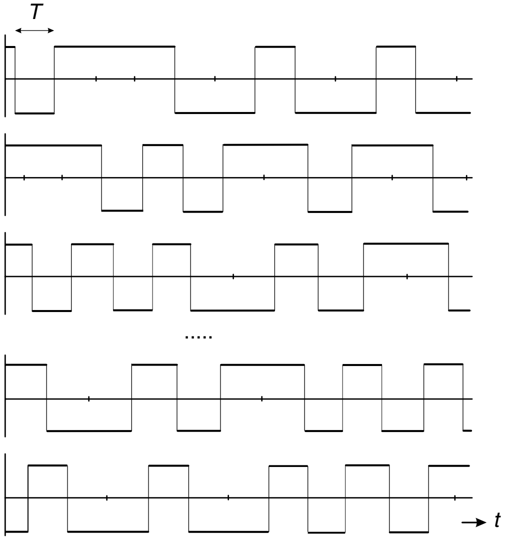

Figure 1.6 Ensemble of sample functions of the stochastic process $\sum_n A_n p(t - nT - \Theta)$, with Θ uniformly distributed on the interval $(0, T]$

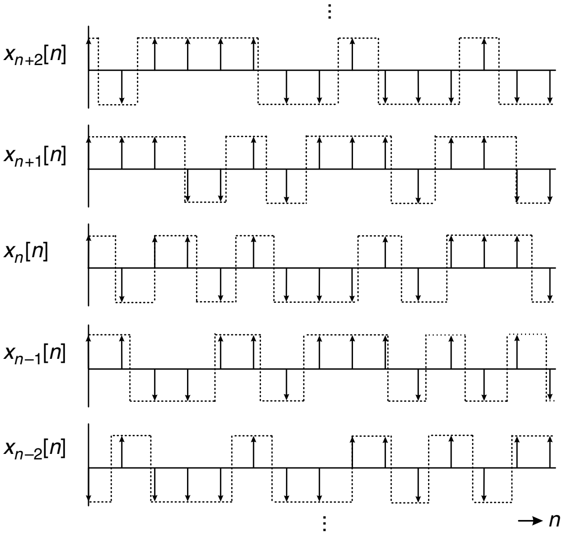

Figure 1.7 Example of a discrete random sequence

are considered as special cases of the former ones by limiting the realization values x to a discrete set.

1.3.5 Deterministic Function versus Stochastic Process

The concept of the stochastic process does not conflict with the theory of deterministic functions. It should be recognized that a deterministic function can be considered as nothing else but a special case of a stochastic process. This is elucidated by considering Example 1.1. If the random variable Θ is given the probability density function $f_\Theta(\theta) = \delta(\theta)$, then the stochastic process reduces to the function $\cos(\omega_0 t)$. The given probability density function is actually a discrete one with a single outcome. In fact, the ensemble of the process reduces in this case to a family comprising merely one member. This is a general rule; when the probability density function of the stochastic process that is governed by a single random variable consists of a single delta function, then a deterministic function results. This way of generalization avoids the often confusing discussion on the difference between a deterministic function on the one hand and a stochastic process on the other hand. In view of the consideration presented here they can actually be considered as members of the same class, namely the class of stochastic processes.

1.4 SUMMARY

Definitions of random signals and noise have been given. A random signal is, as a rule, an information carrying wanted signal that behaves randomly. Noise also behaves randomly but is unwanted and disturbs the signal. A common tool to describe both is the concept of a stochastic process. This concept has been explained and different classes of stochastic processes have been identified. They are distinguished by the behaviour of the time parameter and the values of the process. Both can either be continuous or discrete.

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a.c. component, 27
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