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About convergence of numerical approximations
to homoclinic twist bifurcation points in
 Z_2 -symmetric systems in \mathbb{R}^4

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About convergence of numerical approximations to homoclinic twist bifurcation points in Z_2 -symmetric systems in \mathbb{R}^4 .

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Abstract

An algorithm to detect homoclinic twist bifurcation points in Z_2 -symmetric autonomous systems of ordinary differential equations in \mathbb{R}^4 along curves of symmetric homoclinic orbits to hyperbolic equilibria has been developed. We show convergence of numerical approximations to homoclinic twist bifurcation points in such systems. A test function is defined on the homoclinic solutions, which has a regular zero in the codimension-two bifurcation points. This codimension-two singularity can be continued appending the test function to a three parameter vector field. We demonstrate the use of the test function on several examples of two coupled Josephson junctions.

Key words. homoclinic twist bifurcation, Z_2 -symmetry, continuation

AMS classification: 58F14 34C23 34C37 35B32 58F40

1 Introduction

In a two-parameter family of vector fields, there is generically a branch in the parameter plane along which orbits homoclinic to hyperbolic equilibria occur. Performing numerical experiments, codimension-two homoclinic orbits can be detected along branches of codimension-one homoclinics by locating zeros of a *test function* [4]. This paper presents an algorithm for numerical detection of *homoclinic twist points* in a two-parameter family of Z_2 -symmetric vector fields

in \mathbb{R}^4 with the help of a test function. The codimension-two singularity can be continued appending the test function to the problem in a three parameter vector field.

This codimension-two point can be characterized in the following way. Consider a unique, up to scalar multiplication, solution of the linearized flow around the homoclinic orbit whose direction limits in backward time on the most unstable direction of the saddle. In the forward time limit, this solution approaches precisely the same direction or exactly the opposite direction. It is shown that as parameters vary along the curve of homoclinic solutions a transition occurs between these two cases. At the transition point, solutions of the linearized flow around the homoclinic orbit whose direction limits in backward time on the most unstable direction of the saddle approach in the forward time limit the most stable direction. A point of the transition is called a *homoclinic twist point*.

A similar transition can occur in non-symmetric systems. Then it will be a codimension-three phenomenon. Thus test functions designed by Sandstede [10] for *inclination* bifurcations of vector fields without symmetry and then for general n -dimensional systems in [4] may not have meaning for symmetric systems with a homoclinic twist point. A test function for the homoclinic twist bifurcation in presence of symmetry has been derived for the first time.

The codimension-one homoclinic orbits have been computed using *boundary-value methods*, which have been recently analyzed by Beyn [3] and Schechter [11]. Such methods truncate the homoclinic problem to a finite interval with certain boundary conditions at the end points. In this paper, we prove a theorem about the rate of convergence of the computed parameter values for a codimension-two homoclinic orbit to the true parameter values as the length of the time interval on which the solution is computed increases.

The outline of the rest of the paper is as follows. In Section 1.1, we describe families of vector fields near a twist point as in [1]. In Section 2.1, we describe an algorithm for the computation of hyperbolic homoclinic orbits as in [11]. In section 2.2, we introduce a test function which is used to detect the homoclinic twist point and state the main result about convergence of the test function monitored along numerically computed homoclinic orbits on $[-T_1, T_2]$ as $T_1, T_2 \rightarrow \infty$. In Section 2.3, we prove the main result. Section 3 contains numerical analysis using the algorithm and results on a system of two coupled Josephson junctions. In Section 4, we show how the method for detecting homoclinic twist bifurcations

can be extended to Γ -equivariant vector fields in \mathbb{R}^n , where Γ is a finite subgroup of $O(n)$.

1.1 Homoclinic twist point

First, we formulate the twist bifurcation problem with Z_2 symmetry. We consider a family of ordinary differential equations

$$\dot{x} = F(x, \alpha, \lambda), \quad (1)$$

where $x(t) \in \mathbb{R}^4$, $\alpha, \lambda \in \mathbb{R}$ and $F : \mathbb{R}^4 \times \mathbb{R}^2 \rightarrow \mathbb{R}^4$ is a C^3 smooth function, $F(0, \alpha, \lambda) = 0$ for all $\alpha, \lambda \in \mathbb{R}$.

(H_1) $F(\cdot, \alpha, \lambda)$ commutes with the action of Z_2 on $\mathbb{R}^4 = S \oplus S^\perp$ given by: $\sigma y = y$, $y \in S$ and $\sigma y = -y$, $y \in R = S^\perp$, where σ is the reflection in Z_2 ; $\dim S=2$.

Symmetry implies that $D_1F(q_\lambda(t), 0, \lambda)$ commutes with σ and can be written in appropriate symmetric coordinates in the following form:

$$D_1F(q_\lambda(t), 0, \lambda) = \begin{pmatrix} \mathcal{A}_\lambda(q_\lambda(t)) & 0 \\ 0 & \mathcal{B}_\lambda(q_\lambda(t)) \end{pmatrix},$$

where $\mathcal{A}_\lambda(q_\lambda(t))$, $\mathcal{B}_\lambda(q_\lambda(t))$ are 2×2 blocks. We write \mathcal{A}_λ and \mathcal{B}_λ for $\mathcal{A}_\lambda(p)$ and $\mathcal{B}_\lambda(p)$.

(H_2) We assume that the eigenvectors of $D_1F(p, \alpha, \lambda)$ do not depend on (α, λ) . For $\alpha = 0$, $\lambda \in \Lambda = [\lambda_1, \lambda_2]$, $\lambda_1 < 0 < \lambda_2$, (1) has a smooth family of homoclinic orbits $q_\lambda(t)$ to a saddle $p = 0$ and $q_\lambda(t) \in S = \text{Fix}_{\mathbb{R}^4}(Z_2)$. Thus, \mathcal{A}_λ has one negative eigenvalue $\sigma_S^s(\lambda)$ and one positive eigenvalue $\sigma_S^u(\lambda)$. We assume that $\forall \lambda \in \Lambda$, $\sigma_S^s(\lambda) < 0$ and $\sigma_S^u(\lambda) > 0$.

(H_3) \mathcal{B}_λ has one positive and one negative eigenvalue $\sigma_R^u(\lambda)$ (resp. $\sigma_R^s(\lambda)$) such that $\forall \lambda \in \Lambda$, $\sigma_R^u(\lambda) > 0$ and $\sigma_R^s(\lambda) < 0$, and none of the eigenvalues $\{\sigma_S^s(\lambda), \sigma_S^u(\lambda), \sigma_R^s(\lambda), \sigma_R^u(\lambda)\}$ have equal modulus for $\lambda \in \Lambda$.

