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Packing a bin online to maximize the total number of items

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PACKING A BIN ONLINE TO MAXIMIZE THE TOTAL NUMBER OF ITEMS

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Abstract. A bin of capacity 1 and a finite sequence σ of items of sizes $a_1, a_2, \ldots$ are considered, where the items are given one by one without information about the future. An *online algorithm* A must irrevocably decide whether or not to put an item into the bin whenever it is presented. The goal is to maximize the number of items collected. A is f-*competitive* for some function f if $n^*(\sigma) \le f(n_A(\sigma))$ holds for all sequences σ, where n^* is the (theoretical) optimum and n_A the number of items collected by A.

A necessary condition on f for the existence of an f-competitive (possibly randomized) online algorithm is given. On the other hand, this condition is seen to guarantee the existence of a deterministic online algorithm that is "almost" f-competitive in a well-defined sense.

1. Introduction and Main Results

We consider a binpacking problem in the following setting. There is one bin of capacity 1 into which items i are to be packed. The goal is to fill the bin with as many items as possible. The items are assumed to be presented in a finite sequential order $1\,2\,\ldots\,k$ and have sizes $a_1, a_2, \ldots, a_k$.

We want to analyze the performance of an *online algorithm* A for this problem that must, each time an item i of size a_i is presented, irreversibly decide whether or not i is to be put into the bin. The algorithm A is assumed to know the past but not the future at any moment a decision must be made, *i.e*, A knows all the items seen so far and the current content of the bin – but is without any prior knowledge about the total number k of items that will be presented and the sizes of the items not yet seen at stage i. We make no assumption on the way A reaches a decision. In other words, we also allow the decisions to be randomized.

If such an online algorithm A is applied to the sequence $\sigma = a_1, a_2, \ldots, a_k$ of items, we denote by $n_A(\sigma)$ the number of items collected by A from σ. If A is randomized, $n_A(\sigma)$ is the expected number of items collected.

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It turns out that our problem here can be analyzed with similar methods as employed for a certain seemingly quite different online interval scheduling problem (see Lipton and Tomkins [1994] and Faigle *et al.* [1996]). In contrast to the latter, however, where randomization produces a logarithmic improvement, we will see that randomization does not help much in our current setting.

Let $n^* = n^*(\sigma)$ be the maximal number of items from the sequence σ that fit into the bin. One cannot expect a constant bound on the ratio n^*/n_A for any online algorithm A (see Theorem 1.1 below). We try to relate n^* and n_A by means of functions $f : \mathbb{R} \to \mathbb{R}$.

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an arbitray function. We say that the online algorithm A is *f-competitive* if

$$n^* \leq f(n_A)$$

holds. (Intuitively, $n_A(\sigma)$ is "f-large" with respect to $n^*(\sigma)$ for all possible input sequences σ).

Since n^* is always an integer, we may assume that f is integer-valued (otherwise we round down to the nearest integer). Also note that $f(n_A) \geq n_A$ necessarily must hold if the algorithm A is f-competitive.

Our main results can now be stated. We first present a necessary conditions on f for the existence of a (possibly randomized) f-competitive online algorithm.

Theorem 1.1. *Let $f : \mathbb{R}_+ \to \mathbb{N}_0$ be non-decreasing.*

(a) *If there exists some randomized f-competitive algorithm A, then*

$$\sum_{n \in \mathbb{N}} \frac{1}{1 + f(n)} \leq 1 .$$

(b) *If there exists some deterministic f-competitive algorithm A, then*

$$\sum_{n \in \mathbb{N}_0} \frac{1}{1 + f(n)} \leq 1 .$$

Next we show that the conditions on the function f in Theorem 1.1 are sufficient for the existence of "almost" f-competitive deterministic algorithms.

Theorem 1.2. *Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfy* $\sum_{n \in \mathbb{N}_0} \frac{1}{1 + f(n)} \leq 1$.

Then there exists a deterministic online algorithm A_f such that

$$n^* \leq n_{A_f} + f(n_{A_f}) .$$

As a consequence, Theorem 1.2 guarantees the existence of a deterministic $2f$-competitive algorithm A_f if $f(n) \geq n$ holds for all $n \in \mathbb{N}$. Often, however, the situtation is much better.