(H_4) The linear differential equation $\dot{y} = \mathcal{A}_\lambda(q_\lambda(t))y$ has a unique, up to scalar multiplication, bounded solution \dot{q}_λ . Let $\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}$ be the unique,

up to scalar multiplication, bounded solution of the equation $\dot{\phi} = -\mathcal{A}_0^T(q_0(t))\phi$ then

$$\int_{-\infty}^{\infty} \psi(t) D_2 F(q_0(t), 0, 0) dt \neq 0,$$

where $\psi(t) = (\phi_1(t), \phi_2(t), 0, 0)$.

Let χ_S^u, χ_S^s be the unit length eigenvectors corresponding to the positive (negative) eigenvalue of \mathcal{A}_λ and χ_R^u, χ_R^s be the unit length eigenvectors corresponding to the positive (negative) eigenvalue of \mathcal{B}_λ .

Consider the linearized equation

$$\dot{y} = D_1 F(q_\lambda(t), 0, \lambda)y. \quad (2)$$

For $t \rightarrow \pm\infty$, the behavior of its solutions is qualitatively the same as the behavior of the solutions of

$$\dot{y} = D_1 F(p, 0, \lambda)y.$$

It follows that (2) has a unique solution, y_λ such that $\lim_{t \rightarrow -\infty} \frac{y_\lambda(t)}{|y_\lambda(t)|} = \chi_R^u$ and $|y_\lambda(0)| = 1$. Then, generically [1], $\lim_{t \rightarrow \infty} \frac{y_\lambda(t)}{|y_\lambda(t)|} = \pm\chi_R^u$.

Definition 1.1 *If $\lim_{t \rightarrow \infty} \frac{y_\lambda(t)}{|y_\lambda(t)|}$ points in the opposite directions for $\lambda > 0$ and $\lambda < 0$ then we call the point $(\alpha, \lambda) = (0, 0)$ a homoclinic twist point.*

The point of transition $(\alpha, \lambda) = (0, 0)$ can be characterized as follows. Let W^u (W^s) denote the unstable (resp. stable) manifold of p for $\alpha = 0, \lambda = 0$. Let $T_a W^u$ ($T_b W^s$) denote the tangent space to W^u at the point $a \in W^u$ (resp. the tangent space to W^s at the point $b \in W^s$). The invariance of S implies that at the point $(\alpha, \lambda) = (0, 0)$ the vector χ_R^u is transported in forward time from $-\infty$ to $+\infty$ by the linearized equation into the direction of χ_R^s . We conclude that the homoclinic orbit $q_0(t)$ is *degenerate* in the sense that

$$(H_5) \dim T_{q_0(0)} W^s \cap T_{q_0(0)} W^u = 2.$$

As an example, we consider the following system of two coupled Josephson junctions in the case of a capacitive load written in symmetric coordinates [1]:

$$\begin{aligned} \dot{s}_1 &= s_2, \\ \dot{s}_2 &= \frac{1}{3+\beta} (I - s_2 - \sin(s_1) \cos(r_1)), \\ \dot{r}_1 &= r_2, \\ \dot{r}_2 &= -\frac{1}{\beta} (r_2 + \sin(r_1) \cos(s_1)). \end{aligned} \quad (3)$$

The fixed point subspace of Z_2 is $S = \{(s_1, s_2, r_1, r_2) \in \mathbb{R}^4 : r_1 = r_2 = 0\}$ and $p = (\pi - \sin^{-1}(I), 0, 0, 0)$ is a hyperbolic equilibrium.

D.G.Aronson, M.Golubitsky, M.Krupa [1] proved a theorem about existence of homoclinic orbits of (3) in S and showed that after an appropriate change in the parameters and the state variables this system satisfies the hypotheses $H_1 - H_5$.

2 The Algorithm and the Theorem

2.1 Continuation of codimension-one homoclinic orbits

Generic homoclinic orbits have been computed using boundary-value methods. Such methods truncate the homoclinic problem to a finite interval with certain boundary conditions at the end points.

We consider first (1). We wish to find all values of (α, λ) for which there is an orbit homoclinic to p . A homoclinic solution can be thought of as the solution of an infinite-interval boundary value problem. The boundary conditions are $x(\pm\infty) = p$. However, it is also a solution of a finite-interval boundary value problem. Let $W_{loc}^s(p, \alpha, \lambda)$ ($W_{loc}^u(p, \alpha, \lambda)$) denote the local stable (unstable) manifold of p , and let $J = [-T_1, T_2]$, $T_1, T_2 > 0$. Then a solution of (1) in S and homoclinic to p satisfies the boundary conditions $x(-T_1) \in W_{loc}^u(p, \alpha, \lambda) \cap S$, $x(T_2) \in W_{loc}^s(p, \alpha, \lambda) \cap S$. Since a phase shift of the solution is again a solution, we must add a phase condition to resolve this redundancy. This is achieved by adding the condition in the form of a functional, defined on the solution x , $\Phi_J(x, \alpha, \lambda) = 0$, where $\Phi_J : C^0(J, \mathbb{R}^4) \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. Here we use the following phase condition: we fix the crosssection $\Sigma = q_0(0) + \dot{q}_0(0)^\perp$, which is locally transverse to the flow for all $\lambda \in \Lambda$ and all small α and require that the solution intersects Σ at time zero. We obtain the following finite-interval boundary value problem denoted by P_1 :

$$P_1 \begin{cases} \dot{x}(t) &= F(x(t), \alpha, \lambda), t \in J, \\ x(-T_1) &\in W_{loc}^u(p, \alpha, \lambda) \cap S, \\ x(T_2) &\in W_{loc}^s(p, \alpha, \lambda) \cap S, \\ x(0) &\in \Sigma. \end{cases}$$

Given λ , and assuming $H_1 - H_4$, a solution $(q_\lambda^J, \alpha_\lambda)$ of P_1 is locally unique in

$C^1(J, \mathbb{R}^n) \times \mathbb{R}$. Our assumptions imply that $\alpha_\lambda = 0$ for $\lambda \in \Lambda$. We can assume without loss of generality that $q_\lambda^J(t) = q_\lambda(t)$, $t \in [-T_1, T_2]$ due to the fact that the phase condition is given in the form of a functional independent of J and (α, λ) .

In numerical computations, the local stable and unstable manifolds of p must be approximated, say by $\tilde{W}^s(p, \alpha, \lambda)$ and $\tilde{W}^u(p, \alpha, \lambda)$. One then obtains a boundary value problem denoted by P_2 :

$$P_2 \begin{cases} \dot{x}(t) &= F(x(t), \alpha, \lambda), \quad t \in J, \\ x(-T_1) &\in \tilde{W}^u(p, \alpha, \lambda) \cap S, \\ x(T_2) &\in \tilde{W}^s(p, \alpha, \lambda) \cap S, \\ x(0) &\in \Sigma. \end{cases}$$

In order to compare solutions of the problems P_1 and P_2 we can apply the following theorem. Let $\epsilon > 0$ and $\sigma_S^s, \sigma_R^s < 0$ and $\sigma_S^u, \sigma_R^u > 0$ be such that $\forall \lambda \in \Lambda$, $\sigma_S^s(\lambda) + \epsilon \leq \sigma_S^s$, $\sigma_R^s(\lambda) + \epsilon \leq \sigma_R^s$, $\sigma_S^u(\lambda) - \epsilon \geq \sigma_S^u$ and $\sigma_R^u(\lambda) - \epsilon \geq \sigma_R^u$.

Theorem 2.1 [Theorem 3.3 in [11]] *Let $H_1 - H_4$ hold and $\epsilon > 0$. If T_1, T_2 are sufficiently large then P_2 has a unique solution $(\tilde{q}_\lambda^J, \tilde{\alpha}_\lambda^J)$. There is a constant C which does not depend on J and λ , such that the following estimates hold*

$$\begin{aligned} |\tilde{\alpha}_\lambda| &\leq C(e^{-(-\sigma_S^s + 2\sigma_S^u)T_1} + e^{-(-2\sigma_S^s + \sigma_S^u)T_2}), \\ |\tilde{q}_\lambda^J(t) - q_\lambda^J(t)| &\leq C(e^{-2\sigma_S^u T_1} + e^{2\sigma_S^s T_2}). \end{aligned} \quad (4)$$

Remark: The independence of C of λ is a consequence of H_2 .

2.2 Detection of the homoclinic twist point

First, we compute a branch of symmetric homoclinic orbits. Then, we detect the homoclinic twist point using a test function which is monitored along this branch. In order to detect the homoclinic twist point, the truncated problem should be formulated in such a way that it can be continued through the degenerate point.

Consider the unique solution $u_\lambda(t)$ of the boundary value problem

$$\begin{cases} \dot{u} &= \mathcal{B}_\lambda(q_\lambda(t))u, \\ \lim_{t \rightarrow -\infty} \frac{u(t)}{|u(t)|} &= \chi_R^u, \\ |u(0)| &= 1, \end{cases} \quad (5a)$$

and the unique solution $w_\lambda(t)$ of the boundary value problem

$$\begin{cases} \dot{w} &= \mathcal{B}_\lambda(q_\lambda(t))w, \\ \lim_{t \rightarrow \infty} \frac{w(t)}{|w(t)|} &= \chi_R^s \\ |w(0)| &= 1. \end{cases} \quad (5b)$$

It follows from (5a) and (5b) that

$$w_\lambda(t) \in T_{q_\lambda(t)}W^s(p, 0, \lambda), \quad t \in \mathbb{R}_+, \quad (6a)$$

$$u_\lambda(t) \in T_{q_\lambda(t)}W^u(p, 0, \lambda), \quad t \in \mathbb{R}_-. \quad (6b)$$

By H_5 , at the twist point $u_0(0) = \pm w_0(0)$.

The defining equation for the twist bifurcation is given then by

$$\langle u_\lambda(0), w_\lambda^\perp(0) \rangle = 0.$$

We make the following assumption about the mapping

$$\lambda \mapsto \tau(\lambda) = \langle u_\lambda(0), w_\lambda^\perp(0) \rangle$$

(H_6) $\tau'(0) \neq 0$.

Geometrically, (H_6) means that the ‘‘angle’’ between the stable and unstable manifolds, which is zero at $\lambda = 0$ has non-zero derivative there. Our numerical studies suggest that the system (3) satisfies H_6 .

In order to obtain an approximation for the $\tau(\lambda)$, we replace (5a)-(5b) by the truncation, where $(q_\lambda(t), 0)$ is replaced by the solution $(\tilde{q}_\lambda^J(t), \tilde{\alpha}_\lambda^J)$ to the perturbed truncated problem with $\mathcal{B}_\lambda^J(\tilde{q}_\lambda^J(t)) = D_1F(\tilde{q}_\lambda^J(t), \tilde{\alpha}_\lambda, \lambda)$. Let \tilde{y}_λ^J be the unique solution of the boundary value problem:

$$\begin{cases} \dot{y} &= \mathcal{B}_\lambda^J(\tilde{q}_\lambda^J(t))y, \quad t \in [-T_1, 0], \\ \frac{y(-T_1)}{|y(-T_1)|} &= \chi_R^u, \\ |y(0)| &= 1, \end{cases} \quad (7a)$$

and \tilde{z}_λ^J be the unique solution of the boundary value problem:

$$\begin{cases} \dot{z} &= \mathcal{B}_\lambda^J(\tilde{q}_\lambda^J(t))z, \quad t \in [0, T_2], \\ \frac{z(T_2)}{|z(T_2)|} &= \chi_R^s, \\ |z(0)| &= 1. \end{cases} \quad (7b)$$

Evaluating the solutions at $t = 0$, yields the following test function $\tau_J : \Lambda \rightarrow \mathbb{R}$:

$$\tau_J(\lambda) = \langle \tilde{y}_\lambda^J(0), \tilde{z}_\lambda^J(0)^\perp \rangle. \quad (8)$$

The following theorem shows that the test function is well defined for both the original and the perturbed boundary value problems.