Example: Let $g(n) = n^{(1+\epsilon)}$. Then the function $f(n) = (2+\epsilon^{-1})g(n)$ satisfies the condition of Theorem 1.2. On the other hand, writing

$$n + f(n) \;=\; (1+\delta(n))f(n)\;,$$

one sees that $\delta(n)$ approaches 0 exponentially fast as n gets large.

We say that the online algorithm A is *almost f-competitive* if there exists a constant $N \in \mathbb{N}$ such that

$$n^* \;\leq\; f(n_A)$$

whenever $n^* \geq N$.

Corollary 1.1. *Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be non-decreasing such that*

$$\sum_{n\geq 0} \frac{1}{1+f(n)} \;<\; \infty\;.$$

Then there exists an almost f-competitive deterministic online algorithm A_f.

How much can randomization generally help? Assume that there is some f-competitive randomized algorithm, where f is non-decreasing. By Theorem 1.1, we know that $\sum_{n\geq 1}[1+f(n)]^{-1} \;\leq\; 1$ must hold. Hence

$$\sum_{n\geq 0} \frac{1}{1+f(n)} \;<\; \infty$$

holds, which implies the existence of a deterministic almost f-competitive online algorithm by Corollary 1.1.

On the other hand, we also observe that

$$\sum_{n\geq 0} \frac{1}{2+2f(n)} \;\leq\; 1$$

holds, which implies the existence of a deterministic $(4f+2)$-competitive online algorithm by Theorem 1.2 (assuming, w.l.o.g., $n \leq 2f(n)+1$ to be satisfied).

2. Proofs

For the proof of Theorem 1.1, let $f : \mathbb{R}_+ \to \mathbb{N}_0$ be nondecreasing and assume that A is a feasible (possibly randomized) f-competitive online algorithm. To prove (a), we must show that

$$\sum_{n\geq 1} \frac{1}{1+f(n)} \;\leq\; 1$$

holds.

We define an infinite collection of finite sequences $\sigma^{(i)}$, $i \in \mathbb{N}_0$, relative to g as follows:

$$\begin{array}{lcll} \sigma^{(0)} & := & 1+f(0) & \text{successive items, each of size } [1+f(0)]^{-1} \\ \sigma^{(1)} & := & \sigma^{(0)} & \text{followed by } 1+f(1) \text{ items, each of size } [1+f(1)]^{-1} \\ \vdots & & & \\ \sigma^{(i)} & := & \sigma^{(i-1)} & \text{followed by } 1+f(i) \text{ items, each of size } [1+f(i)]^{-1} \\ \vdots & & & \end{array}$$

We claim that the expected number ϵ_i of items selected by A from $\sigma^{(i)}$ is at least i for all $i \geq 0$.

To establish the claim, observe first that

$$n^*(\sigma^{(i)}) = 1 + f(i)$$

must hold. So, because A is f-competitive, we have

$$f(i) + 1 \;\leq\; f(\epsilon_i)\ .$$

Since f is non-decreasing, the latter inequality yields $\epsilon_i > i$.

To be more precise, let ρ_i denote the expected number of items of size $[1+f(i)]^{-1}$ that A selects when applied to $\sigma^{(i)}$. Because A is an online algorithm, this number remains the same when A is applied to any $\sigma^{(j)}$ with $j \geq i$. Hence we have

$$\epsilon_i = \rho_0 + \rho_1 + \ldots + \rho_i \;>\; i\ .$$

Due to the feasibility of A, we must have

$$\rho_0 \frac{1}{1+f(0)} + \rho_1 \frac{1}{1+f(1)} + \ldots + \rho_i \frac{1}{1+f(i)} \;\leq\; 1$$

for all $i \geq 0$. Hence we obtain the inequality

$$\sum_{i\geq 0} \rho_i \frac{1}{1+f(i)} \;\leq\; 1.$$

In view of $\rho_0 + \rho_1 + \ldots + \rho_i \;\geq\; i$ and the monotonicity of f, it is now straightforward to see that

$$\sum_{i\geq 1} \frac{1}{1+f(i)} \;\leq\; \sum_{i\geq 0} \rho_i \frac{1}{1+f(i)} \;\leq\; 1$$

must hold, which proves (a).