Theorem 2.2 *Suppose that $H_1 - H_6$ hold. For $\lambda \in \Lambda$, let q_λ^J be solution of the problem P_1 . Suppose we use linear approximations to $W_{loc}^s(p, \alpha, \lambda)$ and $W_{loc}^u(p, \alpha, \lambda)$ denoted by $E^s(p, \alpha, \lambda)$ and $E^u(p, \alpha, \lambda)$. Then $\forall \epsilon > 0$ there is a ball whose radius is independent of $J = [-T_1, T_2]$ and λ about $(q_\lambda, 0)$ in $C^0(J, \mathbb{R}^2) \times \mathbb{R}$, and there are two positive constants m and M such that for T_1, T_2 sufficiently large:*

- (i) $(q_\lambda^J(t), 0)$ is the unique solution of (P_1) in that ball;
- (ii) the perturbed problem (P_2) has a unique solution $(\tilde{q}_\lambda^J(t), \tilde{\alpha}_\lambda)$ in that ball;
- (iii) $\exists \tilde{\lambda}$ such that $(\tilde{q}_{\tilde{\lambda}}(t), \tilde{\alpha}_{\tilde{\lambda}})$ is the unique solution of (P_2) , that satisfies also $\tau_J(\tilde{\lambda}) = 0$. Moreover,

$$|\tilde{\lambda}| + |\tilde{\alpha}_{\tilde{\lambda}}| = O\left(e^{(-\sigma_R^u + \sigma_R^s + \epsilon)T_1} + e^{(-\sigma_R^u + \sigma_R^s + \epsilon)T_2} + e^{-(2\sigma_S^u - \epsilon)T_1} + e^{(2\sigma_S^s + \epsilon)T_2}\right),$$

- (iv) Furthermore, $\tilde{\lambda}$ introduced in (iii) is such that $\forall \lambda_1 \in [m\tilde{\lambda}, M]$ and $\forall \lambda_2 \in [-M, -m\tilde{\lambda}]$, $\tau_J(\lambda_1)\tau_J(\lambda_2) < 0$.

The (iv) from the Theorem has a meaning that the function τ_J changes its sign inside a “small” interval containing zero and does not change its sign outside the “small” interval in a “bigger” interval.

2.3 Two lemmas and the proof of the Theorem

In order to prove Theorem 2.2, we shall use estimates (4) from S.Schecter [11] and the roughness properties of exponential dichotomies.

Let $X_\lambda(t, s)$ be the principal matrix solution of the equation

$$\dot{u} = \mathcal{B}_\lambda(q_\lambda(t))u. \quad (9)$$

From assumption H_2 it follows that $X_\lambda(t, s)$ has an exponential dichotomy on $(-\infty, 0]$ with projection operators $P_\lambda^u(t)$ and $P_\lambda^s(t)$, $P_\lambda^u(t) = Id - P_\lambda^s(t)$. It implies that there exists a constant $K > 0$, such that $\forall \lambda \in \Lambda$, $X_\lambda(t, s)$ satisfies the following inequalities:

$$\begin{aligned} |X_\lambda(s, t)P_\lambda^u(t)| &\leq Ke^{-(\sigma_R^u + \epsilon/2)(t-s)}, & s \leq t \leq 0 \\ |X_\lambda(t, s)P_\lambda^s(s)| &\leq Ke^{(\sigma_R^s - \epsilon/2)(t-s)}, & s \leq t \leq 0. \end{aligned} \quad (10)$$

Likewise, $X_\lambda(t, s)$ has an exponential dichotomy on $[0; \infty)$ with the same constant K . Hence, for sufficiently large T_1 and some positive C_{T_1} , u_λ satisfies

$$u(-T_1) = C_{T_1} P_\lambda^u(-T_1) \chi_R^u, \quad (11)$$

where C_{T_1} is positive and bounded away from zero if T_1 is sufficiently large. This is a consequence of the fact that $P_\lambda^u(t) \rightarrow P_\lambda^u$, P_λ^u is the spectral projection of \mathcal{B}_λ corresponding to the eigenvalue $\sigma_R^u(\lambda)$, and $\mathcal{R}P_\lambda^u(t) = \text{span}\{u_\lambda(t)\}$ for $t \in (-\infty, 0]$ (see p. 106 [5]). Thus, if T_1 is sufficiently large, then

$$|P_\lambda^u(t) \chi_R^u| \geq \frac{1}{2}, \quad t \in (-\infty, -T_1]. \quad (12)$$

A similar statement holds for w_λ .

From (4) it follows that for some constants C_1, C_2 which do not depend on λ and J , the following estimate holds:

$$\begin{aligned} \|\mathcal{B}_\lambda(q_\lambda(t)) - \mathcal{B}_\lambda^J(\tilde{q}_\lambda^J(t))\| &\leq C_1(|q_\lambda^J(t) - \tilde{q}_\lambda^J(t)| + |\tilde{\alpha}_\lambda^J|) \\ &\leq C_2(e^{-2\sigma_s^u T_1} + e^{-2\sigma_s^s T_2}). \end{aligned} \quad (13)$$

Then we intend to use roughness properties of the exponential dichotomy to compare solutions of the linear equation

$$\dot{x} = \mathcal{B}_\lambda^J(\tilde{q}_\lambda^J(t))x$$

with those of equation (9).

We introduce $B(t)$ by:

$$B(t) = \begin{cases} D_1 F(\tilde{q}_\lambda^J(t), \tilde{\alpha}_\lambda^J, \lambda), & t \in [-T_1, 0], \\ D_1 F(\tilde{q}_\lambda^J(-T_1), \tilde{\alpha}_\lambda^J, \lambda), & t \in (-\infty, -T_1]. \end{cases}$$

Obviously $B(t) = \mathcal{B}_\lambda^J(\tilde{q}_\lambda^J(t))$ for $t \in [-T_1, 0]$, and for some constant C_3 independent of λ and J we have

$$\|B(t) - \mathcal{B}_\lambda(q_\lambda(t))\| \leq C_3(e^{-2\sigma_s^u T_1} + e^{-2\sigma_s^s T_2}), \quad t \in \mathbb{R}_-.$$

Let $\tilde{X}_\lambda(t, s)$ be for $t, s \in \mathbb{R}_-$ the principal matrix solution of the equation

$$\dot{y} = B(t)y.$$

If T_1, T_2 are sufficiently large then

$$\begin{aligned} e^{-2\sigma_s^u T_1} + e^{2\sigma_s^s T_2} &\leq \frac{\sigma_R^u}{4C_3 K^2}, \\ KC_3(e^{-2\sigma_s^u T_1} + e^{-2\sigma_s^s T_2}) &\leq \frac{\epsilon}{4}. \end{aligned}$$

Then it follows from the roughness theorem [6] that $\tilde{X}_\lambda(t, s)$ satisfies an exponential dichotomy on $(-\infty, 0]$ with projections $\tilde{P}_\lambda^u(t), \tilde{P}_\lambda^s(t)$ satisfying, for some $t \in (-\infty, 0]$, $\mathcal{N}(\tilde{P}_\lambda^u(t)) = \mathcal{R}(\tilde{P}_\lambda^s(t)) = \mathcal{N}(P_\lambda^u(t))$ and

$$\begin{aligned} |\tilde{X}_\lambda(s, t)\tilde{P}_\lambda^u(t)| &\leq K_1 e^{-\sigma_R^u(t-s)}, & s \leq t \leq 0 \\ |\tilde{X}_\lambda(t, s)\tilde{P}_\lambda^s(s)| &\leq K_1 e^{\sigma_R^s(t-s)}, & s \leq t \leq 0. \end{aligned} \quad (14)$$

with constant $K_1 \geq \frac{5K^2}{2}$. Moreover, there is a positive constant C_4 , which can be taken independently of J and λ such that

$$\|\tilde{P}_\lambda^u(t) - P_\lambda^u(t)\| \leq C_4(e^{-2\sigma_s^u T_1} + e^{2\sigma_s^s T_2}), \quad t \in \mathbb{R}_-. \quad (15)$$

Furthermore, it is a consequence of (12) that for T_1 sufficiently large

$$|\tilde{P}_\lambda^u(t)\chi_R^u| \geq \frac{1}{4}, \quad t \in (-\infty, -T_1]. \quad (16)$$

Applying similar arguments we conclude that $\tilde{X}_\lambda(t, s)$ has an exponential dichotomy also on $[0; \infty)$ with the same constants K_1, σ_R^u and σ_R^s .

Now we state a lemma in which solutions of (5a-5b) are compared with solutions of (7a-7b).

Lemma 2.3 *Let $H_1 - H_5$ hold, $J = [-T_1, T_2]$, T_1, T_2 sufficiently large, and \tilde{y}_λ^J , \tilde{z}_λ^J be solutions of (7a-7b) and u_λ , w_λ be solutions of (5a-5b). Then for some constant C , which does not depend on λ and J ,*

$$\begin{aligned} |u_\lambda(0) - \tilde{y}_\lambda^J(0)| &\leq C(e^{(-\sigma_R^u + \sigma_R^s)T_1} + e^{-2\sigma_S^u T_1} + e^{-2\sigma_S^s T_2}), \\ |w_\lambda(0) - \tilde{z}_\lambda^J(0)| &\leq C(e^{(-\sigma_R^u + \sigma_R^s)T_2} + e^{-2\sigma_S^u T_1} + e^{-2\sigma_S^s T_2}). \end{aligned}$$

Proof . We consider $t \leq 0$ and estimate both terms in the right hand side of the inequality

$$\begin{aligned} |u_\lambda(0) - \tilde{y}_\lambda^J(0)| &\leq \left| \tilde{y}_\lambda^J(0) - \frac{\tilde{X}_\lambda(0, -T_1)\tilde{P}^u(-T_1)\chi_R^u}{|\tilde{X}_\lambda(0, -T_1)\tilde{P}^u(-T_1)\chi_R^u|} \right| \\ &\quad + \left| u_\lambda(0) - \frac{\tilde{X}_\lambda(0, -T_1)\tilde{P}^u(-T_1)\chi_R^u}{|\tilde{X}_\lambda(0, -T_1)\tilde{P}^u(-T_1)\chi_R^u|} \right| \end{aligned} \quad (17)$$

We consider the first term. By definition we have

$$\begin{aligned} \tilde{y}_\lambda^J(0) &= \frac{\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u}{|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u + \tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^s(-T_1)\chi_R^u|} \\ &\quad + \frac{\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^s(-T_1)\chi_R^u}{|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u + \tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^s(-T_1)\chi_R^u|}. \end{aligned}$$

From (14) and (16), we see that

$$\begin{aligned} |\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u| &\geq \frac{1}{4K_1}e^{\sigma_R^u T_1}, \\ |\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^s(-T_1)\chi_R^u| &\leq K_1 e^{\sigma_R^s T_1}, \end{aligned}$$

and if T_1 is sufficiently large then

$$|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^s(-T_1)\chi_R^u| \leq \frac{1}{2}|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u|.$$

Then there is a constant C_1 such that

$$\left| \frac{\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^s(-T_1)\chi_R^u}{|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u + \tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^s(-T_1)\chi_R^u|} \right| \leq C_1 e^{(\sigma_R^s - \sigma_R^u)T_1}.$$

Similarly,

$$\begin{aligned} & \left| \frac{\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u}{|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u + \tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^s(-T_1)\chi_R^u|} - \frac{\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u}{|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u|} \right| \\ &= \left| \frac{|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u|}{|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u + \tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^s(-T_1)\chi_R^u|} - 1 \right| \\ &\leq C_1 e^{(\sigma_R^s - \sigma_R^u)T_1}. \end{aligned}$$

Thus, we have shown that there is a constant C_2 such that

$$\left| \tilde{y}_\lambda^J(0) - \frac{\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u}{|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u|} \right| \leq C_2 e^{(\sigma_R^s - \sigma_R^u)T_1}. \quad (18)$$

Now we estimate the second term in the right hand side of (17). Using (15) and (11) we obtain that

$$\left| \frac{\tilde{P}_\lambda^u(-T_1)\chi_R^u}{|\tilde{P}_\lambda^u(-T_1)\chi_R^u|} - \frac{u_\lambda(-T_1)}{|u_\lambda(-T_1)|} \right| \leq C_4 (e^{-2\sigma_S^u T_1} + e^{2\sigma_S^s T_2})$$

and, hence, for T_1, T_2 sufficiently large

$$0 < \left\langle \frac{\tilde{P}_\lambda^u(-T_1)\chi_R^u}{|\tilde{P}_\lambda^u(-T_1)\chi_R^u|}, \frac{u_\lambda(-T_1)}{|u_\lambda(-T_1)|} \right\rangle. \quad (19)$$

Since $\text{span}\{\tilde{X}_\lambda(t, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u\} = \mathcal{R}\tilde{P}_\lambda^u(t)$ and $\text{span}\{u_\lambda(t)\} = \mathcal{R}P_\lambda^u(t)$, (15) implies that

$$\langle \tilde{X}_\lambda(t, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u, u_\lambda(t) \rangle \neq 0$$

for all $t \in [-T_1, 0]$. Thus, (19) implies by continuity of solutions that

$$\langle \tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u, u_\lambda(0) \rangle > 0$$

and since $\mathcal{R}\tilde{P}_\lambda^u(0)$ and $\mathcal{R}P_\lambda^u(0)$ are one-dimensional, we obtain using (15) that

$$\left| \frac{\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u}{|\tilde{X}_\lambda(0, -T_1)\tilde{P}_\lambda^u(-T_1)\chi_R^u|} - u_\lambda(0) \right| \leq C_4 (e^{-2\sigma_S^u T_1} + e^{2\sigma_S^s T_2}). \quad (20)$$