If A is deterministic, n_A is integer-valued. So our argument above implies the stronger inequality $\epsilon_i \geq i+1$. Hence we can conclude

$$\sum_{i\geq 0} \frac{1}{1+f(i)} \;\leq\; \sum_{i\geq 0} \rho_i \frac{1}{1+f(i)} \;\leq\; 1\;,$$

which yields (b).

We prove Theorem 1.2 by exhibiting a suitable deterministic algorithm A_f for every f satisfying the hypothesis of Theorem 1.2.

Consider the following (deterministic) online algorithm A_f with initializing step $A_f(0)$ and iterations $A_f(i)$, where b represents the current content of the bin and n the number of items in the bin:

$A_f(0)$: $b \leftarrow 0\;;\; s \leftarrow 0\;;\; n \leftarrow 0\;;$
$A_f(i)$: IF $b + a_i \;\leq\; s + [1+f(n)]^{-1}$ THEN
put i into the bin;
$b \leftarrow [b+a_i]\;;\; s \leftarrow [s + 1/f(n)]\;;\; n \leftarrow [n+1]\;;$

By our assumption on f, the algorithm A_f is clearly feasible (*i.e.*, A_f does not fill the bin to more than its capacity). Moreover, if A_f selected n_{A_f} items from the sequence σ, then all items not selected must have size strictly larger than $[1+f(n_{A_f})]^{-1}$. Since only (strictly) less than $f(n_{A_f})+1$ of such items fit into the bin, we deduce

$$n_{A_f} \;\leq\; n^* \;\leq\; n_{A_f} + f(n_{A_f})$$

and Theorem 1.2 is seen to hold.

It remains to prove Corollary 1.1. Thus assume that $f : \mathbb{N}_0 \to \mathbb{N}_0$ is non-decreasing and satisfies

$$\sum_{n\geq 0} \frac{1}{1+f(n)} \;<\infty\;.$$

Observe that the inequality $1+f(n) > 6n$ can be violated by only a finite number of integers $n \in \mathbb{N}$. Otherwise, if there were an infinite number of violating integers n_i, we could assume $n_{i+1} > 2n_i$ and thus obtain

$$\sum_{n\geq 0} \frac{1}{1+f(n)} \geq \frac{n_1}{6n_1} + \frac{n_2-n_1}{6n_2} + \frac{n_3-n_2}{6n_3} + \ldots \geq \frac{1}{6} + \frac{1}{12} + \frac{1}{12} + \ldots\;,$$

a contradiction to the assumed convergence of the series.

Define $g(n) := \lfloor f(n)/2 \rfloor$. Then

$$\sum_{n\geq 0} \frac{1}{1+g(n)} \;\leq\; \sum_{n\geq 0} \frac{2}{1+f(n)} \;<\; \infty\ .$$

By the foregoing, we can find $n_0 \in \mathbb{N}$ such that

$$\sum_{n\geq n_0} \frac{1}{1+g(n)} \;\leq\; \frac{1}{2}$$

holds and $g(n) > 2(n+1)$ is valid for every $n \geq n_0$.

We now define $\bar{g} : \mathbb{N}_0 \to \mathbb{N}_0$ via

$$\bar{g}(n) \;:=\; \begin{cases} g(n_0) & \text{for } 0 \leq n \leq n_0 \\ g(n) & \text{otherwise}\ . \end{cases}$$

By definition, we have

$$\sum_{n\geq 0} \frac{1}{1+\bar{g}(n)} \;\leq\; 1\ .$$

Hence Theorem 1.2 implies the existence of a deterministic $2\bar{g}$-competitive algorithm A. Let $N = 2g(n_0)+1$ and consider any $n^* \geq N$. Then we observe

$$2\bar{g}(n_0)+1 \;=\; 2g(n_0)+1 \;\leq\; n^* \;\leq\; 2\bar{g}(n_A)\ .$$

So the monotonicity of $\bar{g}$ yields $n_A \geq n_0$ and, therefore,

$$2\bar{g}(n_A) \;=\; 2g(n_A) \;\leq\; f(n_A)\ ,$$

which proves the Corollary.

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