Combining (18) and (20), we obtain for some constant C that

$$|u_\lambda(0) - \tilde{y}_\lambda^J(0)| \leq C(e^{(-\sigma_R^u + \sigma_R^s)T_1} + e^{-2\sigma_S^u T_1} + e^{2\sigma_S^s T_2}).$$

Similarly we obtain

$$|w_\lambda(0) - \tilde{z}_\lambda^J(0)| \leq C(e^{(-\sigma_R^w + \sigma_R^s)T_2} + e^{-2\sigma_S^w T_1} + e^{-2\sigma_S^s T_2}),$$

which completes the proof. ■

We shall use the following lemma to solve $\tau_J(\lambda) = 0$.

Lemma 2.4 *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be C^2 and $\varphi_J : \mathbb{R} \rightarrow \mathbb{R}$ be C^0 , $0 < b < 1$, $L \geq 0$ and $\Lambda = [-\frac{b}{L}, \frac{b}{L}]$. Assume that*

- (i) $\varphi(0) = 0$,
- (ii) $|\varphi''(\lambda)| \leq L$ for $\lambda \in \Lambda$,
- (iii) $|\varphi'(0)| \geq b > 0$,
- (iv) $|\varphi_J(\lambda) - \varphi(\lambda)| \leq \delta < \frac{b^2}{4L}$, $0 < \delta$, $\lambda \in \Lambda$.

Then there is $\tilde{\lambda}$, $|\tilde{\lambda}| \leq \frac{4\delta}{b}$, such that $\varphi_J(\tilde{\lambda}) = 0$, and $\varphi_J(\lambda) \neq 0$ for $\lambda \in [-\frac{b}{L}, -\frac{4\delta}{b}] \cup [\frac{4\delta}{b}, \frac{b}{L}]$.

Proof. From (i)-(iii), it follows that for $\lambda \in \Lambda$, $|\varphi(\lambda)| \geq |\lambda|(b - |\lambda|L/2)$. By (iii), $\varphi(\lambda)$ has different signs for $\lambda \in [-\frac{b}{L}, 0)$ and for $\lambda \in (0, \frac{b}{L}]$. It follows from (iv) that $|\varphi_J(\lambda)| \geq |\lambda|(b - |\lambda|L/2) - \delta$, and that if $\lambda \in [\frac{4\delta}{b}, \frac{b}{L}]$ or $\lambda \in [-\frac{b}{L}, -\frac{4\delta}{b}]$ then $|\varphi_J(\lambda)| \geq \delta > 0$. Moreover, $\varphi_J(\lambda)$ has different sign in these two intervals. ■

Proof of Theorem 2.2. The first two statements of Theorem 2.2 follow from Theorem 2.1.

To prove (iii) we apply Lemma 2.4 with $\varphi = \tau$ and $\varphi_J = \tau_J$. By H_5 , $\tau(0) = 0$. The mapping $\lambda \mapsto D_1 F(q_\lambda(t), 0, \lambda)$ is C^2 . Consequently, u_λ and w_λ are C^2 in λ , hence we can choose the constant L in Lemma 2.4 as $L = \sup_{\lambda \in \Lambda} \{\tau''(\lambda)\}$.

Hypothesis H_6 gives that assumption (iii) in Lemma 2.4 is satisfied, if we choose $b = \tau'(0)$. From Lemma 2.3, we obtain there is a constant C such that

$$|\tau_J(\lambda) - \tau(\lambda)| \leq C(e^{(-\sigma_R^u + \sigma_R^s)T_1} + e^{-2\sigma_S^u T_1} + e^{2\sigma_S^s T_2}).$$

If we choose $\delta = C(e^{(-\sigma_R^u + \sigma_R^s)T_1} + e^{-2\sigma_S^u T_1} + e^{2\sigma_S^s T_2})$ then we satisfy condition (iv) from Lemma 2.4 if we take T_1 and T_2 sufficiently large.

Part (iii) of Theorem 2.2 follows then from the estimate for $\tilde{\lambda}$ (Lemma 2.4) and the estimate (4) for $\tilde{\alpha}_\lambda^J$. Finally, (iv) follows from Lemma 2.4 with $m = \frac{4\tau_J(0)}{\tau'(0)}$ and $M = \frac{\tau'(0)}{L}$. \blacksquare

Remark: one may ask how the accuracy in the computation of the eigenvectors χ_R^u and χ_R^s influences the error in computations of the test function τ_J . Fortunately, instead of the vectors χ_R^u and χ_R^s in the formulation of the boundary value problems (7a) and (7b), we can take any unit vectors χ_1 and χ_2 , satisfying the conditions $|P_\lambda^u(-t)\chi_1| > 0$, $|P_\lambda^s(t)\chi_2| > 0$, $t \in [T, \infty)$, $\forall \lambda \in \Lambda$ and for T sufficiently large. These conditions are similar to (12).

3 Numerical results

Finally, we report on the numerical simulations obtained for system (3). We have used the continuation package AUTO86 [8]. First, we solve for an orbit satisfying the boundary conditions $x(-T_1) = \delta\chi^u$, where $\delta > 0$ is a small parameter using the homotopy as described in [4]. We begin with continuation in T_1 starting from $T_1 = 0.1$, $\delta = 0.01$ using approximate parameter values for a homoclinic orbit, namely $\beta = 0.125$, $I = 0.6$. At $T_1 = 12.54027$, the output shows that $x(0-) \in \Sigma = \{(s_1, s_2) \in \mathbb{R}^2, s_1 = 3.446276\}$. Then, we solve for an orbit satisfying the boundary conditions $x(T_2) = \delta\chi^s$ and $x(0+) \in \Sigma$ again using the homotopy. At $T_2 = 8.32174$, the output shows that $x(0+) \in \Sigma$. We observe that there is a gap of -0.15 between $x(0-)$ and $x(0+)$ in Σ . An adjustment in I eliminates the gap and produces a numerical solution of P_2 .

At $\beta = 0.125$, $I = 0.6635044$, $T_1 = 12.86013$, $T_2 = 8.481780$ we have a good approximation of a homoclinic orbit. Then we perform continuation in δ , which measures the distance from the points $x(-T_1)$ and $x(T_2)$ to the equilibrium $p(\lambda)$. We obtain at $\delta = 10^{-5}$ that $T_1 = 32.307$, $T_2 = 18.72196$, $\beta = 0.125$, $I =$

0.6635038. This numerical solution will be used to perform a two-parameter continuation in (β, I) .

We note that if the function $v(t)$ satisfies $\dot{v}(t) = \mathcal{B}_\lambda^J(\tilde{q}_\lambda^J(t))v(t)$ then $v_1(t) = v(t)e^{-\sigma t}$ is a solution of the modified variational equation

$$\dot{v}_1(t) = \mathcal{B}_\lambda^J(\tilde{q}_\lambda^J(t))v_1(t) - \sigma v_1(t).$$

We exploit this to compensate for the exponential decay of the norms of the solutions of (7a), (7b) as $t \rightarrow \pm\infty$ and avoid loss of precision due to round off errors in the numerical experiments adjusting the parameter σ . We add the modified linear variational equation to the set of equations and solve them numerically. Then we perform a two-parameter continuation in (β, I) while monitoring the test function τ_J . Output shows the zero of the test function (8), which gives the location of the twist bifurcation:

$$(\beta^*, I^*) = (0.4198656, 0.6389036).$$

It turns out that at these critical parameter values the symmetric eigenvalues are $\alpha_S = 0.350$ and $\beta_S = -0.642$. The eigenvalues in the asymmetric direction are $\alpha_R = 0.612$ and $\beta_R = -2.999$. This is in agreement with the numbers as given in [1].

In Figure 1, $|\beta - \beta^*| + |I - I^*|$ is plotted in the logarithmic scale versus T_1 , fixing T_2 at 30.215. The slope is equal to -1.345 differing from the slope -0.7 , which is the expected rate of convergence of the numerical approximation to the homoclinic orbit, by -0.645 .

In Figure 3, $|\beta - \beta^*| + |I - I^*|$ is plotted in the logarithmic scale versus T_2 , fixing T_1 at 30.38638. The slope is equal to -1.63 differing from the slope -1.284 , which is the expected rate of convergence of the numerical approximation to the homoclinic orbit, by -0.346 . Thus these examples suggest that the true convergent rate might be

$$|I^* - \tilde{I}| + |\beta^* - \tilde{\beta}| = O\left(e^{(-\alpha_R + \beta_R)T_1} + e^{(-\alpha_R + \beta_R)T_2} + e^{(-2\alpha_S + \beta_S)T_1} + e^{(2\beta_S - \alpha_S)T_2}\right).$$

and estimate (15) can be improved. We conjecture that this can be done using a better pointwise estimate in (13).

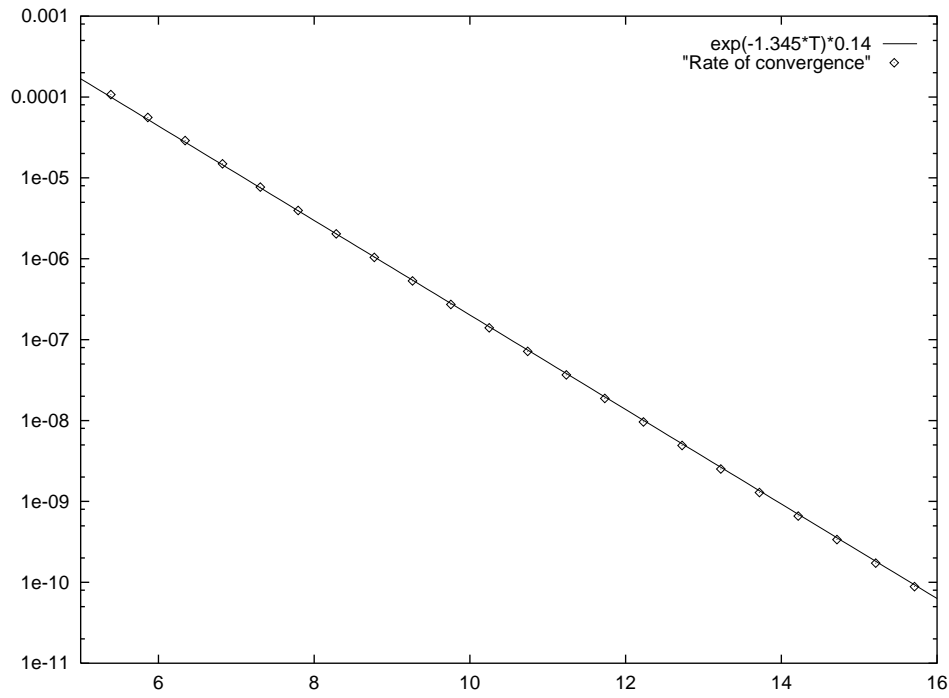


Figure 1: Graph of $|\beta^* - \beta| + |I^* - I|$ versus T_1 with the logarithmic scale in the vertical axis, $T_2 = 30.215$.

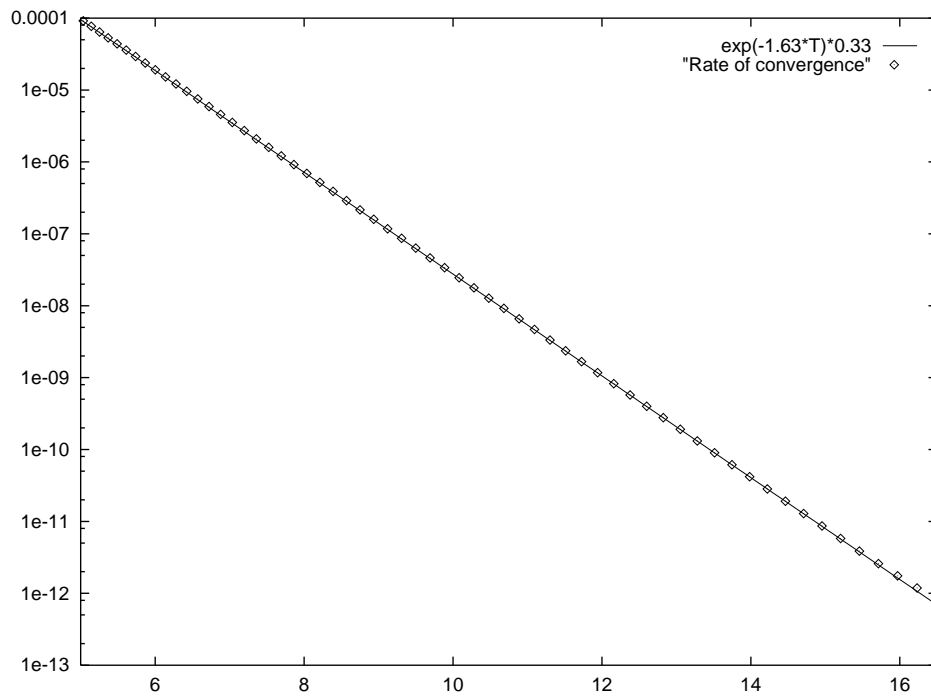


Figure 2: Graph of $|\beta^* - \beta| + |I^* - I|$ versus T_2 with the logarithmic scale in the vertical axis, $T_1 = 30.38638$.

In the next experiment, we examine the following system depending on three parameters (β, I, γ) :

$$\begin{aligned}
\dot{s}_1 &= s_2, \\
\dot{s}_2 &= \frac{1}{3+\beta}(I - s_2(1 + \gamma \cos(s_1)) - \sin(s_1) \cos(r_1)), \\
\dot{r}_1 &= r_2, \\
\dot{r}_2 &= \frac{1}{\beta}(\gamma s_2 \sin(s_1) \sin(r_1) - r_2(1 + \gamma \cos(s_1) \cos(r_1)) - \sin(r_1) \cos(s_1)).
\end{aligned}
\tag{21}$$

This equations results from a resistively and capacitively shunted junction model, see [7], here written in the symmetric coordinates. The terms multiplying γ model the flow of unpaired electrons through the junction. At $\gamma = 0$ equations (21) coincide with equations (3). We performed the continuation of the homoclinic twist bifurcations in the space of three parameters (β, I, γ) for $\gamma \in [-0.261, 13.48]$ (see Figure 3).

4 Conclusions

This paper presents a method for detecting homoclinic twist points for Z_2 -symmetric vector fields. The resulting AUTO driver allows to continue twisted homoclinic orbits satisfying certain conditions in three parameters. Using this driver, a homoclinic twist point has been detected in the system of two coupled Josephson junctions.

As it has been noted in [1], the occurrence of twist points is independent of the number of the junctions in the array. The representation of the group S_n , the symmetry group of the equations governing the evolution of an array of n globally coupled junctions in \mathbb{R}^{2n} , admits decomposition in the direct sum of irreducible representations. The $(n - 1)$ -dimensional irreducible representation and one-dimensional identity representation have multiplicity two in this decomposition. Thus in the symmetric coordinates the matrix of the variational equation along the symmetric homoclinic solution $q_\lambda(t)$ is a block diagonal matrix

$$\begin{pmatrix}
\mathcal{A}_\lambda(q_\lambda(t)) & & & 0 \\
& \mathcal{B}_\lambda(q_\lambda(t)) & & \\
0 & & \ddots & \\
& & & \mathcal{B}_\lambda(q_\lambda(t))
\end{pmatrix},$$

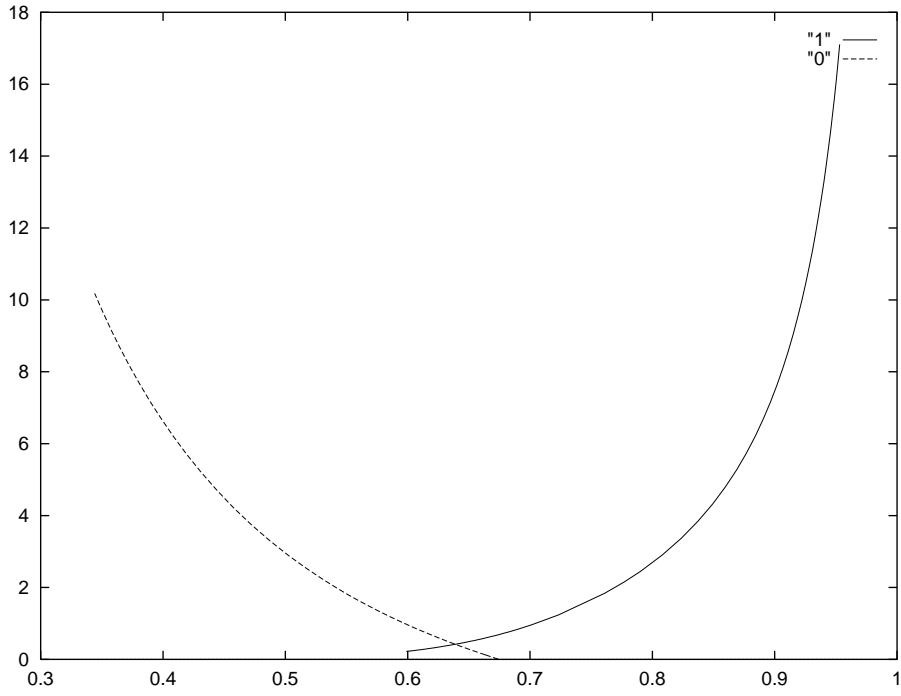


Figure 3: Bifurcation diagram for equation (21) in (I, β) -plane. Along curve “0” there is a branch of symmetric homoclinic connections, at $\gamma = 0$. Homoclinic twist bifurcations occur along curve “1”, the parameter γ is changing in the interval $[-0.261, 13.48]$.

where $\mathcal{A}_\lambda(q_\lambda(t))$, $\mathcal{B}_\lambda(q_\lambda(t))$ are 2×2 blocks. Hence, we can use the same test function τ_J for detecting homoclinic twist points in the system of n -coupled Josephson junctions.

The general case for a smooth vector field $F : \mathbb{R}^n \times \mathbb{R}^2 \rightarrow \mathbb{R}^n$ is C^3 equivariant with respect to the action of some finite group $\Gamma \in O(n)$, and admitting a branch of symmetric homoclinic orbits, requires additional treatment. The codimension-two homoclinic twist bifurcation occurs only if the subspace $\dot{q}_0(0)^\perp \cap \dim T_{q_0(0)}W^s \cap T_{q_0(0)}W^u$ is an irreducible subspace and it has multiplicity two in the decomposition of the state space into irreducible subspaces. Then, $D_1F(q_\lambda, 0, \lambda)$ has a block-diagonal structure in the corresponding coordinates, and we can supply a test function using solutions of the corresponding linear variational equation.

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